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FRANCIANE DE BRITO VIEIRA

CONTROLLABILITY OF SOME NONLINEAR PDES AND DENSITY AND SPECTRUM OF MINIMAL SUBMANIFOLDS IN SPACE FORMS

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Tese apresentada ao Curso de Doutorado em Matemática do Programa de Pós-Graduação em Matemática do Centro de Ciências da Universidade Federal do Ceará, como requisito parcial à obtenção do título de doutor em Matemática. Área de Concentração: Análise

Orientador: Prof. Dr. José Fábio Bezerra Montenegro

Coorientador: Enrique Fernández-Cara

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À minha família.

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"O conhecimento do futuro faz parte do que você sabe."

(G. R.)

RESUMO

Na primeira parte desta tese tratamos dos sistemas 3D de Navier-Stokes e Boussinesq em um cubo. Nós provamos alguns resultados sobre a controlabilidade aproximada global por meio de controles de bordo que agem em uma parte da fronteira. Estes reultados são generalizações e variações de alguns resultados anteriores de Guerrero, Imanuvilov e Puel. Ainda na primeira parte da tese, nós provamos a controlabilidade nula local interna e de bordo de uma EDP parabólica 1D com difusão não linear. Aqui, as ferramentas principais são o teorema da função inversa de Liusternik e desigualdades de Carleman adequadas.

Na segunda parte desta tese, consideramos M^m subvariedades mínimas propriamente imersas em um espaço ambiente completo N^n adequadamente próximo a um espaço forma N^n_k de curvatura $-k \leq 0$. Estamos interessados na relação entre a função densidade $\Theta(r)$ de M^m e o espectro do operador Laplace-Beltrami. Em particular, provamos que se $\Theta(r)$ temum crescimento subexponencial (quando $k < 0$) ou bubpolinomial ($k = 0$) ao longo de uma sequência, então o espectro de M^m é o mesmo do espaço forma $\mathbb{N}^m_k$. Notavelmente, o resultado se aplica a soluções Anderson (suaves) do problema de Plateau no infinito sobre o espaço hiperbólico $\mathbb{H}^n$, independentemente da regularidade dos seus bordos. Nós também fornecemos uma condição simples sobre a segunda forma fundamental que garante que M tem densidade finita. Em particular, mostramos que subvariedades mínimas de $\mathbb{H}^n$ com curvatura total finita te densidade finita.

Palavras-chave: Controlabilidade nula. Controlabilidade aproximada. Sistema de Navier-Stokes. Sistema de Boussinesq. EDPs parabólicas não lineares. Subvariedades mínimas.

ABSTRACT

In the first part of this thesis we deal with the 3D Navier-Stokes and Boussinesq systems in a cube. We prove some results concerning the global approximate controllability by means of boundary controls which act in some part of the boundary. They are generalizations and variants of some previous results by Guerrero, Imanuvilov and Puel. Still in the first part of this Thesis, we prove the internal and boundary local null controllability of a 1D parabolic PDE with nonlinear diffusion. Here, the main tools are Liusternik's inverse function Theorem and appropriate Carleman estimates.

In the second part of this Thesis, we consider M^m minimal properly immersed submanifolds in a complete ambient space N^n suitably close to a space form N^n_k of curvature $-k \leq 0$. We are interested in the relation between the density function $\Theta(r)$ of M^m and the spectrum of the Laplace-Beltrami operator. In particular, we prove that if $\Theta(r)$ has subexponential growth (when $k < 0$) or sub-polynomial growth ($k = 0$) along a sequence, then the spectrum of M^m is the same as that of the space form $\mathbb{N}^m_k$. Notably, the result applies to Anderson's (smooth) solutions of Plateau's problem at infinity on the hyperbolic space $\mathbb{H}^n$, independently of their boundary regularity. We also give a simple condition on the second fundamental form that ensures M to have finite density. In particular, we show that minimal submanifolds of $\mathbb{H}^n$ with finite total curvature have finite density.

Keywords: Null controllability. Approximate controllability. Navier-Stokes system. Boussinesq system. Parabolic nonlinear PDEs. Minimal submanifolds.

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SUMÁRIO

1 INTRODUÇÃO

For a better understanding, this Thesis is divided in two parts. The first part is devoted to the controllability of some initial-boundary value problems for PDEs. The second part brings a study about the relation between the density function $\Theta(r)$ of a submanifold M^m, which is a minimal properly immersed submanifold in a complete ambient space N^n, and the spectrum of its Lapace-Beltrami operador.

About the first part, concerning control theory, we give now a very succinct, but important, prelude.

Roughly speaking, the idea of controllability may be formulated as follows. For an evolution system in a time interval $[0,T]$, the main concern is to act by means of a function v, called the control, in order to drive the solution to a desired state at the final time T. In this framework, we can deal with different notions of controllability, depending on the nature of the problem.

- We say that the system has the property of approximate controllability if, starting from an arbitrary initial state, the system solution can be driven arbitrarily close (with respect to a particular norm) to any desired state.
- We say that the system has the property of exact controllability if, starting from an arbitrary initial state, the system solution can be driven exactly to any desired state.
- We say that the system has the property of null controllability if, starting from an arbitrary initial state, the system solution can always be driven exactly to zero.
- We will say that the system has the property of exact controllability to the trajectories if, starting from an arbitrary initial state, the system solution can be driven exactly to any solution.

The controllability theory for evolution PDEs began to be developed in the $1960's$. The foundations of this theory were laid, among others, by Yegorov (1963), Russell (1973, 1987) and Fattorini (1975). At the time, some well known and well used and improved techniques were introduced. One of them was the moment method, which reduces the exact controllability question to a problem in the theory of exponential series. Also, the duality principle, which reduces the controllability problem to an observability problem concerning the adjoint equation, was introduced. For a general review of the state of the theory up to 1978 we recommend the paper (RUSSELL, 1987). Going on to the mid-$1980's$ and beyond, the interest in controllability theory happen to increased substantially. At this time, J.-L. Lions introduced the Hilbert Uniqueness

Method (HUM), which enables one to drive the solubility of the controllability problem for the original equation from the uniqueness theorem for the adjoint equation, (see (LIONS, 1988b; LIONS, 1988a)). Other important step in the development of controllability theory was made by A. V. Fursikov and O. Yu. Imanuvilov (1996), who used Carleman estimates to manage null controllability problems.

Concerning parabolic equations, we can mention the work of G. Lebeau and L. Robbiano (1995), dedicated to the linear heat equation, which combined the method of Russell, the properties of an integral transform and a Carleman estimate for elliptic equations to deduce the null controllability of the heat equation. On the other hand, the approximate controllability for semilinear heat PDEs was proved by C. Fabre, J.-P. Puel and E. Zuazua (1995).

In the context of the Navier-Stokes equations, Jacques-Louis Lions (1990), made a conjecture on the global, boundary and internal approximate controllability. Since then, the controllability of these equations and its variants has awaken the interest of many researchers, but, until the present moment, only partial results are known. In (1999), Fursikov and Imanuvilov proved a local result on the exact controllability to the C^{∞} trajectories of the Navier-Stokes equations by means of a Carleman inequality and the inverse function theorem. Some years later, E. Fernández-Cara, S. Guerrero, O. Yu Imanuvilov and J.-P. Puel (2004) improved this result, relaxing the regularity of the trajectories to L^{∞}. Some time later, inspired by (FERNÁNDEZ-CARA *et al.*, 2004; FURSIKOV; IMANUVILOV, 1999), Guerrero (2006) proves a local exact controllability result for the Boussinesq system. Then, several authors proved local exact controllability to the trajectories results for the N-dimensional Navier-Stokes and Boussinesq systems with a reduced number of scalar controls under some geometric conditions (see (FERNÁNDEZ-CARA *et al.*, 2006; CORON; GUERRERO, 2009; CORON J.-M., 2014)).

In the second part of this Thesis we deal with another knowledge area: geometry. Here we develop a study concerning a minimal properly immersed submanifold, denoted by M^m, in a complete ambient space N^n suitably close to a space form $\mathbb{N}^n_k$ of curvature $-k \leq 0$.

The Laplacian operator Δ acting on $C_0^{\infty}(M)$ has a unique self-adjoint extension to an unbounded operator acting on $L^2(M)$, also denoted by Δ. Since $-\Delta$ is positive and self-adjoint, we have its spectrum $\sigma(M)$ being the set of $\lambda \geq 0$ such that the operator $\Delta + \lambda I$ does not have a bounded inverse. In particular, Our focus is to study the spectrum $\sigma(M)$ of the Laplace-Beltrami operator $-\Delta$ on M (sometimes called spectrum of M) and its relationship with the density at infinity of M, that is, the limit as $r \to +\infty$ of the (monotone) quantity

$$\Theta(r) \doteq \frac{\mathrm{vol}(M \cap B_r)}{V_k(r)}, \tag{1.1}$$

where B_r indicates a geodesic ball of radius r in N^n and $V_k(r)$ is the volume of a geodesic ball of radius r in $\mathbb{N}_k^m$. Associated to $\Theta(r)$ we have the idea of finite density. We say that the submanifold M has finite density if

$$\Theta(+\infty) \doteq \lim_{r \to +\infty} \Theta(r) < +\infty.$$

In the literature, characterizations of the whole $\sigma(M)$ are known only in few special cases. Among them, we have the spectrum of the Euclidean space $\mathbb{R}^m$, and the hyperbolic space $\mathbb{H}_k^m$, for which, respectively, $\sigma(\mathbb{R}^m) = [0, \infty)$ and

$$\sigma(\mathbb{H}_k^m) = \left[\frac{(m-1)^2 k}{4}, +\infty\right), \tag{1.2}$$

The well-known Weyl's characterization for the case of the spectrum of a self-adjoint operator in a Hilbert space implies the following:

Lemma 1.0.1 *(DAVIES, 1995, Lemma 4.1.2) A number $\lambda \in \mathbb{R}$ lies in $\sigma(M)$ if and only if there exists a sequence of nonzero functions $u_j \in \mathrm{Dom}(-\Delta)$ such that*

$$\|\Delta u_j + \lambda u_j\|_2 = o\big(\|u_j\|_2\big) \qquad \text{as } j \to +\infty. \tag{1.3}$$

The approach to guarantee that $\sigma(M) = [c, +\infty)$, for some $c \geq 0$, usually splits into two parts. The first one is to show that $\inf \sigma(M) \geq c$ via, for instance, the Laplacian comparison theorem, and the second one is to produce a sequence like in lemma 1.0.1 for each $\lambda > c$. This step is accomplished by considering radial functions of compact support, and, at least in the first results on the topic like the one in (DONNELLY, 1981), uses the comparison theorems on both sides for $\Delta\rho$, ρ being the distance from a fixed origin $o \in M$. Therefore, the method needs both a pinching on the sectional curvature and the smoothness of ρ, that is, that o is a pole of M (see (DONNELLY, 1981; ESCOBAR; FREIRE, 1992; LI, 1994) and Corollary 2.17 in (BIANCHINI *et al.*, 2013)), which is a severe topological restriction. Since then, various efforts were made to weaken both the curvature and the topological assumptions. We briefly overview some of the main achievements.

In (1997), Kumura observed that to perform the second step (and just for it) it is enough that there exists a relatively compact, mean convex, smooth open set Ω with the property that the normal exponential map realizes a global diffeomorphism $\partial\Omega \times \mathbb{R}_0^+ \to M\backslash\Omega$. Conditions of this kind seem, however, unavoidable for his techniques to work. On the other hand, in (KUMURA, 2005) the author drastically weakened the curvature requirements needed to establish Step 2, by replacing the two-sided pinching on the sectional curvature with a combination of a lower bound on a suitably weighted volume and an L^p-bound on the Ricci curvature.

Regarding the need for a pole, major recent improvements have been made in a series of papers ((STURM, 1993; WANG, 1997; LU; ZHOU, 2011; CHARALAMBOUS; LU, 2014)): their guiding idea was to replace the L^2-norm in relation (1.3) with the L^1-norm, which, via a trick in (WANG, 1997; LU; ZHOU, 2011), enables to use smoothed distance functions to construct sequences as in Lemma 1.0.1.

Building on deep function-theoretic results due to Sturm (1993) and Charalambous-Lu (2014, in (WANG, 1997; LU; ZHOU, 2011) the authors proved that $\sigma(M) = [0,\infty)$ when

$$\liminf_{\rho(x)\to+\infty} \mathrm{Ricc}_x = 0 \tag{1.4}$$

in the sense of quadratic forms, without any topological assumption. This remarkable result improves on (LI, 1994) and (ESCOBAR; FREIRE, 1992) (see also Corollary 2.17 in (BIANCHINI *et al.*, 2013)), where M was assumed to have a pole. Further refinements of (1.4) have been given in (CHARALAMBOUS; LU, 2014). However, when (1.4) does not hold, the situation is more delicate and is still the subject of an active area of research. In this respect, we also quote the general function-theoretic criteria developed by H. Donnelly (1997), and Elworthy and Wang (2004) to ensure that a half-line belongs to the spectrum of M.

1.1 Main results

This Thesis is divided in two parts. The first part is composed by Chapters 2 and 3, which present the results of the controllability of some incompressible fluids models and the null controllability of a quasi-linear parabolic equation in a bounded domain of with Dirichlet boundary conditions. The second part is formed by Chapter 4, deals with the characterization of the spectrum of the Laplace Beltrami operator $-\Delta$ on minimal submanifolds.

In the sequel, we will present the main results and ideas used in the proofs.

1.1.1 Main results of Chapter 2

In the Chapter 2, we deal with some 3D systems of the Navier-Stokes kind in a cube or a similar set. Let $T > 0$ and let Ω be the open set

$$\Omega = \{ x \in \mathbb{R}^3 : x_i \in (0,1), \ \ 1 \leq i \leq 3 \},$$

whose boundary is denoted by $\partial\Omega$. We will set $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$.

Let us introduce the Hilbert spaces

$$H(\Omega) := \{ w \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) : \nabla \cdot w = 0 \text{ in } \ \Omega, \ w \cdot n = 0 \text{ on } \partial\Omega \}$$

(where $n = n(x)$ is the outward unit normal vector at $x \in \partial\Omega$) and

$$V_0(\Omega) := \{ w \in H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) : \nabla \cdot w = 0 \text{ in } \ \Omega \}.$$

Let $f \in L^2(Q) \times L^2(Q) \times L^2(Q)$ and $u_0 \in H(\Omega)$ be given and let us first consider the 3D Navier-Stokes system

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \ \nabla \cdot u = 0 & \text{in } \ Q \\ u(0,x_2,x_3,t) = 0, & \text{on } \ (0,1)^2 \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{1.5}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity vector field of the fluid, ∇p is the pressure gradient in the fluid, Δ is the laplace operator,

$$(u \cdot \nabla)u = \sum_{i=1}^{3} u_i \frac{\partial_i u}{\partial x_i}, \quad f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$$

is given density of external forces, u_0 is given initial data. From now, in order to simplify notations we will denote by $L^2(\Omega)^3$ and $H^1(\Omega)^3$ the vector spaces $L^2(Q) \times L^2(Q) \times L^2(Q)$ and $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, respectively.

Our first main results concern two generalizations of Theorem 1 in (GUERRERO *et al.*, 2012). In the first one, we prove that the partial approximate controllability can also be obtained with controls acting only on three faces of the unit cube. In the second one, we show that Ω can be a much more general set, namely a bounded domain of $\mathbb{R}^3$ whose boundary contains a piece of a plane and is contained in one of the associated semispaces.

Theorem 1.1.1 *Assume that $u_0 \in H(\Omega)$ and $f \in L^2(Q)^3$ are given. Then, there exists a sequence $\{f_\varepsilon\}$ in $L^2(Q)^3$ such that*

$$f_\varepsilon \to f \ \textit{in} \ \ L^r(0,T;H^{-1}(\Omega)^3)$$

for all $r \in (1,4/3)$ and there exist solutions $(u_\varepsilon, p_\varepsilon)$ to the null controllability problems

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in } Q \\ \nabla \cdot u_\varepsilon = 0 & \text{in } Q \\ u_\varepsilon(0,x_2,x_3,t) = u_\varepsilon(1,x_2,x_3,t) = u_\varepsilon(x_1,x_2,0,t) = 0 & \text{on } (0,1)^2 \times (0,T) \\ u_\varepsilon(x,0) = u_0(x),\ u_\varepsilon(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Now, let π be a plane in $\mathbb{R}^3$, let π^+ be one of the semispaces determined by π and let $\Omega_\pi \subset \mathbb{R}^3$ be a bounded domain satisfying

$$\Omega_\pi \subset \pi^+, \ \ \partial\Omega_\pi \cap \pi \ \text{ is a non-empty open set.}$$

Theorem 1.1.2 *Assume that $u_0 \in H(\Omega_\pi)$ and $f \in L^2(\Omega_\pi \times (0,T))^3$. Then, there exists a sequence $\{f_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega_\pi \times (0,T))^3$ such that*

$$f_\varepsilon \to f \ \text{ in } \ L^r(0,T;H^{-1}(\Omega_\pi)^3)$$

for all $r \in (1,4/3)$ and there exist solutions $(u_\varepsilon, p_\varepsilon)$ to the null controllability problems

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in } \Omega_\pi \times (0,T) \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\pi \times (0,T) \\ u_\varepsilon(x,t) = 0 & \text{on } (\partial\Omega_\pi \cap \pi) \times (0,T) \\ u_\varepsilon(x,0) = u_0(x),\ u_\varepsilon(x,T) = 0 & \text{in } \Omega_\pi. \end{cases}$$

We will also consider a system of the Boussinesq kind:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla p = \theta e_N + f,\ \nabla \cdot u = 0 & \text{in } Q \\ \theta_t - \Delta\theta + u \cdot \nabla\theta = g & \text{in } Q \\ u(0,x_2,x_3,t) = 0,\ \ \theta(0,x_2,x_3,t) = 0 & \text{on } (0,1)^2 \times (0,T) \\ (u(x,0), \theta(x,0)) = (u_0(x), \theta_0(x)) & \text{in } \Omega. \end{cases}$$

Here, $f \in L^2(0,T;L^2(\Omega)^3)$, $g \in L^2(0,T;L^2(\Omega))$ are given source terms, $u_0 \in H(\Omega)$ and $\theta_0 \in L^2(\Omega)$.

Theorem 1.1.3 *Assume that $(u_0, \theta_0) \in V_0(\Omega) \times H^1(\Omega)$ and $(f,g) \in L^2(Q)^3 \times L^2(Q)$. Then, there exists a sequence $\{(f_\varepsilon, g_\varepsilon)\}_{\varepsilon>0}$ in $L^2(Q)^3 \times L^2(Q)$ such that*

$$(f_\varepsilon, g_\varepsilon) \to (f,g) \ \text{ in } \ L^r(0,T;H^{-1}(\Omega)^3) \times L^r(0,T;H^{-1}(\Omega))$$

for all $r \in (1,4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = \theta_\varepsilon e_N + f_\varepsilon, \ \ \nabla \cdot u_\varepsilon = 0 & \textit{in } \ Q \\ \theta_{\varepsilon,t} - \Delta \theta_\varepsilon + u_\varepsilon \cdot \nabla \theta = g_\varepsilon & \textit{in } \ Q \\ u_\varepsilon(0,x_2,x_3,t) = 0, \ \ \theta_\varepsilon(0,x_2,x_3,t) = 0 & \textit{on } \ (0,1)^2 \times (0,T) \\ (u_\varepsilon(x,0), \theta_\varepsilon(x,0)) = (u_0(x), \theta_0(x)) & \textit{in } \ \Omega \\ (u_\varepsilon(x,T), \theta_\varepsilon(x,T)) = (0,0) & \textit{in } \ \Omega, \end{cases}$$

with

$$u_\varepsilon \in L^2(0,T;V(\Omega)) \cap C^0_w([0,T];H(\Omega))$$

and

$$\theta_\varepsilon \in L^2(0,T;H^1_0(\Omega)) \cap C^0_w([0,T];L^2(\Omega)).$$

As in (GUERRERO *et al.*, 2012), the proofs of the previous results consist of four steps. Thus, we divide our time interval $(0,T)$ in four subintervals, where different strategies are used. In fact, nothing is done in a first (large) subinterval; then, we pass from the first final state to a regular, compactly supported, close field in a second step; then, we drive the solution to a particular field that can be viewed as the solution to a simpler parabolic system; finally, in the last step, we introduce controls that drive this parabolic system to zero.

The content of the chapter 2 was taken from the preprint (FERNÁNDEZ-CARA *et al.*, 2017).

1.1.2 Main results of Chapter 3

The third Chapter of this Thesis deals with the distributed and boundary null controllability of a $1D$ nonlinear parabolic system.

Let us consider an open bounded interval $I \subset \mathbb{R}$ and denote by Q the cylinder $Q := I \times (0,T)$ with lateral boundary $\Sigma := \partial I \times (0,T)$. Also, we consider a non-empty open set of $\omega \subset I$. As usual, 1_ω denotes the characteristic function of ω.

We are interested in the null controllability of the nonlinear systems

$$\begin{cases} y_t - (a(y) y_x)_x = v_1 1_\omega & \text{in} \quad Q \\ y(x,t) = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0(x) & \text{in} \quad I \end{cases} \tag{1.6}$$

and

$$\begin{cases} y_t - (a(y)y_x)_x = 0 & \text{in} \quad (0,1)\times(0,T) \\ y(0,t) = v_2(t),\ y(1,t) = 0 & \text{on} \quad (0,T) \\ y(x,0) = y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{1.7}$$

where v_1 and v_2 are control functions and y is the associated state.

It will be assumed that the real function $a = a(r)$ is of class C^1, possesses bounded derivatives and satisfies

$$0 < m \le a(r) \le M,\ \forall r \in \mathbb{R}.$$

Definition 1.1.1 *It will be said that* (1.6) *(resp.* (1.7)*) is locally null-controllable at time T if there exists $\varepsilon > 0$ such that, for any $y_0 \in H_0^1(I)$ with*

$$\|y_0\|_{H_0^1}(I) \le \varepsilon,$$

there exist controls $v_1 \in L^2(\omega \times (0,T))$ (resp. $v_2 \in L^2(0,T)$) such that the associated states y satisfy

$$y(x,T) = 0 \ \ in \ \ I.$$

The main result in the Chapter 3 is the following:

Theorem 1.1.4 *Under the previous assumptions on a, the nonlinear system* (1.6) *is locally null-controllable at any time $T > 0$.*

A consequence of Theorem 1.1.4 is the local null controllability of (1.7). Thus, the second result of Chapter 3 is:

Theorem 1.1.5 *Under the previous assumptions on a, the nonlinear system* (1.7) *is locally null-controllable at any time $T > 0$.*

The proof of Theorem 1.1.4 relies on an application of *Liusternik's Inverse Function Theorem* in Banach spaces (see (ALEKSEEV *et al.*, 1987)).
The content of the chapter 3 was taken from the preprint (FERNÁNDEZ-CARA; VIEIRA, 2017).

1.1.3 Main results of Chapter 4

The present Chapter develops as follows. The first result of Chapter 4 characterize $\sigma(M)$ when the density of M grows subexponentially (respectively, sub-polynomially) along a sequence. In our second result we give a sufficient condition in terms of the decay of the second fundamental form in order to ensure that $\Theta(+\infty) < +\infty$.

Before we deal with the main results, we set some definition.

Definition 1.1.2 *Let N^n possess a pole $\bar{o}$ and denote with $\bar{\rho}$ the distance function from $\bar{o}$. Assume that the radial sectional curvature $\bar{K}_{\mathrm{rad}}$ of N, i.e., the sectional curvature restricted to planes π containing $\bar{\nabla}\bar{\rho}$, satisfies*

$$-G\big(\bar{\rho}(x)\big) \leq \bar{K}_{\mathrm{rad}}(\pi_x) \leq -k \leq 0 \qquad \forall x \in N \backslash \{\bar{o}\}, \tag{1.8}$$

for some $G \in C^0(\mathbb{R}^+_0)$. We say that

(i) N has a pointwise (respectively, integral) pinching to $\mathbb{R}^n$ if $k = 0$ and

$$sG(s) \to 0 \ \text{ as } \ s \to +\infty \qquad \big(\text{respectively, } sG(s) \in L^1(+\infty)\big);$$

(ii) N has a pointwise (respectively, integral) pinching to $\mathbb{H}^n_k$ if $k > 0$ and

$$G(s) - k \to 0 \ \text{ as } \ s \to +\infty \qquad \big(\text{respectively, } G(s) - k \in L^1(+\infty)\big).$$

Now we present the results.

Theorem 1.1.6 *Let $\varphi : M^m \to N^n$ be a minimal properly immersed submanifold and suppose that N has a pointwise or an integral pinching to a space form. If either*

$$N \text{ is pinched to } \mathbb{H}^n_k, \text{ and } \liminf_{s\to+\infty} \frac{\log\Theta(s)}{s} = 0, \quad \text{or}$$

$$N \text{ is pinched to } \mathbb{R}^n, \text{ and } \liminf_{s\to+\infty} \frac{\log\Theta(s)}{\log s} = 0,$$

then

$$\sigma(M) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.9}$$

The proof of of 1.1.6 follow an approach inspired by a general result due to Elworthy and Wang (2004). Because of the upper bound in (1.8), by (CHEUNG; LEUNG, 2001) and (BESSA; MONTENEGRO, 2007) the bottom of $\sigma(M)$ satisfies

$$\inf\sigma(M) \geq \frac{(m-1)^2 k}{4}.$$

To complete the proof of the 1.1.6, since $\sigma(M)$ is closed, it is sufficient to show that each $\lambda > (m-1)^2 k/4$ lies in $\sigma(M)$. To this end, we build a sequence as in Lemma 1.0.1.

Corollary 1.1.1 *Let $\Sigma \subset \partial_\infty \mathbb{H}^n_k$ be a closed, integral $(m-1)$ current in the boundary at infinity of $\mathbb{H}^n_k$ such that, for some neighborhood U of $\operatorname{supp}(\Sigma)$, Σ does not bound in U, and let $M^m \hookrightarrow \mathbb{H}^n_k$ be the solution of Plateau's problem at infinity constructed in (ANDERSON, 1982) for Σ. If M is smooth, then* (1.9) *holds.*

Another result is.

Theorem 1.1.7 *Let $\varphi : M^m \to N^n$ be a minimal immersion and suppose that N has an integral pinching to $\mathbb{R}^n$ or to $\mathbb{H}^n_k$. Let us denote $\rho(x)$ the intrinsic distance from some reference origin $o \in M$. Assume that there exist $c > 0$ and $\alpha > 1$ such that the second fundamental form satisfies, for $\rho(x) >> 1$,*

$$|\mathrm{II}(x)|^2 \leq \frac{c}{\rho(x)\log^\alpha \rho(x)}, \quad \text{if } N \text{ is pinched to } \mathbb{H}^n_k;$$

$$|\mathrm{II}(x)|^2 \leq \frac{c}{\rho(x)^2\log^\alpha \rho(x)}, \quad \text{if } N \text{ is pinched to } \mathbb{R}^n.$$

Then, φ is proper, M is diffeomorphic to the interior of a compact manifold with boundary and $\Theta(+\infty) < +\infty$.

