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ITAMAR SALES DE OLIVEIRA FILHO

TIME-FREQUENCY ANALYSIS

THE BILINEAR HILBERT TRANSFORM AND THE CARLESON THEOREM

FORTALEZA

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Dissertação apresentada ao Programa de Pós-graduação em Matemática do Departamento de Matemática da Universidade Federal do Ceará, como parte dos requisitos necessários para a obtenção do título de Mestre em Matemática. Área de concentração: Análise.

Orientador: Prof. Dr. Diego Ribeiro Moreira
Coorientador: Prof. Dr. Emanuel Augusto de Souza Carneiro.

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To Itamar, Margarete, Heitor, Arthur and Rebeca.

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"Somewhere, something incredible is waiting to be known." Carl Sagan

RESUMO

Em 1966, Lennart Carleson provou que a série de Fourier de uma função periódica, quadrado-integrável em um domínio fundamental na reta converge para a própria função em quase todo ponto. Esse resultado foi revisitado alguns anos depois por Charles Fefferman (1973) e por Lacey e Thiele (2000). É estudado aqui o trabalho desses últimos, onde o problema é abordado através de análise de tempo e frequência. Essa demonstração foi inspirada em um trabalho anterior dos mesmos autores em que estabelecem limitação para a transformada de Hilbert bilinear em espaços de Lebesgue. O estudo da limitação desse operador começou com as tentativas de estabelecer limitação para o primeiro comutador de Calderón. Também sob o ponto de vista da análise de tempo e frequência, será estudado um dos trabalhos de Lacey e Thiele sobre a transformada de Hilbert bilinear.

Palavras-chave: Análise de tempo e frequência. Operador de Carleson. Transformada de Hilbert bilinear.

ABSTRACT

In 1966, Lennart Carleson proved that the Fourier series of a periodic function, square integrable over a fundamental domain of the real line converges to the same function almost everywhere. This result was revisited years later by Charles Fefferman (1973) and by Lacey and Thiele (2000). It is studied here Lacey and Thiele's work, where they approached the problem through time-frequency analysis. This proof was inspired in a previous work of theirs, where they establish boundedness for the bilinear Hilbert transform in Lebesgue spaces. The study of boundedness for this operator started with the attempts to establish boundedness for the first Calderón's commutator. Also through time-frequency analysis, it will be studied one of the works of Lacey and Thiele about the bilinear Hilbert transform.

Keywords: Time-frequency analysis. Carleson operator. Bilinear Hilbert transform.

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1 INTRODUCTION

Let us first introduce two problems: L^2-boundedness of the Calderón's first commutator and convergence of Fourier series.

L^2-BOUNDEDNESS OF THE CALDERÓN'S FIRST COMMUTATOR

Let

$$\gamma(x) = x + iA(x)$$

be a curve in $\mathbb{C}$, where $A' = a \in L^\infty(\mathbb{R})$. The Hilbert transform corresponding to γ is given by:

$$H_\gamma f(x) := \text{p.v.} \int \frac{f(y)(1 + ia(y))dy}{x - y + i(A(x) - A(y))}.$$

It is known (thanks to Calderón, Coifman, McIntosh and Meyer) that H_γ is bounded from L^2 to itself. One can approach this result by doing the following expansion

$$\frac{1}{x - y + i(A(x) - A(y))} = \frac{1}{x - y} \sum_{k=0}^{\infty} (-i)^k \left(\frac{A(x) - A(y)}{x - y} \right)^k.$$

This way, we obtain the operators

$$C_k f(x) = \text{p.v.} \int \frac{(A(x) - A(y))^k}{(x - y)^{k+1}} f(y) dy.$$

Observe that C_0 is the Hilbert transform. C_1 is the *Calderón's first commutator* and it can be rewritten as

$$\begin{aligned} C_1 f(x) &= \text{p.v.} \int \int_0^1 a(x + \alpha(y - x)) \frac{1}{x - y} f(y) d\alpha dy \\ &= \text{p.v.} \int \int_0^1 a(x - \alpha y) f(x - y) \frac{1}{y} d\alpha dy \\ &= \int_0^1 H_\alpha(f, a)(x) d\alpha \end{aligned}$$

where we have the bilinear Hilbert transform in the integrand, i.e.,

$$H_\alpha(f, a)(x) = \text{p.v.} \int a(x - \alpha y) f(x - y) \frac{1}{y} dy$$

for $a \in L^\infty(\mathbb{R})$ fixed, one can ask if C_1 is a bounded operator from $L^2(\mathbb{R})$ to itself. To prove this, Calderón attemped to find appropriate bounds for the bilinear Hilbert transform. However, he was not able to do it and proved boundedness for C_1 by other means. Following LACEY and THIELE (1997) and LACEY and THIELE (1999), we discuss their proof of L^p bounds for the bilinear Hilbert transform.

CONVERGENCE OF FOURIER SERIES

A classical problem in Analysis is to determine when and how a given function f can be approximated (in some sense) by an expression of the form:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi nix} = S[f](x) \tag{1}$$

where $\widehat{f}$ is the Fourier transform of f (if it makes sense for such function). Consider, for instance, the set of *trigonometric polynomials*, i.e., the set of functions given by:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi nix}$$

where all but finitely many a_n are zero. One can easily see by orthogonality properties of $\{e^{2\pi ixn}; n \in \mathbb{Z}\}$ that:

$$a_n = \widehat{f}(n).$$

So (1) holds pointwise with $=$ instead of $\sim$ in this case. But what happens in a more general setting? Let us start with the following result:

Theorem 1.1. *If $f \in C^{\alpha}(\mathbb{T})$ with $0 < \alpha \leq 1$ then $\|S_N[f] - f\|_{\infty} \to 0$ as $N \to 0$, where*

$$S_N[f](x) = \sum_{n=-N}^{N} \widehat{f}(x) e^{2\pi ixn}.$$

This is a good initial answer, but we still do not know what happens if f is, say, continuous. In this case, one can construct a continuous periodic function f such that $S[f]$ diverges at some point (see, for example, ZYGMUND and WHEEDEN (1977), p. 227). In dimension 1, we have the following:

Theorem 1.2. *If f has bounded variation in $\mathbb{T}$, (i.e., a periodic function in $\mathbb{R}$ with bounded variation on the torus), then:*

$$\lim_{N\to\infty} S_N[f](x) = \frac{f(x^+) + f(x^-)}{2}$$

for all $x \in \mathbb{T}$. In particular, $S[f](x) = f(x)$ at every point of continuity.

We would also like to know what happens for $f \in L^p$, $1 \leq p < \infty$. If we allow $\sim$ to be "= almost everywhere" instead of =, we have the following:

Theorem 1.3. *If $f \in L^1(\mathbb{T}^n)$ and $\widehat{f} \in l^1(\mathbb{Z}^n)$, then*

$$f(x) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) e^{2\pi i x \cdot k}$$

for almost every $x \in \mathbb{T}^n$.

However, KOLMOGOROV (1923) showed that there exists $f \in L^1(\mathbb{T})$ whose Fourier series diverges almost everywhere. For $p = 2$ we have:

Theorem 1.4. ***(CARLESON (1966))*** *If $f \in L^2(\mathbb{T})$, then*

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i x n}$$

for almost all $x \in \mathbb{T}$.

A few years later, R. Hunt extended Carleson's result for $1 < p < \infty$. In this work, we will prove a version of Theorem 1.4 proved by LACEY and THIELE (2000). An improvement of this alternative version will imply Theorem 1.4 by transference methods, which we will discuss in the conclusion.

WHAT CONNECTS THESE PROBLEMS?

Our three main references are LACEY and THIELE (1997), LACEY and THIELE (1999) and LACEY and THIELE (2000). What connects them is the technique used: time-frequency analysis. A common feature of the bilinear Hilbert transform and the Carleson's operator that benefits from time-frequency analysis is their *modulation symmetries*. By *modulation* we mean

$$M_\xi f(x) = f(x) e^{2\pi i \xi x}.$$

Carleson's operator can be written as:

$$C(f)(x) = \sup_\xi \left| \text{p.v.} \int f(x-t) \frac{e^{2\pi i \xi t}}{t} dt \right|$$

and it satisfies the modulation symmetry

$$C(M_\eta f) = C(f).$$

The bilinear Hilbert transform with parameter $\alpha \neq 0, -1$ can be written as

$$H_\alpha(f, g)(x) = p.v. \int f(x-t) g(x + \alpha t) \frac{dt}{t}$$

and satisfies the modulation symmetry

$$H(M_{\alpha\eta}f, M_{\eta}g) = M_{(\alpha+1)\eta}H(f, g).$$

The study of these operators is rather difficult because of the behavior of their singularities. The singularities of the bilinear Hilbert transform, for example, spread throughout the frequency space, which makes Calderón-Zygmund's techniques inappropriate. However, time-frequency analysis proves itself to be very well suited in a modulation invariant setting. Two very important references used were GRAFAKOS (2014a) and GRAFAKOS (2014b). We also used FEFFERMAN (1973), STEIN (1993),MUSCALU and SCHLAG (2013a) and MUSCALU and SCHLAG (2013b).

2 NOTATION AND BASIC RESULTS

Let $\mathcal{J}$ be the set of all intervals of type $I = [a, b)$ with $a, b \in \mathbb{R}$ and $a < b$. The center of I will be denoted by $c(I)$ and its length by $|I|$. We say that $[a, b) < [a', b')$ if $b \leq a'$. If $t > 0$, tI denotes an interval of length $t|I|$ and center $c(I)$. A subset $\mathcal{J}' \subset \mathcal{J}$ is a *grid* if for all $J, J' \in \mathcal{J}'$, the following conditions are satisfied:

$$J \cap J' \in \{\emptyset, J, J'\}, \tag{2}$$

$$J \subset J', J \neq J' \Rightarrow 2|J| \leq |J'|. \tag{3}$$

For example, by proposition 4.1, the set of dyadic intervals is a grid. We will denote the set of all rectangles $J \times J'$, with $J, J' \in \mathcal{J}$, by $\mathcal{R}$.

Two rectangles $I \times \omega$ and $I' \times \omega'$ are called *A-separated* if $AI \cap AJ = \emptyset$ or $\omega \cap \omega' = \emptyset$.

We will denote by M and M_2 the following Hardy-Littlewood operators related to a grid $\mathcal{J}$:

$$M(f)(x) := \sup_{J \in \mathcal{J} x \in J} \frac{1}{|J|} \int_J |f(y)| dy,$$

$$M_2(f)(x) := \sup_{J \in \mathcal{J} x \in J} \left(\frac{1}{|J|} \int_J |f(y)|^2 dy \right)^{\frac{1}{2}}.$$

Denote by $\mathcal{S}_u(\mathbb{R})$ the set of all Schwartz functions f with $\|f\|_2 \leq 1$.

Definition 2.1. *For $a > 0$ and $y \in \mathbb{R}$ we define τ^y (translation), D^a (dilation) and M^a (modulation) by:*

$$\tau^y(f)(x) = f(x - y),$$

$$D^y(f)(x) = y^{-\frac{1}{2}} f(y^{-1}x),$$

$$M^y(f)(x) = f(x) e^{2\pi i y x}.$$

Definition 2.2. *Let $f \in L^1(\mathbb{R}^n)$. Define its Fourier transform by:*

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \dot{\xi}} dx.$$

It is immediate that these operators are isometries on $L^2(\mathbb{R})$.

Lemma 2.1. *Let τ, M and D be as in definition 2.1.*

$$\widehat{\tau^y(f)}(\xi) = e^{-2\pi i\xi y}\widehat{f}(\xi) = M^{-y}\widehat{f}(\xi),$$

$$\widehat{D^y(f)}(\xi) = y^{\frac{1}{2}}\widehat{f}(y\xi) = D^{\frac{1}{y}}(\widehat{f})(\xi),$$

$$\widehat{M^y(f)}(\xi) = \widehat{f}(\xi - y) = \tau^y\widehat{f}(\xi).$$

Proof. This is just a simple computation using the definition of Fourier transform and an appropriate change of variables. □

3 THE BILINEAR HILBERT TRANSFORM

Let $f, g \in \mathcal{S}(\mathbb{R})$. The *bilinear Hilbert transform* of f and g is defined by:

$$H(f,g)(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} f(x-t)g(x+t)\frac{1}{t}dt. \tag{4}$$

We first need to show that limit (4) above exists. Using Fourier inversion we can rewrite it as:

$$\lim_{\epsilon\to 0}\int_{\epsilon<|t|<\frac{1}{\epsilon}}\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi)e^{2\pi i\xi(x-t)}\widehat{g}(\eta)e^{2\pi i\eta(x+t)}\frac{1}{t}d\xi d\eta dt =$$
$$\lim_{\epsilon\to 0}\int_{\epsilon<|t|<\frac{1}{\epsilon}}\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix(\xi+\eta)}\frac{1}{t}e^{-2\pi it(\xi-\eta)}d\xi d\eta dt.$$

The function on the integrand above is integrable. Using Fubini's theorem and the dominated convergence theorem we conclude that

$$\begin{aligned} H(f,g)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix(\xi+\eta)}\left(\lim_{\epsilon\to 0}\int_{\epsilon<|t|<\frac{1}{\epsilon}}\frac{e^{-2\pi it(\xi-\eta)}}{t}dt\right)d\xi d\eta \\ &= -i\pi\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix(\xi+\eta)}d\xi d\eta. \end{aligned}$$

Where we used Proposition 5.3. This last integral clearly converges, so our definition makes sense.

We want to find L^p estimates for H. The main objective of this chapter is to prove the following theorem:

Theorem 3.1. *Let $\Lambda : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ be the trilinear operator defined by:*

$$\Lambda(f_1,f_2,f_3) := \int \left[\text{p.v.}\int f_1(x-t)f_2(x+t)\frac{1}{t}dt\right] f_3(x)dx = \int H(f_1,f_2)(x)f_3(x)dx. \tag{5}$$

Then $\forall\ 2 < p_1, p_2, p_3 < \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, there is a constant C such that $\forall$ $f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R})$,

$$|\Lambda(f_1,f_2,f_3)| \le C\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}. \tag{6}$$

Observe that, as we are working on a reflexive Banach space, estimate (6) guarantees (by duality),

$$\|H(f,g)\|_p \le C_{p_1,p_2}\|f\|_{p_1}\|g\|_{p_2} \tag{7}$$

if $p = \frac{p_1 p_2}{p_1+p_2}$.

3.1 The bilinear Hilbert transform through time-frequency analysis

We will make several reductions in this section. Let us start with a very simple lemma:

Lemma 3.1. *The distribution* p.v.$\frac{1}{t}$ *is a linear combination of* δ *and* γ*, where* δ *is the Dirac distribution at the origin and* γ *is such that* $\widehat{\gamma}(\xi) = 0$ *if* $\xi \leq 0$ *and* $\widehat{\gamma}(\xi) = 1$ *if* $\xi > 0$.

Proof. Let γ be as stated. We have:

$$\langle \gamma, f \rangle = \langle \widehat{\gamma}, \check{f} \rangle = \left\langle \frac{\text{sgn} + 1}{2}, \check{f} \right\rangle = \frac{1}{2} \langle \check{\text{sgn}}, f \rangle + \frac{1}{2} \left\langle \check{1}, f \right\rangle = -\frac{1}{2\pi i} \left\langle \text{p.v.}\frac{1}{t}, f \right\rangle + \frac{1}{2} \langle \delta_0, f \rangle .$$

□

Next we observe that (5) follows if we replace p.v.$\frac{1}{t}$ by δ. Indeed,

$$\left\langle \delta, (\tau^{-x} \tilde{f}_1)(\tau^{-x} f_2) \right\rangle = f_1(-x) f_2(x) \Rightarrow$$
$$|\Lambda(f_1, f_2, f_3)| \leq \int |f_1(-x) f_2(x) f_3(x)| dx \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}$$

by Hölder's inequality. So, by Lemma 3.1, it suffices to prove Theorem 3.1 for γ instead of p.v.$\frac{1}{t}$.

Let $\epsilon := 2^{-10}$, $L := \frac{1}{\epsilon}$ and θ be a smooth function which is equal to 1 on the interval $(-\infty, L]$ and 0 on $[L+1, \infty)$. Define ψ by

$$\widehat{\psi}(\xi) := \theta(\xi) - \theta(2^{\epsilon} \xi).$$

Lemma 3.2. *The following hold about* ψ*:*
(a) $\widehat{\psi} \neq 0$ *and* supp $\widehat{\psi} \subset [L-1, L+1]$.
(b) Define, for $k \in \mathbb{Z}$,

$$\psi_k(x) := 2^{\frac{-\epsilon k}{2}} \psi(2^{-\epsilon k} x).$$

Then

$$\gamma = \sum_{k \in \mathbb{Z}} 2^{\frac{-\epsilon k}{2}} \psi_k$$

as tempered distributions

Proof of (a). It is easy to find a ξ such that $\widehat{\psi} \neq 0$. About the support, we have

$$\xi < L - 1 \Rightarrow \theta(\xi) = 1 \text{ and } \theta(2^{\epsilon} \xi) = 1 \text{ since } 2^{\epsilon}(L-1) < L.$$

$$\xi > L + 1 \Rightarrow \theta(\xi) = 0 \text{ and } \theta(2^{\epsilon} \xi) = 0 \text{ since } 2^{\epsilon}(L+1) > L + 1.$$

Then $\widehat{\phi}$ vanishes on $(-\infty, L-1) \cup (L+1, \infty)$. □

Proof of (b). This is equivalent to convergence in the Fourier side:

$$\widehat{\gamma} = \sum_{k\in\mathbb{Z}} 2^{\frac{-\epsilon k}{2}} \widehat{\psi_k}.$$

To prove this, take $f \in \mathcal{S}(\mathbb{R})$ and observe that

$$\sum_{k\in\mathbb{Z}} 2^{\frac{-\epsilon k}{2}} \widehat{\psi_k}(f) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} 2^{\frac{-\epsilon k}{2}} 2^{\frac{\epsilon k}{2}} \widehat{\psi}(2^{\epsilon k}x) f(x) dx = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \widehat{\psi}(2^{\epsilon k}x) f(x) dx. \tag{8}$$

On the other hand,

$$S = \sum_{k\in\mathbb{Z}} \widehat{\psi}(2^{\epsilon k}x) = \sum_{k\in\mathbb{Z}} \theta(2^{\epsilon k}x) - \theta(2^{\epsilon(k+1)}x) = \lim_{k\to-\infty} \theta(2^{\epsilon k}x) - \lim_{k\to\infty} \theta(2^{\epsilon k}x).$$

If $x \leq 0$, $S = 0$ since each limit above is 1. If $x > 0$, $S = 1$ since the first limit is 1 and the second is 0. Going back to (8), we have:

$$\sum_{k\in\mathbb{Z}} 2^{\frac{-\epsilon k}{2}} \widehat{\psi_k}(f) = \int_{x>0} f(x) dx = \int_{\mathbb{R}} \widehat{\gamma}(x) f(x) dx = \widehat{\gamma}(f),$$

which is what we wanted. □

So, in view of what we just explained, to prove Theorem 3.1 it suffices to prove an estimate like (6) to the operator

$$\tilde{\Lambda}(f_1, f_2, f_3) := \sum_{k\in\mathbb{Z}} 2^{\frac{-\epsilon k}{2}} \int\int f_1(x-t) f_2(x+t) f_3(x) \psi_k(t) dt dx. \tag{9}$$

Let $\phi \in \mathcal{S}_u(\mathbb{R})$ such that $\widehat{\phi}$ is supported in $[0, 1]$ and that

$$\sum_{l\in\mathbb{Z}} \left| \hat{\phi}\left(t + \frac{l}{2}\right) \right|^2 = c_0$$

for all $t \in \mathbb{R}$. Define:

$$\phi_{\kappa,n,l}(x) := 2^{\frac{-\epsilon k}{2}} \phi(2^{-\epsilon k}x - n) e^{2\pi i 2^{-\epsilon k} x l}. \tag{10}$$

Computing the Fourier transform we get:

$$\widehat{\phi}_{\kappa,n,l}(\xi) = 2^{\frac{\epsilon k}{2}} \widehat{\phi}(2^{\epsilon k}x - l) e^{-2\pi i (2^{\epsilon k}\xi - l)n}. \tag{11}$$

By an immediate variant of Proposition 5.1, it follows that

$$\sum_{n,l\in\mathbb{Z}} \left\langle f, \phi_{\kappa,n,\frac{l}{2}} \right\rangle \phi_{\kappa,n,\frac{l}{2}} = c_0' f \tag{12}$$

for all $f \in \mathcal{S}(\mathbb{R})$ and $\kappa \in \mathbb{R}$. We now expand f_1, f_2 and f_3 in (9) using (12) above and obtain

$$\tilde{\Lambda}(f_1, f_2, f_3) = \sum_{k,n_1,l_1,n_2,l_2,n_3,l_3 \in \mathbb{Z}} C_{k,n_1,n_2,n_3,l_1,l_2,l_3} \Lambda_{k,n_1,n_2,n_3,l_1,l_2,l_3}(f_1, f_2, f_3),$$

with

$$\Lambda_{k,n_1,n_2,n_3,l_1,l_2,l_3} := 2^{\frac{-\epsilon k}{2}} \left\langle f_1, \phi_{k,n_1,\frac{l_1}{2}} \right\rangle \left\langle f_2, \phi_{k,n_2,\frac{l_2}{2}} \right\rangle \left\langle f_3, \phi_{k,n_3,\frac{l_3}{2}} \right\rangle$$

and

$$C_{k,n_1,n_2,n_3,l_1,l_2,l_3} := C \int\int \phi_{k,n_1,\frac{l_1}{2}}(x-t)\phi_{k,n_2,\frac{l_2}{2}}(x+t)\phi_{k,n_3,\frac{l_3}{2}}(x)\psi_k(t)dtdx.$$

This way, to prove boundedness of (9) it suffices to prove it to

$$\tilde{\tilde{\Lambda}}(f_1, f_2, f_3) := \sum_{k,n_1,l_1,n_2,l_2,n_3,l_3 \in \mathbb{Z}} |C_{k,n_1,n_2,n_3,l_1,l_2,l_3} \Lambda_{k,n_1,n_2,n_3,l_1,l_2,l_3}(f_1, f_2, f_3)| \tag{13}$$

because $|\tilde{\Lambda}(f_1, f_2, f_3)| \leq |\tilde{\tilde{\Lambda}}(f_1, f_2, f_3)|$. We now estimate $C_{k,n_1,n_2,n_3,l_1,l_2,l_3}$ in two different ways.