We say that M has finite total curvature when the second fundamental form II satisfies

$$\int_M |\mathrm{II}|^m < +\infty. \tag{1.10}$$

The relation between (1.10) and the finiteness of $\Theta(+\infty)$ has been investigated in depth for minimal submanifolds of $\mathbb{R}^n$, but the case of $\mathbb{H}^n_k$ seems to be partly unexplored. About this, we have as consequence of Theorem 1.1.7 the following corollary

Corollary 1.1.2 *Let M^m be a minimal properly immersed submanifold in $\mathbb{H}^n_k$. If M has finite total curvature, then $\Theta(+\infty) < +\infty$.*

The content of the chapter 4 was taken from the recent work (LIMA *et al.*, 2016).

2 REMARKS CONCERNING THE APPROXIMATE CONTROLLABILITY OF SYSTEMS OF THE NAVIER-STOKES KIND

2.1 Introduction

Let $T > 0$ and let Ω be the open set

$$\Omega = \{x \in \mathbb{R}^3 : x_i \in (0,1), \ \ 1 \leq i \leq 3\},$$

whose boundary is denoted by $\partial\Omega$. We will set $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$.

Let us introduce the Hilbert spaces

$$H(\Omega) := \{w \in L^2(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \ \Omega, \ w \cdot n = 0 \text{ on } \partial\Omega\}$$

(where $n = n(x)$ is the outward unit normal vector at $x \in \partial\Omega$) and

$$V_0(\Omega) := \{w \in H_0^1(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \ \Omega\}.$$

Let $f \in L^2(Q)^3$ and $u_0 \in H(\Omega)$ be given and let us first consider the three-dimensional Navier-Stokes system

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \ \nabla \cdot u = 0 & \text{in } \ Q \\ u(0,x_2,x_3,t) = 0, & \text{on } \ (0,1)^2 \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \ \Omega. \end{cases} \tag{2.1}$$

In a recent work, Guerrero, Imanuvilov and Puel (GUERRERO *et al.*, 2012) have established some results concerning a "partial" approximate controllability property of (2.1). Specifically, they have proved that, for any u_0 and f, there exists a sequence $\{f_n\}$ in $L^2(0,T;L^2(\Omega)^3)$ such that $f_n \to f$ in an appropriate sense and, for each n, the corresponding system (2.1) is null-controllable, with controls supported by the faces on the boundary where $x_1 \neq 0$. Note that, in view of the time irreversibility of (2.1), we cannot expect the exact controllability to hold to an arbitrary target function. On the other hand, recall that the global approximate controllability is an open question for this system, due to the presence of a Dirichlet condition at $x_1 = 0$.

This paper is devoted to present some extensions and variants of the results in (GUERRERO *et al.*, 2012) that include in particular

- A result concerning boundary controls with support in smaller sets,
- Similar results in more general domains and
- A control result for a Boussinesq system.

Let us recall some (partial) results concerning the controllability of (2.1). Global controllability results can be proved using the arguments in (FURSIKOV; IMANUVILOV, 1999) if the control is exerted on the whole boundary. On the other hand, the local exact controllability to bounded trajectories with distributed controls was established in (FERNÁNDEZ-CARA *et al.*, 2004) and (GUERRERO, 2006), respectively for the Navier-Stokes and Boussinesq systems. This has been revisited and improved in a set of papers, where it was shown that $N-1$ or even less scalar controls suffice; see (FERNÁNDEZ-CARA *et al.*, 2006; NO, 2012; CORON; GUERRERO, 2009; CORON J.-M., 2014). In (CORON, 1996), the global approximate controllability of the 2D Navier-Stokes equations completed with Navier slip boundary conditions was proved. Then, in (CORON; FURSIKOV, 1996), a global exact controllability result was established for the same system in a 2D manifold without boundary.

Our first main results concern two generalizations of Theorem 1 in (GUERRERO *et al.*, 2012). In the first one, we prove that the partial approximate controllability can also be obtained with controls acting only on three faces of the unit cube. In the second one, we show that Ω can be a much more general set, namely a bounded domain of $\mathbb{R}^3$ whose boundary contains a piece of a plane and is contained in one of the associated semispaces, see figure below.

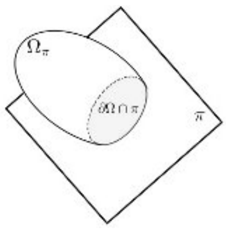

Figura 1 – The situation in Theorem 2.1.2.

Theorem 2.1.1 *Assume that* $u_0 \in H(\Omega)$ *and* $f \in L^2(Q)^3$ *are given. Then, there exists a sequence* $\{f_\varepsilon\}$ *in* $L^2(Q)^3$ *such that*

$$f_\varepsilon \to f \text{ in } L^r(0,T;H^{-1}(\Omega)^3)$$

for all $r \in (1,4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \textit{in } Q \\ \nabla \cdot u_\varepsilon = 0 & \textit{in } Q \\ u_\varepsilon(0,x_2,x_3,t) = u_\varepsilon(1,x_2,x_3,t) = u_\varepsilon(x_1,x_2,0,t) = 0 & \textit{on } (0,1)^2 \times (0,T) \\ u_\varepsilon(x,0) = u_0(x),\ u_\varepsilon(x,T) = 0 & \textit{in } \Omega. \end{cases}$$

Now, let π be a plane in $\mathbb{R}^3$, let π^+ be one of the semispaces determined by π and let $\Omega_\pi \subset \mathbb{R}^3$ be a bounded domain satisfying

$$\Omega_\pi \subset \pi^+, \ \ \partial\Omega_\pi \cap \pi \ \text{ is a non-empty open set.}$$

Theorem 2.1.2 *Assume that* $u_0 \in H(\Omega_\pi)$ *and* $f \in L^2(\Omega_\pi \times (0,T))^3$. *Then, there exists a sequence* $\{f_\varepsilon\}_{\varepsilon>0}$ *in* $L^2(\Omega_\pi \times (0,T))^3$ *such that*

$$f_\varepsilon \to f \ \textit{in} \ \ L^r(0,T;H^{-1}(\Omega_\pi)^3)$$

for all $r \in (1,4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \textit{in } \Omega_\pi \times (0,T) \\ \nabla \cdot u_\varepsilon = 0 & \textit{in } \Omega_\pi \times (0,T) \\ u_\varepsilon(x,t) = 0 & \textit{on } (\partial\Omega_\pi \cap \pi) \times (0,T) \\ u_\varepsilon(x,0) = u_0(x),\ u_\varepsilon(x,T) = 0 & \textit{in } \Omega_\pi. \end{cases}$$

We will also consider a system of the Boussinesq kind:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla p = \theta e_N + f, \ \nabla \cdot u = 0 & \text{in } Q \\ \theta_t - \Delta\theta + u \cdot \nabla\theta = g & \text{in } Q \\ u(0,x_2,x_3,t) = 0, \ \ \theta(0,x_2,x_3,t) = 0 & \text{on } (0,1)^2 \times (0,T) \\ (u(x,0), \theta(x,0)) = (u_0(x), \theta_0(x)) & \text{in } \Omega. \end{cases} \tag{2.2}$$

Here, $f \in L^2(0,T;L^2(\Omega)^3)$, $g \in L^2(0,T;L^2(\Omega))$ are given source terms, $u_0 \in H(\Omega)$ and $\theta_0 \in L^2(\Omega)$.

Theorem 2.1.3 *Assume that* $(u_0, \theta_0) \in V_0(\Omega) \times H^1(\Omega)$ *and* $(f,g) \in L^2(Q)^3 \times L^2(Q)$. *Then, there exists a sequence* $\{(f_\varepsilon, g_\varepsilon)\}_{\varepsilon>0}$ *in* $L^2(Q)^3 \times L^2(Q)$ *such that*

$$(f_\varepsilon, g_\varepsilon) \to (f,g) \ \textit{in} \ \ L^r(0,T;H^{-1}(\Omega)^3) \times L^r(0,T;H^{-1}(\Omega))$$

for all $r \in (1,4/3)$ *and there exist solutions* $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ *to the null controllability problems*

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = \theta_\varepsilon e_N + f_\varepsilon, \;\; \nabla \cdot u_\varepsilon = 0 & \textit{in } \; Q \\ \theta_{\varepsilon,t} - \Delta \theta_\varepsilon + u_\varepsilon \cdot \nabla \theta = g_\varepsilon & \textit{in } \; Q \\ u_\varepsilon(0,x_2,x_3,t) = 0, \;\; \theta_\varepsilon(0,x_2,x_3,t) = 0 & \textit{on } \; (0,1)^2 \times (0,T) \\ (u_\varepsilon(x,0), \theta_\varepsilon(x,0)) = (u_0(x), \theta_0(x)) & \textit{in } \; \Omega \\ (u_\varepsilon(x,T), \theta_\varepsilon(x,T)) = (0,0) & \textit{in } \; \Omega, \end{cases}$$

with

$$u_\varepsilon \in L^2(0,T;V(\Omega)) \cap C^0_w([0,T];H(\Omega))$$

and

$$\theta_\varepsilon \in L^2(0,T;H^1_0(\Omega)) \cap C^0_w([0,T];L^2(\Omega)).$$

As in (GUERRERO *et al.*, 2012), the proofs of the previous results consist of four steps. For instance, in the case of Theorem 2.1.3, we divide the time interval $(0,T)$ in four subintervals, where different strategies are used:

- In the first interval $(0,T_1)$ no control is needed, so we let the system evolve from the initial condition at $t = 0$ to some non-zero state with zero Dirichlet boundary conditions.
- In the second time interval, we give explicitly give our solution. This way, we drive the system to a compactly supported and regular state at a time T_2.
- In the third time interval, we construct our solution in a much more intrinsic way. Indeed, it is split as the sum of three functions: a very particular and simple solution multiplied by a large parameter plus a solution to a transport equation plus a solution to a Stokes system. This allows to drive the system to a solution to a more simple problem.
- In the last time interval, we reduce the question to a standard null controllability problem for a system composed of two coupled 1D parabolic equations. In view of some recent results from (FERNÁNDEZ-CARA *et al.*, 2010), this is easy to achieve and allows to conclude.

This paper is organized as follows. In the next section, we construct some intermediate functions and we prove some crucial estimates. In Section 2.3, the proof of Theorem 2.1.3 is given, following the ideas in (GUERRERO *et al.*, 2012). Section 2.4 deals with the proofs of

Theorems 2.1.1 and 2.1.2. Finally, in Section 2.5, we present some additional comments and questions.

2.2 Some auxiliary problems and estimates

In this section, we will construct a specific solution (U,q,Θ) to the Boussinesq system with boundary conditions, with $(U\cdot\nabla)U\equiv 0$.

2.2.1 The Navier-Stokes system with a boundary control acting on three faces

Let $z=z(x_1,x_3,t)$ be solution to the following system for the 2D heat PDE:

$$\begin{cases} z_t-(\partial^2_{x_1x_1}z+\partial^2_{x_3x_3}z)=c(t),\ (x_1,x_3,t)\in(0,1)^2\times(0,T)\\ z(0,x_3,t)=z(1,x_3,t)=z(x_1,0,t)=0,\ x_1,x_3\in(0,1),t\in(0,T)\\ z(x_1,1,t)=w(t),\ (x_1,t)\in(0,1)\times(0,T)\\ z(x_1,x_3,0)=0,\ (x_1,x_3)\in(0,1)^2. \end{cases}$$

Here, $c\in C^2([0,T])$ is a positive function ($c(0)$ is as large as needed) and w is a nonnegative function satisfying

$$w\in C^\infty([0,T]),\ w(0)=0,\ w'(0)=c(0),\ w''(0)=c'(0). \tag{2.3}$$

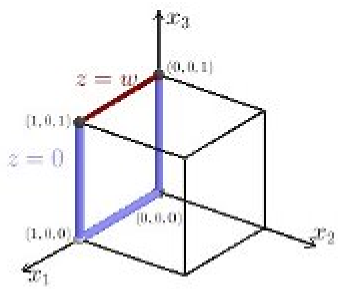

Figura 2 – The situation in Theorem 2.1.1

Thanks to the compatibility condition (2.3), we can argue as in (GUERRERO *et al.*, 2012) and check that

$$z\in C^2([\delta,1-\delta]\times[\delta,1]\times[0,T])\ \forall\delta>0.$$

On the other hand, thanks to Taylor's formula, we can obtain functions β_δ, γ^i_δ, λ_δ and μ^{ij}_δ

in $C^0([\delta,1-\delta]\times[\delta,1]\times[0,T])$ such that

$$\begin{aligned}
&z(x_1,x_3,t)=c(0)t+\beta_\delta(x_1,x_3,t)t^2\\
&\partial_{x_i}z(x_1,x_3,t)=\gamma_\delta^i(x_1,x_3,t)t^2,\ i\in\{1,3\}\\
&\partial_t z(x_1,x_3,t)=c(0)+\lambda_\delta(x_1,x_3,t)t\\
&\partial^2_{x_i,x_j}z(x_1,x_3,t)=\mu_\delta^{ij}(x_1,x_3,t)t,\ i,j\in\{1,3\}.
\end{aligned}\tag{2.4}$$

Let $\mathscr{G}$ and $\mathscr{I}$ be given by

$$\mathscr{G}=\{\,(x_1,x_2,x_3):x_2\in\mathbb{R},\ (x_1,x_3)\in(0,1)^2\,\},$$

$$\mathscr{I}=(\{0,1\}\times\mathbb{R}\times(0,1))\cup((0,1)\times\mathbb{R}\times\{0\}).$$

Now, we introduce the functions U and q, with $U(x,t):=(0,z(x_1,x_3,t),0)$ and $q:=-c(t)x_2$. Note that the couple (U,q) satisfies

$$\begin{cases}
U_t-\Delta U+\nabla q=0 & \text{in } \mathscr{G}\times(0,T)\\
\nabla\cdot U=0 & \text{in } \mathscr{G}\times(0,T)\\
U(x,t)=0 & \text{on } \mathscr{I}\times(0,T)\\
U(x,0)=0 & \text{in } \mathscr{G}.
\end{cases}$$

Later, we will look for a solution to the Navier-Stokes system of the form

$$u=N^2U+y+\xi(t)W,$$

where N is a large constant, y is the solution to a transport equation, W solves a Stokes system and $\xi\in C^2[0,2/N]$ is a cut-off function.

2.2.1.1 *Transport equation*

For an arbitrary initial condition $y_0\in V_0(\Omega)\cap C^1_0(\Omega)$ extended by zero on $\mathscr{G}$ we consider the system

$$\begin{cases}
y_t+N^2(U\cdot\nabla)y+N^2(y\cdot\nabla)U=0 & \text{in } Q_{2/N}\\
y(x,t)=0 & \text{on } \Sigma_{2/N}\\
y(x,0)=y_0(x) & \text{in } \Omega.
\end{cases}\tag{2.5}$$

Here, we have used the notation

$$Q_{2/N}=\mathscr{G}\times(0,2/N),\ \ \Sigma_{2/N}=\mathscr{I}\times(0,2/N).$$

Let us denote by C_δ the maximum of the norms of the functions β_δ, γ_δ^i, λ_δ and μ_δ^{ij} in $C^0([\delta,1-\delta]\times[\delta,1]\times[0,T])$. We will look for a particular estimate for y, with an explicit dependence on N that is satisfied when N is large enough. This is given in the following lemma:

Lemma 2.2.1 *Let $y_0 \in C_0^1(\Omega)\cap V_0(\Omega)$. For each small $\delta > 0$, there exists $N_\delta = N(\delta)$ such that, for any $N \geq N_\delta$, there exist a solution y to (2.5) and a positive constant K_δ (independent of N), with the following properties:*

$$\|y\|_{C^1(\overline{Q}_{2/N})} \leq K_\delta \|y_0\|_{C^1(\overline{\Omega})} \tag{2.6}$$

and

$$y(x,t) = 0 \quad \forall (x,t) \in \Omega \times (1/N, 2/N). \tag{2.7}$$

Proof: Let us consider the Banach space

$$Y = \{y \in C^1(\overline{Q}_{2/N}) : y(x,0) = y_0(x)\}.$$

and the mapping $\Lambda : Y \mapsto Y$, given by

$$\Lambda(y)(x,t) := y_0(x - N^2 Z(x,t)) - N^2 \int_0^t (y\cdot\nabla)U(x - N^2 Z(x,s),s)\,ds,$$

$$Z(x,t) := \left(0, \int_0^t z(x_1,x_3,s)ds, 0\right).$$

Let us assume that $\operatorname{supp} y_0 \subset (\delta, 1-\delta)^3$. It is easy to see that here exists N_δ such that, for any $N \geq N_\delta$, we can appl *Banach's Fixed-Point Theorem* to Λ and deduce the existence and uniqueness of a solution to (2.5).

Let us put $y_0 = (y_{0,1}, y_{0,2}, y_{0,3})$ and $U = (U_1,U_2,U_3)$. Then, we have

$$y_1(x,t) = y_{0,1}(x - N^2 Z(x,t)),$$

$$y_2(x,t) = y_{0,2}(x - N^2 Z(x,t)) - N^2 \int_0^t y\cdot\nabla U_2(x - N^2 Z(s,x),s)\,ds,$$

$$y_3(x,t) = y_{0,3}(x - N^2 Z(x,t)).$$

From these identities, it is easy to check that, for N large enough, one has:

$$\begin{aligned} &\|y\|_{C^0(Q_{2/N})} \leq C\|y_0\|_{C^0(\overline{\Omega})} \\ &\|\nabla y_1\|_{C^0(Q_{2/N})} \leq C\|\nabla y_{0,1}\|_{C^0(\overline{\Omega})} \\ &\|\nabla y_3\|_{C^0(Q_{2/N})} \leq C\|\nabla y_{0,3}\|_{C^0(\overline{\Omega})}. \end{aligned} \tag{2.8}$$

On the other hand, we also have

$$\begin{aligned}\|\nabla y_2\|_{C^0(Q_{2/N})} &\leq 7\|\nabla y_{0,2}\|_{C^0(\overline{\Omega})} + C_\delta \|y_0\|_{C^0(\overline{\Omega})}\\ &+ 3\frac{C_\delta}{N}\|\nabla y\|_{C^0(Q_{2/N})} + 15\frac{C_\delta^2}{N^2}\|\nabla y\|_{C^0(Q_{2/N})}.\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), the inequality (2.6) holds (for N large enough). On the other hand, arguing as in the proof of Lemma 3 of (GUERRERO *et al.*, 2012) (see p. 693–695), we deduce (2.7).

This ends the proof.

2.2.1.2 The solution to a Stokes system with $\nabla \cdot W = -\nabla \cdot y$

Consider the following Stokes problem:

$$\begin{cases} W_t - \Delta W + \nabla r = 0, & \text{in } Q_{2/N}\\ \nabla \cdot W = -\nabla \cdot y, & \text{in } Q_{2/N}\\ W(x,t) = 0, & \text{on } \Sigma_{2/N}\\ W(x,0) = 0, & \text{in } \Omega\\ W(x,t) \to 0 \text{ as } |x_2| \to +\infty. \end{cases} \tag{2.10}$$

The following result holds:

Proposition 2.2.1 *Let W be the solution to* (2.10)*. Then, for any $p \in (1,+\infty)$, there exists C (independent of N) such that*

$$\|W\|_{L^p(Q_{2/N})} \leq C(p)N^{-1/p}\|y_0\|_{C^3(\overline{\Omega})}. \tag{2.11}$$

Furthermore, there exists a positive constant $C > 0$ (again independent of N) such that

$$\begin{cases} \|W\|_{C^0([0,2/N];L^2(\mathscr{G}))} + \|\partial_{x_2} W\|_{C^0([0,2/N];L^2(\mathscr{G}))}\\ \qquad + \|\partial_{x_3} W\|_{C^0([0,2/N];L^2(\mathscr{G}))} \leq CN^{-1/4}. \end{cases} \tag{2.12}$$

The proof is given in (GUERRERO *et al.*, 2012) (see Proposition 1, p. 696).

2.2.2 Boussinesq system

We will construct a specific solution (U,Θ) to the Boussinesq system.

Let us first introduce the functions $z_2 = z_2(x_1,t)$, $z_3 = z_3(x_1,t)$ and $\Theta = \Theta(x_1,t)$ as follows. First, z_2 is the solution to the system

$$\begin{cases} \partial_t z_2 - \partial^2_{x_1x_1} z_2 = c(t) & \text{in } (0,1)\times(0,T) \\ z_2(0,t) = 0, \quad z_2(1,t) = w_2(t) & \text{on } (0,T) \\ z_2(x_1,0) = 0 & \text{in } (0,1), \end{cases} \tag{2.13}$$

where $c \in C^2([0,T])$ is a positive function and w_2 is a nonnegative function satisfying

$$w_2(t) \in C^\infty[0,T], \quad w_2(0) = 0, \quad w_2'(0) = c(0), \quad w_2''(0) = c'(0).$$

Then, (z_3,Θ) solves

$$\begin{cases} \partial_t z_3 - \partial^2_{x_1x_1} z_3 = c(t) + \Theta(x_1,t) & \text{in } (0,1)\times(0,T) \\ \partial_t \Theta - \partial^2_{x_1x_1}\Theta = 0 & \text{in } (0,1)\times(0,T) \\ z_3(0,t) = 0, \quad z_3(1,t) = w_3(t) & \text{on } (0,T) \\ \Theta(0,t) = 0, \quad \Theta(1,t) = m(t) & \text{on } (0,T) \\ z_3(x_1,0) = 0, \quad \Theta(x_1,0) = 0 & \text{in } (0,1). \end{cases} \tag{2.14}$$

with

$$w_3 \in C^\infty[0,T],\ w_3(0) = 0,\ w_3'(0) = c(0),\ w_3''(0) = c'(0),$$
$$m \in C^\infty[0,T],\ m(0) = m'(0) = m''(0) = 0.$$

Proposition 2.2.2 *Under the above assumptions on w_2 and c, there exist a unique solution to* (2.13) *with*

$$z_2 \in L^2(0,T;H^1(0,1)) \cap L^\infty((0,1)\times(0,T)), \quad \partial_t z_2 \in L^2(0,T;H^{-1}(0,1)).$$

Furthermore, for all small $\delta > 0$, we have $z_2 \in C^2([\delta,1]\times[0,T])$ and there exist functions $\beta_{2,\delta}$, $\gamma_{2,\delta}$, $\mu_{2,\delta}$ and $\lambda_{2,\delta}$ such that

(i) $z_2(x_1,t) = c(0)t + \beta_{2,\delta}(x_1,t)t^2$, $|\beta_{2,\delta}| \le C_\delta$,

(ii) $\partial_{x_1} z_2(x_1,t) = \gamma_{2,\delta}(x_1,t)t^2$, $|\gamma_{2,\delta}| \le C_\delta$,

(iii) $\partial_t z_2(x_1,t) = c(0) + \mu_{2,\delta}(x_1,t)t$, $|\mu_{2,\delta}| \le C_\delta$,

(iv) $\partial^2_{x_1x_1} z_2(x_1,t) = \lambda_{2,\delta}(x_1,t)t$, $|\lambda_{2,\delta}| \le C_\delta$.

The proof is not difficult. For instance, let us see how *(i)* can be proved.

We simply write that

$$\begin{aligned} z_2(x,t) &= z_2(x_1,0) + \int_0^t \partial_t z_2(x_1,s)\,ds \\ &= \partial_t z_2(x,0)t + \left(\int_0^t \partial_t z_2(x_1,s)\,ds - \partial_t z_2(x_1,0)\,t\right) \\ &= c(0)t + \beta_{2,\delta}(x_1,t)t^2 \end{aligned}$$

where we have used the notation

$$\begin{aligned}\beta_{2,\delta}(x_1,t) &:= \frac{1}{t^2}\int_0^t \partial_t z_2(x_1,s)\,ds - \frac{1}{t}\partial_t z_2(x_1,0)\\ &= (\partial_t z_2(x,\tilde{t}) - \partial_t z_2(x_1,0))\,t^{-1}\end{aligned}$$

for some $0 < \tilde{t} < t$.

The proof of *(ii)*, *(iii)* and *(iv)* follows through analogous computations.

A similar result can be established for the solution (z_3,Θ) to (2.14):

Proposition 2.2.3 *Under the above assumptions on c, w_2, w_3 and m, there exists a unique solution (z_3,Θ) to* (2.14) *with*

$$z_3 \in L^2(0,T;H^1(0,1)) \cap L^\infty((0,1)\times(0,T)),\, z_{3,t} \in L^2(0,T;H^{-1}(0,1)),$$

$$\Theta \in L^2(0,T;H^1(0,1)) \cap L^\infty((0,1)\times(0,T)),\, \Theta_t \in L^2(0,T;H^{-1}(0,1)).$$

Furthermore,, for all small $\delta > 0$, we have that $z_3,\Theta \in C^2([\delta,1]\times[0,T])^2$ and there exist functions $\beta_{3,\delta}$, $\gamma_{3,\delta}$, $\mu_{3,\delta}$, $\lambda_{3,\delta}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\mu}$ such that

(i) $z_3(x_1,t) = c(0)t + \beta_{3,\delta}(x_1,t)t^2$ *and* $\Theta(x_1,t) = \overline{\beta}(x_1,t)t^2$, *with* $|\overline{\beta}|, |\beta_{3,\delta}| \le C_\delta$,

(ii) $\partial_{x_1} z_2(x_1,t) = \gamma_{3,\delta}(x_1,t)t^2$ *and* $\partial_{x_1}\Theta(x_1,t) = \overline{\gamma}(x_1,t)t^2$, *with* $|\overline{\gamma}|, |\gamma_{3,\delta}| \le C_\delta$,

(iii) $\partial_t z_2(x_1,t) = c(0) + \mu_{3,\delta}(x_1,t)t$ *and* $\partial_t\Theta(x_1,t) = \partial^2_{x_1x_1}\Theta(x_1,t) = \overline{\mu}(x_1,t)t$, *with* $|\overline{\mu}|, |\mu_{3,\delta}| \le C_\delta$,

(iv) $\partial^2_{x_1x_1} z_2(x_1,t) = \lambda_{3,\delta}(x_1,t)t$, *with* $|\lambda_{3,\delta}| \le C_\delta$.

Now, consider the functions $U(x,t) = (0, z_2(x_1,t), z_3(x_1,t))$, $\Theta = \Theta(x_1,t)$ as before and $q(x,t) = -(x_2+x_3)c(t)$. Observe that (U,q,Θ) solves the following Boussinesq problem:

$$\begin{cases} U_t - \Delta U + (u\cdot\nabla)U + \nabla q = \Theta e_3, & \text{in} \quad \mathscr{G}\times(0,T)\\ \nabla\cdot U = 0 & \text{in} \quad \mathscr{G}\times(0,T)\\ \Theta_t - \Delta\Theta + U\cdot\nabla\Theta = 0 & \text{in} \quad \mathscr{G}\times(0,T)\\ U(0,x_2,x_3,t) = 0,\ \ \Theta(0,x_2,x_3,t) = 0 & \text{on} \quad \mathbb{R}^2\times(0,T)\\ U(x,0) = 0,\ \ \Theta(x,0) = 0 & \text{in} \quad \mathscr{G}.\end{cases} \tag{2.15}$$

In the proof of Theorem 2.1.3, the construction of the solution to (2.2) is divided into four steps. In one of them, (u,θ,p) is written in the form

$$\begin{aligned} u(x,t) &= N^2U(x,t)+y(x,t)-\xi(t)W(x,t), && (x,t)\in\Omega\times(T_1,T_2)\\ \theta(x,t) &= N^2\Theta(x,t)+h(x,t), && (x,t)\in\Omega\times(T_1,T_2)\\ p(x,t) &= N^2q(x,t)+\xi(t)r(x,t), && (x,t)\in\Omega\times(T_1,T_2), \end{aligned}$$

where (y,h) is the solution to a transport equation and W is the solution to a linear Stokes system. In the next two paragraphs, we construct (y,h) and W and we prove some estimates.

For any $\delta>0$, we define

$$C^0_\delta(\overline{\mathscr{G}}\times[0,2/N])^4 := \{\,(y,h)\in C^0(\mathscr{G}\times[0,2/N])^4 : y=0,\, h=0 \text{ for } x_1\in[0,\delta]\,\}.$$

2.2.2.1 Another transport system

For an arbitrary initial condition extended by zero on $\mathscr{G}$ and for some $N>0$ large enough (to be defined precisely later), we solve the following null controllability problem for the transport equation

$$\begin{cases} y_t+N^2(U\cdot\nabla)y+N^2(y\cdot\nabla)U=he_3 & \text{in } Q_{2/N}\\ h_t+N^2U\cdot\nabla h+N^2y\cdot\nabla\Theta=0 & \text{in } Q_{2/N}\\ y(0,x_2,x_3,t)=0,\ h(0,x_2,x_3,t)=0 & \text{on } \mathbb{R}^2\times(0,2/N)\\ y(x,0)=y_0(x),\ h(x,0)=h_0(x) & \text{in } \mathscr{G}, \end{cases} \tag{2.16}$$

where $Q_{2/N}=\mathscr{G}\times(0,2/N)$.

Lemma 2.2.2 *Let us assume that* $(y_0,h_0)\in(C^1_0(\Omega)\cap V_0(\Omega))\times C^1_0(\Omega)$. *For each* $\delta>0$, *there exists* $N_0(\delta)$ *such that, for any* $N\geq N_0(\delta)$, *there exist a solution* (y,h) *to* (2.16) *and a positive constant* $C(\delta)$ *(independent of N) such that* $(y,h)\in C^0_\delta(\overline{\mathscr{G}}\times[0,2/N])$,

$$\|y\|_{C^1_\delta(\overline{Q}_{2/N})}+\|h\|_{C^1_\delta(\overline{Q}_{2/N})}\leq C(\delta)\|(y_0,h_0)\|_{C^1(\overline{\Omega})}, \tag{2.17}$$

$$\|y_t\|_{C^1_\delta(\overline{Q}_{2/N})}+\|h_t\|_{C^1_\delta(\overline{Q}_{2/N})}\leq C(\delta)\|(y_0,h_0)\|_{C^1(\overline{\Omega})} \tag{2.18}$$

and

$$y(x,t)=0,\ \ h(x,t)=0\quad \forall(x,t)\in\Omega\times(1/N,2/N).$$

The proof is very similar to the proof of Lemma 2.2.1. For brevity, it is left to the reader.

2.3 Proof of Theorem 2.1.3

Let us first prove the controllability result in Theorem 2.1.3.

As mentioned above, we will closely follow the arguments in the proof of Theorem 1 in (GUERRERO *et al.*, 2012). As there, we consider several steps, each one related to a time subinterval.

- FIRST STEP: We know that there exists at least one weak solution (u,p,θ) to the problem

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla p = \theta e_3 + f, \ \ \nabla\cdot u = 0 & \text{in } \ Q \\ \theta_t - \Delta\theta + u\cdot\nabla\theta = g & \text{in } \ Q \\ u = 0, \ \ \theta = 0 & \text{on } \ \partial\Omega\times(0,T) \\ (u(x,0),\theta(x,0)) = (u_0,\theta_0) & \text{in } \ \Omega, \end{cases}$$

with

$$u\in L^2(0,T;V_0(\Omega))\cap L^\infty(0,T;H(\Omega)), \ \theta\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)).$$

Let $\tilde{T}_1\in(T-\delta_0,T)$ be such that $(\tilde{u}_1,\tilde{\theta}_1) := (u(\cdot\,,\tilde{T}_1),\theta(\cdot\,,\tilde{T}_1))\in V_0(\Omega)\times H_0^1(\Omega)$ and

$$\|f\|_{L^2(T-\delta,T;V_0'(\Omega))} + \|g\|_{L^2(T-\delta,T;H^{-1}(\Omega))} \leq \frac{\varepsilon}{5}.$$

For any small $\eta>0$ with $\tilde{T}_1+\eta<T$, there exists a unique strong solution (u,p,θ) to the Boussinesq problem in $\Omega\times((\tilde{T}_1,\tilde{T}_1+\eta)$, with $(u(\cdot\,,\tilde{T}_1),\theta(\cdot\,,\tilde{T}_1)) = (\tilde{u}_1,\tilde{\theta}_1)$ (see for instance (CONSTANTIN; FOIAS, 1988)) and there exist many $T_1\in(\tilde{T}_1,\tilde{T}_1+\eta)$ with

$$(u(\cdot\,,T_1),\theta(\cdot\,,T_1)) \in (H^2(\Omega)^3\cap V_0(\Omega))\times(H^2(\Omega)\cap H_0^1(\Omega)).$$

On the interval $(0,T_1)$ we do not exert any control and we take

$$u_\varepsilon := u, \quad p_\varepsilon := p, \quad f_\varepsilon := f, \quad \theta_\varepsilon := \theta, \quad g_\varepsilon := g.$$

- SECOND STEP: Let us write $(u_1,\theta_1) := (u(\cdot\,,T_1),\theta(\cdot\,,T_1))$ and let us take $u_{1,\alpha}\in V_0(\Omega)\cap C_0^\infty(\Omega)^3$ and $\theta_{1,\alpha}\in C_0^\infty(\Omega)$ with

$$(u_{1,\alpha},\theta_{1,\alpha}) \to (u_1,\theta_1) \text{ in } \ V_0(\Omega)\times H_0^1(\Omega) \text{ as } \alpha\to 0^+$$

and

$$\|u_{1,\alpha}\|_{V_0(\Omega)} + \|\theta_{1,\alpha}\|_{H_0^1(\Omega)} \leq 2\,(\|u_1\|_{V_0(\Omega)} + \|\theta_1\|_{H_0^1(\Omega)}).$$

Let $T_2 \in (T_1, T)$ be a time; its precise value will be given below. We introduce now $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ in (T_1, T_2), with

$$u_\varepsilon = \frac{t - T_1}{T_2 - T_1}\,u_{1,\alpha} + \frac{T_2 - t}{T_2 - T_1}\,u_1, \quad p_\varepsilon = 0, \quad f_\varepsilon = \mathscr{L}u_\varepsilon - \theta_\varepsilon e_3,$$

$$\theta_\varepsilon = \frac{t - T_1}{T_2 - T_1}\,\theta_{1,\alpha} + \frac{T_2 - t}{T_2 - T_1}\,\theta_1, \quad g_\varepsilon = \mathscr{M}_\varepsilon \theta_\varepsilon,$$

where

$$\begin{aligned} \mathscr{L}u_\varepsilon &:= \partial_t u_\varepsilon - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon \\ \mathscr{M}_\varepsilon \theta_\varepsilon &:= \partial_t \theta_\varepsilon - \Delta \theta_\varepsilon + u_\varepsilon \cdot \nabla \theta_\varepsilon. \end{aligned}$$

It is then clear that

$$(u_\varepsilon(\cdot\,,T_1), \theta_\varepsilon(\cdot\,,T_1)) = (u_1, \theta_1), \;\; (u_\varepsilon(\cdot\,,T_2), \theta_\varepsilon(\cdot\,,T_2)) = (u_{1,\alpha}, \theta_{1,\alpha}), \;\; \nabla \cdot u_\varepsilon = 0$$

and the couple $(f_\varepsilon, g_\varepsilon)$ satisfies $(f_\varepsilon, g_\varepsilon) \in L^2(Q)^3 \times L^2(Q)$,

$$\begin{aligned} \|f_\varepsilon\|_{L^2(T_1,T_2;V_0'(\Omega))} &\leq \frac{C}{\sqrt{T_2 - T_1}} \|u_{1,\alpha} - u_1\|_{V_0(\Omega)} \\ &+ C\sqrt{T_2 - T_1}\left(\|u_1\|_{H_0^1(\Omega)^3} + \|u_1\|^2_{H_0^1(\Omega)^3} + \|\theta_1\|_{H_0^1(\Omega)}\right) \end{aligned}$$

and

$$\begin{aligned} \|g_\varepsilon\|_{L^2(T_1,T_2;H^{-1}(\Omega))} &\leq \frac{C}{\sqrt{T_2 - T_1}} \|\theta_{1,\alpha} - \theta_1\|_{H_0^1(\Omega)} \\ &+ C\sqrt{T_2 - T_1}\left(\|\theta_1\|_{H_0^1(\Omega)} + \|\theta_1\|_{H_0^1(\Omega)} \|u_1\|_{H_0^1(\Omega)^3}\right). \end{aligned}$$

Accordingly, we can choose first T_2 close enough of T_1 and then α small enough to have

$$\|f_\varepsilon\|_{L^2(T_1,T_2;V_0'(\Omega))} + \|g_\varepsilon\|_{L^2(T_1,T_2;H^{-1}(\Omega))} \leq \frac{\varepsilon}{5}.$$

- THIRD STEP: Let us set

$$u_2 := u_\varepsilon(\cdot\,,T_2) \in V_0(\Omega) \cap C_0^\infty(\Omega)^3, \;\; \theta_2 := \theta_\varepsilon(\cdot\,,T_2) \in C_0^\infty(\Omega).$$

We will work in the interval $(T_2, T_2 + 2/N)$, where $N \geq N(\delta)$, $N(\delta)$ is furnished by Lemma 2.2.2 and δ is such that the supports of u_2 and θ_2 are contained in $[\delta, 1-\delta]^3$.

In this step, we will take

$$\begin{aligned} u_\varepsilon(x,t) &= N^2\tilde{U}(x,t)+\tilde{y}(x,t)-\xi(t-T_2)\tilde{W}(x,t), \\ p_\varepsilon(x,t) &= -N^2(x_2+x_3)\,c(t-T_2)+\tilde{r}(x,t), \\ \theta_\varepsilon(x,t) &= N^2\tilde{\Theta}(x,t)+\tilde{h}(x,t) \end{aligned}$$

and

$$\begin{aligned} f_\varepsilon &= -\Delta\tilde{y}+((\tilde{y}-\tilde{W})\cdot\nabla)(\tilde{y}-\tilde{W})-N^2(\tilde{U}\cdot\nabla)\tilde{W}-N^2(\tilde{W}\cdot\nabla)\tilde{U}-\xi_t\tilde{W}, \\ g_\varepsilon &= -\Delta\tilde{h}+(\tilde{y}-\tilde{W})\cdot\nabla\tilde{h}-N^2\tilde{W}\cdot\nabla\tilde{\Theta}. \end{aligned}$$

Here, $\tilde{U}$, $\tilde{\Theta}$, etc. are respectively U, Θ, etc. written at time $t-T_2$, (U,θ) is the solution to (2.15), (y,h) is the solution to (2.16) with initial data $(y_0,h_0)=(u_2,\theta_2)$, (W,r) is the solution to (2.10) and $\xi\in C^2([0,2/N])$ is a cut-off function satisfying

$$\xi(t)=1 \text{ in } (0,1/N) \text{ and } \xi(t)=0 \text{ in a neighborhood of } 2/N.$$

From the properties of (y,h) deduced in Lemma 2.2.2 and the definitions of U and W, we have the following:

$$(u_\varepsilon,\theta_\varepsilon)(x,T_2+2/N)\equiv N^2(U,\Theta)(x_1,2/N), \quad \nabla\cdot u_\varepsilon\equiv 0$$

and

$$u_\varepsilon(0,x_2,x_3,t)=0, \quad \theta_\varepsilon(0,x_2,x_3,t)=0 \text{ in } (0,1)^2\times(T_2,T_2+2/N).$$

Let us check that, for N large enough, we have

$$\|f_\varepsilon\|_{L^2(T_2,T_2+2/N;V_0'(\Omega))}+\|g_\varepsilon\|_{L^2(T_2,T_2+2/N;H^{-1}(\Omega))}\leq\frac{\varepsilon}{5}.$$

First, note that Lemma 2.2.2 yields

$$\|\Delta\tilde{y}\|_{L^2(T_2,T_2+2/N;V_0'(\Omega))}+\|\Delta\tilde{h}\|_{L^2(T_2,T_2+2/N;H^{-1}(\Omega))}\leq\frac{C}{N^{1/2}}.$$

Let us decompose Ω in two sets

$$\Omega_1 := (0,\delta/2)\times(0,1)^2 \text{ and } \Omega_2 := (\delta/2,1)\times(0,1)^2.$$

Recall that $\nabla \cdot y = \nabla \cdot W$ in $Q_{2/N}$ and $y = 0$ in Ω_1. Consequently,

$$\begin{aligned}
\|N^2(\tilde{W}\cdot\nabla)\tilde{U}\|_{V_0'(\Omega)} &= \sup_{\beta\in V_0(\Omega),\|\beta\|_{V_0(\Omega)}=1} \int_{\Omega_1} N^2(\tilde{W}\cdot\nabla)\tilde{U}\beta\,dx \\
&\quad + \sup_{\beta\in V_0(\Omega),\|\beta\|_{V_0(\Omega)}=1} \int_{\Omega_2} N^2(\tilde{W}\cdot\nabla)\tilde{U}\beta\,dx \\
&= - \sup_{\beta\in V_0(\Omega),\|\beta\|_{V_0(\Omega)}=1} \int_{\Omega_1} N^2\tilde{W}\cdot\nabla\beta\,\tilde{U}\,dx \\
&\quad + \sup_{\beta\in V_0(\Omega),\|\beta\|_{V_0(\Omega)}=1} \int_{\Omega_2} N^2(\tilde{W}\cdot\nabla)\tilde{U}\beta\,dx.
\end{aligned}$$

The first term is bounded by $C\|N\tilde{W}\|_{L^2(\Omega)}\|N\tilde{U}\|_{L^\infty(\Omega)}$. On the other hand,

$$\int_{\Omega_2} N^2(\tilde{W}\cdot\nabla)\tilde{U}\beta\,dx \le \|N\nabla\tilde{U}\|_{L^\infty((\delta/2,1)\times\mathbb{R}^2)}\|N\tilde{W}\|_{L^2(\Omega)}.$$

Thanks to Propositions 2.2.2 and 2.2.3, there exists $C(\delta) > 0$ such that

$$\|N\nabla\tilde{U}\|_{L^\infty(T_2,T_2+2/N;L^\infty((\delta/2,1)\times\mathbb{R}^2)} \le C.$$

Therefore, we see from (2.12) that

$$\begin{aligned}
\|N^2(\tilde{W}\cdot\nabla)\tilde{U}\|_{L^r(T_2,T_2+2/N;V_0'(\Omega))} &\le C\left(\int_{T_2}^{T_2+2/N} \|NW\|^r_{L^2(\Omega)}\,dt\right)^{1/r} \\
&\le CN^{3/4-1/r}.
\end{aligned}$$

Similarly, the following estimate can be obtained:

$$\begin{aligned}
\|N^2\tilde{W}\cdot\nabla\Theta\|_{L^r(T_2,T_2+2/N;H^{-1}(\Omega))} &+ \|N^2(\tilde{U}\cdot\nabla)\tilde{W}\|_{L^r(T_2,T_2+2/N;V_0'(\Omega))} \\
&\le CN^{3/4-1/r}.
\end{aligned}$$

Next, using (2.11), we deduce that

$$\|\xi_t W\|_{L^r(T_2,T_2+2/N;L^2(\Omega)^3)} \le \|W\|_{L^r(T_2,T_2+2/N;L^2(\Omega))} \le C(r)N^{-1/r}\|u_2\|_{C^1(\Omega)}.$$

Also,

$$\|((\tilde{y}-\tilde{W})\cdot\nabla)(\tilde{y}-\tilde{W})\|_{L^2(T_2,T_2+2/N;V_0')} \le C\|\tilde{y}-\tilde{W}\|^2_{L^4(T_2,T_2+2/N;L^4(\Omega)^3)}$$

and, from Lemmas 2.2.2 and (2.11), this quantity goes to zero as $N\to+\infty$. Similarly,

$$\begin{aligned}
&\|(\tilde{y}-\tilde{W})\cdot\nabla h\|_{L^2(T_2,T_2+2/N;H^{-1})} \\
&\quad \le C\|\tilde{h}\|_{L^4(T_2,T_2+2/N;L^\infty(\Omega))}\|\tilde{y}-\tilde{W}\|_{L^4(T_2,T_2+2/N;L^2(\Omega)^3)} \to 0
\end{aligned}$$

as $N\to+\infty$. This concludes the step.

• FOURTH STEP:

Finally, we set $T_3 := T_2 + 2/N$ and we work in the interval (T_3, T). Note that $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ arrives at $t = T_3$ with the following structure:

$$\begin{aligned} u_\varepsilon(x, T_3) &= \left(0, N^2 z_2(x_1, 2/N), N^2 z_3(x_1, 2/N)\right), \\ \theta_\varepsilon(x, T_3) &= N^2(\partial_t z_3 + \Delta z_3)(x_1, 2/N). \end{aligned}$$

The second component of u_ε can be driven to zero at time $t = T$ by solving a standard null controllability problem for a linear heat equation. On the other hand, the third component of $u_\varepsilon$and θ_ε can be driven to zero by solving a (less standard) null controllability problem for a system of two coupled 1D parabolic PDEs.

Indeed, let us take $f_\varepsilon = 0$ and $g_\varepsilon = 0$ in (T_3, T). It is well-known that there exists $\rho \in L^\infty(0, T - T_3)$ such that the solution to

$$\begin{cases} \partial_t \bar{z} - \partial^2_{x_1 x_1} \bar{z} = 0, & \text{in } \ (0,1) \times (0, T - T_3) \\ \bar{z}(0,t) = 0, \ \ \bar{z}(1,t) = \rho(t), & \text{on } \ (0, T - T_3) \\ \bar{z}(x_1, 0) = N^2 z_2(x_1, 2/N) & \text{in } \ (0,1) \end{cases}$$

satisfies

$$\bar{z}(x_1, T - T_3) = 0 \ \text{ in } \ (0,1).$$

On the other hand, it is proved in (FERNÁNDEZ-CARA *et al.*, 2010) that, if $A \in \mathscr{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$, μ_1 and μ_2 are the eigenvalues of A, $\text{rank}[B|AB] = 2$, $(T/\pi)(\mu_1 - \mu_2)$ is not a integer of the form $4(m+1)$ or $2m+1$ with $m \geq 1$ and $\bar{y}_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0, T - T_3)$ such that the associated solution to the system

$$\begin{cases} \partial_t \bar{y} - \partial^2_{x_1 x_1} \bar{y} = A\bar{y}, & \text{in } \ (0,1) \times (0, T - T_3) \\ \bar{y}(0,t) = 0, \ \ \bar{y}(1,t) = Bv, & \text{on } \ (0, T - T_3) \\ \bar{y}(x_1, 0) = \bar{y}_0(x_1) & \text{in } \ (0,1) \end{cases} \tag{2.19}$$

satisfies

$$\bar{y}(x_1, T - T_3) = 0 \ \text{ in } \ (0,1). \tag{2.20}$$

Then, it suffices to define u_ε and θ_ε in (T_3, T) as follows:

$$\begin{cases} u_\varepsilon(x,t) = (0, \bar{z}(x_1, t - T_3), \bar{y}_1(x_1, t - T_3)) \\ \theta_\varepsilon(x,t) = \bar{y}_2(x_1, t - T_3), \end{cases}$$

where $(\bar{y}_1, \bar{y}_2)$ is, together with some v, a solution to the problem (2.19)–(2.20) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{y}_0 = N^2(z_3, \partial_t z_3 + \Delta z_3)(x_1, 2/N).$$

Finally,

$$u_\varepsilon(\cdot, T) = 0, \quad \theta_\varepsilon(\cdot, T) = 0$$

and we clearly have

$$\|f - f_\varepsilon\|_{L^r(0,T;V_0'(\Omega))} + \|g - g_\varepsilon\|_{L^r(0,T;H^{-1}(\Omega))} \leq \varepsilon.$$

2.4 Proofs of Theorems 2.1.1 and 2.1.2

The proof of Theorem 2.1.1 is similar (and, again, is inspired by the arguments in (GUERRERO *et al.*, 2012)).

For brevity, we will only give an idea of what is actually different from the proof of Theorem 2.1.3.

The first and second steps are almost identical (of course, there is no θ_ε now). In the third step, we take again $T_3 = T_2 + 2/N$ and we introduce

$$u_\varepsilon(x,t) = N^2 U(x, t - T_2) + y(x, t - T_2) + \theta(t - T_2) W(x, t - T_2),$$

$$p_\varepsilon(x,t) = N^2 x_2 c(x, t - T_2) - r(x, t - T_2),$$

where the functions U, y, θ, W, r and c are, this time, as in Section 2.2.

It is easy to check that $(u_\varepsilon, p_\varepsilon)$ solves

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + p_\varepsilon = f_\varepsilon & \text{in } \Omega \times (T_2, T_3) \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega \times (T_2, T_3) \\ u_\varepsilon(0, x_2, x_3, t) = u_\varepsilon(1, x_2, x_3, t) = u_\varepsilon(x_1, x_2, 0, t) = 0 & \text{on } (0,1)^2 \times (T_2, T_3) \\ u_\varepsilon(x, 0) = u_0(T_2) & \text{in } \Omega, \end{cases}$$

where we have set $T_3 = T_2 + 2/N$ and

$$\begin{aligned} f_\varepsilon(x,t) = {} & (-\Delta y + N^2 (u \cdot \nabla) \theta W + N^2 (\theta W \cdot \nabla) U + \theta_t W)(x, t - T_2) \\ & + ((y + \theta W) \cdot \nabla)(y + \theta W))(x, t - T_2). \end{aligned}$$

From (2.4), Lemma 2.2.1 and Proposition 2.2.1, it can be easily checked that, for N large enough, one has

$$\|f_\varepsilon\|_{L^2(T_2,T_2+2/N;V'(\Omega))} \leq \frac{\varepsilon}{5}$$

and

$$u_\varepsilon(x,T_2+2/N) \equiv N^2 U(x_1,x_3,2/N).$$

In the fourth step, we take $T_3 := T_2+2/N$ and we note that u_ε possesses at time T_3 the structure

$$u_\varepsilon(x,T_2+2/N) = (0,N^2 z(x_1,x_3,2/N),0).$$

The second coordinate of u_ε can be driven to zero at time $t=T$ by solving a null controllability problem for a linear 2D heat equation.

More precisely, let us take $f_\varepsilon = 0$ in (T_3,T). Then, there exist controls $\rho = \rho(x_1,t)$ in $L^\infty((0,1)\times(0,T-T_3))$ such that the associated solution to

$$\begin{cases} \bar{z}_t - (\partial_{x_1x_1}\bar{z} + \partial_{x_3x_3}\bar{z}) = c(t) & \text{in } \ (0,1)^2\times(0,T-T_3) \\ \bar{z}(0,x_3,t) = \bar{z}(1,x_3,t) = \bar{z}(x_1,0,t) = 0 & \text{on } \ (0,1)\times(0,T-T_3) \\ \bar{z}(x_1,1,t) = \rho(x_1,t) & \text{on } \ (0,1)\times(0,T-T_3) \\ \bar{z}(x_1,x_3,0) = N^2 z(x_1,x_3,2/N) & \text{in } \ (0,1)^2 \end{cases}$$

satisfies

$$\bar{z}(x_1,T-T_3) = 0 \ \text{ in } \ (0,1)$$

Then, it is sufficient to take in (T_3,T)

$$u_\varepsilon(x,t) = (0,\bar{z}(x_1,x_3,t-T_3),0).$$

This way, we get

$$u_\varepsilon(\cdot,T) = 0$$

and

$$\|f - f_\varepsilon\|_{L^r(0,T;V_0'(\Omega))} \leq \varepsilon.$$

We now give the proof of Theorem 2.1.2.

In fact, Theorem 2.1.2 can be viewed as a corollary of Theorem 1 in (GUERRERO *et al.*, 2012). Indeed, let $R \in \mathbb{R}^3$ be a cube, with edges not necessarily parallel to the axes and let us denote by Γ_0 one of its faces. It is clear that, after appropriate rotation and translation, we can construct right hand sides $f_\varepsilon \in L^2(R \times (0,T))$ satisfying

$$f_\varepsilon \to f \text{ in } \; L^r(0,T;H^{-1}(R)),$$

for all $r \in (1,4/3)$ and solutions $(v_\varepsilon, p_\varepsilon)$ to the corresponding Navier-Stokes systems

$$\begin{cases} v_{\varepsilon,t} - \Delta v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in } \; R \times (0,T) \\ \nabla \cdot v_\varepsilon = 0 & \text{in } \; R \times (0,T) \\ v_\varepsilon = 0 & \text{on } \Gamma_0 \times (0,T) \\ v_\varepsilon(x,0) = u_0(x) & \text{in } \; R \end{cases}$$

that satisfy

$$u_\varepsilon(x,T) = 0 \;\text{ in } \;\; R.$$

Now, let R be a cube with one face on π such that $\Omega_\pi \subset R$. Then, we just take

$$u_\varepsilon := v_\varepsilon\big|_{\Omega_\pi \times (0,T)}$$

and we immediately conclude.

Remark 2.4.1 *Let us set $\Gamma_1 = \partial\Omega \setminus (\{0\} \times (0,1)^2)$ and let $\mathscr{O}$ be a neighborhood of Γ_1 in Ω. It is not difficult to obtain from Theorem 1 in (GUERRERO et al., 2012) a global partial approximate controllability result of the same kind for the Navier-Stokes system with distributed controls, supported by $\mathscr{O} \times (0,T)$. However, a similar result for the Boussinesq system is, to our knowledge, unknown.* □

2.5 Final comments and questions

Results similar to Theorems 2.1.1 and 2.1.2 can be deduced for the Boussinesq system. We leave the details to the reader.