Proposition 3.1. *The following estimates hold to* $C_{k,n_1,n_2,n_3,l_1,l_2,l_3}$

(a) For every positive integer m *there is a constant* C_m *depending on the Schwartz functions* ϕ *and* ψ *such that*

$$|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| \leq C_m(1 + diam(n_1, n_2, n_3))^{-m}, \tag{14}$$

where $diam\{n_1, n_2, n_3\} := \max_{i,j \in \{1,2,3\}} n_i - n_j$.

(b)

$$|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| \leq C \int\int \left| \widehat{\phi}\left(\frac{\xi - \tau}{2} - \frac{l_1}{2}\right) \widehat{\phi}\left(\frac{\xi + \tau}{2} - \frac{l_2}{2}\right) \widehat{\phi}\left(\xi - \frac{l_3}{2}\right) \widehat{\psi}(\tau) \right| d\tau d\xi. \tag{15}$$

Proof. A change of variables and the triangle inequality give us

$$|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| \leq C \int\int |\phi(x - t - n_1)\phi(x + t - n_2)\phi(x - n_3)\psi(t)| dtdx.$$

We observe next that the differences $n_i - n_j$ can be written as a linear combination of the four arguments of the functions appearing in the integrand above, so at least one of the four arguments is greater than $\frac{1}{k} diam\{n_1, n_2, n_3\}$, where k is a fixed constant. Suppose this argument is $(x-t-n_1)$. As $\phi \in \mathcal{S}$, for each $m \in \mathbb{N}$ we can choose K_m such that

$$|\phi(x-t-n_1)| \leq \frac{K_m}{(1+|x-t-n_1|)^m} \leq \frac{\tilde{K}_m}{(1+diam\{n_1,n_2,n_3\})^m},$$

and then we may conclude that

$$\begin{aligned}|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| &\leq \frac{\tilde{K}_m}{(1+diam\{n_1,n_2,n_3\})^m}\int\int|\phi(x+t-n_2)\phi(x-n_3)\psi(t)|dtdx\\ &= \frac{C_m}{(1+diam\{n_1,n_2,n_3\})^m}\end{aligned}$$

since the integral above clearly converges for ϕ and ψ in $\mathcal{S}$ (observe that C_m depends on these two functions, as stated). This proves (a). Fix k, n_1, l_1, n_2, l_2. Define

$$\Phi(x,t) := \phi_{k,n_1,\frac{l_1}{2}}(x-t)\phi_{k,n_2,\frac{l_2}{2}}(x+t),$$

$$\Psi(x,t) := \phi_{k,n_3,\frac{l_3}{2}}(x)\psi_k(t).$$

Computing Fourier transforms,

$$\begin{aligned}\widehat{\Phi}(\xi,\tau) &= \int_{\mathbb{R}^2}\phi_{k,n_1,\frac{l_1}{2}}(x-t)\phi_{k,n_2,\frac{l_2}{2}}(x+t)e^{-2\pi i(x,t)\cdot(\xi,\tau)}dxdt\\ &= \int_{\mathbb{R}^2}\phi_{k,n_1,\frac{l_1}{2}}(u)\phi_{k,n_2,\frac{l_2}{2}}(v)e^{-2\pi i\left(\frac{u+v}{2},\frac{v-u}{2}\right)\cdot(\xi,\tau)}\frac{1}{2}d\xi d\tau\\ &= \frac{1}{2}\int_{\mathbb{R}^2}\phi_{k,n_1,\frac{l_1}{2}}(u)\phi_{k,n_2,\frac{l_2}{2}}(v)e^{-2\pi iu\left(\frac{\xi-\tau}{2}\right)}e^{-2\pi iv\left(\frac{\xi+\tau}{2}\right)}dudv\\ &= \frac{1}{2}\widehat{\phi}_{k,n_1,\frac{l_1}{2}}\left(\frac{\xi-\tau}{2}\right)\widehat{\phi}_{k,n_2,\frac{l_2}{2}}\left(\frac{\xi+\tau}{2}\right).\end{aligned}$$

$$\begin{aligned}\widehat{\Psi}(\xi,\tau) &= \int_{\mathbb{R}^2}\phi_{k,n_3,\frac{l_3}{2}}(x)\psi_k(t)e^{-2\pi i(x,t)\cdot(\xi,\tau)}dxdt\\ &= \left(\int_{\mathbb{R}}\phi_{k,n_3,\frac{l_3}{2}}(x)e^{-2\pi ix\xi}dx\right)\left(\int_{\mathbb{R}}\psi_k(t)e^{-2\pi it\tau}dt\right)\\ &= \widehat{\phi}_{k,n_3,\frac{l_3}{2}}(\xi)\widehat{\psi}_k(\tau).\end{aligned}$$

Parseval's formula, the calculation above and (11) give us:

$$\begin{aligned}
|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| &= \left| C \int\int \Phi(x,t)\overline{\Psi(x,t)} dt dx \right| \\
&= \left| C \int\int \widehat{\Phi}(\xi,\tau)\overline{\widehat{\Psi}(\xi,\tau)} d\tau d\xi \right| \\
&\leq \tilde{C} \int\int \left| \widehat{\phi}_{k,n_1,\frac{l_1}{2}} \left(\frac{\xi-\tau}{2}\right) \widehat{\phi}_{k,n_2,\frac{l_2}{2}} \left(\frac{\xi+\tau}{2}\right) \widehat{\phi}_{k,n_3,\frac{l_3}{2}}(\xi) \widehat{\psi}_k(\tau) \right| d\tau d\xi \\
&= \tilde{C} \int\int \left| \widehat{\phi}\left(\frac{\xi-\tau}{2} - \frac{l_1}{2}\right) \widehat{\phi}\left(\frac{\xi+\tau}{2} - \frac{l_2}{2}\right) \widehat{\phi}\left(\xi - \frac{l_3}{2}\right) \widehat{\psi}(\tau) \right| d\tau d\xi,
\end{aligned} \tag{16}$$

which is exactly what we wanted for (b). □

Assume that the last integral above is nonzero. Then there is a pair (ξ, τ) for which the integrand is nonzero and we have

$$\widehat{\phi}\left(\xi - \frac{l_3}{2}\right) \widehat{\psi}(\tau) \neq 0 \Rightarrow \begin{cases} \frac{l_3}{2} - 1 \leq \xi \leq 1 + \frac{l_3}{2}, \\ L - 1 \leq \tau \leq L + 1. \end{cases}$$

because supp $\widehat{\phi} \subset [-1,1]$ and supp $\widehat{\psi} \subset [L-1, L+1]$. This way,

$$\begin{aligned}
\min\{\xi - \tau\} &= \frac{l_3}{2} - L - 2, \\
\max\{\xi - \tau\} &= \frac{l_3}{2} - L + 2, \\
\min\{\xi + \tau\} &= \frac{l_3}{2} + L - 2, \\
\max\{\xi + \tau\} &= \frac{l_3}{2} + L + 2.
\end{aligned} \tag{17}$$

Also,

$$\widehat{\phi}\left(\frac{\xi-\tau}{2} - \frac{l_1}{2}\right) \widehat{\phi}\left(\frac{\xi+\tau}{2} - \frac{l_2}{2}\right) \neq 0 \Rightarrow \begin{cases} (\xi-\tau) - 2 \leq l_1 \leq 2 + (\xi - \tau), \\ (\xi+\tau) - 2 \leq l_2 \leq 2 + (\xi+\tau), \end{cases}$$

with (17) this gives us

$$\left(\frac{l_3}{2} - L\right) - 4 \leq l_1 \leq \left(\frac{l_3}{2} - L\right) + 4, \tag{18}$$

$$\left(\frac{l_3}{2} + L\right) - 4 \leq l_2 \leq \left(\frac{l_3}{2} + L\right) + 4. \tag{19}$$

We will use proposition 3.1 to reduce once more the problem.

Proposition 3.2. *To prove theorem 3.1 it suffices to show that for all* $S, \nu_1, \nu_2, \lambda_1, \lambda_2$

there exists C_{p_1,p_2,p_3} *such that*

$$\sum_{(k,n,l)\in S} |\Lambda_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}| \leq C_{p_1,p_2,p_3}\nu^{10}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}, \tag{20}$$

where $\nu_1, \nu_2 \geq 0$ *are integers,* $\nu := 1 + \max\{|v_1|, |v_2|\}$, $\lambda_1, \lambda_2 : \mathbb{Z} \to \mathbb{Z}$ *are functions such that* $l_1 = \lambda_1(l_3)$ *and* $l_2 = \lambda_2(l_3)$ *satisfy* (18) *and* (19), *respectively. Also,* $S \subset \mathbb{Z}^3$ *is finite and such that*

$$k \neq k' \Rightarrow |k - k'| > L^{10}, \tag{21}$$

$$n \neq n' \Rightarrow |n - n'| > L^{10}\nu, \tag{22}$$

$$l \neq l' \Rightarrow |l - l'| > L^{10}. \tag{23}$$

Proof. If (20) holds for every finite $S \subset \mathbb{Z}^3$ such that (21), (22) and (23) hold, then it still holds if S is infinite under the same conditions, otherwise

$$\sum_{\substack{(k,n,l)\in S \\ S \text{ infinite}}} |\Lambda_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}| > C_{p_1,p_2,p_3}\nu^{10}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3} \Rightarrow$$
$$\sum_{(k,n,l)\in S'} |\Lambda_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}| > C_{p_1,p_2,p_3}\nu^{10}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}$$

for some $S' \subset S$ finite, which contradicts (20). By the definition of $\Lambda_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}$, (20) still holds if n_1, n_2 and n_3 are in a different order. Reduce the range Γ of indexes on (13) by eliminating the triples (l_1, l_2, l_3) that do not satisfy 18 and 19. We can then write the sum as

$$\sum_{\Gamma} = \sum_{diam\{n_1,n_2,n_3\}} \sum_k \sum_{l_3} \sum_{l_1,l_2} = A + B + C,$$

where

$$A = \sum_k \sum_{l_3} \sum_{l_1,l_2} \sum_{\substack{diam\{n_1,n_2,n_3\} \\ n_3=\min\{n_1,n_2,n_3\}}},$$

$$B = \sum_k \sum_{l_3} \sum_{l_1,l_2} \sum_{\substack{diam\{n_1,n_2,n_3\} \\ n_2=\min\{n_1,n_2,n_3\} \\ n_2\neq n_3}},$$

$$C = \sum_k \sum_{l_3} \sum_{l_1,l_2} \sum_{\substack{diam\{n_1,n_2,n_3\} \\ n_1=\min\{n_1,n_2,n_3\} \\ n_1\neq n_2,n_3}}.$$

We will focus on A since the argument for B and C is the same. Note that A can be rewritten as

$$\sum_d \left(\sum_{\substack{\nu_1,\nu_2 \\ d=\max\{\nu_1,\nu_2\}}} \sum_k \sum_{l_3} \sum_{l_1,l_2} \sum_n \right). \tag{24}$$

For each d, divide the range of indexes in (24) into $(L^{10}+1)^3(d+1)$ subsets as follows:

$$(k,l,n),(k',l',n') \in S_\alpha \Rightarrow \begin{cases} k \equiv k' & \mod \quad L^{10}+1, \\ l \equiv l' & \mod \quad (L^{10}+1)(d+1), \\ n \equiv n' & \mod \quad L^{10}+1. \end{cases}$$

Once we choose l_α, 18 and 18 tell us that l_1 and l_2 can assume at most 9 different values each. Also, by (20), we can use the same upper bound for $\sum_{k_\alpha,l_\alpha,n_\alpha} |\Lambda_{k_\alpha,n_\alpha+\nu_1,n_\alpha+\nu_2,n_\alpha,l_1,l_2,l_\alpha}|$ regardless the values of l_1 and l_2. This way,

$$\begin{aligned}
A &= \sum_d \sum_\alpha \left(\sum_{\substack{\nu_1,\nu_2 \\ d=\max\{\nu_1,\nu_2\}}} \sum_{k_\alpha} \sum_{l_\alpha} \sum_{l_1,l_2} \sum_{n_\alpha} |C_{k,n_\alpha+\nu_1,n_\alpha+\nu_2,n_\alpha,l_1,l_2,l_\alpha}||\Lambda_{k_\alpha,n_\alpha+\nu_1,n_\alpha+\nu_2,n_\alpha,l_1,l_2,l_\alpha}| \right) \\
&\leq \sum_d \sum_\alpha \left(\frac{9C}{(1+d)^{14}} \sum_{\substack{\nu_1,\nu_2 \\ d=\max\{\nu_1,\nu_2\}}} \sum_{k_\alpha} \sum_{l_\alpha} \sum_{n_\alpha} |\Lambda_{k_\alpha,n_\alpha+\nu_1,n_\alpha+\nu_2,n_\alpha,l_1,l_2,l_\alpha}| \right) \\
&\leq \sum_d \sum_\alpha \left(\frac{9C}{(1+d)^{14}} \sum_{\substack{\nu_1,\nu_2 \\ d=\max\{\nu_1,\nu_2\}}} C_{p_1,p_2,p_3}(1+d)^{10}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3} \right) \\
&\leq \sum_d \sum_\alpha \left(\frac{\tilde{C}}{(1+d)^{14}}(1+d)^{11} C_{p_1,p_2,p_3}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3} \right) \\
&\leq \sum_d \left(\frac{\tilde{C}}{(1+d)^{14}}(1+d)^{11}(L^{10}+1)^3(1+d) C_{p_1,p_2,p_3}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3} \right) \\
&= \tilde{C}_{p_1,p_2,p_3}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3} \sum_d \frac{1}{(1+d)^2} \\
&\leq \tilde{\tilde{C}}_{p_1,p_2,p_3}\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}.
\end{aligned}$$

Using an analogous approach for B and C we conclude (13).

□

Let us work now on the proof of (20).

Definition 3.1. *Let C_m be a constant for each integer $m \geq 0$. A* phase plane representation *of a subset $\Phi \subset \mathcal{S}_u(\mathbb{R})$ is an injective map*

$$\begin{aligned} \rho: \quad & \Phi \to \mathcal{R} \\ & \phi \mapsto I_\phi \times \omega_\phi, \end{aligned}$$

such that the following properties hold for all $\phi, \phi' \in \Phi$

$$\omega_\phi \cap \omega_{\phi'} = \emptyset \Rightarrow \langle \phi, \phi' \rangle = 0, \tag{25}$$

$$|\phi(x)| \leq C_m |I_\phi|^{-1/2} \left(1 + \left|\frac{x - c(I_\phi)}{|I_\phi|}\right|\right)^{-m}, \quad \forall m \in \mathbb{Z}_+, x \in \mathbb{R}, \tag{26}$$

$$2^{-1/2}|I_{\phi'}||\omega_{\phi'}| < |I_\phi||\omega_\phi| < 2^{1/2}|I_{\phi'}||\omega_{\phi'}|. \tag{27}$$

For the rest of this section we fix $2 < p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. We now state a key proposition, which we take for granted in this section, and use it prove Theorem 3.1. We will devote the other sections to prove this proposition.

Proposition 3.3. *As in definition 3.1, for each* $m \geq 0$ *integer let* $C_m > 0$ *be a constant associated. There exists* C *(depending on* p_1, p_2, p_3, C_m*) such that the following holds. Let* S *be a finite set,* $\phi_1, \phi_2, \phi_3 : S \to \mathcal{S}_u(\mathbb{R})$ *be injective maps and let* $I, \omega_1, \omega_2, \omega_3 : S \to \mathcal{J}$ *be maps such that the following are satisfied:*

$$\rho_i : \phi_i(S) \to \mathcal{R}, \phi_i \mapsto I_s \times \omega_s \tag{28}$$

is a phase plane representation with constants C_m $\forall i \in \{1, 2, 3\}$

$$\text{The set } I(S) \text{ is a grid,} \tag{29}$$

$$\text{The set } \mathcal{J}_\omega = \omega_1(S) \cup \omega_2(S) \cup \omega_3(S) \text{ is a grid,} \tag{30}$$

$$\omega_i(s) \cap \omega_j(s) = \emptyset \text{ for all } s \in S, i, j \in \{1, 2, 3\},\ i \neq j, \tag{31}$$

$$\exists s \in S;\ \omega_i(s) \subset J,\ \omega_i(s) \neq J,\ i \in \{1, 2, 3\}, J \in \mathcal{J}_\omega \Rightarrow \omega_j(s) \subset J,\ \forall\ j \in \{1, 2, 3\}. \tag{32}$$

Then the following inequality holds for $f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R})$*:*

$$\sum_{s \in S} |I(s)|^{-1/2} |\langle f_1, \phi_1(s)\rangle \langle f_2, \phi_2(s)\rangle \langle f_3, \phi_3(s)\rangle| \leq C\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}. \tag{33}$$

Fix $S, \nu_1, \nu_2, \nu, \lambda_1, \lambda_2$ as in proposition 3.2. We define $\phi_i : S \to \mathcal{S}_u(\mathbb{R})$ by:

$$\phi_1(k, n, l) := \nu^{-1}\phi_{k, n+\nu_1, \frac{\lambda_1(l)}{2}},$$
$$\phi_2(k, n, l) := \nu^{-1}\phi_{k, n+\nu_2, \frac{\lambda_2(l)}{2}},$$
$$\phi_3(k, n, l) := \nu^{-1}\phi_{k, n, \frac{l}{2}}.$$

Claim 3.1. ϕ_i *is injective for each* $i \in \{1, 2, 3\}$.

Proof. For $i = 1$ (the other two cases are similar),

$$\phi_1(k,n,l) = \phi_1(k',n',l') \Rightarrow$$
$$2^{-\frac{\epsilon k}{2}}\phi(2^{-\epsilon k}x - n - \nu_1)e^{2\pi i 2^{-\epsilon k}x\frac{\lambda_1(l)}{2}} = 2^{-\frac{\epsilon k'}{2}}\phi(2^{-\epsilon k'}x - n' - \nu_1)e^{2\pi i 2^{-\epsilon k'}x\frac{\lambda_1(l')}{2}},$$

for all $x \in \mathbb{R}$. Taking L^1 norms on both sides,

$$2^{\frac{-\epsilon k}{2}}\int_{\mathbb{R}}|\phi(2^{-\epsilon k}x - n - \nu_1)|dx = 2^{\frac{-\epsilon k'}{2}}\int_{\mathbb{R}}|\phi(2^{-\epsilon k'}y - n' - \nu_1)|dy \Rightarrow$$
$$2^{\frac{\epsilon k}{2}}\|\phi\|_1 = 2^{\frac{\epsilon k'}{2}}\|\phi\|_1 \Rightarrow$$
$$k = k'.$$

Complex exponential must be equal, so

$$e^{2\pi i 2^{-\epsilon k}x\frac{\lambda_1(l)}{2}} = e^{2\pi i 2^{-\epsilon k'}x\frac{\lambda_1(l')}{2}} \Rightarrow \frac{2^{-\epsilon k}x}{2}(\lambda_1(l') - \lambda_1(l)) \in \mathbb{Z}, \quad \forall x \in \mathbb{R}$$
$$\Rightarrow \lambda_1(l') = \lambda_1(l)$$
$$\Rightarrow l = l'$$

by 18. Finally,

$$\phi(2^{-\epsilon k}x - n - \nu_1) = \phi(2^{-\epsilon k}x - n' - \nu_1) \Rightarrow n = n',$$

otherwise ϕ would be constant. This proves that ϕ_1 is injective. □

Claim 3.2. *There is a map I that satisfies the following for all $s = (k,n,l) \in S$:*

$$|c(I(s)) - 2^{\epsilon k}\nu| \leq 2^{\epsilon k}\nu, \tag{34}$$

$$2^4 2^{\epsilon k}\nu \leq |I(s)| \leq 2^{\epsilon}2^4 2^{\epsilon k}\nu, \tag{35}$$

$$I(S) \text{ is a grid.} \tag{36}$$

Proof. We proceed by induction on the cardinality of S (which will be denoted by $|S|$). If $|S| = 1$ we can easily choose an interval $[a,b)$ as small as necessary so that (34), (35) and (36) are verified. If $|S| > 1$ pick $s = (k,n,l) \in S$ such that k is maximal (i.e., there is no $s' = (k',n',l') \in S$ such that $k' > k$) and define $S' = S\backslash\{s\}$. By induction there is $I : S' \to \mathcal{J}$ that satisfies (34), (35) and (36). If there is an element $(k,n,l') \in S$ with the same first coordinates as s, define $I(s) := I(k,n,l')$ and it is immediate that this definition is consistent with the three conditions above. If this is not the case, let $[a,b)$ be the interval of length $2^4 2^{\epsilon k}\nu$ centered at $2^{\epsilon k}n$ and define $I(s)$ to be

$$I(s) := [a,b) \cup \bigcup_i [a_i, b_i),$$

where a or b belongs to $[a_i, b_i)$ for each i. This definition of $I(s)$ clearly verifies (36). Since $I(S')$ is a grid, we can take $[a_1, b_1]$ and $[a_2, b_2)$ maximal such that

$$\begin{aligned} I(s) &= [a_1, b_1) \cup [a, b) \cup [a_2, b_2), \\ a &\in [a_1, b_1] \text{ and } b \in [a_2, b_2), \\ a &\in [a_j, b_j) \Rightarrow [a_j, b_j) \subset [a_1, b_1), \\ b &\in [a_j, b_j) \Rightarrow [a_j, b_j) \subset [a_2, b_2). \end{aligned}$$

By (21) and the fact that k is maximal,

$$\begin{aligned} |I(s)| \leq |b_1 - a_1| + |b - a| + |b_2 - a_2| &\leq 2^\epsilon 2^4 2^{\epsilon k'} \nu + 2^4 2^{\epsilon k} \nu + 2^\epsilon 2^4 2^{\epsilon k''} \nu \\ &\leq 2^\epsilon 2^4 2^{\epsilon(k - L^{10})} \nu + 2^4 2^{\epsilon k} \nu + 2^\epsilon 2^4 2^{\epsilon(k - L^{10})} \nu \\ &= \left(\tfrac{2^{\epsilon+1}}{2^{L^{10}}} + 1 \right) 2^4 2^{\epsilon k} \nu \\ &< 2^\epsilon 2^4 2^{\epsilon k} \nu, \end{aligned}$$

which verifies the second inequality in (35). The first inequality of this condition is obviously verified by construction. To verify (34),

$$\begin{aligned} c(I(s)) &\leq 2^{\epsilon k} n + |b_2 - a_2| \leq 2^{\epsilon k} n + 2^\epsilon 2^4 2^{\epsilon k''} \nu, \\ c(I(s)) &\geq 2^{\epsilon k} n - |b_1 - a_1| \geq 2^{\epsilon k} n - 2^4 2^{\epsilon k'} \nu, \end{aligned}$$

so

$$|c(I(s)) - 2^{\epsilon k} n| \leq \max\{2^\epsilon 2^4 2^{\epsilon k''} \nu, 2^4 2^{\epsilon k'} \nu\} < 2^{\epsilon k} \nu,$$

where we used (21) again. This completes the proof of this claim. □

Claim 3.3. *There are three maps $\omega_1, \omega_2, \omega_3 : S \to \mathcal{J}$ that satisfy the following for all $s = (k, n, l) \in S$*

$$\operatorname{supp}(\widehat{\phi}_i(s)) \subset \omega_i(s) \text{ for } i = 1, 2, \tag{37}$$

$$\operatorname{supp}(\widehat{\phi}_3(s)) \subset [2a, 2b) \text{ where } [a, b) = \omega_3(s), \tag{38}$$

$$2^4 2^{-\epsilon k} \leq |\omega_i(s)| \leq 2^\epsilon 2^4 2^{-\epsilon k} \text{ for } i = 1, 2, 3, \tag{39}$$

$$\mathcal{J}_\omega := \omega_1(S) \cup \omega_2(S) \cup \omega_3(S) \text{ is a grid}, \tag{40}$$

and for $i \in 1, 2, 3$,

$$\omega_i(s) \subsetneq J \text{ for some } J \in \mathcal{J}_\omega \Rightarrow \omega_j \subset J, \text{ for all } j \in 1, 2, 3. \tag{41}$$

Proof. We proceed once more by induction on the cardinality of S. Observe that

$$\begin{aligned} \operatorname{supp}(\widehat{\phi_1}(k,n,l)) &\subset \left[2^{-\epsilon k}\left(\frac{\lambda_1(l)}{2}-1\right), 2^{-\epsilon k}\left(\frac{\lambda_1(l)}{2}+1\right)\right] = U, \\ \operatorname{supp}(\widehat{\phi_2}(k,n,l)) &\subset \left[2^{-\epsilon k}\left(\frac{\lambda_2(l)}{2}-1\right), 2^{-\epsilon k}\left(\frac{\lambda_2(l)}{2}+1\right)\right] = V, \\ \operatorname{supp}(\widehat{\phi_3}(k,n,l)) &\subset \left[2^{-\epsilon k}\left(\frac{l}{2}-1\right), 2^{-\epsilon k}\left(\frac{l}{2}+1\right)\right] = W. \end{aligned} \tag{42}$$

If $|S| = 1$ let this element be $s = (k,n,l)$. Define

$$\begin{aligned} \omega_1(s) &= 2U, \\ \omega_2(s) &= 2V, \\ \omega_3(s) &= 2W. \end{aligned}$$

By (42), (18) and (19) we see that ω_i satisfies the conditions of this claim. If $|S| > 1$ pick $s = (k,n,l) \in S$ such that k is minimal and define $S' = S \backslash \{s\}$. By induction we find maps $\omega_1, \omega_2, \omega_3 : S' \to \mathcal{J}$ which satisfy the conditions stated. If there is an element $(k,n',l') \in S'$ then define $\omega_i(s) = \omega_i(k,n',l')$ for $i = 1,2,3$. One can easily see that this definition extends ω_i to S so (37)-(41) are satisfied. Otherwise, for $i = 1,2,3$, let $[a_i, b_i)$ be an interval of length $2^4 2^{-\epsilon k}$ that satisfies $\operatorname{supp}(\widehat{\phi_i}(s)) \subset [a_i, b_i)$ for $i = 1,2$ and $\operatorname{supp}(\widehat{\phi_3}(s)) \subset [2a_i, 2b_i)$ for $i = 3$, which can be done by (42). Define

$$\omega_i(s) := \text{convex hull of } [a_i, b_i) \bigcup_{s' \in S} (\omega_1(s') \cup \omega_2(s') \cup \omega_3(s')),$$

where this union is taken over all sets of the form $\omega_1(s') \cup \omega_2(s') \cup \omega_3(s')$ that intersect $[a_i, b_i)$. As we did in the previous claim, $\mathcal{J}_\omega$ being a grid allows us to take s' and s'' maximal such that

$$\begin{aligned} \omega_i(s) &:= \text{convex hull of } [\omega_1(s') \cup \omega_2(s') \cup \omega_3(s')] \cup [a_i, b_i) \cup [\omega_1(s'') \cup \omega_2(s'') \cup \omega_3(s'')] \\ &= [x,y), \text{ where } [a_i, b_i) \subset [x,y). \end{aligned}$$

(37) and (38) are trivially satisfied by construction. As for (39), the first inequality follows from $|\omega_i(s)| \geq |[a_i, b_i)| = 2^4 2^{-\epsilon k}$ and the second follows from

$$\begin{aligned} |\omega_i(s)| &\leq |\omega_1(s')| + |\omega_2(s')| + |\omega_3(s')| + |[a_i, b_i)| + |\omega_1(s'')| + |\omega_2(s'')| + |\omega_3(s'')| \\ &\leq 3.2^\epsilon 2^4 2^{-\epsilon k'} + 2^4 2^{-\epsilon k} + 3.2^4 2^{-\epsilon k''} \\ &\leq 3.2^\epsilon 2^4 2^{-\epsilon(k+L^{10})} + 2^4 2^{-\epsilon k} + 3.2^4 2^{-\epsilon(k+L^{10})} \\ &\leq 2^\epsilon 2^4 2^{-\epsilon k}. \end{aligned}$$

(40) and (41) are also easily seen to be satisfied by our construction. □

We have defined the ϕ_i, I and ω_i needed in proposition 3.3.

Claim 3.4. *ϕ_i, I and ω_i satisfy the conditions of proposition 3.3.*

Proof. Conditions (29), (30) and (32) are easily seen to be satisfied by the way we defined ϕ_i, I and ω_i. Let's prove that $\omega_1(k,n,l) \cap \omega_2(k,n,l) = \emptyset$. We have:

$$|\omega_1(k,n,l) \cup \omega_2(k,n,l)| \leq |\omega_1(k,n,l)| + |\omega_2(k,n,l)| \leq 2^{\epsilon} 2^5 2^{-\epsilon k} \tag{43}$$

by (39). Also,

$$2^{-\epsilon k}\left(\frac{\lambda_1(l)}{2}+1\right) < 2^{-\epsilon k}\left(\frac{\lambda_2(l)}{2}-1\right)$$

by (18) and (19) and

$$\operatorname{supp}(\widehat{\phi}_1(s)) \cup \operatorname{supp}(\widehat{\phi}_2(s)) \subset \omega_1(s) \cup \omega_2(s)$$

by (37). If $\omega_1(k,n,l) \cap \omega_2(k,n,l) \neq \emptyset$, then $\omega_1(k,n,l) \cup \omega_2(k,n,l)$ would be an interval and

$$\omega_1(k,n,l) \cup \omega_2(k,n,l) \supset \left[2^{-\epsilon k}\left(\frac{\lambda_1(l)}{2}-1\right), 2^{-\epsilon k}\left(\frac{\lambda_2(l)}{2}+1\right)\right]$$

$$|\omega_1(k,n,l) \cup \omega_2(k,n,l)| \geq 2^{-\epsilon k}\left(\frac{\lambda_2(l)-\lambda_1(l)}{2}-2\right) > 2^{-\epsilon k} 2^5 2^{\epsilon}$$

by (18) and (19), which contradicts (43) and verifies condition (31) for $i=1$ and $j=2$ (the other cases are similar). It remains to verify that the maps ρ_i defined in (28) are phase plane representations (remember definition 3.1). For $i=1$ we have

$$\begin{aligned} \rho_1: \quad & \phi_1(S) \to \mathcal{R} \\ & \phi_1(s) \mapsto I(s) \times \omega_1(s) \end{aligned}$$

Let us first verify that ρ_1 is injective. For $s=(k,n,l)$ and $s'=(k',n',l')$,

$$\rho_1(\phi_1(s)) = \rho_1(\phi_1(s')) \Rightarrow \left\{ \begin{array}{c} I(s) = I(s') \\ \omega_1(s) = \omega_1(s') \end{array} \right. \Rightarrow \left\{ \begin{array}{c} k = k', \text{ by (35) and (21)} \\ n = n', \text{ by (34) and (22)} \\ l = l', \text{ by (37), (23) and (18)} \end{array} \right.$$

which means $s = s'$. If $\omega_1(s) \cap \omega_1(s') = \emptyset$,

$$\langle \phi_1(s), \phi_1(s') \rangle = \left\langle \widehat{\phi}_1(s), \widehat{\phi}_1(s') \right\rangle = 0$$

by Parseval and (37). If $i=2$ same argument applies. If $i=3$ we use the same argument and the fact that

$$\omega_3(s) \cap \omega_3(s') = \emptyset \Rightarrow 2\omega_3(s) \cap 2\omega_3(s') = \emptyset$$

by (39). This proves (25). Finally, as $\phi \in \mathcal{S}$,

$$|\phi(x)| \leq C_m(1+|x|)^{-m},$$

Then

$$|\phi_1(k,n,l)(x)| \leq \nu^{-1}C_m 2^{\frac{-\epsilon k}{2}}(1+|2^{-\epsilon k}x-(n+\nu_1)|)^{-m}$$

For $m=0$,

$$\begin{aligned}|\phi_1(k,n,l)(x)| &\leq \nu^{-1}C_0 2^{\frac{-\epsilon k}{2}}\\ &\leq \tilde{C}_0|I(s)|^{-\frac{1}{2}}\end{aligned}$$

by (35). Observe that it suffices to prove (26) for $x \notin I(s)$. Indeed, we can choose $\tilde{C}_m$ such that (26) holds for all $x \in I(s)$ and, if it also holds for $x \notin I(s)$ with constant $\tilde{\tilde{C}}_m$, we make $C_m := \max\{\tilde{C}_m, \tilde{\tilde{C}}_m\}$ and prove (26). In the case mentioned,

$$\begin{aligned}|x-c(I(s))| &\leq |x-2^{\epsilon k}(n+\nu_1)|+|2^{\epsilon k}(n+\nu_1)-c(I(s))|\\ &\leq |x-2^{\epsilon k}n|+|2^{\epsilon k}n-c(I(s))|+2.2^{\epsilon k}\nu_1\\ &\leq |x-2^{\epsilon k}n|+2^{\epsilon k}(\nu+2\nu_1)\\ &\leq |x-2^{\epsilon k}n|+3\nu 2^{\epsilon k}\\ &\leq |x-2^{\epsilon k}n|+\frac{|I(s)|}{4}\\ &\leq |x-2^{\epsilon k}n|+\frac{|x-c(I(s))|}{2} \Rightarrow\\ |2^{-\epsilon k}x-2^{-\epsilon k}c(I(s))| &\leq 2|2^{-\epsilon k}x-n|\end{aligned}$$

where we used (34) in the third inequality, (35) in the fifth and the fact that $x \notin I(s)$ in the sixth. This way,

$$\begin{aligned}|\phi_1(k,n,l)(x)| &\leq \nu^{-1}C_m 2^{\frac{-\epsilon k(m-\frac{1}{2})}{2}}|x-c(I(s))|^{-m},\\ |\phi_1(k,n,l)(x)| &\leq C_m|I(s)|^{m-\frac{1}{2}}|x-c(I(s))|^{-m}.\end{aligned}$$

and this proves (26) since $x \notin I(s)$. Finally, (27) follows immediately from (35) and (39). □

Now we conclude the proof of theorem 3.1 by using proposition 3.3 and 3.2.

We have:

$$
\begin{aligned}
\sum_{s=(k,n,l)\in S} |\Lambda_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}| &= \sum_{s=(k,n,l)\in S} \left| 2^{\frac{-\epsilon k}{2}} \left\langle f_1, \phi_{k,n_1,\frac{l_1}{2}} \right\rangle \left\langle f_2, \phi_{k,n_2,\frac{l_2}{2}} \right\rangle \left\langle f_3, \phi_{k,n_3,\frac{l_3}{2}} \right\rangle \right| \\
&= \sum_{s=(k,n,l)\in S} \left| 2^{\frac{-\epsilon k}{2}} \langle f_1, \nu\phi_1(s)\rangle \langle f_2, \nu\phi_2(s)\rangle \langle f_3, \nu\phi_3(s)\rangle \right| \\
&\leq 2^{\frac{\epsilon}{2}} 2^2 v^{\frac{5}{2}} \sum_{s=(k,n,l)\in S} |I(s)|^{-\frac{1}{2}} |\langle f_1, \phi_1(s)\rangle \langle f_2, \phi_2(s)\rangle \langle f_3, \phi_3(s)\rangle| \\
&\leq C_{p_1,p_2,p_3} \nu^{10} \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}
\end{aligned}
$$

And this is enough by proposition 3.2.

3.2 Boundedness of the model form

Our purpose in this section is to prove proposition 3.3. Let us establish some notations and conventions before. For each $m \geq 0$ fix a constant C_m and $2 < p_1, p_2, p_3 < \infty$ such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

From now on, C will denote a constant depending on these data and its value may change from line to line. The indices i and j run over $\{1, 2, 3\}$ unless we say otherwise. Define η to be the greatest number for which η^{-1} is an integer and

$$\eta \leq 2^{-100} \left(\min_i \{p_i\} - 2 \right) \left(\max_j p_j \right)^{-1}$$

Define:

$$t := \left(\frac{1}{2} \min_i \{p_i\} \right) - \eta. \tag{44}$$

Observe that

$$t - 1 = \left(\frac{1}{2} \min_i \{p_i\} - 1 \right) - \eta \geq \left(\frac{1}{2} \min_i \{p_i\} - 1 \right) - 2^{-99} \left(\frac{1}{2} \min_i \{p_i\} - 1 \right) > 0$$

so $t \geq 1$. Fix the notation of proposition 3.3 and assume that its hypothesis are satisfied. Since the proposition is invariant under a permutation of the indexes $1, 2$ and 3, we can strengthen (31) to

$$\omega_1(s) < \omega_2(s) < \omega_3(s) \text{ for all } s \in S \tag{45}$$

by partitioning S into at most six subsets and proving proposition 3.3 to each such subset. For the rest of this section we assume (45). Define

$$\mathcal{G}_S(f_1, f_2, f_3)(x) := \sum_{s\in S} |I(s)|^{-\frac{3}{2}} \left(\prod_{i=1}^{3} |\langle f_i, \phi_i(s)\rangle| \right) \chi_{I(s)}(x).$$

which clearly makes sense since S is finite.

Claim 3.5. *For $f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R})$, the L^1 norm of $\mathcal{G}_S(f_1, f_2, f_3)$ is equal to the left-hand side of* (33).

Proof.

$$\begin{aligned} \|\mathcal{G}_S(f_1, f_2, f_3)\|_1 &= \int_{\mathbb{R}} \sum_{s\in S} |I(s)|^{-\frac{3}{2}} \left(\prod_{i=1}^{3} |\langle f_i, \phi_i(s)\rangle| \right) \chi_{I(s)}(x)dx \\ &= \sum_{s\in S} |I(s)|^{-\frac{3}{2}} \left(\prod_{i=1}^{3} |\langle f_i, \phi_i(s)\rangle| \right) \int_{\mathbb{R}} \chi_{I(s)}(x)dx \\ &= \sum_{s\in S} |I(s)|^{-\frac{1}{2}} \left(\prod_{i=1}^{3} |\langle f_i, \phi_i(s)\rangle| \right) \end{aligned}$$

□

Proposition 3.4. *For all r_1, r_2, r_3 with $|r_i - p_i| \le \eta$ and all functions $f_1, f_2, f_3 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\|f_i\|_{r_i} = 1$,*

$$|\{x \in \mathbb{R} : \mathcal{G}_S(f_1, f_2, f_3)(x) \ge 1\}| \le C \tag{46}$$

Before we prove this proposition, let us first prove:

Claim 3.6. *Proposition 3.4 implies* (33).

Proof. Consider the following data:

$$\begin{aligned} I_\lambda(s) &:= [\lambda^r a, \lambda^r b) \text{ where } [a, b) = I(s) \\ \phi_{i,\lambda}(s)(x) &:= \lambda^{-\frac{r}{2}} \phi_i(s)(\lambda^{-r} x) \\ f_{i,\lambda}(x) &:= \lambda^{-\frac{r}{r_i}} f_i(\lambda^{-r} x) \\ \omega_{i,\lambda} &:= \omega_i \end{aligned}$$

Observe that $\|f_i\|_{r_i} = 1$. This way,

$$\begin{aligned} \sum_{s\in S} |I_\lambda(s)|^{-\frac{3}{2}} \left(\prod_{i=1}^{3} |\langle f_{i,\lambda}, \phi_{i,\lambda}\rangle| \right) \chi_{I_\lambda(s)}(x) &= \lambda^{-1} \sum_{s\in S} |I(s)|^{-\frac{3}{2}} \left(\prod_{i=1}^{3} |\langle f_i, \phi_i\rangle| \right) \chi_{I(s)}(\lambda^{-r} x) \\ &= \lambda^{-1} \mathcal{G}(f_1, f_2, f_3)(\lambda^{-r} x) \end{aligned}$$

By scaling invariance,

$$
\begin{aligned}
|\{x \in \mathbb{R}^n : \lambda^{-1}\mathcal{G}_S(f_1, f_2, f_3)(\lambda^{-r}x) \geq 1| \leq C \Rightarrow \\
|\{x \in \mathbb{R}^n : \mathcal{G}_S(f_1, f_2, f_3)(\lambda^{-r}x) \geq \lambda| \leq C \Rightarrow \\
|\{x \in \mathbb{R}^n : \mathcal{G}_S(f_1, f_2, f_3)(x) \geq \lambda| \leq C\lambda^r
\end{aligned} \tag{47}
$$

Since the last inequality in (47) holds for every (r_1, r_2, r_3) in a small neighborhood of (p_1, p_2, p_3), by the multilinear Marcinkiewicz interpolation theorem we have:

$$\|\mathcal{G}_S(f_1, f_2, f_3)\|_1 \leq C \prod_i \|f_i\|_{p_i}$$

which is exactly what we wanted. Therefore it suffices to prove (46). □

For the rest of this section, fix r_1, r_2 and r_3 with $|r_i - p_1| < \eta$ and define r as the sum of its inverses, as above. Some of the following arguments will be explained as if C did depend on r_i, but this constant can always be picked uniformly in r_i as long as $|r_i - p_i| \leq \eta$. Fix f_1, f_2, $f_3 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\|f_i\|_{r_i} = 1$ and define $F_i : S \to \mathbb{R}$ by:

$$F_i(s) := |\langle f_i, \phi_i(s)\rangle|$$

Define also:

$$F := F_1 F_2 F_3$$

We now pass to the proof of (46). Define a partial order $<$ on the set of rectangles by

$$J_1 \times J_2 < J_1' \times J_2' \Leftrightarrow J_1 \subset J_1' \text{ and } J_2' \subset J_2. \tag{48}$$

$T \subset S$ is called a *CF set of type i* (Carleson-Fefferman sets) if the set $\rho_i(\phi_i(T))$ has a unique maximal element with respect to $<$. The unique element $s \in T$ such that $\rho_i(\phi_i(s))$ is maximal is called the *base* of the CF set T and denoted by s_T.