Actually, the previous results do not imply global approximate controllability, since the right hand sides f and g have to be modified (the same can be said on the results in (GUERRERO *et al.*, 2012)). What we would need is a uniform bound of the controls in some Banach space B allowing to take limits as ε to 0. But, at present, this is missing.

Thus, it would be interesting to be able to modify the constructions of u_ε and θ_ε paying special attention to the behavior of their traces.

Another possible approach relies on the following idea:

1. Solve the extremal problems

$$\begin{cases} \text{Minimize } J_\varepsilon(h) = \|h\|_B \\ \text{Subject to } h \in \mathscr{B} \end{cases}$$

   where $\mathscr{B}$ is the family of boundary null controls for the Boussinesq system with f replaced by f_ε that belong to a suitable Banach space B.

2. Then, prove that the solutions satisfy

$$\|\tilde{h}_\varepsilon\|_B \leq C.$$

Observe that, with a good choice of B, all these problems are solvable. Therefore, one can probably use an optimality characterization to get some information. f

3 LOCAL NULL CONTROLLABILITY OF A NONLINEAR PARABOLIC SYSTEM IN DIMENSION 1

3.1 Introduction

Let $I \subset \mathbb{R}$ be an open bounded interval. Let us denote by Q the cylinder $Q := I \times (0,T)$, with lateral boundary $\Sigma := \partial I \times (0,T)$. Also consider a non-empty open set of $\omega \subset I$; as usual, 1_ω denotes the characteristic function of ω.

We will be concerned with the null controllability of the nonlinear systems

$$\begin{cases} y_t - (a(y)y_x)_x = v_1 1_\omega & \text{in} \quad Q \\ y(x,t) = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0(x) & \text{in} \quad I \end{cases} \tag{3.1}$$

and

$$\begin{cases} y_t - (a(y)y_x)_x = 0 & \text{in} \quad (0,1)\times(0,T) \\ y(0,t) = v_2(t),\ y(1,t) = 0 & \text{on} \quad (0,T) \\ y(x,0) = y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.2}$$

where v_1 and v_2 are the controls and y is the associated state. It will be assumed that the real function $a = a(r)$ is of class C^1, possesses bounded derivatives and satisfies

$$0 < m \leq a(r) \leq M \quad \forall r \in \mathbb{R}.$$

Definition 3.1.1 *It will be said that* (3.1) *(resp.* (3.2)*) is locally null-controllable at time T if there exists $\varepsilon > 0$ such that, for any $y_0 \in H_0^1(I)$ with*

$$\|y_0\|_{H_0^1}(I) \leq \varepsilon,$$

there exist controls $v_1 \in L^2(\omega \times (0,T))$ (resp., $v_2 \in L^2(0,T)$) such that the associated states y satisfy

$$y(x,T) = 0 \ \text{ in } \ I. \tag{3.3}$$

The controllability of linear and nonlinear parabolic PDEs and systems has been the objective of a lot of work the last decades. Some relevant contributions on the subject are (FABRE C.; ZUAZUA, 1995; FURSIKOV; IMANUVILOV, 1996; DOUBOVA *et al.*, 2002; FERNÁNDEZ-CARA; ZUAZUA, 2000). Our main result in this paper is the following:

Theorem 3.1.1 *Under the previous assumptions on a, the nonlinear system* (3.1) *is locally null-controllable at any time* $T > 0$.

A consequence of Theorem 3.1.1 is the local null controllability of (3.2). Thus, our second result is the following:

Theorem 3.1.2 *Under the previous assumptions on a, the nonlinear system* (3.2) *is locally null-controllable at any time* $T > 0$.

The proof of Theorem 3.1.1 relies on an application of *Liusternik's Inverse Function Theorem* in Banach spaces, see (ALEKSEEV *et al.*, 1987). We will follow the ideas of (CLARK *et al.*, 2013); this paper is in turn inspired by the works of Fursikov and Imanuvilov (FURSIKOV; IMANUVILOV, 1996) and Imanuvilov and Yamamoto (IMANUVILOV; YAMAMOTO, 2003).

Thus, in a first step, we consider the following linearized system at zero

$$\begin{cases} y_t - a(0)y_{xx} = v_1 1_\omega + h & \text{in} \quad Q \\ y(x,t) = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0(x), & \text{in} \quad I. \end{cases} \tag{3.4}$$

The adjoint of (3.4) is given by

$$\begin{cases} -\varphi_t - a(0)\varphi_{xx} = F & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi_0(x), & \text{in} \quad I. \end{cases} \tag{3.5}$$

The null controllability of (3.4) (for appropriate h) will be obtained as a consequence of a suitable Carleman inequality for the solutions to (3.5).

In a second step, we rewrite the null controllability property of (3.1) as an equation of the form

$$H(y,v) = (0, y_0)$$

in a well chosen space Y of "admissible" state-controls, see the definitions of Y and H in Section 3.

The paper is organized as follows. Section 2 is devoted to prove the null controllability of linearized system (3.4). In Section 3, we will prove Theorem 3.1.1. Section 4 is devoted to the proof of Theorem 3.1.2. Finally, in Section 5 we will present some additional comments and questions.

3.2 Carleman inequalities and the null controllability of (3.4)

We will now recall a Carleman inequality satisfied by the solutions to (3.5). It will be convenient to introduce a new non-empty open set ω_0, with $\omega_0 \Subset \omega$.

The following technical result is fundamental:

Lemma 3.2.1 *There exists a function $\alpha_0 \in C^2(\bar{I})$ satisfying*

$$\begin{cases} \alpha_0(x) > 0 & \forall x \in I \\ \alpha_0 = 0 & \forall x \in \partial I \\ |\alpha_{0,x}(x)| > 0 & \forall x \in \bar{I} \setminus \omega_0. \end{cases}$$

Let us introduce the functions

$$\beta(t) := t(T-t), \quad \phi(x,t) := \frac{e^{\lambda \alpha_0(x)}}{\beta(t)}, \quad \alpha(x,t) := \frac{e^{R\lambda} - e^{\lambda \alpha_0(x)}}{\beta(t)},$$

where $R > \|\alpha_0\|_{L^\infty}$ and $\lambda > 0$.

Then one has the following:

Proposition 3.2.1 *There exist positive constants λ_0, s_0 and c_0 such that, for any $s \geq s_0$ and $\lambda \geq \lambda_0$, any $F \in L^2(Q)$ and any $\varphi_0 \in L^2(I)$, the associated solution to* (3.5) *satisfies*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}[(s\phi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\phi)|\varphi_x|^2 + \lambda^4(s\phi)^3|\varphi|^2]\,dx\,dt \\ &\quad \leq C_0 \left(\iint_Q e^{-2s\alpha}|F|^2\,dx\,dt + \iint_{\omega_0 \times (0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\varphi|^2]\,dx\,dt \right). \end{aligned}$$

Furthermore, C_0 and λ_0 only depend on I and ω.

The next result contains a Carleman inequality for the solution to (3.5) with weights not vanishing at zero. Let m be a function satisfying

$$m \in C^\infty([0,T]), \quad m(t) \geq \frac{T^2}{8} \quad \text{in} \quad [0, T/2], \quad m(t) = t(T-t) \quad \text{in} \quad [T/2, T],$$

let us set

$$\zeta(x,t) := \frac{e^{\lambda \alpha_0(x)}}{m(t)}, \quad A(x,t) := \frac{e^{R\lambda} - e^{\lambda \alpha_0(x)}}{m(t)} \quad \text{with} \quad R > \|\alpha_0\|_{L^\infty}, \quad \lambda > 0$$

and let us introduce the notation

$$\Gamma(s,\lambda,\varphi) := \iint_Q e^{-2sA}[(s\zeta)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\zeta)|\varphi_x|^2 + \lambda^4(s\zeta)^3|\varphi|^2]\,dx\,dt.$$

Proposition 3.2.2 *There exist positive constants λ_1, s_1 and C_1 such that, for any $s \geq s_1$ and $\lambda \geq \lambda_1$, any $F \in L^2(Q)$ and any $\varphi_T \in L^2(I)$, the associated solution to (3.5) satisfies*

$$\Gamma(s,\lambda,\varphi) \leq C_1(s,\lambda)\left(\iint_Q e^{-2sA}\zeta^3|F|^2\,dx\,dt + \iint_{\omega\times(0,T)} e^{-2s\alpha}\phi^7|\varphi|^2\,dx\,dt\right).$$

Furthermore, λ_1 and s_1 only depend on I, ω, T and $a(0)$.

See the detailed proof of Proposition 3.2.2 in (CLARK *et al.*, 2013).

In order to simplify the notation, we fix $\lambda = \lambda_1$ and $s = s_1$ and we set

$$\rho := e^{sA}, \quad \rho_0 := \zeta^{-3/2}e^{sA}, \quad \widehat{\rho} := \zeta^{-5/2}e^{sA}, \quad \rho_* := \zeta^{-7/2}e^{sA}.$$

With Proposition 3.2.2, we are able to show the null controllability of (3.4) for right hand sides h that decay sufficiently fast to zero as $t \to T$. More precisely, one has:

Proposition 3.2.3 *Assume that the function h satisfy*

$$\iint_Q \rho^2\zeta^{-3}|h|^2\,dx\,dt < +\infty.$$

Then (3.4) is null controllable. More precisely, for any $y_0 \in L^2(I)$, there exist controls $v_1 \in L^2(\omega\times(0,T))$ and associated states y satisfying

$$\iint_{\omega\times(0,T)} \rho_*^2|v_1|^2\,dx\,dt < +\infty, \quad \iint_Q \rho_0^2|y|^2\,dx\,dt < +\infty, \tag{3.6}$$

whence, in particular , $y(x,T) = 0$.

The proof of this result is classical; see (FURSIKOV; IMANUVILOV, 1996).

The next results furnish additional properties of the state found in Proposition 3.2.3. The will be needed in Section 3.

Proposition 3.2.4 *Let the hypotheses in Proposition 3.2.3 be satisfied and let v_1 and y satisfy (3.4) and (3.6). Then*

$$\left\{\begin{aligned} &\iint_Q \widehat{\rho}^2|y_x|^2\,dx\,dt \leq C\left(\iint_Q \rho_0^2|y|^2\,dx\,dt\right.\\ &\qquad\left.+\iint_{\omega\times(0,T)} \rho_*^2|v_1|^2\,dx\,dt + \iint_Q \rho_0^2|h|^2\,dx\,dt\right). \end{aligned}\right. \tag{3.7}$$

Proof: Multiplying (3.4) by $\widehat{\rho}^2 y$ and integrating in I, we get:

$$\int_I \widehat{\rho}^2(y_t - a(0)y_{xx})y\,dx = \int_I \widehat{\rho}^2(v_1 1_\omega + h)y\,dx.$$

Notice that

- $\int_I \widehat{\rho}^2 y_t y\,dx = \frac{1}{2}\frac{d}{dt}\left(\int_I \widehat{\rho}^2 |y|^2\,dx\right) - \int_I \widehat{\rho}\widehat{\rho}_t |y|^2\,dx$
- $-\int_I \widehat{\rho}^2 a(0) y_{xx} y\,dx = -\int_I a(0)(\widehat{\rho}^2)_{xx}|y|^2\,dx + \int_I \widehat{\rho}^2 a(0)|y_x|^2\,dx$
- $\int_I \widehat{\rho}^2 v_1 1_\omega y\,dx \le \frac{1}{2}\int_\omega \rho_0^2 |y|^2\,dx + \frac{1}{2}\int_\omega \rho_*^2 |v_1|^2\,dx$
- $\int_I \widehat{\rho}^2 (hy)\,dx \le \frac{1}{2}\int_I \widehat{\rho}^4 \rho_0^{-2}|y|^2\,dx + \frac{1}{2}\int_I \rho_0^2 |h|^2\,dx.$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\int_\omega \widehat{\rho}^2|y|^2\,dx\right) + \int_\omega \widehat{\rho}^2 a(0)|y_x|^2 \\ &\le C\left(\int_\omega (\widehat{\rho}^2 + \widehat{\rho}|\widehat{\rho}_t| + |(\widehat{\rho}^2)_{xx}| + \widehat{\rho}^4\rho_0^{-2})|y|^2\,dx\right. \\ &\quad \left. + \tfrac{1}{2}\int_\omega \rho_*^2|v_1|^2\,dx + \frac{1}{2}\int_\omega \rho_0^2|h|^2\,dx\right),\end{aligned}$$

whence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\int_\omega \widehat{\rho}^2|y|^2\,dx\right) + \int_\omega \widehat{\rho}^2 a(0)|y_x|^2 \\ \le C&\left(\int_\omega \rho_0^2\rho_0^{-2})|y|^2\,dx + \frac{1}{2}\int_\omega \rho_*^2|v|^2\,dx + \frac{1}{2}\int_\omega \rho_0^2|h|^2\,dx\right).\end{aligned}$$

Now, integrating in time, we get the desired estimate.

Proposition 3.2.5 *Let the hypotheses in Proposition 3.2.3 be satisfied, let v_1 and y satisfy* (3.4) *and* (3.6) *and let us assume that*

$$y_0 \in H_0^1(I). \tag{3.8}$$

Then one has

$$\begin{aligned}\iint_Q \rho_*^2(|y_t|^2 + |y_{xx}|^2)\,dx\,dt &\le C\left(\iint_Q \rho_0^2|y|^2\,dx\,dt\right. \\ &\left. + \iint_{\omega\times(0,T)} \rho_*^2|v_1|^2\,dx\,dt + \|y_0\|^2_{H_0^1(I)} + \iint_Q \rho_0^2|h|^2\,dx\,dt\right)\end{aligned} \tag{3.9}$$

Proof:

Multiplying (3.4) by $\rho_*^2 y_t$ and integrating in I

$$\int_I \rho_*^2(y_t - a(0)y_{xx})y_t\,dx = \int_I \rho_*^2(v_1 1_\omega + h)y_t\,dx. \tag{3.10}$$

Notice that

- $\int_I \rho_*^2 v_1 1_\omega y_t\,dx \le \frac{1}{8}\int_\omega \rho_*^2|y_t|^2\,dx + 2\int_I \rho_*^2|v_1|^2\,dx$
- $\int_I \rho_*^2 h y_t\,dx \le \frac{1}{8}\int_I \rho_*^2|y_t|^2\,dx + 2\int_I \rho_*^2|h|^2\,dx,$

- Also,

$$
\begin{aligned}
-\int_I \rho_*^2 a(0) y_{xx} y_t\,dx = \frac{1}{2}\frac{d}{dt}\int_I \rho_*^2 a(0)|y_x|^2\,dx \\
-\tfrac{1}{2}\int_I (\rho_*^2)_t a(0)|y_x|^2\,dx + \int_I (\rho_*^2)_x a(0) y_x y_t\,dx.
\end{aligned}
$$

Using the last equality in (3.10), we obtain that

$$
\begin{aligned}
&\int_I \rho_*^2 |y_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_I \rho_*^2 a(0)|y_x|^2\,dx \\
&\quad = \frac{1}{2}\int_I (\rho_*^2)_t a(0)|y_x|^2\,dx - \int_I (\rho_*^2)_x a(0) y_x y_t\,dx \\
&\quad + \int_I \rho_*^2 v_1 1_\omega y_t\,dx + \int_I \rho_*^2 h y_t\,dx.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\frac{1}{2}\int_I (\rho_*^2)_t a(0)|y_x|^2\,dx - \int_I (\rho_*^2)_x a(0) y_x y_t\,dx \\
&\qquad \le C\left(\int_I [(\rho_*^2)_t + (\rho_*^2)_x \rho_*^{-2}]|y_x|^2\,dx + \frac{1}{8}\int_I \rho_*^2 |y_t|^2\,dx\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\tfrac{1}{2}\int_I \rho_*^2 |y_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_I \rho_*^2 a(0)|y_x|^2\,dx \\
&\qquad \le C\left(\int_I [(\rho_*^2)_t + (\rho_*^2)_x \rho_*^{-2}]|y_x|^2\,dx + \int_\omega \rho_*^2 |v_1|^2\,dx + \int_I \rho_*^2 |h|^2\,dx\right).
\end{aligned}
$$

From the definition of the weight ρ_*, we have

$$
\begin{aligned}
&\frac{1}{2}\int_I \rho_*^2 |y_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_I \rho_*^2 a(0)|y_x|^2\,dx \\
&\qquad \le C\left(\int_I \widehat{\rho}_*^2 |y_x|^2\,dx + \int_\omega \rho_*^2 |v_1|\,dx + \int_I \rho_*^2 |h|^2\,dx\right).
\end{aligned}
$$

Integrating in time and recalling (3.8) and (3.7), we obtain the following estimate

$$
\begin{aligned}
\iint_Q \rho_*^2 |y_t|^2\,dx\,dt \le C\Bigg(&\iint_Q \rho_0^2 |y|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_*^2 |v_1|^2\,dx\,dt \\
&+ \|y_0\|^2_{H_0^1(I)} + \iint_Q \rho_0^2 |h|^2\,dx\,dt\Bigg).
\end{aligned}
\tag{3.11}
$$

Now, let us multiply (3.4) by $-\rho_*^2 y_{xx}$ and let us integrate in I. We find that

$$
\int_I \rho_*^2 (y_t - a(0) y_{xx})(-y_{xx})\,dx = \int_I \rho_*^2 (v 1_\omega + h)(-y_{xx})\,dx.
$$

Note that

$$
-\int_I \rho_*^2 y_t (y_{xx})\,dx = \frac{1}{2}\frac{d}{dt}\int_I \rho_*^2 |y_x|^2\,dx - \frac{1}{2}\int_I (\rho_*^2)_t |y_x|^2\,dx + \int_I (\rho_*^2)_x y_x y_t\,dx.
$$

Then we have

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\int_I \rho_*^2|y_x|^2\,dx+\int_I \rho_*^2 a(0)|y_{xx}|^2\,dx &= \frac{1}{2}\int_I (\rho_*^2)_t|y_x|^2\,dx \\
&-\int_I(\rho_*^2)_x y_t y_x\,dx-\int_I \rho_*^2 v_1 1_\omega y_{xx}\,dx-\int_I \rho_*^2 h y_{xx}\,dx\,dt
\end{aligned}
$$

We also have

- $\displaystyle\int_I (\rho_*^2)_x y_t y_x\,dx \le C\left(\int_I [(\rho_*^2)_x]^2\rho_*^{-2}|y_x|^2\,dx+\int_I \rho_*^2|y_t|^2\,dx\right),$
- $\displaystyle\int_\omega \rho_*^2 v_1 y_{xx}\,dx \le \frac{2}{a(0)}\int_\omega \rho_*^2|v_1|^2\,dx+\frac{a(0)}{8}\int_\omega \rho_*^2|y_{xx}|^2\,dx,$
- $\displaystyle-\int_I \rho_*^2 h y_{xx}\,dx \le C\int_I \rho_*^2|h|^2\,dx+\frac{a(0)}{8}\int_I \rho_*^2|v|^2\,dx.$

Consequently,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho_*^2|y_x|^2\,dx+\frac{1}{2}\int_I \rho_*^2 a(0)|y_{xx}|^2\,dx \\
&\quad\le C\left(\int_I [(\rho_*^2)_t+(\rho_*^2)_x^2\rho_*^{-2}]|y_x|^2\,dx+\int_I \rho_*^2|y_t|^2\,dx\right. \\
&\quad\left.+\int_\omega \rho_*^2|v_1|^2\,dx+\int_I \rho_*^2|h|^2\,dx\right).
\end{aligned}
$$

From the definition of the weight ρ_*, we obtain:

$$
\begin{aligned}
\tfrac{1}{2}\int_I \rho_*^2 a(0)|y_{xx}|^2 &\le C\left(\int_I \widehat{\rho}^2|y_x|^2\,dx+\int_I \rho_*^2|y_t|^2\,dx\right. \\
&\left.+\int_I \rho_*^2|v_1|^2\,dx+\int_I \rho_0^2|h|^2\,dx\right)
\end{aligned}
$$

and, integrating in time, the following is found:

$$
\begin{aligned}
\iint_Q \rho_*^2|y_{xx}|^2\,dx\,dt &\le C\left(\iint_Q \widehat{\rho}^2|y_x|^2\,dx\,dt+\iint_Q \rho_*^2|y_t|^2\,dx\,dt\right. \\
&\left.+\iint_{\omega\times(0,T)} \rho_*^2|v_1|^2\,dx\,dt+\iint_Q \rho_0^2|h|^2\,dx\,dt\right).
\end{aligned} \tag{3.12}
$$

Combining (3.11) and (3.12) have the desired estimates for y_t and y_{xx}.

3.3 Proof of Theorem 3.1.1

This section is devoted to prove the local null controllability of (3.1).

Let us set

$$
\begin{aligned}
Y := \{\, (y,v) : {}& y,\, y_x,\, y_t - a(0)y_{xx} \in L^2(Q),\ v \in L^2(\omega \times (0,T)),\\
& \iint_{\omega\times(0,T)} \rho_*^2 |v|^2\,dx\,dt < +\infty,\ \iint_Q \rho_0^2 |y|^2\,dx\,dt < +\infty,\\
& \iint_Q \rho_0^2 |y_t - a(0)y_{xx} - v1_\omega|^2\,dx\,dt < +\infty,\ \sup_{[0,T]} \int_I \eta^2 |y_x|^2\,dx < +\infty,\\
& \iint_Q \eta^2 |y_{xx}|^2\,dx\,dt < +\infty,\ y(\cdot,0) \in H_0^1(I)\,\},
\end{aligned}
$$

with $\eta := m(t)^{3/4} e^{2s\frac{e^{\lambda R}}{4m(t)}}$,

$$
F := \{\, g \in L^2(Q) : \iint_Q \rho_0^2 |g|^2\,dx\,dt < +\infty \,\}
$$

and

$$
Z := F \times H_0^1(I).
$$

Note that the space Y is well defined, in view of Propositions 3.2.3 and 3.2.4.

We will use the following norms in Y, F and Z:

$$
\begin{aligned}
\|(y,v)\|_Y^2 := {}& \iint_Q \rho_0^2 |y|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_*^2 |v|^2\,dx\,dt\\
& + \iint_Q \rho_0^2 |y_t - a(0)y_{xx} - v1_\omega|^2\,dx\,dt\\
& + \sup_{[0,T]} \int_I \eta^2 |y_x|^2 + \iint_Q \eta^2 |y_{xx}|^2\,dx\,dt + \|y(\cdot,0)\|_{H_0^1}^2,
\end{aligned}
$$

$$
\|g\|_F^2 := \iint_Q \rho_0^2 |g|^2\,dx\,dt.
$$

and

$$
\|(g,z)\|_Z^2 := \|g\|_F^2 + \|z\|_{H_0^1}^2.
$$

Let us consider the mapping $H : Y \mapsto Z$, with

$$
H(y,v) = (y_t - (a(y)y_x)_x - v1_\omega, y(\cdot,0)). \tag{3.13}
$$

We will use Liusternik's Theorem to prove that there exists $\varepsilon > 0$ such that, if $(h,y_0) \in Z$ and $\|(h,y_0)\|_Z \le \varepsilon$, then the equation

$$
H(y,v) = (h,y_0),\ (y,v) \in Y,
$$

possesses at least one solution. In particular, this will show that (3.1) is locally null-controllable, with controls v and associated states y satisfying $(y,v) \in Y$.

The following result can be found for instance in (ALEKSEEV *et al.*, 1987):

Theorem 3.3.1 *Let Y and Z be Banach spaces and let $H : B_r(0) \subset Y \mapsto Z$ be a C^1 mapping. Let us assume that $H'(0)$ is onto and let us set $\zeta_0 = H(0)$. Then, there exist $\varepsilon > 0$, a mapping $W : B_\varepsilon(\zeta_0) \subset Z \mapsto Y$ and a constant $K > 0$ satisfying*

$$\begin{cases} W(z) \in B_r(0) \ \text{ and } \ H(W(z)) = z \ \ \forall z \in B_\varepsilon(\zeta_0), \\ \|W(z)\|_Y \leq K\|z - H(0)\|_Z \ \ \forall z \in B_\varepsilon(\zeta_0). \end{cases}$$

In particular, W is the inverse-to-the-right of H.

Now, our goal is to prove that we can apply this result to the mapping H in (3.13). We will use following lemmas:

Lemma 3.3.1 *For any $(y,v) \in Y$, one has $y_t, y_{xx} \in L^2(Q)$. Furthermore, there exists $C > 0$ such that*

$$\iint_Q \rho_*^2(|y_t|^2 + |y_{xx}|^2)\,dx\,dt \leq C\|(y,v)\|_Y^2$$

for all $(y,v) \in Y$.

Proof: This is an almost immediate consequence of Proposition 3.2.5. Indeed, from (3.9) we get:

$$\begin{aligned} \iint_Q \rho_*^2(|y_t|^2 + |y_{xx}|^2)\,dx\,dt &\leq C\left(\iint_Q \rho_0^2|y|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_*^2|v|^2\,dx\,dt \right. \\ &\qquad \left. + \|y_0\|_{H_0^1(I)}^2 + \iint_Q \rho_0^2|h|^2\,dx\,dt\right) \\ &\leq C\|(y,v)\|_Y^2. \end{aligned}$$

Lemma 3.3.2 *Let $H : Y \mapsto Z$ be the mapping defined by (3.13). Then H is well defined and continuous.*

Proof: Let us assume that $(y,v) \in Y$, let us set $H(y,v) = (H_1(y,v), H_2(y,v))$ and let us see that $H_1(y,v)$ and $H_2(y,v)$ make sense and belong to F and $H_0^1(I)$, respectively.