Claim 3.7. *Let T be a CF set of type i. If $j \neq i$, then for all $s, s' \in T$ with $s \neq s'$ we have $\rho_j(\phi_j(s)) \cap \rho_j(\phi_j(s')) = \emptyset$.*

Proof. Let $s, s' \in T$ and assume $\rho_j(\phi_j(s)) \cap \rho_j(\phi_j(s')) \neq \emptyset$. By (30), we can assume without loss of generality that $\omega_j(s) \subset \omega_j(s')$ (the argument for the other inclusion is the same). If we had $\omega_j(s) \subsetneq \omega_j(s')$, (32) would give us

$$\omega_i(s) \subset \omega_j(s') \text{ for } i = 1, 2 \text{ and } 3.$$

then by (31)

$$\omega_j(s') \cap \omega_i(s') = \emptyset \Rightarrow \omega_i(s) \cap \omega_i(s') = \emptyset$$

Which contradicts the CF property that $\rho_i(\phi_i(s))$ and $\rho_i(\phi_i(s'))$ are dominated by the same maximal element in $\rho_i(\phi_i(T))$. Therefore we must have $\omega_j(s) = \omega_j(s')$. Also, by (29) we may assume without loss of generality that $I(s) \subset I(s')$. By (27),

$$|I(s)| \geq 2^{-\frac{1}{2}}|I(s')| \Rightarrow I(s) = I(s')$$

by (3). By injectivity of ρ_i and ϕ_i we conclude that $s = s'$, which proves the claim □

Now, let T be a CF set. By Cauchy-Schwarz,

$$\sum_{s\in T} \frac{F_1(s)F_2(s)F_3(s)|I(s)|^{-\frac{1}{2}}}{|I(s_T)|} \leq \left(\sup_{s\in T} \frac{F_i(s)}{|I(s)|^{\frac{1}{2}}}\right) \prod_{j\neq i} \left(\frac{\sum_{s\in T} F_j(s)^2}{|I(s_T)|}\right)^{\frac{1}{2}} \tag{49}$$

Our objective now is to decompose S into CF sets in a way that for each CF set there are upper bounds for all three factors on the right-hand side of 49 and a lower bound for one of them. Start by enumerating the three factors above according to the index of the function F_l appearing in each factor and say that a CF set has *color* j if there is a lower bound for the j^{th} factor. The value of such factor will be called the *charge* of the CF set. As we will see, the upper and lower bounds on the factors will depend on a parameter k. We will then decompose the CF sets into collections $\mathcal{F}_{k,i,j}$ where k determines the bounds, i is the type and j is the color of the CF sets in this collection.

Define $S_{-1} := S$ and, assuming that S_{k-1} is already constructed for some integer $k \geq 0$, define recursively

$$S_k := S_{k-1} \backslash \bigcup_{i,j} S_{k,i,j}$$

As one can see by looking at the range of i and j, there are 9 sets $S_{k,i,j}$ to construct. With k fixed, the construction of S_{k,i_1,j_1} will be independent of S_{k,i_2,j_2}. However, we observe that their construction do depend on the previously constructed $S_{k',i',j'}$ with $k' < k$. For i and j fixed, we will pick a collection $\mathcal{F}_{k,i,j}$ of CF sets $T \subset S_{k-1}$. With this collection we define

$$S_{k,i,j} := \bigcup_{T\in\mathcal{F}_{k,i,j}} T$$

This collection $\mathcal{F}_{k,i,j}$ is constructed recursively. Let $\mathcal{F}^{\text{sold}}$ denote the collection of CF sets which have already been picked in the process of constructing $\mathcal{F}_{k,i,j}$. Let $\mathcal{F}^{\text{stock}}$ denote the set of CF sets T of type i which satisfy the following properties:

$$T \subset S_{k-1} \backslash \bigcup_{T'\in\mathcal{F}^{\text{sold}}} T'. \tag{50}$$

$$\text{If } i = j, \text{ then } F_j(s) \geq 2^{-\eta k} 2^{-\frac{k}{r_j}} |I(s)|^{\frac{1}{2}} \text{ for all } s \in T. \tag{51}$$

$$\text{If } i \neq j, \text{ then } \left(\sum_{s\in T} F_j(s)^2\right)^{\frac{1}{2}} \geq 2^4 2^{-\frac{k}{r_j}} |I(s_T)|^{\frac{1}{2}} \text{ for all } s \in T. \tag{52}$$

If $\mathcal{F}^{\text{stock}} = \emptyset$, then we stop the process and set $\mathcal{F}_{k,i,j} := \mathcal{F}^{\text{sold}}$. Otherwise,

Claim 3.8. *If $\mathcal{F}^{\text{stock}} \neq \emptyset$, there is a CF set T which satisfies the following for all $T' \in \mathcal{F}^{\text{stock}}$:*

$$\text{If } T \subset T', \text{ then } T = T' \tag{53}$$

$$\text{If } i < j, \text{ then } \omega_i(s_T) \not< \omega_i(s_{T'}) \tag{54}$$

$$\text{If } i > j, \text{ then } \omega_i(s_{T'}) \not< \omega_i(s_T) \tag{55}$$

Proof. Consider the set $\tilde{\mathcal{F}}$ of all $T \in \mathcal{F}^{\text{stock}}$ which satisfy (53). This set is nonempty because, for example, a T such that $|T|$ is maximum belongs to it. Inside $\tilde{\mathcal{F}}$ there is a T maximal with respect to (54) or (55), depending on i and j. If $i < j$ (the other case is analogous),

$$\omega_i(s_T) \not< \omega_i(s_{T'}) \text{ for all } T' \in \tilde{\mathcal{F}}$$

If there was $T'' \in \mathcal{F}^{\text{stock}} \backslash \tilde{\mathcal{F}}$ such that

$$\omega_i(s_T) < \omega_i(s_{T''})$$

then, as we have $\omega_i(s_{\tilde{T}}) \subset \omega_i(s_{T''})$ for some $\tilde{T} \in \tilde{\mathcal{F}}$ that contains T'' (from the hypothesis that $\mathcal{F}^{\text{stock}}$ is a collection of CF sets of type i),

$$\omega_i(s_T) < \omega_i(s_{\tilde{T}})$$

which contradicts our previous conclusion. So

$$\omega_i(s_T) \not< \omega_i(s_{T'}) \text{ for all } T' \in \mathcal{F}^{\text{stock}}$$

This T will do the job. □

We pick such T, add it to $\mathcal{F}^{\text{sold}}$ and delete it from $\mathcal{F}^{\text{stock}}$. We continue this process and, since S is finite, this recursion stops after finitely many steps. We therefore finish the construction of $S_{k,i,j}$ and S_k. Observe that each $s \in S_k$ satisfies

$$F_j(s) \leq 2^{-\eta k} 2^{-\frac{k}{r_j}} |I(s)|^{\frac{1}{2}} \tag{56}$$

for all j, since the CF set {s} of type j does not satisfy (51). In particular, if $s \in S_k$ for all $k \in \mathbb{N}$ then $F_j(s) = 0$ by (56).

Also, each CF set $T \subset S_k$ of type i satisfies for $j \neq i$

$$\left(\sum_{s \in T} F_j(s)^2\right)^{\frac{1}{2}} < 2^4 2^{-\frac{k}{r_j}} |I(s_T)|^{\frac{1}{2}} \tag{57}$$

We now finish the proof of (46). Define the *counting function* $N_{k,i,j}$ as

$$N_{k,i,j}(x) := \sum_{T \in \mathcal{F}_{k,i,j}} \chi_{I(s_T)}(x)$$

As the name already suggests, $N_{k,i,j}(x)$ returns the number of CF sets $T \in \mathcal{F}_{k,i,j}$ such that x belongs to $I(s_T)$. For each subset $S' \subset S$ define

$$\mathcal{G}_{S'}(x) := \sum_{s \in S'} |I(s)|^{-\frac{3}{2}} F(s) \chi_{I(s)}(x)$$

Claim 3.9. *To prove* (46)*, it suffices to show that there is a set E with $|E| \leq C$ such that*

$$\|\mathcal{G}_S\|_{L^t(E^c)} \leq C \tag{58}$$

with $t > 1$ given by (44)*.*

Proof. If there is such set,

$$\begin{aligned}
|\{x \in \mathbb{R}; \mathcal{G}_S(x) \geq 1\}| &= |\{x \in \mathbb{R}; \mathcal{G}_S(x) \geq 1\} \cap E| + |\{x \in \mathbb{R}; \mathcal{G}_S(x) \geq 1\} \cap E^c| \\
&\leq |E| + \int\limits_{\{x \in \mathbb{R}; \mathcal{G}_S(x) \geq 1\} \cap E^c} 1 dx \\
&\leq C + \int\limits_{\{x \in \mathbb{R}; \mathcal{G}_S(x) \geq 1\} \cap E^c} \mathcal{G}_S(x) dx \\
&\leq C + \int\limits_{\{x \in \mathbb{R}; \mathcal{G}_S(x) \geq 1\} \cap E^c} \mathcal{G}_S^t(x) dx \\
&\leq C + \int_{E^c} \mathcal{G}_S^t(x) dx \\
&\leq C + C^t \\
&= \tilde{C}
\end{aligned}$$

□

Such E will be given by $E = E_1 \cup E_2 \cup E_3$ where

$$E_1 := \bigcup_{k<\eta^{-2}} \bigcup_{i,j=1}^{3} \bigcup_{T \in \mathcal{F}_{k,i,j}} I(s_T),$$
$$E_2 := \bigcup_{k\geq\eta^{-2}} \bigcup_{i,j=1}^{3} \bigcup_{T \in \mathcal{F}_{k,i,j}} \left\{x \in \mathbb{R}; \mathcal{G}_T(x) \geq 2^{-\frac{k}{r}}\right\},$$
$$E_3 := \bigcup_{k\geq\eta^{-2}} \bigcup_{i,j=1}^{3} \left\{x \in \mathbb{R}; N_{k,i,j}(x) \geq 2^{\frac{k}{n}}\right\}$$

If $x \notin E_1$ then

$$\chi_{I_{s(T)}}(x) = 0 \text{ for all } T \in \mathcal{F}_{k,i,j} \text{ with } k < \eta^{-2},\ i,j \in \{1,2,3\}$$

Since

$$S = \left(\bigcup_{k}\bigcup_{i,j}\bigcup_{T\in\mathcal{F}_{k,i,j}}\right) \cup \left(\bigcap_{k} S_k\right)$$

and (as already observed)

$$F(s) = 0 \text{ for all } s \in \bigcap_k S_k$$

we can write for this x:

$$\mathcal{G}_S(x) = \sum_{k\geq\eta^{-2}} \sum_{i,j} \sum_{T\in\mathcal{F}_{k,i,j}} \mathcal{G}_T(x)$$

For a single CF set T, the function $\mathcal{G}_T$ is supported on $I(s_T)$ and it is bounded by $2^{-\frac{k}{r}}$ if $x \notin E_2$ (by the definition of E_2). Therefore for $x \notin E_1 \cup E_2$,

$$\begin{aligned}\mathcal{G}_S(x) &\leq \sum_{k\geq\eta^{-2}} \sum_{i,j} \sum_{T\in\mathcal{F}_{k,i,j}} 2^{-\frac{k}{r}} \chi_{I(s_T)}(x) \\ &= \sum_{k\geq\eta^{-2}} \sum_{i,j} 2^{-\frac{k}{r}} N_{k,i,j}(x)\end{aligned} \tag{59}$$

Claim 3.10. *For all $k \geq \eta^{-2}$ and $i,j \in \{1,2,3\}$ we have*

$$\|N_{k,i,j}\|_{L^t(E_3^c)} \leq C 2^{10\eta r_j k} 2^{\frac{k}{t}}$$

This claim will be proved soon. Applying this estimate to (59) we obtain

$$\begin{aligned}\|\mathcal{G}_S\|_{L^t(E^c)} &\leq \sum_{k\geq\eta^{-2}}\sum_{i,j} 2^{-\frac{k}{r}}\|N_{k,i,j}\|_{L^t(E^c)} \\ &\leq \sum_{k\geq\eta^{-2}}\sum_{i,j} C2^{-\frac{k}{r}}2^{10\eta r_j k}2^{\frac{k}{t}} \\ &\leq \tilde{C}\sum_{k\geq\eta^{-2}} 2^{-\frac{k}{r}}2^{10\eta r_j k}2^{\frac{k}{t}}\end{aligned} \tag{60}$$

Since r is close to 1, t is close to $\frac{1}{2}\min_i\{p_i\} > 1$ and η is very small, the right-hand side of (60) converges and we obtain (58). In order to use 3.9 we will devote the next three subsection to estimate the measures of E_1, E_2 and E_3 and prove the counting function estimate. This finishes the proof of proposition 3.3.

3.2.1 *The set* E_1

Define:

$$E_{k,i,j} := \bigcup_{T\in\mathcal{F}_{k,i,j}} I(s_T)$$

We have two cases:

1. $i = j$. In this case, if $T \in \mathcal{F}_{k,i,j}$, then s_T satisfies

$$|\langle f_j, \phi_j(s_T)\rangle| = F_j(s_T) \geq 2^{-\eta k}2^{-\frac{k}{r_j}}|I(s_T)|^{\frac{1}{2}}$$

By the construction of the sets $\mathcal{F}_{k,i,j}$ done previously. By lemma 3.3,

$$|\langle f_j, \phi_j(s_T)\rangle| \leq C|I(s_T)|^{\frac{1}{2}} \inf_{x\in I(s_T)} M(f_j)(x)$$

This way,

$$\inf_{x\in I(s_T)} M(f_j)(x) \geq \tilde{C}2^{-\eta k}2^{-\frac{k}{r_j}} \Rightarrow M(f_j)(x) \geq \tilde{C}2^{-\eta k}2^{-\frac{k}{r_j}}$$

for all $x \in I(s_T)$. Then we have

$$E_{k,i,j} \subset \{x \in \mathbb{R}; M(f_j)(x) \geq C2^{-\eta k}2^{-\frac{k}{r_j}}\} \tag{61}$$

2. $i \neq j$. Pick $T \in \mathcal{F}_{k,i,j}$. Take $\Phi = \phi_j(T)$ and $J = I(s_T)$ and pick $s, s' \in T$. Hypothesis (115) is satisfied. To verify (116), use the same argument done in the beginning of the proof of claim 3.7. To check (117), assume $I(s) = I(s')$. Since $\omega_i(s_T) \subset \omega_i(s)\cap\omega_i(s') \neq \emptyset$, assume without loss of generality that $\omega_i(s) \subset \omega_i(s')$. If

the inclusion was strict, by (27),

$$2^{-\frac{1}{2}}|\omega(s)| < |\omega(s')| < 2^{\frac{1}{2}}|\omega(s)| \Rightarrow \frac{|\omega(s')|}{|\omega(s)|} < 2^{\frac{1}{2}}$$

But, by (3),

$$\frac{|\omega(s')|}{|\omega(s)|} \geq 2$$

a contradiction. Then we must have $\omega_i(s) = \omega_i(s')$. By injectivity of the phase plane representation ρ_i, $s = s'$. By (118),

$$\sum_{\phi\in\phi_j(T)} |\langle f_j, \phi\rangle|^2 \leq C|I(s_T)| \left(\inf_{x\in I(s_T)} M_2(M(f_j))(x)\right)^2$$

On the other hand, by (52),

$$\left(\sum_{\phi\in\phi_j(T)} |\langle f_j, \phi\rangle|^2\right)^{\frac{1}{2}} \geq 2^4 2^{-\frac{k}{r_j}}|I(s_T)|^{\frac{1}{2}}$$

Combining these inequalities, we obtain:

$$E_{k,i,j} \subset \{x \in \mathbb{R}; M_2(M(f_j))(x) \geq C2^{-\frac{k}{r_j}}\} \tag{62}$$

Take $\epsilon > 0$. If $i = j$, using (r_j, r_j)-weak boundedness of the maximal operators, we obtain:

$$\begin{aligned}|E_{k,i,j}| &\leq |\{x \in \mathbb{R}; M(f_j)(x) \geq C2^{-\eta k}2^{-\frac{k}{r_j}}\}| \\ &\leq |\{x \in \mathbb{R}; M(f_j)(x) > C2^{-\eta k}2^{-\frac{k}{r_j}} - \epsilon\}| \\ &\leq \frac{\|f_j\|_{r_j}^{r_j}}{\left(C2^{-\eta k}2^{-\frac{k}{r_j}} - \epsilon\right)^{r_j}} \\ &\leq \tilde{C}2^{\eta k r_j}2^k\end{aligned}$$

If $i \neq j$,

$$\begin{aligned}|E_{k,i,j}| &\leq |\{x \in \mathbb{R}; M_2(M(f_j))(x) \geq C2^{-\frac{k}{r_j}}\}| \\ &\leq |\{x \in \mathbb{R}; M_2(M(f_j))(x) > C2^{-\frac{k}{r_j}} - \epsilon\}| \\ &\leq \frac{\|M(f_j)\|_{r_j}^{r_j}}{\left(C2^{-\frac{k}{r_j}} - \epsilon\right)^{r_j}} \\ &\leq \tilde{\tilde{C}}\|f_j\|_{r_j}^{r_j}2^k \\ &= \tilde{\tilde{C}}2^k\end{aligned}$$

In any case, we have that for all $k \geq 0$, i, j,

$$|E_{k,i,j}| \leq C 2^k 2^{\eta k r_j} \tag{63}$$

Summing $|E_{k,i,j}|$ over finitely many terms, we conclude that

$$|E_1| \leq C$$

3.2.2 *The set* E_2

Fix $k \geq \eta^{-2}, i, j$ and $T \in \mathcal{F}_{k,i,j}$. By (56) and (57), we have:

$$\begin{aligned}
\|\mathcal{G}_T\|_1 &= \sum_{s \in T} |I(s)| \left(\prod_{l=1}^{3} |\langle f_l, \phi_l(s) \rangle| \right) \\
&\leq \left(\sup_{s \in T} F_i(s) |I(s)|^{-\frac{1}{2}} \right) \prod_{j \neq i} \left(\sum_{s \in T} F_j(s)^2 \right)^{\frac{1}{2}} \\
&\leq 2^{-\eta k} 2^{-\frac{k}{r_i}} |I(s)|^{\frac{1}{2}} |I(s)|^{-\frac{1}{2}} \prod_{j \neq i} 2^4 2^{-\frac{k}{r_j}} |I(s_T)|^{\frac{1}{2}} \\
&= C 2^{-\eta k} 2^{-k\left(\frac{1}{r_i} + \sum_{j \neq i} \frac{1}{r_j}\right)} |I(s_T)| \\
&= C 2^{-\eta k} 2^{-\frac{k}{r}} |I(s_T)|
\end{aligned} \tag{64}$$

Let $J \in I(T)$ and pick $s_J \in T$ such that $J = I(s_J)$. Let T_J be the set of all $s \in T$ such that $I(s) \subset J$, then of course T_J is a CF set of type i and base s_J. Using again Hölder's inequality, (56) and (57), we get:

$$\sum_{s \in T_J} F_1(s) F_2(s) F_3(s) |I(s)|^{-\frac{1}{2}} \leq C 2^{-\eta k} 2^{-\frac{k}{r}} |J| \tag{65}$$

which amounts to a BMO estimate on $\mathcal{G}_T$, with respect to the grid $I(T)$. **By the analogue of the John-Nirenberg inequality** we obtain:

$$|\{x \in \mathbb{R} : \mathcal{G}_T(x) \geq 2^{-\frac{k}{r}}\}| \leq C 2^{-\eta k} |I(s_T)| \tag{66}$$

Define $\mathcal{F} := \{T \in \mathcal{F}_{k,i,j} : I(s_T) \not\subset E_3\}$. Then

$$\sum_{T \in \mathcal{F}_{k,i,j}} |\{x \in \mathbb{R} \backslash E_3 : \mathcal{G}_T(x) \geq 2^{-\frac{k}{r}}\}| \leq \sum_{T \in \mathcal{F}} |\{x \in \mathbb{R} : \mathcal{G}_T(x) \geq 2^{-\frac{k}{r}}\}|$$

Using (66) we can estimate this by

$$\sum_{T\in\mathcal{F}_{k,i,j}} |\{x \in \mathbb{R}\backslash E_3 : \mathcal{G}_T(x) \geq 2^{-\frac{k}{r}}\}| \leq C2^{-\frac{k}{\eta}} \|N_{\mathcal{F}}\|_1 \tag{67}$$

where $N_{\mathcal{F}} := \sum_{T\in\mathcal{F}} \chi_{I(s_T)}$. Since the counting function is integer valued, we have

$$\|N_{\mathcal{F}}\|_1 \leq \|N_{\mathcal{F}}\|_t^t$$

As a consequence of claim 3.12, we have

$$\|N_{\mathcal{F}}\|_t \leq C2^{10\eta r_j k} 2^{\frac{k}{t}} \tag{68}$$

Summing the first member of (67) and using (68) (observe that this is summable since η is small), we obtain

$$|E_2\backslash E_3| \leq C$$

3.2.3 *The counting function estimate and the set* E_3

We will prove here that there are not too many CF sets in $\mathcal{F}_{k,i,j}$. Fix $k \geq \eta^{-2}$, i and j. For each subset $\mathcal{F} \subset \mathcal{F}_{k,i,j}$ define

$$N_{\mathcal{F}} := \sum_{T\in\mathcal{F}} \chi_{I(s_T)}$$

and

$$N_{\mathcal{F}}^{\sharp}(x) := \sup_{J\in\{I(s_T):T\in\mathcal{F}\}:\quad x\in J} \frac{1}{|J|} \left(\sum_{T\in\mathcal{F}:I(s_T)\subset J} |I(s_T)| \right)$$

We then have $\|N_{\mathcal{F}}\|_t \leq C\|N_{\mathcal{F}}^{\sharp}\|_t$. Pick a minimal subset $\mathcal{F} \subset \mathcal{F}_{k,i,j}$ such that for all $x \in \mathbb{R}$,

$$N_{\mathcal{F}}(x) \geq \min\{N_{k,i,j}(x), 2^{\frac{k}{\eta}}\} \tag{69}$$

Claim 3.11. *Equality holds in* (69) *for this minimal* $\mathcal{F}$.

Proof. Assume otherwise, i.e., the inequality in (69) is strict for this $\mathcal{F}$ at some x. **Since** $N_{k,i,j} \geq N_{\mathcal{F}}$ we had $N_{\mathcal{F}}(x) > 2^{\frac{k}{\eta}}$ and moreover $N_{\mathcal{F}}(x) \geq 2^{\frac{k}{\eta}} + 1$, since $2^{\frac{k}{\eta}}$ is an integer. Pick a $T \in \mathcal{F}$ such that $x \in I(s_T)$ and $I(s_T)$ has minimal length. Since $\{I(s_T) : T \in \mathcal{F}\}$ is a grid, $I(s_T) \subset I(s_{T'})$ for all $T' \in \mathcal{F}$ with $x \in I(s_{T'})$. Then $\mathcal{F}\backslash T$ also satisfies inequality (69), which contradicts the minimality of $\mathcal{F}$. □

For reference in the previous subsection we note that $\{T \in \mathcal{F}_{k,i,j} : I(s_T) \not\subset E_3\} \subset \mathcal{F}$.

Claim 3.12.

$$\|N_{\mathcal{F}}\|_t \leq C 2^{10\eta r_j k} 2^{\frac{k}{t}} \tag{70}$$

Note that (70) implies

$$\|N_{k,i,j}\|_{L^t(E_3^c)} \leq C 2^{10\eta r_j k} 2^{\frac{k}{t}} \tag{71}$$

Observe that

$$\{x \in \mathbb{R} : N_{\mathcal{F}}(x) \geq 2^{\frac{k}{r}}\} = \{x \in \mathbb{R} : N_{k,i,j}(x) \geq 2^{\frac{k}{r}}\} \tag{72}$$

Therefore (70) also implies $|E_3| \leq C$ when we pass from the strong-type estimate to the weak-type estimate and sum a geometric series.

Proof of claim 3.12. Assume first $i = j$. Let σ be the map which assigns to each CF set T its base s_T. We want to use lemma 3.6 with the set $\mathcal{F}$, the maps $I \circ \sigma$ and $\omega_i \circ \sigma$ and $A = 2^{\eta k}$. Hypothesis (96) follows from condition (53) in the construction of $\mathcal{F}_{k,i,j}$. Lemma 3.6 partitions $\mathcal{F}$ into subsets $\mathcal{F}_l$, $l \in \mathbb{N}$, each being separated in the sense of (97) and **satisfying**

$$\|N_{\mathcal{F}_l}\|_t \leq C 2^{\frac{k}{\eta}} \left(\exp(-l 2^{-10} 2^{-3\eta k}) 2^{\frac{k}{\eta}} |E_{k,i,j}|\right)^{\frac{1}{t}} \tag{73}$$

By (63) and the fact that η is small, we obtain

$$\|N_{\mathcal{F}_l}\|_t \leq C 2^{\frac{k}{\eta^2}} \exp(-l 2^{-12} 2^{-3\eta k}) \tag{74}$$

(74) is useful for large l. To obtain a good estimate for small l, we apply lemma 3.5 with

$$\Phi := \{\phi_i(s_T) : T \in \mathcal{F}_l\}$$

$$\rho := \rho_i$$

$$\mu := -\eta^2$$

$$A := 2^{\eta k}$$

Since by (69) we have $\|N_{\mathcal{F}}\|_\infty \leq 2^{\frac{k}{\eta}}$, the lemma gives for all $J \in \{I(s_T) : T \in \mathcal{F}_l\}$

$$\sum_{T \in \mathcal{F}_l : I(s_T) \subset J} F_i(s_T)^2 \leq C |J| 2^{\eta k} \left(\inf_{x \in J} M_2(M(f_j))(x)\right)^2 \tag{75}$$

This gives with (51), for $x \in J$

$$
\begin{aligned}
N^{\sharp}_{\mathcal{F}_l}(x) &\le \frac{1}{|J|} \sum_{T \in \mathcal{F}_l : I(s_T) \subset J} |I(s_T)| \\
&\le 2^{2\eta k} 2^{\frac{2k}{r_j}} \frac{1}{|J|} \sum_{T \in \mathcal{F}_l : I(s_T) \subset J} F_i(s_T)^2 \\
&\le C 2^{2\eta k} 2^{\frac{2k}{r_j}} 2^{\eta k} \left(M_2(M(f_j))(x)\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|N_{\mathcal{F}_l}\|_t &\le C \|N^{\sharp}_{\mathcal{F}_l}\|_t \\
&\le C 2^{\eta k} 2^{\frac{2k}{r_j}} 2^{\eta k} \|(M_2(M(f_j)))^2 \chi_{E_{k,i,j}}\|_t
\end{aligned}
\tag{76}
$$

This last norm can be estimated using Hölder (observe that $\frac{r_j}{2t} > 1$, so we can use Hölder for this exponent)

$$
\begin{aligned}
\|(M_2(M(f_j)))^2 \chi_{E_{k,i,j}}\|_t &= \left(\int [M_2(M(f_j))]^{2t} \chi_{E_{k,i,j}} \right)^{\frac{1}{t}} \\
&\le \left[\left(\int \{[M_2(M(f_j))]^{2t}\}^{\frac{r_j}{2t}} \right)^{\frac{2t}{r_j}} \left(\int \chi_{E_{k,i,j}}^{\frac{r_j}{r_j - 2t}} \right)^{1 - \frac{2t}{r_j}} \right]^{\frac{1}{t}} \\
&= \|M_2(M(f_j))\|_{r_j}^2 |E_{k,i,j}|^{\frac{1}{t} - \frac{2}{r_j}} \\
&\le \tilde{C} |E_{k,i,j}|^{\frac{1}{t} - \frac{2}{r_j}}
\end{aligned}
\tag{77}
$$

where we used the L^{r_j}-boundedness of M_2 and M and the fact that $\|f_j\|_{r_j} = 1$. Using (77) in (76) we get

$$
\|N_{\mathcal{F}_l}\|_t \le C 2^{\eta k} 2^{\frac{2k}{r_j}} 2^{\eta k} |E_{k,i,j}|^{\frac{1}{t} - \frac{2}{r_j}}
$$

By (63) we have

$$
\|N_{\mathcal{F}_l}\|_t \le C 2^{4\eta r_j k} 2^{\frac{k}{t}} \tag{78}
$$

Picking the minimum between (74) and (78) for each l and summing over l give claim 3.12 for $i = j$. Let us now focus on the proof of the case $i < j$. For $T \in \mathcal{F}$ define

$$
\begin{aligned}
T^{\min} &:= \{s \in T : \rho_i(s) \text{ is minimal in } \rho_i(T) \text{ with respect to the order } <\} \\
T^{\text{fat}} &:= \{s \in T : 2^5 2^{\eta k} |I(s)| \ge |I(s_T)|\} \\
T^{\partial} &:= \{s \in T : I(s) \cap (1 - 2^{-4}) I(s_T) = \emptyset\} \\
T^{\partial \max} &:= \{s \in T^{\partial} : \rho_i(s) \text{ is maximal in } \rho_i(T^{\partial}) \text{ with respect to the order } <\} \\
T^{\text{nice}} &:= T \backslash (T^{\min} \cup T^{\text{fat}} \cup T^{\partial})
\end{aligned}
$$

It will be clear from (81) that T^{nice} is not empty. We will apply lemma 3.7 with $\Phi := \phi_j(\cup_{T \in \mathcal{F}} T^{\text{nice}})$, $\rho := \rho_j$, $\mathcal{F}$ as above, the map τ mapping $\phi_j(s)$ to the CF set T which contains s, the map $\mathcal{I}$ assigning to each $T \in \mathcal{F}$ the interval $I(s_T)$, $\mu := \eta^{-2}$ and $A := 2^{\eta k}$. Let us check hypotheses (105), (106) and (107). The subtraction of T^{fat} and T^{∂} makes all $s \in T^{\text{nice}}$ satisfy

$$2^{\eta k} I(s) \subset I(s_T)$$

which is hypothesis (105).

Let $s, s' \in T^{\text{nice}}$. **This implies** $\omega_i(s) \cap \omega_i(s') \neq \emptyset$. If $I(s) = I(s')$, then (3) and (27) imply $\omega_i(s) = \omega_i(s')$. This gives hypothesis (107).

To prove hypothesis (106), assume that $T, T' \in \mathcal{F}$ and $s \in T^{\text{nice}}$, $s' \in (T')^{\text{nice}}$ with $\omega_j(s) \subset \omega_j(s')$ and $\omega_j(s) \neq \omega_j(s')$. Property (32) implies

$$\omega_i(s) \subset \omega_j(s') \tag{79}$$

Together with (45) and $i < j$ implies

$$\omega_i(s') < \omega_i(s)$$

Therefore $T \neq T'$, and from (54) we conclude that T has been picked before T' in the recursive construction of $\mathcal{F}_{k,i,j}$. Pick an $s'' \in (T')^{\min}$ such that $\rho_i(s'') \leq \rho_i(s')$. We have that $\omega_i(s')$ is strictly contained in $\omega_i(s'')$, and with (32) and (79) we obtain

$$\omega_i(s_T) \subset \omega_i(s) \subset \omega_j(s') \subset \omega_i(s'')$$

Since $s'' \notin T$, we conclude from (53) that $\rho_i(s'') \not< \rho_i(s_T)$, and therefore by the previous chain of inclusions:

$$I(s_T) \cap I(s'') = \emptyset$$

Observe that $I(s'') \subset I(s')$ and $|I(s')| \leq |I(s_T)|$ by (79). Therefore by the previous chain of inclusions and (2):

$$I(s_T) \cap I(s') = \emptyset$$

This proves hypothesis (106).

Lemma 3.7 gives for all $J \in \{I(s_T) : T \in \mathcal{F}\}$,

$$\sum_{T \in \mathcal{F}: I(s_T) \subset J} \sum_{s \in T^{\text{nice}}} F_j(s)^2 \leq C|J| 2^{\eta k} \left(\inf_{x \in J} M_2(M(f))(x) \right)^2 \tag{80}$$

Observe that the intervals $I(s)$ for $s \in T^{\min}$ are pairwise disjoint and contained in $I(s_T)$, so by (56) we have

$$\sum_{s \in T^{\min}} F_j(s)^2 \leq \sum_{s \in T^{\min}} 2^{-2\eta(k-1)} 2^{-2\frac{k-1}{r_j}} |I(s)|$$

Since $k \geq \eta^{-2} > 2^{100}$, this is bounded by

$$2^{-2\frac{k}{r_j}}|I(s_T)|$$

We observe from (3) that for fixed $x \in I(s_T)$ there can be at most $10\eta k$ elements $s \in T^{\text{fat}}$ such that $x \in I(s)$. Again by (56)

$$\begin{aligned}\sum_{s\in T^{\text{fat}}} F_j(s)^2 &\leq 10\eta k 2^{-2\eta(k-1)} 2^{-2\frac{k-1}{r_j}}|I(s_T)| \\ &\leq 2^{-2\frac{k}{r_j}}|I(s_T)|\end{aligned}$$

We can write T^{∂} as a union of CF sets with bases in $T^{\partial\,\max}$. The sets $I(s)$ with $s \in T^{\partial\,\max}$ are pairwise disjoint and contained in a set of measure $2^{-4}|I(s_T)|$, so by (57)

$$\begin{aligned}\sum_{s\in T^{\partial}} F_j(s)^2 &\leq \textstyle\sum_{s\in T^{\partial\,\max}} 2^4 2^{-2\frac{k-1}{r_j}}|I(s)| \\ &\leq 2^2 2^{-2\frac{k}{r_j}}|I(s_T)|\end{aligned}$$

These three estimates together with (52) for T prove

$$\sum_{s\in T^{\text{nice}}} F_j(s)^2 \geq 2^{-2\frac{k}{r_j}}|I(s_T)| \tag{81}$$

This inequality turns (80) into

$$N^{\sharp}_{\mathcal{F}}(x) \leq C 2^{2\frac{k}{r_j}} 2^{\eta k}(M_2(M(f)))(x)^2$$

for all $x \in \mathbb{R}$. As in the case $i = j$ we obtain

$$\|N^{\sharp}_{\mathcal{F}}\|_t \leq C 2^{2\eta k r_j} 2^{\frac{k}{t}}$$

This proves the claim for $i < j$. The case $i > j$ is analogous. □

We have the bound $|E_3| \leq C$.

3.3 Almost orthogonality

During this section, we may use C to denote different constants whose values are not important. Our first two results help us control certain scalar products. We will also need this kind of control in the next chapter when dealing with the Carleson operator. Let $\Phi \subset \mathcal{S}_u(\mathbb{R})$ and

$$\begin{aligned}\rho:\quad &\Phi \to \mathcal{R} \\ &\phi \mapsto I(\phi)\times\omega(\phi)\end{aligned}$$

be a phase plane representation with constants C_m

Lemma 3.3. *There exists C depending on C_2 such that for all L^1_{loc} tempered distributions and all $\phi \in \Phi$,*

$$|\langle f, \phi\rangle| \leq C|I(s)|^{\frac{1}{2}} \inf_{y\in I(\phi)} M(f)(y) \tag{82}$$

Proof. For all $x \in I(\phi)$ we have $\frac{|c(I(\phi))-x|}{|I(\phi)|} \leq \frac{1}{2}$ so:

$$\begin{aligned}
|\langle f, \phi\rangle| &\leq C|I(\phi)|^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{|f(y)|}{\left(1+\frac{|y-c(I(\phi))|}{|I(\phi)|}\right)^2} dy \\
&\leq C|I(\phi)|^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{|f(y)|}{\left(1+\frac{|y-x|-|c(I(\phi))-x|}{|I(\phi)|}\right)^2} dy \\
&\leq \tilde{C}_1|I(\phi)|^{\frac{1}{2}} \int_{\mathbb{R}} \frac{|f(y)|}{|I(\phi)|\left(1+\frac{|y-x|}{|I(\phi)|}\right)^2} dy \\
&\leq \tilde{C}_2|I(\phi)|^{\frac{1}{2}} M(f)(x)
\end{aligned}$$

by lemma 5.1. As the above holds for every $x \in I(\phi)$, we have the desired result.

□

Lemma 3.4. *For all $m \geq 1$ integer there is a constant C depending on C_m and C_0 such that*

$$|\langle \phi, \phi'\rangle| \leq C|I(\phi')|^{\frac{1}{2}}|I(\phi)|^{m-\frac{1}{2}}|c(I(\phi)) - c(I(\phi'))|^{-m}$$

for all $\phi, \phi' \in \Phi$ with $c(I(\phi)) \neq c(I(\phi'))$ and $2|I(\phi)| \geq |I(\phi')|$.

Proof. As $|\langle \phi, \phi'\rangle| = |\langle \phi', \phi\rangle|$, we can interchange ϕ and ϕ' if necessary and assume by symmetry that $c(I(\phi)) < c(I(\phi'))$. Let c be the midpoint of the segment $[c(I(\phi)), c(I(\phi'))]$, i.e.,

$$c = \frac{c(I(\phi)) + c(I(\phi'))}{2}$$

By Hölder's inequality,

$$\begin{aligned}
|\langle \phi, \phi'\rangle| &\leq |\langle \phi, \phi'\chi_{(-\infty,c]}\rangle| + |\langle \phi, \phi'\chi_{[c,+\infty)}\rangle| \\
&\leq \|\phi\|_1\|\phi'\chi_{(-\infty,c]}\|_\infty + \|\phi'\|_1\|\phi\chi_{[c,+\infty)}\|_\infty
\end{aligned}$$

By (26) we have

$$\begin{aligned}\|\phi\|_1 &\leq C_m|I(\phi)|^{-\frac{1}{2}}\int_{\mathbb{R}}\left(1+\left|\frac{x-c(I(\phi))}{|I(\phi)|}\right|\right)^{-m}dx\\ &= C_m|I(\phi)|^{\frac{1}{2}}\int_{\mathbb{R}}\left(1+\left|y-\frac{c(I(\phi))}{|I(\phi)|}\right|\right)^{-m}dy\\ &\leq \tilde{C}_m|I(\phi)|^{\frac{1}{2}}\end{aligned}$$

Then by our previous calculation an by (26)

$$\begin{aligned}|\langle\phi,\phi'\rangle| &\leq C|I(\phi)|^{\frac{1}{2}}|I(\phi')|^{-\frac{1}{2}}\left(1+\frac{|c-c(I(\phi'))|}{|I(\phi')|}\right)^{-m}+C|I(\phi')|^{\frac{1}{2}}|I(\phi)|^{-\frac{1}{2}}\left(1+\frac{|c-c(I(\phi))|}{|I(\phi)|}\right)^{-m}\\ &\leq C|I(\phi)|^{\frac{1}{2}}|I(\phi')|^{m-\frac{1}{2}}|c(I(\phi'))-c(I(\phi))|^{-m}+C|I(\phi')|^{\frac{1}{2}}|I(\phi)|^{m-\frac{1}{2}}|c(I(\phi'))-c(I(\phi))|^{-m}\\ &\leq \tilde{C}|I(\phi')|^{\frac{1}{2}}|I(\phi)|^{m-\frac{1}{2}}|c(I(\phi))-c(I(\phi'))|^{-m}\end{aligned}$$

Since $2|I(\phi)| \geq |I(\phi')|$ and $m \geq 1$.

□

Assume now we are given an integer $\mu > 1$ and a real $C_m > 0$ for each integer $m \geq 0$. Then there is a constant C such that the following holds for all numbers $A \geq 1$.