One has:

$$\begin{aligned} \iint_Q \rho_0^2|H_1(y,v)|^2\,dx\,dt &= \iint_Q \rho_0^2|y_t - (a(y)y_x)_x - v1_\omega|^2\,dx\,dt \\ &\leq C\iint_Q \rho_0^2|y_t - a(0)y_{xx} - v1_\omega|^2\,dx\,dt \\ &\quad + C\iint_Q \rho_0^2|(a(y)y_x)_x - a(0)y_{xx}|^2\,dx\,dt \\ &= A_1 + A_2. \end{aligned}$$

From the definition of Y, we see that

$$A_1 = C \iint_Q \rho_0^2 |y_t - a(0)y_{xx} - v1_\omega|^2 \, dx\, dt \le C\|(y,v)\|_Y^2.$$

On the other hand, since $a \in C^1(\mathbb{R})$ and is (globally) Lipschitz-continuous, one also has:

$$\begin{aligned}
A_2 &= C \iint_Q \rho_0^2 |(a(y)y_x)_x - a(0)y_{xx}|^2 \, dx\, dt \\
&\le C \iint_Q \rho_0^2 |a(y) - a(0)|^2 |y_{xx}|^2 \, dx\, dt + C \iint_Q \rho_0^2 |a'(y)|^2 |y_x|^2 \, dx\, dt \\
&\le C \iint_Q \rho_0^2 |y|^2 |y_{xx}|^2 \, dx\, dt + C \iint_Q \rho_0^2 |y_x|^4 \, dx\, dt \\
&\le C \left(\sup_Q \rho_0^2 \eta^{-2} |y|^2 \right) \iint_Q \eta^2 |y_{xx}|^2 \, dx\, dt + C \iint_Q \rho_0^2 |y_x|^4 \, dx\, dt.
\end{aligned}$$

From the definitions of ρ_0 and η, we have $\rho_0^2 \eta^{-2} \le \eta^2$ and, consequently,

$$\sup_{(0,T)} \left(\sup_I \rho_0^2 \eta^{-2} |y|^2 \right) \le C \sup_{(0,T)} \int_I \eta^2 |y_x|^2 dx \le C\|(y,v)\|_Y^2. \tag{3.14}$$

Moreover,

$$\begin{aligned}
\iint_Q \rho_0^2 |y_x|^4 \, dx\, dt &\le C \int_0^T \left(\int_I \eta^2 |y_x|^2 \, dx \right) \left(\sup_I \eta^2 |y_x|^2 \right) dt \\
&\le C \left(\int_0^T \sup_I \eta^2 |y_x|^2 \, dt \right) \left(\sup_{(0,T)} \int_I \eta^2 |y_x|^2 \, dx \right) \\
&\le C \left(\iint_Q \eta^2 |y_{xx}|^2 \, dx\, dt \right) \left(\sup_{(0,T)} \int_I \eta^2 |y_x|^2 \, dx \right) \\
&\le C\|(y,v)\|_Y^4.
\end{aligned} \tag{3.15}$$

Note that, in these inequalities, it is crucial that the spatial domain is one-dimensional. Combining (3.14) and (3.15), the following is obtained:

$$\begin{aligned}
A_2 &\le C\|(y,v)\|_Y^2 \iint_Q \eta^2 |y_{xx}|^2 \, dx\, dt + C\|(y,v)\|_Y^4 \\
&\le C\|(y,v)\|_Y^4.
\end{aligned}$$

Therefore, H is well defined.

Furthermore, using similar argument is easy to check that H is continuous.

Lemma 3.3.3 *The mapping $H : Y \mapsto Z$ is continuously differentiable.*

Proof: Let us fix (y,v) in Y and let us choose arbitrary $(y',v')\in Y$ and $\sigma>0$. We have:

$$\begin{aligned}\frac{1}{\sigma}\left[H_1((y,v)+\sigma(y',v'))-H_1(y,v)\right] = y'_t &- \frac{1}{\sigma}\left[a'(y+\sigma y')((y+\sigma y')^2_x - y^2_x)\right]\\ &-\frac{1}{\sigma}\left[a'(y+\sigma)-a'(y)\right]y^2_x - a(y+\sigma y')y'_{xx}\\ &-\frac{1}{\sigma}\left[a(y+\sigma y')-a(y)\right]y_{xx} - v'1_\omega.\end{aligned}$$

Let us consider the linear mapping $DH : Y \mapsto Z$ given by

$$\begin{aligned} &DH=(DH_1,DH_2)\\ &DH_1(y',v') := y'_t - 2a'(y)y_xy'_x - a''(y)y'y^2_x\\ &\qquad\qquad - a(y)y'_{xx} - a'(y)y'y_{xx} - v'1_\omega\\ &DH_2(y',v') := y'(\cdot,0),\end{aligned}$$

for all $(y',v')\in Y$. We claim that

$$\frac{1}{\sigma}[H_1((y,v)-\sigma(y',v'))-H_1(y,v)] \to DH_1(y',v') \text{ strongly in } F \tag{3.16}$$

as $\sigma\to 0$.

Indeed,

$$\begin{aligned}&\|\frac{1}{\sigma}[H_1((y,v)+\sigma(y',v'))-H_1(y,v)]-DH_1(y',v')\|_F\\ &\qquad\le \|2a'(y)y_xy'_x - \frac{1}{\sigma}\left[a'(y+\sigma y')((y+\sigma y')^2_x - y^2_x)\right]\|_F\\ &\qquad+\|a''(y)y'y^2_x - \frac{1}{\sigma}\left[a'(y+\sigma)-a'(y)\right]y^2_x\|_F\\ &\qquad+\|a'(y)y'y_{xx} - -\frac{1}{\sigma}\left[a(y+\sigma y')-a(y)\right]y_{xx}\|\\ &\qquad+\|a(y)y'_{xx}-a(y+\sigma y')y'_{xx}\|_F\\ &\qquad= B_1+B_2+B_3+B_4\end{aligned}$$

Let us check that the $B_i\to 0$ as $\sigma\to 0$. First, one has

$$B_1^2 = \iint_Q \rho_0^2(2a'(y)y_xy'_x - a'(y+\sigma y')(2y'_xy_x - \sigma y'_x))^2\,dx\,dt \to 0,$$

as a consequence of Lebesgue's Theorem and the fact that $a\in C^1(\mathbb{R})$.

Let us denote by a'_* and a'_{**} the derivatives of a at some intermediate points. Using now that $a\in C^2(\mathbb{R})$ and, again, Lebesgue's Theorem, we have:

$$\begin{aligned}B_2^2 &= \iint_Q \rho_0^2(a''(y)yy^2_x - \frac{1}{\sigma}[a'(y+\sigma y')-a'(y)]y^2_x)^2\,dx\,dt\\ &= \iint_Q \rho_0^2(a''(y)-a'_*)^2(y'y^2_x)^2\,dx\,dt \to 0\end{aligned}$$

and

$$B_3^2 = \iint_Q \rho_0^2 (a'(y)y'y_{xx} - \frac{1}{\sigma}[a(y+\sigma y') - a(y)]y_{xx})^2 \, dx\, dt$$
$$= \iint_Q \rho_0^2 ((a'(y) - a'_{**})y'y_{xx}) \, dx\, dt \to 0$$

A similar argument shows that B_4^2 also converges to zero as $\sigma \to 0$. Thus, (3.16) holds.

Let us denote by $H'(y,v)$ the linear mapping DH. It is clear that $H'(y,v) \in \mathscr{L}(Y;Z)$. Let us prove that $(y,v) \mapsto H'(y,v)$ is a continuous mapping from Y into $\mathscr{L}(Y:Z)$. This will be sufficient to achieve the proof.

Thus, let us assume that $(y^n, v^n) \to (y,v)$ in Y and let us check that

$$\|(DH(y^n,v^n) - DH(y,v))(y',v')\|_Z \le \varepsilon_n \|(y',v')\|_Y \text{ for some } \varepsilon_n \to 0. \tag{3.17}$$

Observe that

$$\begin{aligned}
&\|(DH_1(y^n,v^n) - DH_1(y,v))(y',v')\|_F^2 \\
&\qquad \le C \iint_Q \rho_0^2 [a'(y^n)y_x^n y_x' - a'(y)y_x y_x']^2 \, dx\, dt \\
&\qquad + C \iint_Q \rho_0^2 [a''(y^n)y'(y_x^n)^2 - a''(y)y' y_x^2]^2 \, dx\, dt \\
&\qquad + C \iint_Q \rho_0^2 [a(y^n)y'_{xx} - a(y)y'_{xx}]^2 \, dx\, dt \\
&\qquad + C \iint_Q \rho_0^2 [a'(y^n)y'y^n_{xx} - a'(y)y'y_{xx}]^2 \, dx\, dt \\
&\qquad = D_1 + D_2 + D_3 + D_4.
\end{aligned}$$

Then, after some tedious but straightforward computations, we see that

$$D_1 \le C\|(y^n,v^n) - (y,v)\|_Y^2 (1 + \|(y,v)\|_Y^2) \|(y',v')\|_Y^2,$$

$$D_2 \le C\|(y^n,v^n) - (y,v)\|_Y^2 (1 + \|(y,v)\|_Y^2) \|(y,v)\|_Y^2 \|(y',v')\|_Y^2$$

and similar estimates hold to D_3 and D_4.

Accordingly, (3.17) is satisfied and the proof is done.

Lemma 3.3.4 *Let H be the mapping defined by (3.13). Then $H'(0,0) \in \mathscr{L}(Y;Z)$ is onto.*

Proof: Let us introduce the linear mapping $H'(0,0) = (K_1, K_2)$, where

$$\begin{cases} K_1(y',v') = y'_t - a(0)y'_{xx} - v' 1_\omega \\ K_2(y',v') = y'(\cdot,0) \end{cases} \tag{3.18}$$

for all $(y',v') \in Y$. Observe that $H'(0,0)$ is onto if and only if for each $(g,y_0) \in Z$ there exist $(y,v) \in Y$ satisfying

$$\begin{cases} y_t - a(0)y_{xx} = v1_\omega + g & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0(x) & \text{in} \quad I. \end{cases}$$

From Proposition 3.2.3, there exists a couple (y,v) with the desired properties. Consequently, the lemma holds.

From the previous lemmas, we see that, in the present context, all the assumptions in Theorem 3.3.1 are satisfied. Thus, this result can be applied, (3.1) is locally null-controllable and Theorem 3.1.1 holds.

3.4 Proof of Theorem 3.1.2

For example, let us assume that $I = (0,1)$, let us set $I_\delta = (-\delta, 1)$ with $\delta > 0$ and let $\tilde{\omega} \subset I_\delta$ be a non-empty open set.

let us consider the following auxiliar system:

$$\begin{cases} \tilde{y}_t - (a(\tilde{y})\tilde{y}_x)_x = \tilde{v}1_{\tilde{\omega}} & \text{in} \quad I_\delta \times (0,T) \\ \tilde{y}(x,t) = 0 & \text{on} \quad \partial I_\delta \times (0,T) \\ \tilde{y}(x,0) = \tilde{y}_0(x) & \text{in} \quad I_\delta, \end{cases} \tag{3.19}$$

where $\tilde{y}_0 \in H_0^1(I_\delta)$ is the extension-by-zero of y_0 to I_δ.

From Theorem 3.1.1, we deduce the existence of a control $\tilde{v} \in L^2(\tilde{\omega} \times (0,T))$ and an associated state $\tilde{y}$ solving (3.19) and satisfying

$$\tilde{y}(x,T) = 0 \ \text{ in } \ I_\delta.$$

Let v_2 be the trace of $\tilde{y}$ on $\partial I \times (0,T)$. Then, the couple (y, v_2), where y is the restriction of $\tilde{y}$ to $I \times (0,T)$, solves the corresponding system (3.2).

This proves Theorem 3.1.2.

3.5 Some additional comments and questions

3.5.1 Other nonlinear control problems

The local null controllability of the system

$$\begin{cases} y_t - (a(y_x)y_x)_x = v1_\omega, & \text{in} \quad Q \\ y(x,t) = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0, & \text{in} \quad I \end{cases} \tag{3.20}$$

is an open question.

If we try to apply the same technique, the main difficulty is found in the proof that, for any $(y,v) \in Y$ (defined in Section 3), $H(y,v) = (y_t - (a(y_x)y_x)_x - v1_\omega, y(\cdot\,,0))$ is well defined.

On the other hand, as we have already mentioned, the proof of Lemma 3.3.2 uses in a fundamental way that the problem is one-dimensional. Thus, it is not clear whether similar arguments can be applied to problems of the kind (3.1) if the spatial dimension is ≥ 2.

A related (but different) situation is found in (CLARK *et al.*, 2013), where the diffusion coefficient depends on the state through quantities that are global in space.

3.5.2 Nonlinear parabolic systems with radial symmetry

Under radial symmetry, it is possible to find control results similar to Theorems 3.1.1 and 3.1.2. Let us indicate briefly the situation.

Let us assume that $\Omega = \{\, x \in \mathbb{R}^2;\ |x| < \tilde{R} \,\}$ and $\omega = \{\, x \in \mathbb{R}^2;\ |x| < \ell \,\}$, with $0 < \ell < R$ and let us consider the nonlinear system

$$\begin{cases} y_t - (a(y)y_x)_x = v1_\omega, & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(x,0) = y_0, & \text{in} \quad I, \end{cases} \tag{3.21}$$

where y_0 is a radial function in $H^1_0(I)$. The following holds:

Theorem 3.5.1 *Assume that a is as in Theorem 3.1.1. Then,* (3.21) *is locally radially null-controllable at any time* $T > 0$. *In other words, there exists* $\varepsilon > 0$ *such that, if* y_0 *is a rdial atet satisfying* $y_0 \in H^1_0(I)$ *and*

$$\|y_0\|_{H^1_0} \leq \varepsilon,$$

there exist radial controls $v \in L^2(\omega \times (0,T))$ and associated states y satisfying

$$y(x,T) = 0 \ \ in \ \ \Omega.$$

3.5.3 An iterative algorithm

Arguing as in (CLARK *et al.*, 2013) and (FERNÁNDEZ-CARA *et al.*, 2015), an iterative algorithm of the quasi-Newton kind can be introduced for the computation of a solution to the null control problem:

ALG 1:

1. Choose $(y^0, v^0) \in Y$.
2. Then, for given $n \geq 0$ and $(y^n, v^n) \in Y$, compute

$$(y^{n+1}, v^{n+1}) = (y^n, v^n) - H'(0,0)^{-1}(H(y^n, v^n) - (0, y_0)).$$

Here $H'(0,0)^{-1}$ is an inverse to $H'(0,0)$ (as in (3.18)).

Notice that, for each n, the task reduces to the solution of a null controllability problem for the linear problem

$$\begin{cases} z^n_t - a(0) z^n_{xx} = w^n 1_\omega - (y^n_t - (a(y^n) y^n_x)_x - v 1_\omega), & \text{in} \quad Q \\ z^n(x,t) = 0 & \text{on} \quad \Sigma \\ z^n(x,0) = -y^n(x,t) + y_0(x) & \text{in} \quad I. \end{cases}$$

and then take $(y^{n+1}, v^{n+1}) = (y^n, v^n) + (z^n, w^n)$.

The following result can be established:

Theorem 3.5.2 *Let ε be given by Theorem 3.1.1. Assume that $\|y_0\|_{H^1_0(I)} \leq \varepsilon$, (y,v) satisfies (3.1) and (3.3) and $\|(y,v)\|_Y$ is sufficiently small. There exists $\kappa \in (0,1)$ such that, if $(y^0, v^0) \in Y$ and*

$$\|(y^0, v^0) - (y, v)\|_Y \leq \kappa,$$

then the (y^n, v^n) converge to (y,v) and satisfy

$$\|(y^{n+1}, v^{n+1}) - (y, v)\|_Y \leq \theta \|(y^n, v^n) - (y, z)\|_Y$$

for some $\theta \in (0,1)$ for all $n \geq 0$.

4 DENSITY AND SPECTRUM OF MINIMAL SUBMANIFOLDS IN SPACE FORMS

4.1 Introduction

The main concern in this paper is to achieve, in the above-mentioned setting of minimal submanifolds $\varphi : M \to N$, a characterization of the whole $\sigma(M)$ free from curvature or topological conditions on M(in this respect, observe that the completeness of M follows from that of N and the properness of φ). It is known by (CHEUNG; LEUNG, 2001) and (BESSA; MONTENEGRO, 2007) that for a minimal immersion $\varphi : M^m \to \mathbb{N}^n_k$ the fundamental tone of M, $\inf \sigma(M)$, is at least that of $\mathbb{N}^m_k$, i.e.,

$$\inf \sigma(M) \geq \frac{(m-1)^2 k}{4}. \tag{4.1}$$

Moreover, as a corollary of (KUMURA, 1997) and (BESSA *et al.*, 2007; BESSA; COSTA, 2009), if the second fundamental form II satisfies the decay estimate

$$\begin{aligned} &\lim_{\rho(x)\to+\infty} \rho(x)|\mathrm{II}(x)| = 0 \qquad \text{if } k = 0 \\ &\lim_{\rho(x)\to+\infty} |\mathrm{II}(x)| = 0 \qquad\qquad \text{if } k > 0 \end{aligned} \tag{4.2}$$

($\rho(x)$ being the intrinsic distance with respect to some fixed origin $o \in M$), then M has the same spectrum that a totally geodesic submanifold $\mathbb{N}^m_k \subset \mathbb{N}^n_k$, that is,

$$\sigma(M) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{4.3}$$

According to (ANDERSON, 1984; FILHO, 1993), (4.2) is ensured when M has finite total curvature, that is, when

$$\int_M |\mathrm{II}|^m < +\infty. \tag{4.4}$$

Remark 4.1.1 A characterization of the essential spectrum, similar to (4.3), also holds for submanifolds of the hyperbolic space $\mathbb{H}^n_k$ with constant (normalized) mean curvature $H < \sqrt{k}$. There, condition (4.4) is replaced by the finiteness of the L^m-norm of the traceless second fundamental form. For deepening, see (CASTILLON, 1999).

Inspecting the proofs of the above results it seemed to us that, for (4.3) to hold, condition (4.4) and more generally (4.2) could be substantially weakened. Here, we identify a suitable growth condition on the density function $\Theta(r)$ *along a sequence* as a natural candidate to replace them, see (4.6). As a very special case, (4.3) holds when M has finite density. We feel quite surprising that just a volume growth condition along a sequence could control the whole spectrum of M; clearly, for this to happen, the minimality condition enters in a crucial way.

Regarding the relation between (4.4) and the finiteness of $\Theta(+\infty)$, we remark that their interplay has been investigated in depth for minimal submanifolds of $\mathbb{R}^n$, but the case of $\mathbb{H}^n_k$ seems to be partly unexplored. In the next section, we will briefly discuss the state of the art, to the best of our knowledge. As a corollary of Theorem 4.1.2 below, we will show the following

Corollary 4.1.1 *Let M^m be a minimal properly immersed submanifold in $\mathbb{H}^n_k$. If M has finite total curvature, then $\Theta(+\infty) < +\infty$.*

As far as we know, this result was previously known just in dimension $m = 2$ via a Chern-Osserman type inequality, see the next section for further details.
We now come to our results, beginning with defining the ambient spaces which we are interested in: these are manifolds with a pole, whose radial sectional curvature is suitably pinched to that of the model $\mathbb{N}^n_k$.

Definition 4.1.1 *Let N^n possess a pole $\bar{o}$ and denote with $\bar{\rho}$ the distance function from $\bar{o}$. Assume that the radial sectional curvature $\bar{K}_{\mathrm{rad}}$ of N, that is, the sectional curvature restricted to planes π containing $\bar{\nabla}\bar{\rho}$, satisfies*

$$-G\big(\bar{\rho}(x)\big) \le \bar{K}_{\mathrm{rad}}(\pi_x) \le -k \le 0 \qquad \forall x \in N\backslash\{\bar{o}\}, \tag{4.5}$$

for some $G \in C^0(\mathbb{R}^+_0)$. We say that

(i) N has a pointwise (respectively, integral) pinching to $\mathbb{R}^n$ if $k = 0$ and

$$sG(s) \to 0 \;\text{ as }\; s \to +\infty \qquad \big(\text{respectively, } sG(s) \in L^1(+\infty)\big);$$

(ii) N has a pointwise (respectively, integral) pinching to $\mathbb{H}^n_k$ if $k > 0$ and

$$G(s) - k \to 0 \;\text{ as }\; s \to +\infty \qquad \big(\text{respectively, } G(s) - k \in L^1(+\infty)\big).$$

Hereafter, given an ambient manifold N with a pole $\bar{o}$, the density function $\Theta(r)$ will always be computed by taking extrinsic balls centered at $\bar{o}$.

Our main achievements are the following two theorems. The first one characterizes $\sigma(M)$ when the density of M grows subexponentially (respectively, sub-polynomially) along a sequence. Condition (4.6) below is very much in the spirit of a classical volume growth hypothesis due to R. Brooks (BROOKS, 1981) and Y. Higuchi (HIGUCHI, 2001) to bound from above the infimum of the essential spectrum of $-\Delta$. However, we stress that Theorem 4.1.1 below seems to be the first result in the literature characterizing the whole spectrum of M under just a mild volume growth assumption.

Theorem 4.1.1 *Let $\varphi : M^m \to N^n$ be a minimal properly immersed submanifold, and suppose that N has a pointwise or an integral pinching to a space form. If either*

$$\begin{array}{lll} N \text{ is pinched to } \mathbb{H}^n_k, \text{ and} & \displaystyle\liminf_{s\to+\infty} \frac{\log\Theta(s)}{s} = 0, & \text{or} \\ N \text{ is pinched to } \mathbb{R}^n, \text{ and} & \displaystyle\liminf_{s\to+\infty} \frac{\log\Theta(s)}{\log s} = 0. & \end{array} \tag{4.6}$$

then

$$\sigma(M) = \left[\frac{(m-1)^2k}{4}, +\infty\right). \tag{4.7}$$

The above theorem is well suited for minimal submanifolds constructed via Geometric Measure Theory since, typically, their existence is guaranteed by controlling the density function $\Theta(r)$. As an important example, Theorem 4.1.1 applies to all solutions of Plateau's problem at infinity $M^m \to \mathbb{H}^n_k$ constructed in (ANDERSON, 1982), provided that they are smooth. Indeed, because of their construction, $\Theta(+\infty) < +\infty$ (see (ANDERSON, 1982), part [A] at p. 485) and they are proper (it can also be deduced as a consequence of $\Theta(+\infty) < +\infty$, see Remark 4.3.2). By standard regularity theory, smoothness of M^m is automatic if $m \leq 6$.

Corollary 4.1.2 *Let $\Sigma \subset \partial_\infty \mathbb{H}^n_k$ be a closed, integral $(m-1)$ current in the boundary at infinity of $\mathbb{H}^n_k$ such that, for some neighborhood U of* $\mathrm{supp}(\Sigma)$, *Σ does not bound in U, and let $M^m \hookrightarrow \mathbb{H}^n_k$ be the solution of Plateau's problem at infinity constructed in (ANDERSON, 1982) for Σ. If M is smooth, then* (4.7) *holds.*

An interesting fact of Corollary 4.1.2 is that M is *not* required to be regular up to $\partial_\infty \mathbb{H}^n_k$, in particular it might have infinite total curvature. In this respect, we observe that if M be C^2 up to $\partial_\infty \mathbb{H}^n$, then M would have finite total curvature (Lemma 4.6.1 in Appendix 1). By deep regularity results, this is the case if, for instance, $M^m \to \mathbb{H}^{m+1}$ is a smooth hypersurface that solves Plateau's problem for Σ, and Σ is a $C^{2,\alpha}$ (for $\alpha > 0$), embedded compact submanifold of $\partial_\infty \mathbb{H}^n$. See Appendix 1 for details.

The spectrum of solutions of Plateau's problems has also been considered in (BESSA *et al.*, 2015) for minimal surfaces in $\mathbb{R}^3$. In this respect, it is interesting to compare Corollary 4.1.2 with (3) of Corollary 2.6 therein.

Remark 4.1.2 The solution M of Plateau's problem in (ANDERSON, 1982) is constructed as a weak limit of a sequence M_j of minimizing currents for suitable boundaries Σ_j converging to Σ. and property $\Theta(+\infty) < +\infty$ is a consequence of a uniform upper bound for the mass of a sequence M_j (part [A], p. 485 in (ANDERSON, 1982)). Such a bound is achieved because of the way the boundaries Σ_j are constructed, in particular, since they are all sections of the same cone. One might wonder whether $\Theta(+\infty) < +\infty$, or at least the subexponential growth in (4.6), is satisfied by all solutions of Plateau's problem. In this respect, we just make this simple observation: in the hypersurface case $n = m+1$, if $M \cap B_r^{m+1}$ is volume-minimizing then clearly

$$\Theta(r) = \frac{\mathrm{vol}(M \cap B_r^{m+1})}{V_k(r)} \leq \frac{\mathrm{vol}(\partial B_r^{m+1} \subset \mathbb{H}_k^{m+1})}{V_k(r)} = c_k \frac{\sinh^m(\sqrt{k}r)}{V_k(r)},$$

but this last expression diverges exponentially fast as $r \to +\infty$ (differently from its Euclidean analogous, which is finite). This might suggest that a general solution of Plateau's problem does not automatically satisfies $\Theta(+\infty) < +\infty$, and maybe not even (4.6).

In our second result we focus on the particular case when $\Theta(+\infty) < +\infty$, and we give a sufficient condition for its validity in terms of the decay of the second fundamental form. Towards this aim, we shall restrict to ambient spaces with an integral pinching.

Theorem 4.1.2 *Let $\varphi : M^m \to N^n$ be a minimal immersion, and suppose that N has an integral pinching to $\mathbb{R}^n$ or to $\mathbb{H}_k^n$. Denote with $\rho(x)$ the intrinsic distance from some reference origin $o \in M$. Assume that there exist $c > 0$ and $\alpha > 1$ such that the second fundamental form satisfies, for $\rho(x) >> 1$,*

$$\begin{aligned} |\mathrm{II}(x)|^2 &\leq \frac{c}{\rho(x)\log^\alpha \rho(x)} && \text{if } N \text{ is pinched to } \mathbb{H}_k^n; \\ |\mathrm{II}(x)|^2 &\leq \frac{c}{\rho(x)^2\log^\alpha \rho(x)} && \text{if } N \text{ is pinched to } \mathbb{R}^n. \end{aligned} \tag{4.8}$$

Then, φ is proper, M is diffeomorphic to the interior of a compact manifold with boundary, and $\Theta(+\infty) < +\infty$.

Because of a result in (FILHO, 1993; PIGOLA; VERONELLI, 2011), if $\varphi : M \to \mathbb{H}^n_k$ has finite total curvature then $|II(x)| = o(\rho(x)^{-1})$ as $\rho(x) \to +\infty$. Hence, (4.8) is met and Corollary 4.1.1 follows at once.

We briefly describe the strategy of the proof of Theorem 4.1.1. In view of (4.1), it is enough to show that each $\lambda > (m-1)^2k/4$ lies in $\sigma(M)$. To this end, we follow an approach inspired by a general result due to K.D. Elworthy and F-Y. Wang (ELWORTHY; WANG, 2004). However, Elworthy-Wang's theorem is not sufficient to conclude, and we need to considerably refine the criterion in order to fit in the present setting. To construct the sequence as in Lemma 1.0.1, a key step is to couple the volume growth requirement (4.6) with a sharpened form of the monotonicity formula for minimal submanifolds, which improves on the classical ones in (SIMON, 1983; ANDERSON, 1982). Indeed, in Proposition 4.3.1 we describe three monotone quantities other than $\Theta(s)$, and we expect these to be useful beyond the purpose of the present paper. For example, in the very recent (GIMENO; MARKVOSEN, 2015) the authors discovered and used some of the relations in Proposition 4.3.1 to show interesting comparison results for the capacity and the first eigenvalue of minimal submanifolds.

4.1.1 Finite density and finite total curvature in $\mathbb{R}^n$ and $\mathbb{H}^n$

The first attempt to extend the classical theory of finite total curvature surfaces in $\mathbb{R}^n$ (see (OSSERMAN, 1986; JORGE; MEEKS, 1983; CHERN; OSSERMAN, 1967; CHERN; OSSERMAN, 1984)) to the higher-dimensional case is due to Anderson. In (ANDERSON, 1984), the author drew from (4.4) a number of topological and geometric consequences, and here we focus on those useful to highlight the relationship between total curvature and density. First, he showed that (4.4) implies an uniform decay of the second fundamental form II to zero which is faster that power ρ^{-1} in the intrinsic distance function ρ on M from an origin $o \in M$:

$$|\mathrm{II}(x)| \le \frac{\eta(\rho(x))}{\rho(x)} \qquad \text{with } \eta(t) \to 0 \text{ as } t \to +\infty, \tag{4.9}$$

and as a consequence M is proper, the extrinsic distance function r has no critical points outside some compact set and $|\nabla r| \to 1$ as r diverges, so by Morse theory M is diffeomorphic to the interior of a compact manifold with boundary. Moreover, he proved that M has finite density via a higher-dimensional extension of the Chern-Osserman identity (CHERN; OSSERMAN, 1967; CHERN; OSSERMAN, 1984), namely the following relation linking the Euler characteristic

$\chi(M)$ and the Pfaffian form Ω ((ANDERSON, 1984), Theorem 4.1):

$$\chi(M) = \int_M \Omega + \lim_{r\to+\infty} \frac{\mathrm{vol}(M\cap \partial B_r)}{V_0'(r)}. \tag{4.10}$$

Observe that, since $|\nabla r| \to 1$, by coarea's formula the limit in the right hand-side coincides with $\Theta(+\infty)$. We underline that property $\Theta(+\infty) < +\infty$ plays a fundamental role to apply the machinery of manifold convergence to get information on the limit structure of the ends of M ((ANDERSON, 1984; SHEN; ZHU, 1998; TYSK, 1989)). For instance, $\Theta(+\infty)$ is related to the number $\mathscr{E}(M)$ of ends of M: if we denote with $V_1,\ldots,V_{\mathscr{E}(M)}$ the (finitely many) ends of M, (4.4) implies for $m \geq 3$ the identities

$$\Theta(+\infty) = \sum_{i=1}^{\mathscr{E}(M)} \lim_{r\to+\infty} \frac{\mathrm{vol}(V_i\cap \partial B_r)}{V_0'(r)} \equiv \mathscr{E}(M), \tag{4.11}$$

and thus M is totally geodesic provided that it has only one end and finite total curvature ((ANDERSON, 1984), Thm 5.1 and its proof). Further information on the mutual relationship between the finiteness of the total curvature and $\Theta(+\infty) < +\infty$ can be deduced under the additional requirement that M is stable or it has finite stability index. For example, by work of J. Tysk (TYSK, 1989), if M^m has finite index and $m \leq 6$, then

$$\Theta(+\infty) < +\infty \qquad \text{if and only if} \qquad \int_M |\mathrm{II}|^m < +\infty. \tag{4.12}$$

Remark 4.1.3 Indeed, the main result in (TYSK, 1989) states that, when $\Theta(+\infty) < +\infty$ and $m \leq 6$, M has finite index if and only if it has finite total curvature. However, since the finite total curvature condition alone implies both that M has finite index and $\Theta(+\infty) < +\infty$ (in any dimension), the characterization in (4.12) is equivalent to Tysk's theorem. We underline that it is still a deep open problem whether or not, for $m \geq 3$, stability or finite index alone implies the finiteness of the density at infinity.

Since then, efforts were made to investigate analogous properties for minimal submanifolds of finite total curvature immersed in $\mathbb{H}^n_k$. There, some aspects show strong analogy with the $\mathbb{R}^n$ case, while others are strikingly different. For instance, minimal immersions $\varphi : M^m \to \mathbb{H}^n_k$ with finite total curvature enjoy the same decay property (4.9) with respect to the intrinsic distance $\rho(x)$ ((FILHO, 1993), see also (PIGOLA; VERONELLI, 2011)), which is enough to deduce that they are properly immersed and diffeomorphic to the interior of a compact manifold with boundary.

Moreover, as already noticed, Anderson (ANDERSON, 1982) proved the monotonicity of $\Theta(r)$ in (1.1). In order to show (among other things) that complete, finite total curvature surfaces $M^2 \hookrightarrow \mathbb{H}^n$ have finite density, in (CHEN, 1999; CHEN; CHENG, 2000) the authors obtained the following Chern-Osserman type (in)equality:

$$\chi(M) \geq -\frac{1}{4\pi}\int_M |\mathrm{II}|^2 + \Theta(+\infty), \tag{4.13}$$

see also (GIMENO; PALMER, 2013). However, in the higher dimensional case we found no analogous of (4.10), (4.13) in the literature, and adapting the proof of (4.10) to the hyperbolic ambient space seems to be subtler than what we expected. In fact, an *equality* like (4.10) is not even possible to obtain, since there exist minimal submanifolds of $\mathbb{H}^n_k$ with finite density but whose density at infinity depends on the chosen reference origin (Gimeno, V., Private communication). We point out that, on the contrary, inequality (4.13) holds for each choice of the reference origin in $\mathbb{H}^n$. This motivated the different route that we follow to prove Theorem 4.1.2 and Corollary 4.1.1. Among the results in (ANDERSON, 1984) that could not admit a corresponding one in $\mathbb{H}^n_k$, in view of the solvability of Plateau's problem at infinity on $\mathbb{H}^n_k$ we stress that a relation like (4.11) cannot hold for each minimal submanifold of $\mathbb{H}^n_k$ with finite total curvature. Indeed, there exist a wealth of properly immersed minimal submanifolds in $\mathbb{H}^n_k$ with finite total curvature and one end, as the example in Appendix 1 shows. It shall be observed, however, that when II decays sufficiently fast at infinity with respect to the extrinsic distance function $r(x)$:

$$|\mathrm{II}(x)| \leq \frac{\eta(r(x))}{e^{2\sqrt{k}r(x)}} \qquad \text{with } \eta(t) \to 0 \text{ as } t \to +\infty, \tag{4.14}$$

then the inequality $\Theta(+\infty) \leq \mathscr{E}(M)$ still holds for minimal hypersurfaces in $\mathbb{H}^n_k$ as shown in (GIMENO; PALMER, 2012), and in particular M is totally geodesic provided that it has only one end, as first observed in (KASUE; SUGAHARA, 1987; KASUE; SUGAHARA, 1986). We remark that there exists an infinite family of complete minimal cylinders $\varphi_\lambda : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{H}^3$ whose second fundamental form II_λ decays exactly of order $\exp\{-2r(x)\}$, see (MORI, 1981).

4.2 Preliminaries

Let $\varphi : (M^m, \langle\, , \rangle) \to (N^n, (\, ,))$ be an isometric immersion of a complete m-dimensional Riemannian manifold M into an ambient manifold N of dimension n and possessing a pole $\bar{o}$. We denote with $\nabla, \mathrm{Hess}\,, \Delta$ the connection, the Riemannian Hessian and the Laplace-Beltrami operator on M, while quantities related to N will be marked with a bar. For instance, $\bar{\nabla}, \overline{\mathrm{dist}}, \overline{\mathrm{Hess}}$ will identify the connection, the distance function and the Hessian in N. Let $\bar{\rho}(x) = \overline{\mathrm{dist}}(x, \bar{o})$ be the distance function from $\bar{o}$. Geodesic balls in N of radius R and center y will be denoted with $B^N_R(y)$. Moreover, set

$$r \,:\, M \to \mathbb{R}, \qquad r(x) = \bar{\rho}\big(\varphi(x)\big), \tag{4.15}$$

for the extrinsic distance from $\bar{o}$. We will indicate with Γ_s the extrinsic geodesic spheres restricted to M: $\Gamma_s \doteq \{x \in M;\ r(x) = s\}$. Fix a base point $o \in M$. In what follows, we shall also consider the intrinsic distance function $\rho(x) = \mathrm{dist}(x, o)$ from a reference origin $o \in M$.

4.2.1 Target spaces

Hereafter, we consider an ambient space N possessing a pole $\bar{o}$ and satisfying (4.5) for some $k \geq 0$ and some $G \in C^0(\mathbb{R}^+_0)$. Let $\mathrm{sn}_k(t)$ be the solution of

$$\begin{cases} \mathrm{sn}_k'' - k\,\mathrm{sn}_k = 0 \quad \text{on } \mathbb{R}^+, \\ \mathrm{sn}_k(0) = 0, \quad \mathrm{sn}_k'(0) = 1, \end{cases} \tag{4.16}$$

that is

$$\mathrm{sn}_k(t) = \begin{cases} t & \text{if } k = 0, \\ \sinh(\sqrt{k}t)/\sqrt{k} & \text{if } k > 0. \end{cases} \tag{4.17}$$

Observe that $\mathbb{R}^n$ and $\mathbb{H}^n_k$ can be written as the differentiable manifold $\mathbb{R}^n$ equipped with the metric given, in polar geodesic coordinates $(\rho, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ centered at some origin, by

$$\mathrm{d}s_k^2 = \mathrm{d}\rho^2 + \mathrm{sn}_k^2(\rho)\,\mathrm{d}\theta^2,$$

$\mathrm{d}\theta^2$ being the metric on the unit sphere $\mathbb{S}^{n-1}$.

We also consider the model M^n_g associated with the lower bound $-G$ for $\bar{K}_{\mathrm{rad}}$, that is, we let $g \in C^2(\mathbb{R}^+_0)$ be the solution of

$$\begin{cases} g'' - Gg = 0 \quad \text{on } \mathbb{R}^+, \\ g(0) = 0, \quad g'(0) = 1, \end{cases} \tag{4.18}$$

and we define M^n_g as being $(\mathbb{R}^n, \mathrm{d}s_g^2)$ with the C^2-metric $\mathrm{d}s_g^2 = \mathrm{d}\rho^2 + g^2(\rho)\mathrm{d}\theta^2$ in polar coordinates. Because of (4.5), by the Hessian comparison theorem (Theorem 2.3 in (PIGOLA *et al.*, 2008), or Theorem 1.15 in (BIANCHINI *et al.*, 2013)) it holds

$$\frac{\mathrm{sn}_k'(\bar{\rho})}{\mathrm{sn}_k(\bar{\rho})}\Big((\,,\,) - \mathrm{d}\bar{\rho} \otimes \mathrm{d}\bar{\rho}\Big) \leq \overline{\mathrm{Hess}}(\bar{\rho}) \leq \frac{g'(\bar{\rho})}{g(\bar{\rho})}\Big((\,,\,) - \mathrm{d}\bar{\rho} \otimes \mathrm{d}\bar{\rho}\Big). \tag{4.19}$$

The usefulness of our pinching conditions on N depends on the following

Proposition 4.2.1 *Let N satisfy* (4.5)*, and let* sn_k, g *be solutions of* (4.17), (4.18). *Define*

$$\zeta(s) \doteq \frac{g'(s)}{g(s)} - \frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}. \tag{4.20}$$

Then, $\zeta(0^+) = 0$, $\zeta \geq 0$ *on* $\mathbb{R}^+$. *Moreover,*

(i) *If N has a pointwise pinching to $\mathbb{H}^n_k$ or $\mathbb{R}^n$, then $\zeta(s) \to 0$ as $s \to +\infty$.*

(ii) *If N has an integral pinching to $\mathbb{H}^n_k$ or $\mathbb{R}^n$, then $g/\mathrm{sn}_k \to C$ as $t \to +\infty$ for some $C \in \mathbb{R}^+$, and*

$$\zeta(s) \in L^1(\mathbb{R}^+), \qquad \zeta(s)\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} \to 0 \ \text{ as } \ s \to +\infty. \tag{4.21}$$

Proof: The non-negativity of ζ, which in particular implies that g/sn_k is non-decreasing, follows from $G \geq k$ via Sturm comparison, and $\zeta(0^+) = 0$ depends on the asymptotic relations $\mathrm{sn}_k'/\mathrm{sn}_k = s^{-1} + o(1)$ and $g'/g = s^{-1} + o(1)$ as $s \to 0^+$, which directly follow from the ODEs satisfied by sn_k and g. To show (i), differentiating ζ we get

$$\zeta'(s) = R(s) - \zeta(s)B(s), \tag{4.22}$$

where $R(s) \doteq G(s) - k$ and $B(s) = \dfrac{g'(s)}{g(s)} + \dfrac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}$. Thus, integrating on $[1,s]$, we can rewrite ζ as follows:

$$\zeta(s) = \zeta(1)e^{-\int_1^s B} + e^{-\int_1^s B}\int_1^s R(\sigma)e^{\int_1^\sigma B}\mathrm{d}\sigma \tag{4.23}$$

Using that $B \notin L^1([1,+\infty))$, and applying de l'Hopital's theorem, we infer

$$\lim_{s\to+\infty} \zeta(s) = \lim_{s\to+\infty} \frac{R(s)}{B(s)} \leq \lim_{s\to+\infty} \frac{\mathrm{sn}_k(s)[G(s)-k]}{\mathrm{sn}_k'(s)}.$$

In our pointwise pinching assumptions on $G(s)$, for both $k = 0$ and $k > 0$ the last limit is zero, hence $\zeta(s) \to 0$ as s diverges. To show (ii), suppose that N has an integral pinching to $\mathbb{H}^n_k$ or to $\mathbb{R}^n$. We first observe that the boundedness of g/sn_k on $\mathbb{R}^+$ equivalent to the property $\zeta \in L^1(+\infty)$, as it follows from

$$\log\frac{g(s)}{\mathrm{sn}_k(s)} = \int_0^s \frac{\mathrm{d}}{\mathrm{d}\sigma}\log\left(\frac{g(\sigma)}{\mathrm{sn}_k(\sigma)}\right)\mathrm{d}s = \int_0^s \zeta \tag{4.24}$$

(we used that $(g/\mathrm{sn}_k)(0^+) = 1$). The boundedness of g/sn_k is the content of Corollary 4 and Remark 16 in (BIANCHINI *et al.*, 2015), but we prefer here to present a direct proof. Integrating

(4.23) on $[1,s]$ and using Fubini's theorem, the monotonicity of g/sn_k and the expression of B we obtain

$$\begin{aligned}
\int_1^s \zeta &= \zeta(1)\int_1^s \frac{g(1)\mathrm{sn}_k(1)}{g(\sigma)\mathrm{sn}_k(\sigma)}\mathrm{d}\sigma + \int_1^s e^{-\int_1^\sigma B}\int_1^\sigma R(\tau)e^{\int_1^\tau B}\mathrm{d}\tau\mathrm{d}\sigma \\
&\le \zeta(1)\mathrm{sn}_k(1)^2\int_1^s \frac{\mathrm{d}\sigma}{\mathrm{sn}_k^2(\sigma)} + \int_1^s\left[\int_\tau^s e^{-\int_1^\sigma B}R(\tau)e^{\int_1^\tau B}\mathrm{d}\sigma\right]\mathrm{d}\tau \\
&\le C + \int_1^s R(\tau)g(\tau)\mathrm{sn}_k(\tau)\left[\int_\tau^s \frac{\mathrm{d}\sigma}{g(\sigma)\mathrm{sn}_k(\sigma)}\right]\mathrm{d}\tau \\
&\le C + \int_1^s R(\tau)g(\tau)\mathrm{sn}_k(\tau)\left[\int_\tau^{+\infty} \frac{\mathrm{d}\sigma}{g(\sigma)\mathrm{sn}_k(\sigma)}\right]\mathrm{d}\tau
\end{aligned} \tag{4.25}$$

for some $C>0$, where we have used that $\mathrm{sn}_k^{-2}, g^{-1}\mathrm{sn}_k^{-1}\in L^1(+\infty)$. Next, since $g\,\mathrm{sn}_k/\mathrm{sn}_k^2$ is non-decreasing, Proposition 3.12 in (BIANCHINI *et al.*, 2013) ensures the validity of the following inequality:

$$g(\tau)\mathrm{sn}_k(\tau)\left[\int_\tau^{+\infty}\frac{\mathrm{d}\sigma}{g(\sigma)\mathrm{sn}_k(\sigma)}\right] \le \mathrm{sn}_k^2(\tau)\left[\int_\tau^{+\infty}\frac{\mathrm{d}\sigma}{\mathrm{sn}_k^2(\sigma)}\right].$$

It is easy to show that the last expression is bounded if $k>0$, and diverges at the order of τ if $k=0$. In other words, it can be bounded by $C_1\mathrm{sn}_k/\mathrm{sn}_k'$ on $[1,+\infty)$, for some large $C_1>0$. Therefore, by (4.25)

$$\int_1^s \zeta \le C + C_1\int_1^s R(\tau)\frac{\mathrm{sn}_k(\tau)}{\mathrm{sn}_k'(\tau)}\mathrm{d}\tau = C + C_1\int_1^s \left[G(\tau)-k\right]\frac{\mathrm{sn}_k(\tau)}{\mathrm{sn}_k'(\tau)}\mathrm{d}\tau.$$

In our integral pinching assumptions, both for $k=0$ and for $k>0$ it holds $(G-k)\mathrm{sn}_k/\mathrm{sn}_k'\in L^1(+\infty)$, and thus $\zeta\in L^1(+\infty)$. Next, we use (4.22) and the non-negativity of ζ, B to obtain

$$\begin{aligned}
\left(\frac{\zeta(s)\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)}\right)' &= \left[G(s)-k-\zeta(s)B(s)\right]\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} + \zeta(s)\left[1-k\left(\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)}\right)^2\right] \\
&\le \frac{\left[G(s)-k\right]\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} + \zeta(s)\in L^1(+\infty),
\end{aligned}$$

hence $\zeta\,\mathrm{sn}_k/\mathrm{sn}_k'\in L^\infty(\mathbb{R}^+)$. This implies that the function B in (4.22) satisfies $B\le C\mathrm{sn}_k'/\mathrm{sn}_k$ for some constant $C>0$. Therefore, from (4.22) we get $\zeta'\ge -\zeta B\ge -C\zeta\,\mathrm{sn}_k'/\mathrm{sn}_k$. Integrating on $[t,s]$ and using the monotonicity of $\mathrm{sn}_k'/\mathrm{sn}_k$ we obtain

$$-C\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\int_s^t \zeta \le \zeta(t)-\zeta(s).$$

Since $\zeta\in L^1(\mathbb{R}^+)$, we can choose a divergent sequence $\{s_j\}$ such that $\zeta(s_j)\to 0$ as $j\to+\infty$. Setting $s=s_j$ into the above inequality and taking limits we deduce

$$\zeta(s)\le C\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\int_s^{+\infty}\zeta,$$

thus letting $s \to +\infty$ we get the second relation in (4.21).

4.2.2 A transversality lemma

This subsection is devoted to an estimate of the measure of the critical set

$$S_{t,s} = \Big\{ x \in M \ : \ t \le r(x) \le s, \ |\nabla r(x)| = 0 \Big\},$$

with the purpose of justifying some coarea's formulas for integrals over extrinsic annuli. We begin with the next

Lemma 4.2.1 *Let $\varphi : M^m \to N^n$ be an isometric immersion, and let $r(x) = \overline{\mathrm{dist}}(\varphi(x), \bar{o})$ be the extrinsic distance function from $\bar{o} \in N$. Denote with $\Gamma_\sigma \doteq \{x \in M;\ r(x) = \sigma\}$. Then, for each $f \in L^1(\{t \le r \le s\})$,*

$$\int_{\{t \le r \le s\}} f \,\mathrm{d}x = \int_{S_{t,s}} f \,\mathrm{d}x + \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|} \right] \mathrm{d}\sigma. \tag{4.26}$$

In particular, if

$$\mathrm{vol}(S_{t,s}) = 0, \tag{4.27}$$

then

$$\int_{\{t \le r \le s\}} f \,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|} \right] \mathrm{d}\sigma. \tag{4.28}$$

Proof: We prove (4.26) for $f \ge 0$, and the general case follows by considering the positive and negative part of f. By the coarea's formula, we know that for each $g \in L^1(\{t \le r \le s\})$,

$$\int_{\{t \le r \le s\}} g |\nabla r| \,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma} g \right] \mathrm{d}\sigma. \tag{4.29}$$

Fix j and consider $A_j = \{|\nabla r| > 1/j\}$ and the function

$$g = f 1_{A_j} / |\nabla r| \in L^1(\{t \le r \le s\}).$$

Applying (4.29), letting $j \to +\infty$ and using the monotone convergence theorem we deduce

$$\int_{\{t \le r \le s\} \setminus S_{t,s}} f \,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma \setminus S_{t,s}} \frac{f}{|\nabla r|} \right] \mathrm{d}\sigma = \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|} \right] \mathrm{d}\sigma, \tag{4.30}$$

where the last equality follows since $\Gamma_\sigma \cap S_{t,s} = \emptyset$ for a.e. $\sigma \in [t,s]$, in view of Sard's theorem. Formula (4.26) follows at once.

Let now N possess a pole $\bar{o}$ and satisfy (4.5), and consider a minimal immersion $\varphi : M \to N$. Since, by the Hessian comparison theorem, geodesic spheres in N centered at $\bar{o}$ are positively curved, it is reasonable to expect that the “transversality"condition (4.27) holds. This is the content of the next

Proposition 4.2.2 *Let $\varphi : M^m \to N^n$ be a minimal immersion, where N possesses a pole $\bar{o}$ and satisfies* (4.5). *Then,*

$$\mathrm{vol}(S_{0,+\infty}) = 0. \tag{4.31}$$

Proof: Suppose by contradiction that $\mathrm{vol}(S_{0,+\infty}) > 0$. By Stampacchia and Rademacher's theorems,

$$\nabla|\nabla r|(x) = 0 \qquad \text{for a.e. } x \in S_{0,+\infty}. \tag{4.32}$$

Pick one such x and a local Darboux frame $\{e_i\}, \{e_\alpha\}$, $1 \le i \le m$, $m+1 \le \alpha \le n$ around x, that is, $\{e_i\}$ is a local orthonormal frame for TM and $\{e_\alpha\}$ is a local orthonormal frame for the normal bundle $TM^\perp$. Since $\nabla r(x) = 0$, then $\bar{\nabla}\bar{\rho}(x) \in T_xM^\perp$. Up to rotating $\{e_\alpha\}$, we can suppose that $\bar{\nabla}\bar{\rho}(x) = e_n(x)$. Fix i and consider a unit speed geodesics $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = e_i$. Identify γ with its image $\varphi \circ \gamma$ in N. By Taylor's formula and (4.32),

$$|\nabla r|(\gamma(t)) = o(t) \qquad \text{as } t \to 0^+.$$

Using that $|\nabla r| = \sqrt{1 - \sum_\alpha (\bar{\nabla}\bar{\rho}, e_\alpha)^2}$, we deduce

$$1 - \sum_\alpha (\bar{\nabla}\bar{\rho}, e_\alpha)^2_{\gamma(t)} = o(t^2). \tag{4.33}$$

Since $\bar{\nabla}\bar{\rho}(x) = e_n(x)$, we deduce from (4.34) that also

$$u(t) \doteq 1 - (\bar{\nabla}\bar{\rho}, e_n)^2_{\gamma(t)} = o(t^2), \tag{4.34}$$

thus $\dot{u}(0) = \ddot{u}(0) = 0$. Computing,

$$\begin{aligned}
\dot{u}(t) &= 2(\bar{\nabla}\bar{\rho}, e_n)\left[(\bar{\nabla}_{\dot{\gamma}}\bar{\nabla}\bar{\rho}, e_n) + (\bar{\nabla}\bar{\rho}, \bar{\nabla}_{\dot{\gamma}} e_n)\right] \\
\ddot{u}(t) &= 2\left[(\bar{\nabla}_{\dot{\gamma}}\bar{\nabla}\bar{\rho}, e_n) + (\bar{\nabla}\bar{\rho}, \bar{\nabla}_{\dot{\gamma}} e_n)\right]^2 \\
&\quad +2(\bar{\nabla}\bar{\rho}, e_n)\left[(\bar{\nabla}_{\dot{\gamma}}\bar{\nabla}_{\dot{\gamma}}\bar{\nabla}\bar{\rho}, e_n) + 2(\bar{\nabla}_{\dot{\gamma}}\bar{\nabla}\bar{\rho}, \bar{\nabla}_{\dot{\gamma}} e_n) + (\bar{\nabla}\bar{\rho}, \bar{\nabla}_{\dot{\gamma}}\bar{\nabla}_{\dot{\gamma}} e_n)\right].
\end{aligned}$$

Evaluating at $t = 0$ we deduce

$$0 = \ddot{u}(0)/2 = (\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}, \bar{\nabla}\bar{\rho}) + 2(\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}, \bar{\nabla}_{e_i} e_n) + (e_n, \bar{\nabla}_{e_i}\bar{\nabla}_{e_i} e_n).$$

Differentiating twice $1 = |e_n|^2 = |\bar{\nabla}\bar{\rho}|^2$ along e_i we deduce the identities $(e_n, \bar{\nabla}_{e_i}\bar{\nabla}_{e_i}e_n) = -|\bar{\nabla}_{e_i}e_n|^2$ and $(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}, \bar{\nabla}\bar{\rho}) = -|\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}|^2$, hence

$$0 = \ddot{u}(0)/2 = -|\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}|^2 + 2(\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}, \bar{\nabla}_{e_i}e_n) - |\bar{\nabla}_{e_i}e_n|^2 = -\left|\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho} - \bar{\nabla}_{e_i}e_n\right|^2,$$

which implies $\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho} = \bar{\nabla}_{e_i}e_n$. Therefore, at x,

$$(\mathrm{II}(e_i,e_i), e_n) = -(\bar{\nabla}_{e_i}e_n, e_i) = -(\bar{\nabla}_{e_i}\bar{\nabla}\bar{\rho}, e_i) = \overline{\mathrm{Hess}}(\bar{\rho})(e_i,e_i).$$

Tracing with respect to i, using that M is minimal and (4.19) we conclude that

$$0 \geq \frac{\mathrm{sn}_k'(r(x))}{\mathrm{sn}_k(r(x))}(m - |\nabla r(x)|^2) = m\frac{\mathrm{sn}_k'(r(x))}{\mathrm{sn}_k(r(x))} > 0,$$

a contradiction.