Lemma 3.5. *Let $\Phi \subset \mathcal{S}_u(\mathbb{R})$ be a nonempty finite set and ρ be a phase plane representation of Φ with constants C_m, such that $I(\Phi)$ and $\omega(\Phi)$ are grids and the following holds for all $\phi, \phi' \in \Phi$:*

$$\phi \neq \phi' \Rightarrow \rho(\phi) \text{ and } \rho(\phi') \text{ are A-separated} \tag{83}$$

Define $N_\Phi := \sum_{\phi\in\Phi}\chi_{I(\phi)}(\xi)$. Under these hypotheses, for $f \in L^2(\mathbb{R})$ we have

$$\sum_{\phi\in\Phi}|\langle f,\phi\rangle|^2 \leq C(1+\|N_\Phi\|_\infty A^{-\mu})\|f\|_2^2 \tag{84}$$

Also, if f is a locally square integrable tempered distribution, then for all $J \in I(\Phi)$,

$$\sum_{\substack{\phi\in\Phi\\ I(\phi)\subset J}}|\langle f,\phi\rangle|^2 \leq C|J|\|N_\Phi\|_\infty^{\frac{1}{\mu}}(1+\|N_\Phi\|_\infty A^{-\mu})\left(\inf_{x\in J}M_2(M(f))(x)\right) \tag{85}$$

Proof. Let us first prove (83). Define the following operator $\mathcal{T}$ on $L^2(\mathbb{R})$ by

$$\mathcal{T}f := \sum_{\phi\in\Phi}\langle f,\phi\rangle\phi$$

Observe that $\mathcal{T}(L^2(\mathbb{R})) \subset \text{span}\{\mathcal{T}(\phi)\}$, hence $\mathcal{T}$ has finite rank and is a compact operator.

It is also self-adjoint because

$$\langle \mathcal{T} f, g\rangle = \sum_{\phi\in\Phi} \langle f, \phi\rangle\ \langle \phi, g\rangle = \sum_{\phi\in\Phi} \langle f, \phi\rangle\ \overline{\langle g, \phi\rangle} = \sum_{\phi\in\Phi} \langle f, \langle g, \phi\rangle \phi\rangle = \langle f, \mathcal{T} g\rangle$$

Finally, it is obviously positive. This way, we can select its the largest eigenvalue B. We claim that it suffices to prove that

$$B \leq C\left(1 + \|N_\Phi\|_\infty A^{-\mu}\right)$$

Indeed, if this is the case, for all $f \in L^2(\mathbb{R})$ we have:

$$\sum_{\phi\in\Phi} |\langle f, \phi\rangle|^2 = |\langle \mathcal{T} f, f\rangle| \leq B\|f\|_2^2 \leq C\left(1 + \|N_\Phi\|_\infty A^{-\mu}\right)\|f\|_2^2$$

By the spectral theorem, we can take a normal eigenvector f of $\mathcal{T}$ corresponding to B, so $B^2 = \|\mathcal{T} f\|_2^2$. Expanding:

$$\|\mathcal{T} f\|_2^2 = \langle \mathcal{T} f, \mathcal{T} f\rangle = \left\langle \sum_{\phi\in\Phi} \langle f, \phi\rangle \phi, \sum_{\phi'\in\Phi} \langle f, \phi'\rangle \phi' \right\rangle = \sum_{\phi\in\Phi} |\langle f, \phi\rangle|^2 + \sum_{\substack{\phi,\phi'\in\Phi \\ \phi\neq\phi'}} \langle f, \phi\rangle \langle f, \phi'\rangle \langle \phi, \phi'\rangle$$

so

$$B^2 = \|\mathcal{T} f\|_2^2 \leq \underbrace{\sum_{\phi\in\Phi} |\langle f, \phi\rangle|^2}_{\textbf{(I)}} + \underbrace{\sum_{\substack{\phi,\phi'\in\Phi \\ \phi\neq\phi'}} |\langle f, \phi\rangle \langle f, \phi'\rangle \langle \phi, \phi'\rangle|}_{\textbf{(II)}} \tag{86}$$

For **(I)** we have:

$$\sum_{\phi\in\Phi} |\langle f, \phi\rangle|^2 = |\langle \mathcal{T} f, f\rangle| = B$$

Since $\omega(\Phi)$ is a grid, $\langle \phi, \phi'\rangle \Leftrightarrow \omega(\phi) \cap \omega(\phi') \neq \emptyset \Leftrightarrow \omega(\phi) \subset \omega(\phi')$ or $\omega(\phi') \subset \omega(\phi)$. This way, by symmetry we have the following estimate for **(II)**:

$$\sum_{\substack{\phi,\phi'\in\Phi \\ \phi\neq\phi'}} |\langle f, \phi\rangle \langle f, \phi'\rangle \langle \phi, \phi'\rangle| \leq 2 \sum_{\phi\in\Phi} |\langle f, \phi\rangle| \sum_{\phi'\in\Phi_\phi} |\langle \phi, \phi'\rangle \langle \phi', f\rangle|$$

where Φ_ϕ is the set of all ϕ' such that $\omega_\phi \subset \omega_{\phi'}$. Using Cauchy-Schwarz in the right-hand side of the inequality above:

$$\sum_{\substack{\phi,\phi'\in\Phi\\ \phi\neq\phi'}} |\langle f,\phi\rangle\langle f,\phi'\rangle\langle\phi,\phi'\rangle| \leq 2\left(\sum_{\phi\in\Phi}|\langle f,\phi\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{\phi\in\Phi}\left(\sum_{\phi'\in\Phi_\phi}|\langle\phi,\phi'\rangle\langle\phi',f\rangle|\right)^2\right)^{\frac{1}{2}}$$
$$= B^{\frac{1}{2}}\left(\sum_{\phi\in\Phi}\left(\sum_{\phi'\in\Phi_\phi}|\langle\phi,\phi'\rangle\langle\phi',f\rangle|\right)^2\right)^{\frac{1}{2}} \tag{87}$$

Note that (27) and $\omega_\phi \subset \omega_{\phi'}$ give us:

$$|I(\phi')||\omega(\phi')| \leq 2^{\frac{1}{2}}|I(\phi)||\omega(\phi)| \leq 2|I(\phi)||\omega(\phi)|$$
$$\Rightarrow |I(\phi')| \leq 2|I(\phi)|\frac{|\omega(\phi)|}{|\omega(\phi')|} \leq 2|I(\phi)|$$

Also, $I(\phi)\cap I(\phi') = \emptyset$ by (83). We can then use lemmas 3.3 and 3.4 for $m = \mu+1$ to estimate the following:

$$\sum_{\phi'\in\Phi_\phi}|\langle\phi,\phi'\rangle\langle\phi',f\rangle| \leq C\sum_{\phi'\in\Phi_\phi}|I(\phi')||I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-c(I(\phi'))|^{-\mu-1}\inf_{x\in I(\phi')}M(f)(x)$$
$$\leq \tilde{C}\sum_{\phi'\in\Phi_\phi}|I(\phi')||I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x'|^{-\mu-1}\inf_{x\in I(\phi')}M(f)(x)$$

for all $x'\in I(\phi')$. Observe that we used above that $2|c(I(\phi))-c(I(\phi'))| \geq |c(I(\phi))-y|$ for all $y\in I(\phi')$. By (83), all $I(\phi')$ for $\phi'\in\Phi_\phi$ are pairwise disjoint and contained in $(AI(\phi))^c$. We can then compare the expression above with the following integral:

$$\tilde{C}\sum_{\phi'\in\Phi_\phi}|I(\phi')||I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x'|^{-\mu-1}\inf_{x\in I(\phi')}M(f)(x)$$
$$\leq \tilde{C}\int_{(AI(\phi))^c}|I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x|^{-\mu-1}M(f)(x)dx \tag{88}$$

Let us make a few observations before we proceed. For each $y\in I(\phi)$ we have:

$$|I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x|^{-\mu-1}\chi_{(AI(\phi))^c}(x) \leq C|I(\phi)|^{\mu+\frac{1}{2}}|y-x|^{-\mu-1}$$

by an argument already used above. Also,

$$|I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x|^{-\mu-1}\chi_{(AI(\phi))^c}(x) = |I(\phi)|^{-\frac{1}{2}}\left(\frac{|I(\phi)|}{|c(I(\phi))-x|}\right)^{\mu+1}\chi_{(AI(\phi))^c}(x)$$
$$\leq CA^{-\mu-1}|I(\phi)|^{-\frac{1}{2}}$$

from the A-separability condition. This way,

$$|I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x|^{-\mu-1}\chi_{(AI(\phi))^c}(x) \leq \min\{C|I(\phi)|^{\mu+\frac{1}{2}}|y-x|^{-\mu-1}, CA^{-\mu-1}|I(\phi)|^{-\frac{1}{2}}\}$$

Let $U(x)$ be the upper bound above. We have:

$$\begin{aligned}
(88) &\leq \int_{\mathbb{R}} U(x)M(f)(x)dx \\
&= \int_{|y-x|\geq|I(\phi)|A} U(x)M(f)(x)dx + \int_{|y-x|<|I(\phi)|A} U(x)M(f)(x)dx \\
&\leq \int_{|y-x|\geq|I(\phi)|A} C|I(\phi)|^{\mu+\frac{1}{2}}|y-x|^{-\mu-1}M(f)(x)dx \\
&\qquad + \int_{|y-x|<|I(\phi)|A} CA^{-\mu-1}|I(\phi)|^{-\frac{1}{2}}M(f)(x)dx \\
&= \int_{|z|\geq|I(\phi)|A} C|I(\phi)|^{\mu+\frac{1}{2}}|z|^{-\mu-1}M(f)(y-z)dz \\
&\qquad + \int_{|y-x|<|I(\phi)|A} CA^{-\mu-1}|I(\phi)|^{-\frac{1}{2}}M(f)(x)dx \\
&\leq C|I(\phi)|^{\mu+\frac{1}{2}}[(|z|^{-\mu-1}\chi_{|z|\geq|I(\phi)|A}) * M(f)](y) \\
&\qquad + CA^{-\mu}|I(\phi)|^{\frac{1}{2}}\frac{1}{|I(\phi)|A}\int_{|y-x|<|I(\phi)|A} M(f)(x)dx \\
&\leq C|I(\phi)|^{\mu+\frac{1}{2}}\left\||z|^{-\mu-1}\chi_{|z|\geq|I(\phi)|A}\right\|_{L^1} M(M(f))(y) + CA^{-\mu}|I(\phi)|^{\frac{1}{2}}M(M(f))(y) \\
&\leq CA^{-\mu}\frac{|I(\phi)|^{\frac{1}{2}}}{\mu}M(M(f))(y) + CA^{-\mu}|I(\phi)|^{\frac{1}{2}}M(M(f))(y) \\
&\leq \tilde{C}A^{-\mu}|I(\phi)|^{\frac{1}{2}}M(M(f))(y)
\end{aligned} \tag{89}$$

where we used lemma 5.1. As this holds for each $y \in I(\phi)$,

$$(88) \leq CA^{-\mu}|I(\phi)|^{\frac{1}{2}} \inf_{y\in I(\phi)} M(M(f))(y)$$

Going back to (87),

$$\begin{aligned}
\sum_{\substack{\phi,\phi'\in\Phi \\ \phi\neq\phi'}} |\langle f,\phi\rangle\langle f,\phi'\rangle\langle\phi,\phi'\rangle| &\leq CA^{-\mu}B^{\frac{1}{2}}\left(\sum_{\phi\in\Phi}|I(\phi)|\inf_{y\in I(\phi)}(M(M(f))(y))^2\right)^{\frac{1}{2}} \\
&\leq CA^{-\mu}B^{\frac{1}{2}}\left(\int_{\mathbb{R}}\sum_{\phi\in\Phi}\chi_{I(\phi)}(y)(M(M(f))(y))^2dy\right)^{\frac{1}{2}} \\
&\leq CA^{-\mu}B^{\frac{1}{2}}\|N_\Phi\|_\infty^{\frac{1}{2}}\|M(M(f))\|_2 \\
&\leq \tilde{C}A^{-\mu}B^{\frac{1}{2}}\|N_\Phi\|_\infty^{\frac{1}{2}}\|f\|_2
\end{aligned} \tag{90}$$

Going back to (86),

$$B^2 \leq B + CA^{-\mu}B^{\frac{1}{2}}\|N_\Phi\|_\infty^{\frac{1}{2}}$$

We have $B \leq 1$ or $B^{\frac{1}{2}} \leq B$. It is immediate that in any case we have

$$B \leq 1 + CA^{-\mu}\|N_\Phi\|_\infty^{\frac{1}{2}} \leq 1 + CA^{-\mu}\|N_\Phi\|_\infty$$

since $\|N_\Phi\|_\infty \geq 1$. This proves (84).
Let f be a locally square integrable tempered distribution and let $J \in I(\Phi)$. Define:

$$J_\mu := 2\|N_\Phi\|_\infty^{\frac{1}{\mu}} J$$

Using the elementary fact $\dfrac{(a+b)^2}{2} \leq a^2 + b^2$, we have:

$$\frac{1}{2}\sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} |\langle f,\phi\rangle|^2 \leq \underbrace{\sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} |\langle f\chi_{J_\mu},\phi\rangle|^2}_{\textbf{(III)}} + \underbrace{\sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} |\langle f,\phi\chi_{(J_\mu)^c}\rangle|^2}_{\textbf{(IV)}} \tag{91}$$

We estimate **(III)** with (84):

$$\sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} |\langle f\chi_{J_\mu},\phi\rangle|^2 \leq C(1 + \|N_\Phi\|_\infty A^{-\mu})\|f\chi_{J_\mu}\|_2^2$$

Observe that for all $x \in J$:

$$\begin{aligned} &\|f\chi_{J_\mu}\|_2^2 = \int_{J_\mu} f^2(y)dy = |J_\mu|\frac{1}{|J_\mu|}\int_{J_\mu} f^2(y)dy \leq |J_\mu|(M_2(f)(x))^2 \Rightarrow \\ &\|f\chi_{J_\mu}\|_2^2 \leq |J_\mu|\inf_{x\in J}(M_2(f)(x))^2 = 2\|N_\Phi\|_\infty^{\frac{1}{\mu}}|J|\inf_{x\in J}(M_2(f)(x))^2 \end{aligned} \tag{92}$$

This way,

$$\sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} |\langle f\chi_{J_\mu},\phi\rangle|^2 \leq \tilde{C}(1 + \|N_\Phi\|_\infty A^{-\mu})\|N_\Phi\|_\infty^{\frac{1}{\mu}}|J|\inf_{x\in J}(M_2(f)(x))^2 \tag{93}$$

As for **(IV)**, using (26) for $m = \mu + 1$:

$$\begin{aligned} \int_{\mathbb{R}} |\phi(x)\chi_{(J_\mu)^c}| &\leq C_{\mu+1}|I_\phi|^{-\frac{1}{2}}\int_{J_\mu^c - c(I(\phi))}\left(1 + \left|\frac{y}{|I(\phi)|}\right|\right)^{-\mu-1} dy \\ &= C_{\mu+1}|I_\phi|^{\frac{1}{2}}\int_{\frac{1}{|I(\phi)|}[J_\mu^c - c(I(\phi))]} (1+|z|)^{-\mu-1}\, dz \\ &\leq C|I(\phi)|^{\mu+\frac{1}{2}}|J_\mu|^{-\mu} \end{aligned}$$

Also, for all $y \in I(\phi)$:

$$|\langle f, \phi\chi_{(J_\mu)^c}\rangle| \leq \int_{\mathbb{R}} |f|(x)\phi\chi_{(J_\mu)^c}(x)dx = \int_{\mathbb{R}} |f|(x)\tau_{-y}\widetilde{\phi\chi_{(J_\mu)^c}}(y-x)dx = |f| * \tau_{-y}\widetilde{\phi\chi_{(J_\mu)^c}}(y)$$

Using (26), one can find a radially decreasing majorant for $\tau_{-y}\widetilde{\phi\chi_{(J_\mu)^c}}$ with L^1 norm less than $C|I(\phi)|^{\mu+\frac{1}{2}}|J_\mu|^{-\mu}$. Using lemma 5.1 to conclude that for every $y \in I(\phi)$:

$$|\langle f, \phi\chi_{(J_\mu)^c}\rangle| \leq C|I(\phi)|^{\mu+\frac{1}{2}}|J_\mu|^{-\mu}M(f)(y)$$

Therefore:

$$\begin{aligned}
\sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} |\langle f, \phi\chi_{(J_\mu)^c}\rangle|^2 &\leq \sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} C|I(\phi)|^{2\mu+1}|J_\mu|^{-2\mu} \inf_{y\in I(\phi)} (M(f)(y))^2 \\
&\leq \sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} C|I(\phi)|\|N_\Phi\|_\infty^{-2} \left(\frac{|J|}{|J_\mu|}\right)^{2\mu} \inf_{y\in I(\phi)} (M(f)(y))^2 \\
&\leq \sum_{\substack{\phi\in\Phi \\ I(\phi)\subset J}} C|I(\phi)|\|N_\Phi\|_\infty^{-2} \inf_{y\in I(\phi)} (M(f)(y))^2 \\
&\leq C\int_J N_\Phi(y)\|N_\Phi\|_\infty^{-2}(M(f)(y))^2 \\
&\leq C|J| \left(\inf_{x\in J} M_2(M(f))(x)\right)^2
\end{aligned}$$

We conclude (85) by adding this to the estimate of **(III)**. □

In what follows, let $A \geq 1$, $L = 2^{10}A^3$ and $C = 3$.

Lemma 3.6. *Let S be a finite set and let $I, \omega : S \mapsto \mathcal{J}$ be two maps such that the following hold for all $s, s' \in S$:*

$$\omega(S) \text{ is a grid.} \tag{94}$$

$$2^{-1}|I(s')||\omega(s')| < |I(s)||\omega(s)| < 2|I(s')||\omega(s')| \tag{95}$$

$$\text{If } s \neq s', \text{ then } I(s) \cap I(s') = \emptyset \text{ or } \omega(s) \cap \omega(s') = \emptyset. \tag{96}$$

Then there is a sequence of subsets $S_l \subset S$, $l \in \mathbb{N}$ such that the following hold for all $l \in \mathbb{N}$:

$$\text{If } s, s' \in S_l \text{ are different, then } AI(s) \cap AI(s') = \emptyset \text{ or } \omega(s) \cap \omega(s') = \emptyset. \tag{97}$$

$$\sum_{s\in S_l} |I(s)| \leq Ce^{-\frac{l}{L}} \sum_{s\in S} |I(s)| \tag{98}$$

$$\bigcup_{l' \in \mathbb{N}} S_{l'} = S \tag{99}$$

Proof. Let $S_{\max}$ be the subset of S such that $\omega(s)$ is maximal with respect to inclusion. Since $\omega(S)$ is a grid, all elements of $S_{\max}$ are disjoint. We will break down this proof into several steps.

Step 1. Divide S into fewer than $\log_2(100A)$ subsets $S' \subset S$ such that for each S' and each $s, s' \in S'$:

$$\text{If } \omega(s) \subset \omega(s') \text{ and } \omega(s) \neq \omega(s'), \text{ then } 32A|\omega(s)| < |\omega(s')| \tag{100}$$

Define for each $s \in S$, $k \geq 1$ natural:

$$P_{s,k} := \{s'; \, \omega(s') \subset \omega(s) \text{ and } 2^{k-1}|\omega(s')| < |\omega(s)| \leq 2^k|\omega(s')| \,\}$$

Note that $u, u' \in P_{s,k} \Rightarrow \omega(u) \cap \omega(u') = \emptyset$. On the contrary, we would have $\omega(u) \subset \omega(u')$ or $\omega(u') \subset \omega(u)$. In the first case, by the definition of grid:

$$2|\omega(u')| \leq |\omega(u)| \Rightarrow |\omega(s)| \leq 2^k|\omega(u')| \leq 2^{k-1}|\omega(u)|$$

which contradicts $u \in P_{s,k}$. The other case is identical. Define:

$$\begin{aligned}
S_1 &= S_{\max} \cup \bigcup_{\substack{l=1 \\ s \in S_{\max}}}^{\infty} P_{s,l\log_2(2^6A)} \\
S_2 &= \bigcup_{\substack{l=0 \\ s \in S_{\max}}}^{\infty} P_{s,l\log_2(2^6A)+1} \\
&\vdots \\
S_{\log_2(2^6A)} &= \bigcup_{\substack{l=0 \\ s \in S_{\max}}}^{\infty} P_{s,l\log_2(2^6A)+(\log_2(2^6A)-1)}
\end{aligned}$$

Since every $s' \in S$ is such that $\omega(s') \subset \omega(s)$ for some $s \in S_{\max}$ and since there is $j \in \mathbb{N}$ such that $2^{j-1}|\omega(s')| < |\omega(s)| \leq 2^j|\omega(s')|$, it follows that

$$S = \bigcup_{n=1}^{\log_2(2^6A)} S_n$$

We must verify that each S_n satisfies (100). Observe that $P_{s,k} \cap P_{s',k'} = \emptyset$ for all $s, s' \in S_{\max}$ with $s \neq s'$, so we must investigate only $s = s'$. Suppose $u, u' \in S_n$ for $n > 1$ (the case n=1 is pratically the same) and satisfy the conditions of (100) with $\omega(u') \subset \omega(u)$. We can then assume that:

$$u \in P_{s,k\log_2(2^6A)+(n-1)}$$
$$u' \in P_{s,k'\log_2(2^6A)+(n-1)}$$

for $k' > k$. This way,

$$2^{k'\log_2(2^6A)+(n-1)}|\omega(u')| < |\omega(s)| \leq 2^{k\log_2(2^6A)+n}|\omega(u)|$$

which implies:

$$32A|\omega(u')| < 2^{(k'-k)\log_2(2^6A)-1}|\omega(u')| \leq |\omega(u)|$$

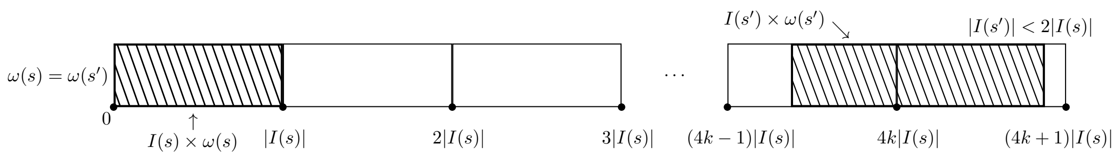
This concludes the first step.