4.3 Monotonicity formulae and conditions equivalent to $\Theta(+\infty) < +\infty$

Our first step is to improve the classical monotonicity formula for $\Theta(r)$, that can be found in (SIMON, 1983) (for $N = \mathbb{R}^m$) and (ANDERSON, 1982) (for $N = \mathbb{H}^n_k$). For $k \geq 0$, let v_k, V_k denote the volume function, respectively, of geodesic spheres and balls in the space form of sectional curvature $-k$, i.e.,

$$v_k(s) = \omega_{m-1}\mathrm{sn}_k(s)^{m-1}, \qquad V_k(s) = \int_0^s v_k(\sigma)\mathrm{d}\sigma, \tag{4.35}$$

where ω_{m-1} is the volume of the unit sphere $\mathbb{S}^{m-1}$. Although we shall not use all the four monotone quantities in (4.37) below, nevertheless they have independent interest, and for this reason we state the result in its full strength. We define the *flux* $J(s)$ of ∇r over the extrinsic sphere Γ_s:

$$J(s) \doteq \frac{1}{v_k(s)}\int_{\Gamma_s}|\nabla r|. \tag{4.36}$$

Proposition 4.3.1 (The monotonicity formulae) *Suppose that N has a pole $\bar{o}$ and satisfies* (4.5)*, and let $\varphi : M^m \to N^n$ be a proper minimal immersion. Then, the functions*

$$\Theta(s), \qquad \frac{1}{V_k(s)}\int_{\{0\leq r\leq s\}}|\nabla r|^2 \tag{4.37}$$

are absolutely continuous and monotone non-decreasing. Moreover, $J(s)$ coincides, on an open set of full measure, with the absolutely continuous function

$$\bar{J}(s) \doteq \frac{1}{v_k(s)}\int_{\{r\leq s\}}\Delta r$$

and $\bar{J}(s)$, $V_k(s)\left[\bar{J}(s) - \Theta(s)\right]$ are non-decreasing. In particular, $J(s) \geq \Theta(s)$ a.e. on $\mathbb{R}^+$.

Remark 4.3.1 To the best of our knowledge, the monotonicity of $J(s)$ (aside from its differentiability properties) has first been shown, in the Euclidean setting, in a paper by V. Tkachev (TKACHEV, 1989).

Proof: We first observe that, in view of Lemma 4.2.1 and Proposition 4.2.2 applied with $f = \Delta r$,

$$v_k(s)\bar{J}(s) \doteq \int_{\{r\le s\}} \Delta r \equiv \int_0^s \left[\int_{\Gamma_\sigma} \frac{\Delta r}{|\nabla r|}\right] \mathrm{d}\sigma \tag{4.38}$$

is absolutely continuous, and by the divergence theorem it coincides with $v_k(s)J(s)$ for regular values of s. Consider

$$f(s) = \int_0^s \frac{V_k(\sigma)}{v_k(\sigma)}\mathrm{d}\sigma = \int_0^s \frac{1}{v_k(\sigma)}\left[\int_0^\sigma v_k(\tau)\mathrm{d}\tau\right]\mathrm{d}\sigma \tag{4.39}$$

which is a C^2 solution of

$$f'' + (m-1)\frac{\mathrm{sn}_k'}{\mathrm{sn}_k} f' = 1 \quad \text{on } \mathbb{R}^+, \quad f(0) = 0, \quad f'(0) = 0,$$

and define $\psi(x) = f(r(x)) \in C^2(M)$. Let $\{e_i\}$ be a local orthonormal frame on M. Since φ is minimal, by the chain rule and the lower bound in the Hessian comparison theorem 4.19

$$\Delta r = \sum_{j=1}^m \overline{\mathrm{Hess}}\,(\bar{\rho})\big(\mathrm{d}\varphi(e_j), \mathrm{d}\varphi(e_j)\big) \ge \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\big(m - |\nabla r|^2\big). \tag{4.40}$$

We then compute

$$\begin{aligned}\Delta\psi &= f''|\nabla r|^2 + f'\Delta r \ge f''|\nabla r|^2 + f'\frac{\mathrm{sn}_k'}{\mathrm{sn}_k}(m - |\nabla r|^2) \\ &= 1 + \big(1 - |\nabla r|^2\big)\left(f'(r)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} - f''(r)\right).\end{aligned} \tag{4.41}$$

It is not hard to show that the function

$$z(s) \doteq f'(s)\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} - f''(s) = \frac{m}{m-1}\frac{V_k(s)v_k'(s)}{v_k^2(s)} - 1.$$

is non-negative and non-decreasing on $\mathbb{R}^+$. Indeed, from

$$z(0) = 0, \qquad z'(s) = \frac{m}{v_k(s)}\left[kV_k(s) - \frac{1}{m-1}v_k'(s)z(s)\right] \tag{4.42}$$

we deduce that $z' > 0$ when $z < 0$, which proves that $z \ge 0$ on $\mathbb{R}^+$. Fix $0 < t < s$ regular values for r. Integrating (4.41) on the smooth compact set $\{t \le r \le s\}$ and using the divergence theorem we deduce

$$\frac{V_k(s)}{v_k(s)}\int_{\Gamma_s} |\nabla r| - \frac{V_k(t)}{v_k(t)}\int_{\Gamma_t} |\nabla r| \ge \mathrm{vol}\big(\{t \le r \le s\}\big). \tag{4.43}$$

By the definition of $J(s)$ and $\Theta(s)$, and since $J(s) \equiv \bar{J}(s)$ for regular values, the above inequality rewrites as follows:

$$V_k(s)\bar{J}(s) - V_k(t)\bar{J}(t) \geq V_k(s)\Theta(s) - V_k(t)\Theta(t),$$

or in other words,

$$V_k(s)\big[\bar{J}(s) - \Theta(s)\big] \geq V_k(t)\big[\bar{J}(t) - \Theta(t)\big].$$

Since all the quantities involved are continuous, the above relation extends to all $t, s \in \mathbb{R}^+$, which proves the monotonicity of $V_k[\bar{J} - \Theta]$. Letting $t \to 0$ we then deduce that $\bar{J}(s) \geq \Theta(s)$ on $\mathbb{R}^+$. Next, by using $f \equiv 1$ and $f \equiv |\nabla r|^2$ in Lemma 4.2.1 and exploiting again Proposition 4.2.2 we get

$$\mathrm{vol}\big(\{t \leq r \leq s\}\big) = \int_t^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma, \qquad \int_{\{0 \leq r \leq s\}} |\nabla r|^2 = \int_0^s \left[\int_{\Gamma_\sigma} |\nabla r|\right] \mathrm{d}\sigma, \tag{4.44}$$

showing that the two quantities in (4.37) are absolutely continuous. Plugging into (4.43), letting $t \to 0$ and using that $z \geq 0$ we deduce

$$\frac{V_k(s)}{v_k(s)} \int_{\Gamma_s} |\nabla r| \geq \int_0^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma, \tag{4.45}$$

for regular s, which together with the trivial inequality $|\nabla r|^{-1} \geq |\nabla r|$ and with (4.44) gives

$$\begin{aligned} V_k(s) \int_{\Gamma_s} |\nabla r| &\geq v_k(s) \int_0^s \left[\int_{\Gamma_\sigma} |\nabla r|\right] \mathrm{d}\sigma, \\ V_k(s) \left[\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{vol}\big(\{r \leq s\}\big)\right] &\geq v_k(s) \mathrm{vol}\big(\{r \leq s\}\big). \end{aligned} \tag{4.46}$$

Integrating the second inequality we obtain the monotonicity of $\Theta(s)$, while integrating the first one and using (4.44) we obtain the monotonicity of the second quantity in (4.37). To show the monotonicity of $\bar{J}(s)$, by (4.40) and using the full information coming from (4.19) we obtain

$$\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\big(m - |\nabla r|^2\big) \leq \Delta r \leq \frac{g'(r)}{g(r)}\big(m - |\nabla r|^2\big). \tag{4.47}$$

In view of the identity (4.38), we consider regular $s > 0$, we divide (4.47) by $|\nabla r|$ and integrate on Γ_s to get

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \int_{\Gamma_s} \frac{m - |\nabla r|^2}{|\nabla r|} \leq \big(v_k(s)\bar{J}(s)\big)' \leq \frac{g'(s)}{g(s)} \int_{\Gamma_s} \frac{m - |\nabla r|^2}{|\nabla r|} \tag{4.48}$$

Writing $m - |\nabla r|^2 = m(1 - |\nabla r|^2) + (m-1)|\nabla r|^2$, setting for convenience

$$v_g(s) = \omega_{m-1} g(s)^{m-1}, \qquad T(s) \doteq \frac{\int_{\Gamma_s} |\nabla r|^{-1}}{\int_{\Gamma_s} |\nabla r|} - 1, \tag{4.49}$$

rearranging we deduce the two inequalities

$$\begin{aligned} (v_k(s)\bar{J}(s))' &\geq v_k'(s)\bar{J}(s) + m\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}T(s)v_k(s)\bar{J}(s) \\ (v_k(s)\bar{J}(s))' &\leq \frac{v_g'(s)}{v_g(s)}v_k(s)\bar{J}(s) + m\frac{g'(s)}{g(s)}T(s)v_k(s)\bar{J}(s). \end{aligned} \tag{4.50}$$

Expanding the derivate on the left-hand side, we deduce

$$\begin{aligned} \bar{J}'(s) &\geq m\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}T(s)\bar{J}(s), \\ \left(\frac{v_k(s)}{v_g(s)}\bar{J}(s)\right)' &\leq m\frac{g'(s)}{g(s)}T(s)\left(\frac{v_k(s)}{v_g(s)}\bar{J}(s)\right). \end{aligned} \tag{4.51}$$

The first inequality together with the non-negativity of T implies the desired $\bar{J}' \geq 0$, concluding the proof. The second inequality in (4.51), on the other hand, will be useful in awhile.

Remark 4.3.2 The properness of φ is essential in the above proof to justify integrations by parts. However, if φ is non-proper, at least when N is Cartan-Hadamard with sectional curvature $\bar{K} \leq -k$ the function Θ is still monotone in an extended sense. In fact, as it has been observed in (TYSK, 1989) for $N = \mathbb{R}^{m+1}$, $\Theta(s) = +\infty$ for each s such that $\{r < s\}$ contains a limit point of φ. Briefly, if $\bar{x} \in N$ is a limit point with $\bar{\rho}(\bar{x}) < s$, choose $\varepsilon > 0$ such that $2\varepsilon < s - \bar{\rho}(\bar{x})$, and a diverging sequence $\{x_j\} \subset M$ such that $\varphi(x_j) \to \bar{x}$. We can assume that the balls $B_\varepsilon(x_j) \subset M$ are pairwise disjoint. Since $\overline{\mathrm{dist}}(\varphi(x), \varphi(x_j)) \leq \mathrm{dist}(x, x_j)$, we deduce that $\varphi(B_\varepsilon(x_j)) \subset \{r < s\}$ for j large enough, and thus

$$\mathrm{vol}\big(\{r \leq s\}\big) \geq \sum_j \mathrm{vol}(B_\varepsilon(x_j)).$$

However, using that $\bar{K} \leq -k$ and since N is Cartan-Hadamard, we can apply the intrinsic monotonicity formula (see Proposition 4.7.2 in Appendix 2 below) with chosen origin $\varphi(x_j)$ to deduce that $\mathrm{vol}(B_\varepsilon(x_j)) \geq V_k(\varepsilon)$ for each j, whence $\mathrm{vol}(\{r \leq s\}) = +\infty$.

We next investigate conditions equivalent to the finiteness of the density.

Proposition 4.3.2 *Suppose that N has a pole and satisfies* (4.5). *Let $\varphi : M^m \to N^n$ be a proper minimal immersion. Then, the following properties are equivalent:*

(1) $\Theta(+\infty) < +\infty$;

(2) $\bar{J}(+\infty) < +\infty$.

Moreover, both (1) *and* (2) *imply that*

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\left[\frac{\int_{\Gamma_s}|\nabla r|^{-1}}{\int_{\Gamma_s}|\nabla r|}-1\right]\in L^1(\mathbb{R}^+). \tag{3}$$

If further N has an integral pinching to $\mathbb{R}^n$ *or* $\mathbb{H}^n_k$*, then* $(1)\Leftrightarrow(2)\Leftrightarrow(3)$.

Proof: We refer to the proof of the previous proposition for notation and formulas. $(2)\Rightarrow(1)$ is obvious since, by the previous proposition, $\bar{J}(s)\geq\Theta(s)$.
$(1)\Rightarrow(2)$. Note that the limit in (2) exists since $\bar{J}$ is monotone. Suppose by contradiction that $\bar{J}(+\infty)=+\infty$, let $c>0$ and fix s_c large enough that $\bar{J}(s)\geq c$ for $s\geq s_c$. From (4.44) and (4.36), and since $\bar{J}\equiv J$ a.e.,

$$\begin{aligned}\Theta(s) &= \frac{1}{V_k(s)}\int_0^s\left[\int_{\Gamma_\sigma}\frac{1}{|\nabla r|}\right]\mathrm{d}\sigma\geq\frac{1}{V_k(s)}\int_0^s v_k(\sigma)J(\sigma)\mathrm{d}\sigma\\ &\geq \frac{1}{V_k(s)}\int_{s_c}^s v_k(\sigma)J(\sigma)\mathrm{d}\sigma\geq c\frac{V_k(s)-V_k(s_c)}{V_k(s)}.\end{aligned}$$

Letting $s\to+\infty$ we get $\Theta(+\infty)\geq c$, hence $\Theta(+\infty)=+\infty$ by the arbitrariness of c, contradicting (1).
$(2)\Rightarrow(3)$. Integrating (4.51) on $[1,s]$ we obtain

$$c_1\exp\left\{m\int_1^s\frac{\mathrm{sn}_k'(\sigma)}{\mathrm{sn}_k(\sigma)}T(\sigma)\mathrm{d}\sigma\right\}\leq\bar{J}(s)\leq c_2\frac{v_g(s)}{v_k(s)}\exp\left\{m\int_1^s\left[\frac{g'(\sigma)}{g(\sigma)}\right]T(\sigma)\mathrm{d}\sigma\right\}, \tag{4.52}$$

for some constants $c_1,c_2>0$, where $v_g(s)$, $T(s)$ is as in (4.49). The validity of (2) and the first inequality show that $\mathrm{sn}_k'T/\mathrm{sn}_k\in L^1(+\infty)$, that is, (3) is satisfied.
$(3)\Rightarrow(2)$. In our pinching assumptions on N, (ii) in Proposition 4.2.1 gives

$$\frac{g'}{g}=\frac{sn_k'}{sn_k}+\zeta,\ \text{with}\ \zeta\leq C\frac{sn_k'}{sn_k}\ \text{on}\ \mathbb{R}^+,\ \text{and}\ g\leq Csn_k\ \text{on}\ \mathbb{R}^+,$$

for some $C>0$. Plugging in (4.52) and recalling the definition of v_g we obtain

$$\bar{J}(s)\leq c_3\exp\left\{c_4\int_1^s\frac{\mathrm{sn}_k'(\sigma)}{\mathrm{sn}_k(\sigma)}T(\sigma)\mathrm{d}\sigma\right\},$$

for some c_3, c_4, and $(3)\Rightarrow(2)$ follows by letting $s\to+\infty$.

Remark 4.3.3 It is worth to observe that a version of Propositions 4.3.1 and 4.3.2 that covers most of the material presented above has also been independently proved in the very recent (GIMENO; MARKVOSEN, 2015), see Theorems 2.1 and 6.1 therein. We mention that their results are stated for more general ambient spaces subjected to specific function-theoretic requirements, and that, in Proposition 4.3.2, it holds in fact $\bar{J}(+\infty) \equiv \Theta(+\infty)$. For an interesting characterization, when $N = \mathbb{R}^n$, of the limit $\bar{J}(+\infty)$ in terms of an invariant called the projective volume of M we refer to (TKACHEV, 1989).

4.4 Proof of Theorem 1

Let M^m be a minimal properly immersed submanifold in N^n, and suppose that N has a pointwise or integral pinching towards a space form. Because of the upper bound in (4.5), by (CHEUNG; LEUNG, 2001) and (BESSA; MONTENEGRO, 2007) the bottom of $\sigma(M)$ satisfies

$$\inf \sigma(M) \geq \frac{(m-1)^2 k}{4}. \tag{4.53}$$

Briefly, the lower bound in (4.47) implies

$$\Delta r \geq (m-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \geq (m-1)\sqrt{k} \qquad \text{on } M.$$

Integrating on a relatively compact, smooth open set Ω and using the divergence theorem and $|\nabla r| \leq 1$, we deduce $\mathscr{H}^{m-1}(\partial\Omega) \geq (m-1)\sqrt{k}\mathrm{vol}(\Omega)$. The desired (4.53) then follows from Cheeger's inequality:

$$\inf \sigma(M) \geq \frac{1}{4}\left(\inf_{\Omega \Subset M} \frac{\mathscr{H}^{m-1}(\partial\Omega)}{\mathrm{vol}(\Omega)}\right)^2 \geq \frac{(m-1)^2 k}{4}.$$

To complete the proof of the theorem, since $\sigma(M)$ is closed it is sufficient to show that each $\lambda > (m-1)^2 k/4$ lies in $\sigma(M)$.

Set for convenience $\beta \doteq \sqrt{\lambda - (m-1)^2 k/4}$ and, for $0 \leq t < s$, let $A_{t,s}$ denote the extrinsic annulus

$$A_{t,s} \doteq \big\{x \in M \ : \ r(x) \in [t,s]\big\}.$$

Define the weighted measure $\mathrm{d}\mu_k \doteq v_k(r)^{-1}\mathrm{d}x$ on $\{r \geq 1\}$. Hereafter, we will always restrict to this set. Consider

$$\psi(s) \doteq \frac{e^{i\beta s}}{\sqrt{v_k(s)}}, \qquad \text{which solves} \qquad \psi'' + \psi'\frac{v_k'}{v_k} + \lambda\psi = a(s)\psi, \tag{4.54}$$

where

$$a(s) \doteq \frac{(m-1)^2 k}{4} + \frac{1}{4}\left(\frac{v_k'(s)}{v_k(s)}\right)^2 - \frac{1}{2}\frac{v_k''(s)}{v_k(s)} \to 0 \tag{4.55}$$

as $s \to +\infty$. For technical reasons, fix $R > 1$ large such that $\Theta(R) > 0$. Fix t, s, S such that

$$R + 1 < t < s < S - 1,$$

and let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 0 \text{ outside of } (t-1,S), \quad \eta \equiv 1 \text{ on } (t,s),$$

$$|\eta'| + |\eta''| \le C_0 \text{ on } [t-1,s], \qquad |\eta'| + |\eta''| \le \tfrac{C_0}{S-s} \text{ on } [s,S]$$

for some absolute constant C_0 (the last relation is possible since $S-s \ge 1$). The value S will be chosen later in dependence of s. Set $u_{t,s} \doteq \eta(r)\psi(r) \in C_c^\infty(M)$. Then, by (4.54),

$$\begin{aligned}
\Delta u_{t,s} + \lambda u_{t,s} &= (\eta''\psi + 2\eta'\psi' + \eta\psi'')|\nabla r|^2 + (\eta'\psi + \eta\psi')\Delta r + \lambda\eta\psi \\
&= \left(\eta''\psi + 2\eta'\psi' - \frac{v_k'}{v_k}\eta\psi' - \lambda\eta\psi + a\eta\psi\right)(|\nabla r|^2 - 1) + a\eta\psi \\
&\quad + (\eta'\psi + \eta\psi')\left(\Delta r - \frac{v_k'}{v_k}\right) + \left(\eta''\psi + 2\eta'\psi' + \eta'\psi\frac{v_k'}{v_k}\right).
\end{aligned}$$

Using that there exists an absolute constant c for which $|\psi| + |\psi'| \le c/\sqrt{v_k}$, the following inequality holds:

$$\begin{aligned}
\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2 &\le C\left(\int_{A_{t-1,S}} \left[(1-|\nabla r|^2)^2 + \left(\Delta r - \frac{v_k'}{v_k}\right)^2 + a(r)^2\right] \mathrm{d}\mu_k \right. \\
&\quad \left. + \frac{\mu_k(A_{s,S})}{(S-s)^2} + \mu_k(A_{t-1,t})\right),
\end{aligned}$$

for some suitable C depending on c, C_0. Since $\|u_{t,s}\|_2^2 \ge \mu_k(A_{t,s})$ and $(1-|\nabla r|^2)^2 \le 1 - |\nabla r|^2$, we obtain

$$\begin{aligned}
\frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} &\le C\left(\frac{1}{\mu_k(A_{t,s})}\int_{A_{t-1,S}} \left[1-|\nabla r|^2 + \left(\Delta r - \frac{v_k'}{v_k}\right)^2 + a(r)^2\right] \mathrm{d}\mu_k \right. \\
&\quad \left. + \frac{1}{(S-s)^2}\frac{\mu_k(A_{s,S})}{\mu_k(A_{t,s})} + \frac{\mu_k(A_{t-1,t})}{\mu_k(A_{t,s})}\right)
\end{aligned} \tag{4.56}$$

Next, using (4.19),

$$\Delta r = \sum_{j=1}^m \overline{\mathrm{Hess}}(\bar\rho)(e_i,e_i) = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(m - |\nabla r|^2) + T(x) = \frac{v_k'(r)}{v_k(r)} + \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + T(x),$$

where, by Proposition 4.2.1,

$$\begin{aligned}
0 \le T(x) &\doteq \sum_{j=1}^m \overline{\mathrm{Hess}}(\bar\rho)(e_i,e_i) - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(m-|\nabla r|^2) \\
&\le \left(\frac{g'(r)}{g(r)} - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\right)(m-|\nabla r|^2) = \zeta(r)(m-|\nabla r|^2) \le m\zeta(r).
\end{aligned} \tag{4.57}$$

We thus obtain, on the set $\{r \ge 1\}$,

$$\begin{aligned}
\left(\Delta r - \frac{v_k'}{v_k}\right)^2 + 1 - |\nabla r|^2 + a(r)^2 &\le \left[\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + m\zeta(r)\right]^2 \\
&\quad + 1 - |\nabla r|^2 + a(r)^2 \\
&\le C\Big(\zeta(r)^2 + 1 - |\nabla r|^2 + a(r)^2\Big)
\end{aligned} \tag{4.58}$$

for some absolute constant C. Note that, in both our pointwise or integral pinching assumptions on N, by Proposition 4.2.1 it holds $\zeta(s) \to 0$ as $s \to +\infty$. Set

$$F(t) \doteq \sup_{\sigma \in [t-1,+\infty)} [a(\sigma)^2 + \zeta(\sigma)^2],$$

and note that $F(t) \to 0$ monotonically as $t \to +\infty$. Integrating (4.58) we get the existence of $C > 0$ independent of s,t such that

$$\begin{aligned} &\int_{A_{t-1,S}} \left[\left(\Delta r - \frac{v_k'}{v_k} \right)^2 + 1 - |\nabla r|^2 + a(r)^2 \right] \mathrm{d}\mu_k \\ &\qquad \le C \left(F(t) \int_{A_{t-1,S}} \frac{1}{v_k(r)} + \int_{A_{t-1,S}} \frac{1 - |\nabla r|^2}{v_k(r)} \right). \end{aligned} \tag{4.59}$$

Using the coarea's formula and the transversality lemma, for each $0 \le a < b$

$$\mu_k(A_{a,b}) = \int_{A_{a,b}} \frac{1}{v_k(r)} = \int_a^b J\big[1+T\big], \qquad \int_{A_{a,b}} \frac{1 - |\nabla r|^2}{v_k(r)} = \int_a^b JT, \tag{4.60}$$

where J and T are defined, respectively, in (4.36) and (4.49). Summarizing, in view of (4.59) and (4.60) we deduce from (4.56) the following inequalities:

$$\begin{aligned} \frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} \ \le \ & C \left(\frac{1}{\int_t^s J\big[1+T\big]} \left[F(t) \int_{t-1}^S J\big[1+T\big] + \int_{t-1}^S JT \right] \right. \\ & \left. + \frac{\int_s^S J\big[1+T\big]}{(S-s)^2 \int_t^s J\big[1+T\big]} + \frac{\int_{t-1}^t J\big[1+T\big]}{\int_t^s J\big[1+T\big]} \right) \doteq \mathscr{Q}(t,s). \end{aligned} \tag{4.61}$$

If we can guarantee that

$$\liminf_{t \to +\infty} \liminf_{s \to +\infty} \frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} = 0, \tag{4.62}$$

then we are able to construct a sequence of approximating eigenfunctions for λ as follows: fix $\varepsilon > 0$. By (4.62) there exists a divergent sequence $\{t_i\}$ such that, for $i \ge i_\varepsilon$,

$$\liminf_{s \to +\infty} \frac{\|\Delta u_{t_i,s} + \lambda u_{t_i,s}\|_2^2}{\|u_{t_i,s}\|_2^2} < \varepsilon/2.$$

For $i = i_\varepsilon$, pick then a sequence $\{s_j\}$ realizing the liminf. For $j \ge j_\varepsilon(i_\varepsilon, \varepsilon)$

$$\|\Delta u_{t_i,s_j} + \lambda u_{t_i,s_j}\|_2^2 < \varepsilon \|u_{t_i,s_j}\|_2^2, \tag{4.63}$$

Writing $u_\varepsilon \doteq u_{t_{i_\varepsilon}, s_{j_\varepsilon}}$, by (4.63) from the set $\{u_\varepsilon\}$ we can extract a sequence of approximating eigenfunctions for λ, concluding the proof that $\lambda \in \sigma(M)$. To show (4.62), by (4.61) it is enough to prove that

$$\liminf_{t \to +\infty} \liminf_{s \to +\infty} \mathscr{Q}(t,s) = 0. \tag{4.64}$$

Suppose, by contradiction, that (4.64) were not true. Then, there exists a constant $\delta > 0$ such that, for each $t \geq t_\delta$, $\liminf_{s\to+\infty} \mathscr{Q}(t,s) \geq 2\delta$, and thus for $t \geq t_\delta$ and $s \geq s_\delta(t)$

$$F(t)\int_{t-1}^{S} J\big[1+T\big] + \int_{t-1}^{S} JT + \int_{s}^{S} \frac{J\big[1+T\big]}{(S-s)^2} + \int_{t-1}^{t} J\big[1+T\big] \geq \delta \int_{t}^{s} J\big[1+T\big], \tag{4.65}$$

and rearranging

$$(F(t)+1)\int_{t-1}^{S} J\big[1+T\big] - \int_{t-1}^{S} J + \int_{s}^{S} \frac{J\big[1+T\big]}{(S-s)^2} + \int_{t-1}^{t} J\big[1+T\big] \geq \delta \int_{t}^{s} J\big[1+T\big]. \tag{4.66}$$

We rewrite the above integrals in order to make $\Theta(s)$ appear. Integrating by parts and using again the coarea's formula and the transversality lemma,

$$\begin{aligned} \int_a^b J\big[1+T\big] &= \int_{A_{a,b}} \frac{1}{v_k(r)} = \int_a^b \frac{1}{v_k(\sigma)}\left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma = \int_a^b \frac{\big(V_k(\sigma)\Theta(\sigma)\big)'}{v_k(\sigma)} \mathrm{d}\sigma \\ &= \frac{V_k(b)}{v_k(b)}\Theta(b) - \frac{V_k(a)}{v_k(a)}\Theta(a) + \int_a^b \frac{V_k v_k'}{v_k^2}\Theta. \end{aligned} \tag{4.67}$$

To deal with the term containing the integral of J alone in (4.66), we use the inequality $J(s) \geq \Theta(s)$ coming from the monotonicity formulae in Proposition 4.3.1. This passage is crucial for us to conclude. Inserting (4.67) and $J \geq \Theta$ into (4.66) we get

$$\begin{aligned} &(F(t)+1)\frac{V_k(S)}{v_k(S)}\Theta(S) - (F(t)+1)\frac{V_k(t-1)}{v_k(t-1)}\Theta(t-1) + (F(t)+1)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta \\ &- \int_{t-1}^{S} \Theta + \frac{1}{(S-s)^2}\left[\frac{V_k(S)}{v_k(S)}\Theta(S) - \frac{V_k(s)}{v_k(s)}\Theta(s) + \int_s^S \frac{V_k v_k'}{v_k^2}\Theta\right] + \frac{V_k(t)}{v_k(t)}\Theta(t) \\ &- \frac{V_k(t-1)}{v_k(t-1)}\Theta(t-1) + \int_{t-1}^{t} \frac{V_k v_k'}{v_k^2}\Theta \geq \delta\frac{V_k(s)}{v_k(s)}\Theta(s) - \delta\frac{V_k(t)}{v_k(t)}\Theta(t) + \delta\int_t^s \frac{V_k v_k'}{v_k^2}\Theta. \end{aligned} \tag{4.68}$$

To reach the desired contradiction, the idea is to prove that (4.6) cannot hold by showing that

$$\int_{t-1}^{S} \Theta \tag{4.69}$$

must grow sufficiently fast as $S \to +\infty$. To do so, we need to simplify (4.68) in order to find a suitable differential inequality for (4.69).

We first observe that, both for $k > 0$ and for $k = 0$, there exists an absolute constant $\hat{c}$ such that $\hat{c}^{-1} \leq V_k v_k'/v_k^2 \leq \hat{c}$ on $[1,+\infty)$. Furthermore, by the monotonicity of Θ,

$$\int_s^S \frac{V_k v_k'}{v_k^2}\Theta \leq \hat{c}(S-s)\Theta(S). \tag{4.70}$$

Next, we deal with the two terms in the left-hand side of (4.68) that involve (4.69):

$$\begin{aligned} (F(t)+1)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta - \int_{t-1}^{S}\Theta &= F(t)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta + \int_{t-1}^{S} \frac{V_k v_k' - v_k^2}{v_k^2}\Theta \\ &\leq \hat{c}F(t)\int_{t-1}^{S}\Theta + \int_{t-1}^{S} \frac{V_k v_k' - v_k^2}{v_k^2}\Theta. \end{aligned}$$

The key point is the following relation:

$$\frac{V_k(s)v_k'(s) - v_k(s)^2}{v_k(s)^2} \begin{cases} = -1/m & \text{if } k = 0; \\ \to 0 \text{ as } s \to +\infty, & \text{if } k > 0. \end{cases} \tag{4.71}$$

Define

$$\omega(t) \doteq \sup_{[t-1,+\infty)} \frac{V_k v_k' - v_k^2}{v_k^2}, \qquad \chi(t) \doteq \hat{c}F(t) + \omega(t).$$

Again by the monotonicity of Θ,

$$\begin{aligned}(F(t)+1)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta - \int_{t-1}^{S}\Theta \;&\leq\; \left[\hat{c}F(t)+\omega(t)\right]\int_{t-1}^{S}\Theta = \chi(t)\int_{t-1}^{S}\Theta \\ &\leq\; \chi(t)\Theta(t) + \chi(t)\int_{t}^{S}\Theta.\end{aligned} \tag{4.72}$$

For simplicity, hereafter we collect all the terms independent of s in a function that we call $h(t)$, which may vary from line to line. Inserting (4.70) and (4.72) into (4.68) we infer

$$\begin{aligned}&\left[\left(F(t)+1+\frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)} + \frac{\hat{c}}{S-s}\right]\Theta(S) + \chi(t)\int_t^S\Theta \\ &\geq h(t) + \left(\delta + \frac{1}{(S-s)^2}\right)\frac{V_k(s)}{v_k(s)}\Theta(s) + \delta\hat{c}^{-1}\int_t^s\Theta.\end{aligned} \tag{4.73}$$

Summing $\delta\hat{c}^{-1}(S-s)\Theta(S)$ to the two sides of the above inequality, using the monotonicity of Θ and getting rid of the term containing $\Theta(s)$ we obtain

$$\begin{aligned}&\left[\left(F(t)+1+\frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)} + \frac{\hat{c}}{S-s} + \delta\hat{c}^{-1}(S-s)\right]\Theta(S) + \chi(t)\int_t^S\Theta \\ &\geq h(t) + \delta\hat{c}^{-1}\int_t^S\Theta.\end{aligned} \tag{4.74}$$

Using (4.71), the definition of $\chi(t)$ and the properties of $\omega(t), F(t)$, we can choose t_δ sufficiently large to guarantee that

$$\delta\hat{c}^{-1} - \chi(t) \geq c_k \doteq \begin{cases} \frac{1}{m} + \frac{\delta\hat{c}^{-1}}{2} & \text{if } k = 0, \\ \frac{\delta\hat{c}^{-1}}{2} & \text{if } k > 0, \end{cases} \tag{4.75}$$

hence

$$\left[\left(F(t)+1+\frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)} + \frac{\hat{c}}{S-s} + \delta\hat{c}^{-1}(S-s)\right]\Theta(S) \geq h(t) + c_k\int_t^S\Theta. \tag{4.76}$$

We now specify $S(s)$ depending on whether $k > 0$ or $k = 0$.

The case $k>0$.

We choose $S \doteq s+1$. In view of the fact that V_k/v_k is bounded above on $\mathbb{R}^+$, (4.76) becomes

$$\bar{c}\Theta(s+1) \geq h(t)+c_k\int_t^{s+1}\Theta \geq \frac{c_k}{2}\int_t^{s+1}\Theta, \tag{4.77}$$

for some $\bar{c}$ independent of t,s. Note that the last inequality is satisfied provided $s \geq s_\delta(t)$ is chosen to be sufficiently large, since the monotonicity of Θ implies that $\Theta \notin L^1(\mathbb{R}^+)$. Integrating and using again the monotonicity of Θ, we get

$$(s+1-t)\Theta(s+1) \geq \int_t^{s+1}\Theta \geq \left[\int_t^{s_0+1}\Theta\right]\exp\left\{\frac{c_k}{2\bar{c}}(s-s_0)\right\},$$

hence $\Theta(s)$ grows exponentially. Ultimately, this contradicts our assumption (4.6).

The case $k=0$.

We choose $S \doteq s+\sqrt{s}$. Since $V_k(S)/v_k(S)=S/m$, from (4.76) we infer

$$\left[\left(F(t)+1+\frac{1}{s}\right)\frac{S}{m}+\frac{\hat{c}}{\sqrt{s}}+\delta\hat{c}^{-1}\sqrt{s}\right]\Theta(S) \geq h(t)+c_k\int_t^S\Theta. \tag{4.78}$$

Using the expression of c_k and the fact that $F(t)\to 0$, up to choosing t_δ and then $s_\delta(t)$ large enough we can ensure the validity of the following inequality:

$$\left[\left(F(t)+1+\frac{1}{s}\right)\frac{S}{m}+\frac{\hat{c}}{\sqrt{s}}+\delta\hat{c}^{-1}\sqrt{s}\right] < \left[\frac{1}{m}+\frac{\delta\hat{c}^{-1}}{4}\right]S = \left[c_k-\frac{\delta\hat{c}^{-1}}{4}\right]S$$

for $t \geq t_\delta$ and $s \geq s_\delta(t)$. Plugging into (4.76), and using that $\Theta \notin L^1(\mathbb{R}^+)$,

$$S\Theta(S) \geq h(t)+\frac{c_k}{c_k-\delta\hat{c}^{-1}/4}\int_t^S\Theta \geq (1+\varepsilon)\int_t^S\Theta,$$

for a suitable $\varepsilon>0$ independent of t,S, and provided that $S \geq s_\delta(t)$ is large enough. Integrating and using again the monotonicity of Θ,

$$S\Theta(S) \geq (S-t)\Theta(S) \geq \int_t^S\Theta \geq \left[\int_t^{S_0}\Theta\right]\left(\frac{S}{S_0}\right)^{1+\varepsilon},$$

hence $\Theta(S)$ grows polynomially at least with power ε, contradicting (4.6).

Concluding, both for $k>0$ and for $k=0$ assuming (4.65) leads to a contradiction with our assumption (4.6), hence (4.62) holds, as required.

4.5 Proof of Theorem 2

We first show that φ is proper and that M is diffeomorphic to the interior of a compact manifold with boundary. Both the properties are consequence of the following lemma due to (BESSA; COSTA, 2009), which improves on (ANDERSON, 1984; CASTILLON, 1999).

Lemma 4.5.1 *Let $\varphi : M^m \to N^n$ be an immersed submanifold into an ambient manifold N with a pole and suppose that N satisfies* (4.5) *for some $k \geq 0$. Denote by $B_s = \{x \in M;\ \rho(x) \leq s\}$ the intrinsic ball on M. Assume that*

$$\begin{array}{lll} (i) & \limsup_{s\to+\infty} s\|\mathrm{II}\|_{L^\infty(\partial B_s)} < 1 & \text{if } k=0 \text{ in (4.5), or} \\ (ii) & \limsup_{s\to+\infty} \|\mathrm{II}\|_{L^\infty(\partial B_s)} < \sqrt{k} & \text{if } k>0 \text{ in (4.5).} \end{array} \tag{4.79}$$

Then, φ is proper and there exists $R > 0$ such that $|\nabla r| > 0$ on $\{r \geq R\}$, where r is the extrinsic distance function. Consequently, the flow

$$\Phi : \mathbb{R}^+ \times \{r = R\} \to \{r \geq R\}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(x) = \frac{\nabla r}{|\nabla r|^2}\big(\Phi_s(x)\big) \tag{4.80}$$

is well defined, and M is diffeomorphic to the interior of a compact manifold with boundary.

The properness of φ enables us to apply Proposition 4.3.2. Therefore, to show that $\Theta(+\infty) < +\infty$ it is enough to check that

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \frac{\int_{\Gamma_s}\big[|\nabla r|^{-1} - |\nabla r|\big]}{\int_{\Gamma_s}|\nabla r|} \in L^1(+\infty). \tag{4.81}$$

To achieve (4.81), we need to bound from above the rate of approaching of $|\nabla r|$ to 1 along the flow Φ in Lemma 4.5.1. We begin with the following

Lemma 4.5.2 *Suppose that N has a pole and radial sectional curvature satisfying* (4.5)*, and that $\varphi : M^m \to N^n$ is a proper minimal immersion such that $|\nabla r| > 0$ outside of some compact set $\{r \leq R\}$. Let Φ denote the flow of $\nabla r/|\nabla r|^2$ as in* (4.80) *and let $\gamma : [R, +\infty) \to M$ be a flow line starting from some $x_0 \in \{r = R\}$. Then, along γ,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{sn}_k(r)\sqrt{1-|\nabla r|^2}\big) \leq \mathrm{sn}_k(r)|\mathrm{II}(\gamma(s))| \tag{4.82}$$

Proof: Observe that $r(\gamma(s)) = s - R$. By the chain rule and the Hessian comparison theorem 4.19,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}|\nabla r|^2 &= 2\mathrm{Hess}\, r(\nabla r, \dot\gamma) = \frac{2}{|\nabla r|^2}\mathrm{Hess}\, r(\nabla r, \nabla r) \\ &= \frac{2}{|\nabla r|^2}\overline{\mathrm{Hess}}\,(\bar\rho)\big(\mathrm{d}\varphi(\nabla r), \mathrm{d}\varphi(\nabla r)\big) + \frac{2}{|\nabla r|^2}\big(\bar\nabla\bar\rho, \mathrm{II}(\nabla r, \nabla r)\big) \\ &\geq 2\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + 2|\bar\nabla^\perp\bar\rho||\mathrm{II}|, \end{aligned}$$

where $\bar{\nabla}^{\perp}\bar{\rho}$ is the component of $\bar{\rho}$ perpendicular to $\mathrm{d}\varphi(TM)$ and $|\bar{\nabla}^{\perp}\rho| = \sqrt{1-|\nabla r|^2}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}s}|\nabla r|^2 \geq 2\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + 2|\mathrm{II}|\sqrt{1-|\nabla r|^2}.$$

Multiplying by $\mathrm{sn}_k^2(r)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{sn}_k^2(r)(1-|\nabla r|^2)\big) \leq 2\mathrm{sn}_k^2(r)|\mathrm{II}|\sqrt{1-|\nabla r|^2},$$

which implies (4.82).

The above lemma relates the behavior of $|\nabla r|$ to that of the second fundamental form. The next result makes this relation explicit in the two cases considered in Theorem 4.1.2.

Proposition 4.5.1 *In the assumptions of the above proposition, suppose further that either*

$$\begin{array}{lll} (i) & \|\mathrm{II}\|_{L^\infty(\partial B_s)} \leq \dfrac{C}{s\log^{\alpha/2}s} & \text{if } k=0 \text{ in (4.5), or} \\ (ii) & \|\mathrm{II}\|_{L^\infty(\partial B_s)} \leq \dfrac{C}{\sqrt{s}\log^{\alpha/2}s} & \text{if } k>0 \text{ in (4.5).} \end{array} \tag{4.83}$$

for $s \geq 1$ and some constants $C>0$ and $\alpha>0$. Here, ∂B_s is the boundary of the intrinsic ball $B_s(o)$. Then, $|\nabla r|(\gamma(s)) \to 1$ as s diverges, and if $s>2R$ and R is sufficiently large,

$$\begin{array}{ll} \text{in the case } (i), & 1-|\nabla r(\gamma(s))|^2 \leq \dfrac{\hat{C}}{\log^{\alpha}s} \\ \text{in the case } (ii), & 1-|\nabla r(\gamma(s))|^2 \leq \dfrac{\hat{C}}{s\log^{\alpha}s} \end{array} \tag{4.84}$$

for some constant $\hat{C}$ depending on C.

Proof: We begin by observing that, in (4.83), ∂B_s can be replaced by Γ_s. Indeed, since $r(x) \leq r(o)+\rho(x)$, we can choose R large enough depending on $r(o),\alpha$ in such a way that, for instance in (i),

$$|\mathrm{II}(x)| \leq \frac{C}{\rho(x)\log^{\alpha/2}\rho(x)} \leq \frac{C_1}{r(x)\log^{\alpha/2}r(x)}$$

for some absolute C_1 and for each $r \geq R$. Thus, from (i) and (ii) we infer the bounds

$$\|\mathrm{II}\|_{L^\infty(\Gamma_s)} \leq \frac{C_1}{s\log^{\alpha/2}s} \quad \text{for } (i), \qquad \|\mathrm{II}\|_{L^\infty(\Gamma_s)} \leq \frac{C_1}{\sqrt{s}\log^{\alpha/2}s} \quad \text{for } (ii). \tag{4.85}$$

Because of (4.85), up to enlarging R further there exists a uniform constant $C_2>0$ such that, on $[R,+\infty)$,

$$\mathrm{sn}_k(s)|\mathrm{II}(\gamma(s))| \leq \begin{cases} \dfrac{C_1}{\log^{\alpha/2}s} \leq C_2\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{s}{\log^{\alpha/2}s}\right) & \text{if } k=0; \\[2ex] \dfrac{C_1\mathrm{sn}_k(s)}{\sqrt{s}\log^{\alpha/2}s} \leq C_2\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\mathrm{sn}_k(s)}{\sqrt{s}\log^{\alpha/2}s}\right) & \text{if } k>0. \end{cases} \tag{4.86}$$

Integrating on $[R,s]$ and using (4.82) we get

$$\sqrt{1-|\nabla r(\gamma(s))|^2} \le \begin{cases} \dfrac{C_3(R)}{s} + \dfrac{C_4}{\log^{\alpha/2} s} \le \dfrac{C_5}{\log^{\alpha/2} s} & \text{if } k=0, \\ \dfrac{C_3(R)}{\mathrm{sn}_k(s)} + \dfrac{C_4}{\sqrt{s}\log^{\alpha/2} s} \le \dfrac{C_5}{\sqrt{s}\log^{\alpha/2} s} & \text{if } k>0, \end{cases}$$

for some absolute constants $C_4, C_5 > 0$ and if $s > 2R$ and R is large enough. The desired (4.84) follows by taking squares.

We are now ready to conclude the proof of Theorem 4.1.2 by showing that M has finite density or, equivalently, that (4.81) holds.

Let $\eta(s)$ be either

$$\frac{C}{\log^{\alpha} s} \quad \text{when } k=0, \text{ or } \quad \frac{C}{s\log^{\alpha} s} \quad \text{when } k>0, \tag{4.87}$$

where $\alpha > 1$ and C is a large constant. In our assumptions, we can apply Lemma 4.5.2 and Proposition 4.5.1 to deduce, according to (4.84), that

$$1-|\nabla r(\gamma(s))|^2 \le \eta(s) \qquad \text{on } (R,+\infty),$$

where $\gamma(s)$ is a flow curve of Φ in (4.80) and C in (4.87) is large enough. In particular, $|\nabla r(\gamma(s))| \to 1$ as $s \to +\infty$. We therefore deduce the existence of a constant $c > 0$ such that, if $s \ge R$ and R is large enough,

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \frac{\int_{\Gamma_s} \left[|\nabla r|^{-1} - |\nabla r|\right]}{\int_{\Gamma_s} |\nabla r|} \le \frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \eta(s) \frac{\int_{\Gamma_s} |\nabla r|^{-1}}{\int_{\Gamma_s} |\nabla r|} \le c\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \eta(s).$$

In both our cases $k=0$ and $k>0$, it is immediate to check that $\mathrm{sn}_k'\eta/\mathrm{sn}_k \in L^1(+\infty)$, proving (4.81).

4.6 Appendix 1: finite total curvature solutions of Plateau's problem

In this appendix, we show that (smooth) solutions of Plateau's problem at infinity $M^m \to \mathbb{H}^n$ have finite total curvature whenever M is a hypersurface and the boundary datum $\Sigma \subset \partial_\infty \mathbb{H}^n$ is sufficiently regular. Consider the Poincaré model of $\mathbb{H}^n$, let $\partial_\infty \mathbb{H}^n$ be its sphere at infinity, and let $M \to \mathbb{H}^n$ be a proper minimal submanifold. We say that M is $C^{k,\alpha}$ up to $\partial_\infty \mathbb{H}^n$ if its closure $\overline{M}$ in the topology of the closed unit ball $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ is a $C^{k,\alpha}$-manifold with boundary. We begin with a lemma, whose proof have been suggested to the second author by L. Mazet.

Lemma 4.6.1 *Let $\varphi : M^m \to \mathbb{H}^n$ be a proper minimal submanifold. If M is of class C^2 up to $\partial_\infty \mathbb{H}^n$, then M has finite total curvature.*

Proof: Let $(\mathbb{D}^n, \langle\,,\rangle)$ be the Poincaré model of $\mathbb{H}^n$, and write the Euclidean metric $\overline{\langle\,,\rangle}$ on $\mathbb{D}^n$ as

$$\overline{\langle\,,\rangle} = \lambda^2 \langle\,,\rangle, \qquad \text{with} \quad \lambda = \frac{1-|x|^2}{2}.$$

Given a proper, minimal submanifold $\varphi : (M^m, g) \to (\mathbb{D}^n, \langle\,,\rangle)$, we associate the isometric immersion $\bar{\varphi} : (M, (\lambda^2 \circ \varphi) g) \to (\mathbb{D}^n, \overline{\langle\,,\rangle})$, $\bar{\varphi}(x) \doteq \varphi(x)$. Fix a local Darboux frame $\{e_i, e_\alpha\}$ on (M, g) for φ, with $\{e_i\}$ tangent to M and $\{e_\alpha\}$ in the normal bundle, and let $\bar{e}_i = e_i/\lambda$, $\bar{e}_\alpha = e_\alpha/\lambda$ be the corresponding Darboux frame on $(M, \lambda^2 g)$ for $\bar{\varphi}$. Let $\mathrm{d}V$ and $\mathrm{d}\bar{V} = \lambda^m \mathrm{d}V$ be the volume forms of (M, g) and $(M, \lambda^2 g)$, and denote with h^α_{ij} and $\bar{h}^\alpha_{ij}$ the coefficients of the second fundamental forms of φ and $\bar{\varphi}$, respectively. A standard computation shows that

$$\bar{h}^\alpha_{ij} = \frac{1}{\lambda} h^\alpha_{ij} - \frac{\lambda_\alpha}{\lambda} \delta_{ij},$$

where $\lambda_\alpha = e_\alpha(\lambda)$. Evaluating the norms of II and $\bar{\mathrm{II}}$, since h^α_{ij} is trace-free by minimality we obtain

$$|\bar{\mathrm{II}}|^2 = \lambda^{-2} |\mathrm{II}|^2 + m |\nabla^\perp \log \lambda|^2 \geq \lambda^{-2} |\mathrm{II}|^2,$$

and thus $|\bar{\mathrm{II}}|^m \mathrm{d}\bar{V} \geq |\mathrm{II}|^m \mathrm{d}V$. Integrating on M it holds

$$\int_M |\mathrm{II}|^m \mathrm{d}V \leq \int_M |\bar{\mathrm{II}}|^m \mathrm{d}\bar{V}.$$

However, the last integral is finite since M is C^2 up to $\partial_\infty \mathbb{H}^n$, and thus φ has finite total curvature.
In view of Lemma 4.6.1, we briefly survey on some boundary regularity results for solutions of Plateau's problem. To the best of our knowledge, we just found regularity results *for hypersurfaces*. Let $M^m \to \mathbb{H}^{m+1}$ be a solution of Plateau's problem for a compact, $(m-1)$-dimensional submanifold $\Sigma^{m-1} \subset \partial_\infty \mathbb{H}^{m+1}$. Then, a classical result of Hardt and Lin (HARDT; LIN, 1987) states that if $\Sigma^{m-1} \hookrightarrow \partial_\infty \mathbb{H}^n$ is properly embedded and $C^{1,\alpha}$, with $0 \leq \alpha \leq 1$, near Σ each solution $M^m \to \mathbb{H}^n$ of Plateau's problem is a finite collection of $C^{1,\alpha}$-manifolds with boundary, which are disjoint except at the boundary. Therefore, near Σ, M can locally be described as a graph, and the higher regularity theory in (LIN, 1989; LIN, 2012; TONEGAWA, 1993; TONEGAWA, 1996), applies to give the following: if Σ is $C^{j,\alpha}$, then M is $C^{j,\alpha}$ up to $\partial_\infty \mathbb{H}^n$ whenever

- $1 \leq j \leq m-1$ and $0 \leq \alpha \leq 1$, or

- $j = m$ and $0 < \alpha < 1$, or
- $j \geq m+1$ and $0 < \alpha < 1$, under a further condition on Σ if j is odd.

(see the statement and references in (LIN, 2012)). In particular, because of Lemma 4.6.1, if Σ is $C^{2,\alpha}$ for some $0 < \alpha < 1$ then M has finite total curvature (provided that it is smooth).

4.7 Appendix 2: the intrinsic monotonicity formula

We conclude by recalling an intrinsic version of the monotonicity formula. To state it, we premit the following observation due to H. Donnelly and N. Garofalo, Proposition 3.6 in (DONNELLY; GAROFALO, 1992).

Proposition 4.7.1 *For $k \geq 0$, the function*

$$\frac{V_k(s)}{v_k(s)} \qquad \textit{is non-decreasing on } \mathbb{R}^+. \tag{4.88}$$

Proof: The ratio v_k'/v_k is monotone decreasing by the very definition of v_k. Then, since $v_k' > 0$, the desired monotonicity follows from a lemma at p. 42 of (CHEEGER *et al.*, 1982).

Proposition 4.7.2 (The intrinsic monotonicity formula) *Suppose that N has a pole $\bar{o}$ and satisfies* (4.5)*, and let $\varphi : M^m \to N^n$ be a complete, minimal immersion. Suppose that $\bar{o} \in \varphi(M)$, and choose $o \in M$ be such that $\varphi(o) = \bar{o}$. Then, denoting with ρ the intrinsic distance function from o and with $B_s = \{\rho \leq s\}$,*

$$\frac{\mathrm{vol}(B_s)}{V_k(s)} \tag{4.89}$$

is monotone non-decreasing on $\mathbb{R}^+$.

Proof: We refer to Proposition 4.3.1 for definitions and computations. We know that the function $\psi = f \circ r$, with f as in (4.39), solves $\Delta\psi \geq 1$ on M. Integrating on B_s and using the definition of ψ we obtain

$$\mathrm{vol}(B_s) \leq \int_{B_s} \Delta\psi = \int_{\partial B_s} \langle \nabla\psi, \nabla\rho \rangle \leq \int_{\partial B_s} \frac{V_k(r)}{v_k(r)}.$$

Next, since $\bar{o} = \varphi(o)$, it holds $r(x) \leq \rho(x)$ on M. Using then Proposition 4.7.1, we deduce

$$\mathrm{vol}(B_s) \leq \frac{V_k(s)}{v_k(s)} \mathrm{vol}(\partial B_s).$$

Integrating we obtain the monotonicity of the desired (4.89).

5 CONCLUSÃO

In the first part of this thesis we dealt with the 3D Navier-Stokes and Boussinesq systems in a cube. We proved some results concerning the global approximate controllability by means of boundary controls which act in some part of the boundary. Still we proved the internal and boundary local null controllability of a 1D parabolic PDE with nonlinear diffusion. In the second part of this Thesis, we considered M^m minimal properly immersed submanifolds in a complete ambient space N^n suitably close to a space form N^n_k of curvature $-k \leq 0$ and we proved that if the density function $\Theta(r)$ has subexponential growth (when $k < 0$) or sub-polynomial growth ($k = 0$) along a sequence, then the spectrum of M^m is the same as that of the space form $\mathbb{N}^m_k$. Thus, we have that the applies to Anderson's (smooth) solutions of Plateau's problem at infinity on the hyperbolic space $\mathbb{H}^n$, independently of their boundary regularity. Finally, we also give a simple condition on the second fundamental form that ensures M to have finite density. In particular, we showed that minimal submanifolds of $\mathbb{H}^n$ with finite total curvature have finite density.

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