Step 2. Divide each S_n obtained in the previous step into at most $4A$ subsets S'' such that for each $s, s' \in S''$:

$$\text{If } \omega(s) = \omega(s') \text{ and } s \neq s', \text{ then } AI(s) \cap AI(s') = \emptyset. \tag{101}$$

Choose one interval $\omega(s)$ such that $|\omega(s)|$ is maximal, translate the origin to the lower left corner of $I(s) \times \omega(s)$, and divide the x-axis in intervals of length $|I(s)|$ in a way such that $I(s)$ is one of these intervals (see the figure below).

Figure 1: Step 2



Define:

$$\begin{aligned}
T_{1,s} &:= \{s' \in S;\, \omega(s) = \omega(s') \text{ and } 4kAI(s) \leq c(I(s')) < (4k+1)AI(s),\, k \in \mathbb{Z}\ \} \\
T_{2,s} &:= \{s' \in S;\, \omega(s) = \omega(s') \text{ and } (4k+1)AI(s) \leq c(I(s')) < (4k+2)AI(s),\, k \in \mathbb{Z}\ \} \\
&\ \vdots \\
T_{4A,s} &:= \{s' \in S;\, \omega(s) = \omega(s') \text{ and } (4(k+1)-1)AI(s) \leq c(I(s')) < 4(k+1)AI(s),\, k \in \mathbb{Z}\ \}
\end{aligned}$$

All these sets are disjoint by construction. Also, by (96) we have $I(s) \cap I(s') = \emptyset$ for all s, s' such that $\omega(s) = \omega(s')$, so there is at most one $I(s')$ such that $4kAI(s) \leq c(I(s')) < (4k+1)AI(s)$ for each $k \in \mathbb{Z}$. By (95):

$$2^{-1}|I(s)| < |I(s')| < 2|I(s)|$$

as shown above. This way, if $AI(u) \cap AI(u') \neq \emptyset$ for $u, u' \in T_{1,s}$:

$$x \in AI(u) \cap AI(u') \Rightarrow x \in \left[4kA|I(s)| - |I(s)|, 4lA|I(s)| + \frac{|I(s)|}{2}\right)$$
$$\Rightarrow 4kA|I(s)| - |I(s)| < 4lA|I(s)| + \frac{|I(s)|}{2}, \text{ for } k > l$$

but this last inequality implies $k < l$, which is a contradiction. So $AI(u) \cap AI(u') = \emptyset$ for $u, u' \in T_{1,s}$. Now select $\tilde{s}$ such that $|\omega(\tilde{s})|$ is maximal and $\tilde{s} \in S := S \backslash \bigcup_{k=1}^{4A} T_{k,s}$ and repeat the procedure above. We will obtain sets $T_{1,\tilde{s}}$, $T_{2,\tilde{s}}$, ..., $T_{4A,\tilde{s}}$ as above. Keep doing this ultil S is exhausted. Define

$$T_1 := \bigcup_{s \text{ selected}} T_{1,s}$$
$$T_2 := \bigcup_{s \text{ selected}} T_{2,s}$$
$$\vdots$$
$$T_{4A} := \bigcup_{s \text{ selected}} T_{4A,s}$$

these sets have the desired property since each $T_{i,s}$ has.

Step 3. Reduce the proof to the case where $L = 2^5 A$, $C = 1$ and all s, s' satisfy (100) and (101).

By steps 1 and 2, we have $k = 4A \log_2(100A)$ subsets of S that satisfy (100) and (101). Label them from R_1 to R_k and suppose this lemma is true for R_i if $L = 2^5 A$ and $C = 1$. This way, for each i there is a sequence $R_{i,l}$ such that an obvious analog of (97) and (99) hold and:

$$\sum_{s \in R_{i,l}} |I(s)| \leq e^{-\frac{l}{2^5 A}} \sum_{s \in R_{i,l}} |I(s)|$$

Given $n \in \mathbb{N}$, there are $i \in \mathbb{N}$ and $0 \leq r < k$ such that $n = ki + r$. Define $S_n = S_{r,i}$. This way,

$$\sum_{s \in S_n} |I(s)| = \sum_{s \in S_{r,i}} |I(s)| \leq e^{-\frac{l}{2^5 A}} \sum_{s \in S_i} |I(s)|$$

Let us now work on the right-hand side of the inequality above:

$$\begin{aligned} e^{-\frac{l}{2^5 A}} \sum_{s\in S_i} |I(s)| &\leq e^{\frac{-2^5 A^2 i + r - r}{2^{10}A^3}} \sum_{s\in S} |I(s)| \\ &\leq e^{\frac{-4A\log_2(100A)i - r}{2^{10}A^3}} e^{\frac{r}{2^{10}A^3}} \sum_{s\in S} |I(s)| \\ &\leq e^{-\frac{n}{L}} e^{\frac{4A\log_2(100A)}{L}} \sum_{s\in S} |I(s)| \\ &\leq e^{-\frac{n}{L}} e \sum_{s\in S} |I(s)| \\ &\leq C e^{-\frac{n}{L}} \sum_{s\in S} |I(s)| \end{aligned}$$

where we used that $8A > \log_2(100A)$ in the second inequality above. Therefore, there is a sequence of subsets $S_n \subset S$ such that (97), (98) and (99) hold.

Step 4. Prove the lemma in the case mentioned in Step 3.
Define S_l recursively for $l \in \mathbb{N}$ to be the set of all $s \in S \backslash \bigcup_{l'<l} S_{l'}$ which satisfy the following:

$$\text{For all } s' \in S \backslash \bigcup_{l'<l} S_{l'}, \text{ if } \omega(s') \subset \omega(s) \text{ and } AI(s) \cap AI(s') \neq \emptyset, \text{ then } s' = s \tag{102}$$

(101) garantees that $S_l \neq \emptyset$ if $S \backslash \bigcup_{l'<l} S_{l'} \neq \emptyset$. (99) follows easily from the fact that S is finite. Also, (97) is clear by construction. For each $s \in S_l$, define S_s to be the set of all $s' \in S_{l+1}$ such that $\omega(s) \subset \omega(s')$ and $AI(s) \cap AI(s') \neq \emptyset$. For $s' \in S_s$ fixed, (101) gives us $\omega(s) \neq \omega(s')$. $AI(s) \cap AI(s') \neq \emptyset$ implies:

$$|c(AI(s)) - c(AI(s'))| \leq \frac{|AI(s)| + |AI(s')|}{2}$$

On the other hand, by (95) and (100):

$$\begin{aligned} |I(s')||\omega(s')| < 2|I(s)||\omega(s)| &\Rightarrow 2|I(s')| < \frac{|I(s)|}{8A} \\ &\Rightarrow 2|AI(s')| + |AI(s)| < \frac{|I(s)|}{8} + |AI(s)| \\ &\Rightarrow |AI(s')| + |AI(s)| < \left|\left(A + \frac{1}{4}\right) I(s)\right| - |AI(s')| \end{aligned}$$

Combining this last inequality with our previous one:

$$|c(AI(s)) - c(AI(s'))| < \frac{\left|\left(A + \frac{1}{4}\right) I(s)\right| - |AI(s')|}{2}$$

From the geometric point of view, this implies:

$$AI(s') \subset \left(A + \frac{1}{4}\right) I(s) \tag{103}$$

By (96) we have $I(s) \cap I(s') = \emptyset$, which gives:

$$|c(I(s)) - c(I(s'))| \geq \frac{|I(s)|}{2}$$

By (95) and (100):

$$\left|\frac{1}{2}I(s)\right| > 8A|I(s')| > |AI(s')|$$

Therefore:

$$|c(I(s)) - c(I(s'))| \geq \frac{\left|\frac{1}{2}I(s)\right| + \left|\frac{1}{2}I(s)\right|}{2} \geq \frac{\left|\frac{1}{2}I(s)\right| + |AI(s')|}{2}$$

$$\frac{1}{2}I(s) \cap AI(s') = \emptyset \tag{104}$$

For $s', s'' \in S_s$, $AI(s') \cap AI(s'') = \emptyset$ since $\omega(s) \subset \omega(s') \cap \omega(s'') \neq \emptyset$, (102) and since $\omega(S)$ is a grid. We conclude from (103) and (104) that:

$$\sum_{s' \in S_s} |AI(s')| \leq \left|\left(A + \frac{1}{4}\right) I(s)\right| - \left|\frac{1}{2}I(s)\right| \Rightarrow$$
$$\sum_{s' \in S_s} |I(s')| \leq \left(1 - \frac{1}{4A}\right) |I(s)|$$

Each $s' \in S_{l+1}$ is contained in S_s for some $s \in S_l$, so summing the last inequality over all $s \in S_l$:

$$\sum_{s' \in S_{l+1}} |I(s')| \leq \left(1 - \frac{1}{4A}\right) \sum_{s \in S_l} |I(s)|$$

By recursion:

$$\begin{aligned}
\sum_{s' \in S_l} |I(s')| &\leq \left(1 - \frac{1}{4A}\right)^l \sum_{s \in S} |I(s)| \\
&\leq e^{-\frac{l}{4A}} \sum_{s \in S} |I(s)| \\
&\leq e^{-\frac{l}{2^5 A}} \sum_{s \in S} |I(s)|
\end{aligned}$$

This completes this proof. □

Assume we are given $\mu \geq 1$ and a constant C_m for each integer $m \geq 0$. Then there exists a constant C depending on these data such that the following lemma holds for every $A \geq 2$.

Lemma 3.7. *Let $\Phi \subset \mathcal{S}_u(\mathbb{R})$ be a nonempty finite set. Let $I, \omega : \Phi \mapsto \mathcal{J}$ be maps such that $\rho : \phi \mapsto I(\phi) \times \omega(\phi)$ is a phase plane representation with constants C_m, and $I(\phi)$ and $\omega(\Phi)$ are grids. Let $\mathcal{F}$ be a set, $\tau : \Phi \to \mathcal{F}$ a surjective map, and $\mathcal{I} : \mathcal{F} \to \mathcal{J}$ a map such that $\mathcal{I}(\mathcal{F})$ is a grid and the following properties hold for all $\phi, \phi' \in \Phi$ (from now on, we will write $\mathcal{I}(\phi)$ instead of $\mathcal{I}(\tau(\phi))$).*

$$AI(\phi) \subset \mathcal{I}(\phi) \tag{105}$$

$$\text{If } \omega(\phi) \subset \omega(\phi'), \text{ then } \omega(\phi) = \omega(\phi') \text{ or } \mathcal{I}(\phi) \cap I(\phi') = \emptyset. \tag{106}$$

$$\text{If } \tau(\phi) = \tau(\phi'), \text{ then } \phi = \phi' \text{ or } I(\phi) \neq I(\phi'). \tag{107}$$

Define $N_{\mathcal{F}} = \sum_{T \in \mathcal{F}} \chi_{\mathcal{I}(T)}$. Under these hypothesis for all $f \in L^2(\mathbb{R})$:

$$\sum_{\phi \in \Phi} |\langle f, \phi \rangle|^2 \leq C(1 + \|N_{\mathcal{F}}\|_\infty A^{-\mu}) \|f\|_2^2 \tag{108}$$

and for all locally square integrable tempered distributions f and all $J \in \mathcal{I}(\mathcal{F})$,

$$\sum_{\substack{\phi \in \Phi \\ \mathcal{I}(\phi) \subset J}} |\langle f, \phi \rangle|^2 \leq C|J| \|N_{\mathcal{F}}\|_\infty^{\frac{1}{\mu}} (1 + \|N_{\mathcal{F}}\|_\infty A^{-\mu}) \left(\inf_{x \in J} M_2(M(f))(x) \right)^2. \tag{109}$$

Proof. The idea here is similar to that of lemma 3.5. Take B to be the largest eigenvalue of

$$\mathcal{T} f := \sum_{\phi \in \Phi} \langle f, \phi \rangle \phi$$

and take f to be a corresponding eigenvector with $\|f\|_2 = 1$. This way,

$$B^2 = \|\mathcal{T} f\|_2^2 = \langle \mathcal{T} f, \mathcal{T} f \rangle = \left\langle \sum_{\phi \in \Phi} \langle f, \phi \rangle \phi, \sum_{\phi' \in \Phi} \langle f, \phi' \rangle \phi' \right\rangle = \sum_{\phi, \phi' \in \Phi} \langle f, \phi \rangle \langle f, \phi' \rangle \langle \phi, \phi' \rangle \tag{110}$$

We will break the last sum above in two pieces:

$$\left| \sum_{\phi, \phi' \in \Phi} \langle f, \phi \rangle \langle f, \phi' \rangle \langle \phi, \phi' \rangle \right| \leq \underbrace{\sum_{\substack{\phi, \phi' \in \Phi \\ \omega(\phi) = \omega(\phi')}} |\langle f, \phi \rangle \langle f, \phi' \rangle \langle \phi, \phi' \rangle|}_{\textbf{(I)}} + \underbrace{\sum_{\substack{\phi, \phi' \in \Phi \\ \omega(\phi) \neq \omega(\phi')}} |\langle f, \phi \rangle \langle f, \phi' \rangle \langle \phi, \phi' \rangle|}_{\textbf{(II)}}$$

For **(I)**:

$$\sum_{\substack{\phi,\phi'\in\Phi\\ \omega(\phi)=\omega(\phi')}} |\langle f,\phi\rangle\langle f,\phi'\rangle\langle\phi,\phi'\rangle| \leq \left[\sum_{\substack{\phi,\phi'\in\Phi\\ \omega(\phi)=\omega(\phi')}} |\langle f,\phi\rangle|^2|\langle\phi,\phi'\rangle|\right]^{\frac{1}{2}} \left[\sum_{\substack{\phi,\phi'\in\Phi\\ \omega(\phi)=\omega(\phi')}} |\langle f,\phi'\rangle|^2|\langle\phi,\phi'\rangle|\right]^{\frac{1}{2}}$$

$$= \sum_{\substack{\phi,\phi'\in\Phi\\ \omega(\phi)=\omega(\phi')}} |\langle f,\phi\rangle|^2|\langle\phi,\phi'\rangle|$$

$$= \sum_{\phi\in\Phi} |\langle f,\phi\rangle|^2 \sum_{\substack{\phi'\in\Phi\\ \omega(\phi)=\omega(\phi')}} |\langle\phi,\phi'\rangle|$$

Also, following the lines of the proof of lemma 3.4, we have:

$$\sum_{\substack{\phi'\in\Phi\\ \omega(\phi)=\omega(\phi')}} |\langle\phi,\phi'\rangle| \leq C \sum_{\substack{\phi'\in\Phi\\ \omega(\phi')=\omega(\phi)}} \left(1+\frac{|c(I(\phi))-c(I(\phi'))|}{|I(\phi)|}\right)^{-2}$$

By (27) and by the definition of grid, if $\omega(\phi')=\omega(\phi'')$ and $\phi'\neq\phi''$, then $I(\phi')\cap I(\phi'')=\emptyset$. Therefore the following holds:

$$\sum_{\substack{\phi'\in\Phi\\ \omega(\phi')=\omega(\phi)}} \left(1+\frac{|c(I(\phi))-c(I(\phi'))|}{|I(\phi)|}\right)^{-2} \leq \tilde{C} \sum_{\substack{\phi'\in\Phi\\ \omega(\phi')=\omega(\phi)}} \int_{I(\phi')} \frac{1}{|I(\phi')|}\left(1+\frac{|c(I(\phi))-x|}{|I(\phi)|}\right)^{-2} dx$$

$$\leq \tilde{\tilde{C}} \int_{\mathbb{R}} \frac{1}{|I(\phi)|}\left(1+\frac{|c(I(\phi))-x|}{|I(\phi)|}\right)^{-2} dx$$

$$\leq C$$

Then:

$$\textbf{(I)} \leq C\sum_{\phi} |\langle f,\phi\rangle|^2 = CB \tag{111}$$

Observe now that:

$$\textbf{(II)} = \sum_{\substack{\phi,\phi'\in\Phi\\ \omega(\phi)\neq\omega(\phi')}} |\langle f,\phi\rangle\langle f,\phi'\rangle\langle\phi,\phi'\rangle| = 2\sum_{\phi\in\Phi} |\langle f,\phi\rangle| \sum_{\phi'\Phi_\phi} |\langle\phi,\phi'\rangle\langle\phi',f\rangle| \tag{112}$$

where $\phi\in\Phi$ is the set of all ϕ' for which $\omega(\phi)\subset\omega(\phi')$ and $\omega(\phi)\neq\omega(\phi')$. Using

Cauchy-Schwarz,

$$\begin{aligned}\textbf{(II)} &\leq 2\left(\sum_{\phi\in\Phi}|\langle f,\phi\rangle|^2\right)^{\frac{1}{2}}\left[\sum_{\phi\in\Phi}\left(\sum_{\phi'\in\Phi_\phi}|\langle\phi,\phi'\rangle\langle\phi',f\rangle|\right)^2\right]^{\frac{1}{2}}\\ &= 2B^{\frac{1}{2}}\left[\sum_{\phi\in\Phi}\left(\sum_{\phi'\in\Phi_\phi}|\langle\phi,\phi'\rangle\langle\phi',f\rangle|\right)^2\right]^{\frac{1}{2}}\end{aligned} \tag{113}$$

For $\phi\in\Phi$, define $A(\phi)$ to be the largest number such that $A(\phi)I(\phi)\subset\mathcal{I}(\phi)$. By lemmas 3.3 and 3.4 (with $m=\mu+1$) and by (106):

$$\begin{aligned}\sum_{\phi'\in\Phi_\phi}|\langle\phi,\phi'\rangle\langle\phi',f\rangle| &\leq C\sum_{\phi'\in\Phi_\phi}|I(\phi')||I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-c(I(\phi'))|^{-\mu-1}\inf_{x\in I(\phi')}M(f)(x)\\ &\leq C\int_{(\mathcal{I}(\phi))^c}|I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x|^{-\mu-1}M(f)(x)dx\\ &\leq C\int_{(A(\phi)I(\phi))^c}|I(\phi)|^{\mu+\frac{1}{2}}|c(I(\phi))-x|^{-\mu-1}M(f)(x)dx\\ &\leq CA(\phi)^{-\mu}|I(\phi)|^{\frac{1}{2}}\inf_{y\in I(\phi)}M(M(f))(y)\end{aligned}$$

where the last inequality above is justified by a similar computation as the one done in (89). This way, we estimate (113) by:

$$\begin{aligned}\textbf{(II)} &\leq CB^{\frac{1}{2}}\left(\sum_{\phi\in\Phi}|I(\phi)|A(\phi)^{-2\mu}\inf_{y\in I(\phi)}(M(M(f))(y))^2\right)^{\frac{1}{2}}\\ &= CB^{\frac{1}{2}}\left(\sum_{T\in\mathcal{F}}\sum_{\phi\in\tau^{-1}(T)}|I(\phi)|A(\phi)^{-2\mu}\inf_{y\in I(\phi)}(M(M(f))(y))^2\right)^{\frac{1}{2}}\\ &\leq CB^{\frac{1}{2}}\left(\int_{\mathbb{R}}\sum_{T\in\mathcal{F}}\left(\sum_{\phi\in\tau^{-1}(T)}A(\phi)^{-2\mu}\chi_{I(\phi)}(x)\right)(M(M(f))(x))^2dx\right)^{\frac{1}{2}}\end{aligned}$$

Fix $T\in\mathcal{F}$. Observe that for each $x\in\mathcal{I}(T)$, the numbers $A(\phi)$ for those elements $\phi\in\tau^{-1}(T)$ for which $x\in I(\phi)$ satisfy the following holds:

$$\begin{aligned}\phi,\phi'\in\tau^{-1}(T)\Rightarrow I(\phi)\neq I(\phi') &\Rightarrow I(\phi)\subset I(\phi') \text{ or } I(\phi')\subset I(\phi)\\ &\Rightarrow 2|I(\phi)|\leq|I(\phi')| \text{ or } |2I(\phi')|\leq|I(\phi)|\\ &\Rightarrow 2A(\phi)\leq A(\phi') \text{ or } 2A(\phi')\leq A(\phi)\end{aligned}$$

i.e., these numbers grow “at least” geometrically. As $A(\phi)\geq A\geq 2$, we can estimate the expression above by:

$$\begin{aligned}\textbf{(II)} &\leq CB^{\frac{1}{2}}A^{-\mu}\left(\int_{\mathbb{R}}\sum_{T\in\mathcal{F}}\chi_{\mathcal{I}(T)}(x)(M(M(f))(x))^2dx\right)^{\frac{1}{2}}\\ &\leq CB^{\frac{1}{2}}A^{-\mu}\|N_{\mathcal{F}}\|_{\infty}^{\frac{1}{2}}\|f\|_2\end{aligned} \tag{114}$$

By Hölder and by the L^2-boundedness of M. Combining (110), (111) and (114), we have:

$$B^2 \leq CB + CB^{\frac{1}{2}}A^{-\mu}\|N_{\mathcal{F}}\|_{\infty}^{\frac{1}{2}}$$

Either $B \leq 1$ or $B^{\frac{1}{2}} \leq B$. In both cases,

$$B \leq C(1+\|N_{\mathcal{F}}\|_{\infty}^{\frac{1}{2}}A^{-\mu})$$

which gives us:

$$\sum_{\phi\in\Phi}|\langle f,\phi\rangle|^2 \leq B\|f\|_2^2 \leq C(1+\|N_{\mathcal{F}}\|_{\infty}A^{-\mu})\|f\|_2^2$$

and proves (108). Now let f be a locally square integrable tempered distribution, and let $J \in \mathcal{I}(\mathcal{F})$. Define:

$$J_\mu := 2\|N_{\mathcal{F}}\|_{\infty}^{\frac{1}{\mu}}J$$

Using again the elementary fact that $\dfrac{(a+b)^2}{2} \leq a^2+b^2$ for $a,b\in\mathbb{R}$,

$$\frac{1}{2}\sum_{\substack{T\in\mathcal{F}\\ \mathcal{I}(T)\subset J}}\left(\sum_{\phi\in\tau^{-1}(T)}|\langle f,\phi\rangle|^2\right) \leq \underbrace{\sum_{\substack{T\in\mathcal{F}\\ \mathcal{I}(T)\subset J}}\left(\sum_{\phi\in\tau^{-1}(T)}|\langle f\chi_{J_\mu},\phi\rangle|^2\right)}_{\textbf{(III)}} + \underbrace{\sum_{\substack{T\in\mathcal{F}\\ \mathcal{I}(T)\subset J}}\left(\sum_{\phi\in\tau^{-1}(T)}|\langle f,\phi\chi_{(J_\mu)^c}\rangle|^2\right)}_{\textbf{(IV)}}$$

As we did in lemma 3.5, by (108) and (92):

$$\textbf{(III)} \leq C(1+\|N_{\mathcal{F}}\|_{\infty}A^{-\mu})\|f\chi_{J_\mu}\|_2^2 \leq C|J|\|N_{\mathcal{F}}\|_{\infty}^{\frac{1}{\mu}}(1+\|N_{\mathcal{F}}\|_{\infty}A^{-\mu})\inf_{x\in J}(M_2(f)(x))^2$$

Following the same steps of the proof of lemma 3.5, we have:

$$\begin{aligned}|\langle f,\phi\chi_{(J_\mu)^c}\rangle|^2 &\leq C|I(\phi)|^{2\mu+1}|J_\mu|^{-2\mu}(\inf_{y\in I(\phi)}M(f)(y))^2\\ &= C|I(\phi)|^{2\mu+1}|J|^{-2\mu}\|N_{\mathcal{F}}\|_{\infty}^{-2}(\inf_{y\in I(\phi)}M(f)(y))^2\\ &\leq CA(\phi)^{-2}|I(\phi)|\|N_{\mathcal{F}}\|_{\infty}^{-2}(\inf_{y\in I(\phi)}M(f)(y))^2\end{aligned}$$

where we used that $A(\phi)|I(\phi)| \leq |J|$ in the last inequality. Also, using the growth properties we obtained for $A(\phi)$:

$$
\begin{aligned}
\textbf{(IV)} &\leq \sum_{\substack{T\in\mathcal{F}\\ \mathcal{I}(T)\subset J}} \left(\sum_{\phi\in\tau^{-1}(T)} CA(\phi)^{-2}|I(\phi)|\|N_{\mathcal{F}}\|_\infty^{-2}(\inf_{y\in I(\phi)} M(f)(y))^2 \right) \\
&\leq C \sum_{\substack{T\in\mathcal{F}\\ \mathcal{I}(T)\subset J}} \int_J A^{-2}\chi_{\mathcal{I}(\mathcal{F})}(y)\|N_{\mathcal{F}}\|_\infty^{-2}(M(f)(y))^2 dy \\
&= CA^{-2}\|N_{\mathcal{F}}\|_\infty^{-2} \int_J \sum_{\substack{T\in\mathcal{F}\\ \mathcal{I}(T)\subset J}} \chi_{\mathcal{I}(\mathcal{F})}(y)(M(f)(y))^2 dy \\
&\leq CA^{-2}\|N_{\mathcal{F}}\|_\infty^{-1} \int_J (M(f)(y))^2 dy \\
&\leq CA^{-2}|J|\|N_{\mathcal{F}}\|_\infty^{-1} \left(\inf_{x\in J} M_2(M(f))(x) \right)^2 \\
&\leq C|J| \left(\inf_{x\in J} M_2(M(f))(x) \right)^2
\end{aligned}
$$

We conclude (109) by adding this to the estimate obtained for **(III)**. □

We finish this section by stating a particular case of the lemma just proved:

Corollary 3.1. *Let $\Phi \subset \mathcal{S}_u(\mathbb{R})$ be a nonempty finite set. Let $I, \omega : \Phi \mapsto \mathcal{J}$ be maps such that $\rho : \phi \mapsto I(\phi) \times \omega(\phi)$ is a phase plane representation with constants C_m and such that the following properties are satisfied for all $\phi, \phi' \in \Phi$:*

$$I(\Phi) \text{ and } \omega(\Phi) \text{ are grids.} \tag{115}$$

$$\omega(\phi) \cap \omega(\phi') \neq \emptyset \Rightarrow \omega(\phi) = \omega(\phi') \tag{116}$$

$$\phi \neq \phi' \Rightarrow I(\phi) \neq I(\phi') \tag{117}$$

Let J be an interval such that $I(\phi) \subset J$ for all $\phi \in \Phi$. Then for all locally square integrable tempered distributions f,

$$\sum_{\phi\in\Phi} |\langle f, \phi\rangle|^2 \leq C|J| \left(\inf_{x\in J} M_2(M(f))(x) \right)^2 \tag{118}$$

Proof. Take $A = 2$, $\mathcal{F} = \{T\}$, $\tau : \Phi \mapsto \mathcal{F}$ such that $\tau(\phi) = T$ for every $\phi \in \Phi$ and $\mathcal{I}(T) = 2J$. The conditions of lemma 3.7 are obviously satisfied, so the conclusion follows. □

4 CARLESON'S THEOREM

Let $f \in L^2(\mathbb{R})$. We define the *Carleson operator* by

$$\mathcal{C}(f)(x) = \sup_{N>0} |(\widehat{f}\chi_{[-N,N]})^\vee| = \sup_{N>0} \left| \int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \tag{119}$$

Note that $(\widehat{f}\chi_{[-N,N]})^\vee$ is well-defined for $f \in L^2(\mathbb{R})$ (because Fourier transform is an unitary operator on this space), so $\mathcal{C}$ also is. We have the following result concerning $\mathcal{C}$:

Theorem 4.1. *There exists $C > 0$ such that the following estimate is valid for all $f \in L^2(\mathbb{R})$*

$$\|\mathcal{C}(f)\|_{L^{2,\infty}} \leq C\|f\|_{L^2} \tag{120}$$

Define for each $f \in L^2(\mathbb{R})$:

$$f_N(x) := \int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$

It follows that for all $f \in L^2(\mathbb{R})$ we have

$$\lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} \int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi = f(x) \tag{121}$$

for almost all $x \in \mathbb{R}$.

Note that (121) holds for $f \in \mathcal{S}(\mathbb{R})$ by the dominated convergence theorem and Fourier inversion.

Claim 4.1. (120) *implies* (121)

Proof. Define the *oscillation* of f by:

$$O_f(y) := \limsup_{\epsilon\to 0} \limsup_{\theta\to 0} |f_\epsilon(y) - f_\theta(y)|$$

Take $g \in \mathcal{S}(\mathbb{R})$ such that $\|f - g\|_2 < \eta$. Since $g_N(y) \to g$ a.e., it follows that $O_g = 0$ a.e. This way,

$$O_f(y) \leq O_g(y) + O_{f-g}(y) \leq O_{f-g}(y) \quad \text{a.e.}$$

For any $\delta > 0$ we have:

$$\begin{aligned}
|\{y \in \mathbb{R}; O_f(y) > \delta\}| &\leq |\{y \in \mathbb{R}; O_{f-g}(y) > \delta\}| \\
&\leq \left|\left\{y \in \mathbb{R}; \mathcal{C}(f-g)(y) > \frac{\delta}{2}\right\}\right| \\
&\leq \left(\frac{2B\|f-g\|_2}{\delta}\right)^2 \\
&\leq \left(\frac{2B\eta}{\delta}\right)^2
\end{aligned}$$

Taking $\eta \to 0$, we conclude that $O_f = 0$ a.e., so $f_\epsilon(y)$ is a Cauchy sequence for almost all y when $\epsilon \to 0$, so $f_\epsilon(y)$ converges when $\epsilon \to 0$. We know that $f_\epsilon \to f$ when $\epsilon \to 0$ in L^2 (Plancherel), then there is a subsequence ϵ_k such that $f_{\epsilon_k} \to f$ pointwise a.e. By uniqueness of limits, $f_\epsilon(y) \to f(y)$ a.e.

□

Claim 4.2. *It suffices to prove* (120) *for functions belonging to* $\mathcal{S}(\mathbb{R})$.

Proof. In fact, given $f \in L^2$, if the estimate holds for this class, take $f_j \in \mathcal{S}(\mathbb{R})$ with $f_j \to f$ em L^2. By Cauchy-Schwarz, $(\widehat{f_j}\chi_{[-N,N]})^\vee \to (\widehat{f}\chi_{[-N,N]})^\vee$ pointwise. This way,

$$\begin{aligned}
\|\mathcal{C}(f)\|_{L^{2,\infty}} &= \left\|\sup_{N>0} |(\widehat{f}\chi_{[-N,N]})^\vee|\right\|_{L^{2,\infty}} \\
&= \left\|\sup_{N>0} |\lim_{j\to\infty}(\widehat{f_j}\chi_{[-N,N]})^\vee|\right\|_{L^{2,\infty}} \\
&\leq \left\|\sup_{N>0} \liminf_{j\to\infty} |(\widehat{f_j}\chi_{[-N,N]})^\vee|\right\|_{L^{2,\infty}} \\
&\leq \left\|\liminf_{j\to\infty} \sup_{N>0} |(\widehat{f_j}\chi_{[-N,N]})^\vee|\right\|_{L^{2,\infty}} \\
&\leq \liminf_{j\to\infty} \left\|\sup_{N>0} |(\widehat{f_j}\chi_{[-N,N]})^\vee|\right\|_{L^{2,\infty}} \\
&\leq \liminf_{j\to\infty} C\|f_j\|_{L^2} \\
&= C\|f\|_{L^2}
\end{aligned}$$

(observe that we used Fatou's lemma 5.3 for L^p-weak spaces in the third inequality).

□

Note that

$$\int_{|\xi|\leq N} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi = \int_{-\infty}^{N} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi - \int_{-\infty}^{-N} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi,$$

We claim that it suffices to obtain (2,2)-weak estimates for the *one-sided maximal operators*

$$\mathcal{C}_1(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right|$$

$$\mathcal{C}_2(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{-N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right|$$

Indeed, note that

$$\mathcal{C}_2(f)(x) \leq |f(x)| + \mathcal{C}_1(\tilde{f})(-x)$$

where $\widetilde{f}(x) = f(-x)$ is the reflexion of f. This way, it suffices to study $\mathcal{C}_1$ over $\mathcal{S}(\mathbb{R})$. Note also that the operators $\mathcal{C}_1$ and $\mathcal{C}_2$ are well-defined on $L^2(\mathbb{R})$ (for the same reason $\mathcal{C}$ are). The proof of theorem 4.1 will be completed after several reductions.

4.1 Preliminaries

Denote rectangles of area 1 (with sides parallel to the coordinate axes) in the (x, ξ) plane by s, t, u, etc. We will refer to x as *time* and to ξ as *frequency.* A rectangle $s = I_s \times \omega_s$ of this kind is called a *tile.*

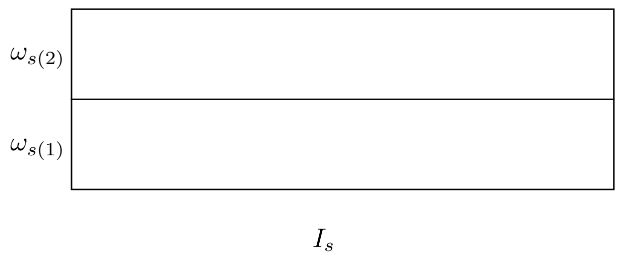
Given a tile s, define:

$$s(1) := I_s \times (\omega_s \cap (-\infty, c(\omega_s))),$$

$$s(2) := I_s \times (\omega_s \cap [c(\omega_s), +\infty)).$$

These sets are called *semi-tiles.* The projections of these sets on the frequency axis will be denoted by $\omega_{s(1)}$ and $\omega_{s(2)}$, respectively.

Figure 2: Tile



A *dyadic interval* has the form $[m2^k, (m+1)2^k)$, where k and m are integers. Denote by $\mathbf{D}$ the set of all *dyadic tiles*, i.e., rectangles $I \times \omega$ with I, ω dyadic intervals and $|I||\omega| = 1$.

Proposition 4.1. *Let I and J be dyadic intervals. If $I \cap J \neq \emptyset$, then either $I \subset J$ or $J \subset I$.*

Proof. Let $I = [m2^k, (m+1)2^k)$ and $J = [m'2^{k'}, (m'+1)2^{k'})$. If we had

$$m2^k < m'2^{k'} < (m+1)2^k < (m'+1)2^{k'},$$

then

$$m < 2^{k'-k}m' < m+1 \Rightarrow k' < k.$$

$$m' < 2^{k-k'}(m+1) < m'+1 \Rightarrow k < k',$$

contradiction. □

Lemma 4.1. *There is a function $\phi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\phi}$ is real, non-negative and is supported on $[-1/10, 1/10]$.*

Proof. We know that there exists $g \in C_c^\infty$ real, non-negative and supported on $[-1/10, 1/10]$. As $C_c^\infty \subset L^2(\mathbb{R})$ and the Fourier transform is an isometry on this space, there exists $\check{g}$. Take $\phi = \check{g}$. □

Definition 4.1. *For each tile s define ϕ_s by*

$$\phi_s(x) = |I_s|^{-\frac{1}{2}}\phi\left(\frac{x - c(I_s)}{|I_s|}\right)e^{2\pi i c(\omega_{s(1)})x}. \tag{122}$$

where ϕ is given by lemma 4.1.

Proposition 4.2. *Using the notation of definition 2.1, we have:*
(a) ϕ_s is given by:

$$\phi_s = M^{c(\omega_{s(1)})}\tau^{c(I_s)}D^{|I_s|}(\phi).$$

(b) The Fourier transform of ϕ_s is given by:

$$\widehat{\phi_s}(\xi) = |\omega_s|^{-\frac{1}{2}}\widehat{\phi}\left(\frac{\xi - c(\omega_{s(1)})}{|\omega_s|}\right)e^{2\pi i(c(\omega_{s(1)})-\xi)c(I_s)}.$$

(c) supp$(\widehat{\phi_s}) \subset \frac{1}{5}\omega_{s(1)}$.
(d) The functions ϕ_s have the same L^2 norm, $\forall\ f \in L^2(\mathbb{R})$.

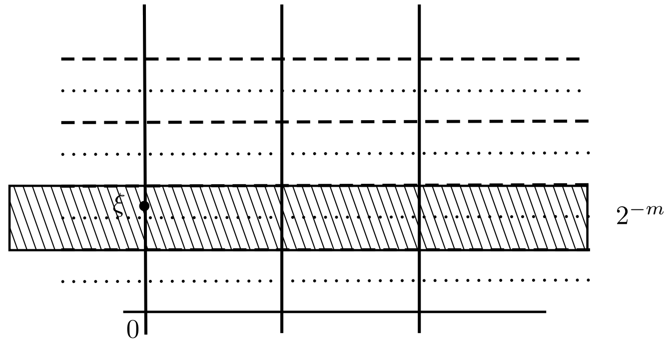
Proof. All items are proved by straightforward computations. □

Definition 4.2. *For all integer m, denote by $\mathbf{D}_m$ the set of all tiles $s \in \mathbf{D}$ such that $|I_s| = 2^m$ (dyadic tiles of order m). Given nonzero $\xi \in \mathbb{R}$, we define the operator A_ξ^m by:*

$$A_\xi^m(f) = \sum_{s \in \mathbf{D}_m} \chi_{\omega_{s(2)}}(\xi)\,\langle f|\phi_s\rangle\,\phi_s \tag{123}$$

for all $f \in \mathcal{S}(\mathbb{R})$.

Figure 3: Relevant tiles



Proposition 4.3. *The series in* (123) *converges absolutely and in* $L^2(\mathbb{R})$ *if* $f \in \mathcal{S}(\mathbb{R})$*.*

Proof. The domains of ϕ_s and ϕ'_s are disjoint if $s \neq s'$ (look at the picture above). Also, there is a constant C such that $\|\phi\|_\infty \leq C$. This way,

$$\begin{aligned}\sum_{s\in\mathbf{D}_m} |\chi_{\omega_{s(2)}}(\xi)\,\langle f|\phi_s\rangle\,\phi_s| &\leq C2^{-\frac{m}{2}} \sum_{s\in\mathbf{D}_m} |\chi_{\omega_{s(2)}}(\xi)\,\langle f|\phi_s\rangle\,| \\ &\leq C2^{-\frac{m}{2}} \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(\xi) \int_{\mathbb{R}} |f||\phi_s| \\ &\leq C^2 2^{-m} \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(\xi) \int_{I_s} |f|\chi_{I_s} \\ &\leq C^2 2^{-m} \int_{\mathbb{R}} |f| \\ &< \infty\end{aligned}$$

Then the series in (123) converges absolutely. The L^2 convergence is an immediate consequence of item (b) of proposition 5.1. □

Thus we can define the operator A_ξ:

$$A_\xi(f) = \sum_{m\in\mathbb{Z}} A_\xi^m(f) = \sum_{s\in\mathbf{D}} \chi_{\omega_{s(2)}}(\xi)\,\langle f|\phi_s\rangle\,\phi_s, \tag{124}$$

for $f \in \mathcal{S}(\mathbb{R})$ and $\xi \in \mathbb{R}$.

Let us discuss our plan before we proceed. We want to move the problem to a discretized version of the Carleson operator. In other words, define formally the operator

$$\Pi_\xi(h) = \lim_{\substack{K\to\infty \\ L\to\infty}} \frac{1}{4KL} \int_{-L}^{L} \int_{-K}^{K} \int_0^1 M^{-\eta}\tau^{-y} D^{2^{-\lambda}} A_{\frac{\xi+\eta}{2^\lambda}} D^{2^\lambda} \tau^y M^\eta(h) d\lambda dy d\eta \tag{125}$$

We will prove that Π_ξ is well defined on $\mathcal{S}$. We will also prove that proposition 4.7

identifies $M^{-\xi}\Pi_\xi M^\xi$ as

$$M^{-\xi}\Pi_\xi M^\xi f(x) = c\int_{-\infty}^{0} \hat{f}(y)e^{2\pi ixy}dy$$

Indeed,

$$M^{-\xi}\Pi_\xi M^\xi(f)(x) = c\int_{-\infty}^{0} \widehat{f}(y)e^{2\pi ixy}dy \Rightarrow \Pi_\xi M^\xi(f)(x)e^{-2\pi ix\xi} = c\int_{-\infty}^{0} \widehat{f}(y)e^{2\pi ixy}dy$$

Replacing f by $M^{-\xi}f$ and changing $y+\xi \mapsto z$,

$$\begin{aligned}\Pi_\xi f(x) &= c\int_{-\infty}^{0} \widehat{M^{-\xi}f}(y)e^{2\pi ix(y+\xi)}dy \\ &= c\int_{-\infty}^{0} \widehat{f}(y+\xi)e^{2\pi ix(y+\xi)}dy \\ &= c\int_{-\infty}^{\xi} \widehat{f}(z)e^{2\pi ixz}dz\end{aligned}$$

This way $\mathcal{C}_1$ can be written as

$$\mathcal{C}_1(f) = \frac{1}{c}\sup_\xi |\Pi_\xi(f)|$$

so it suffices to study Π_ξ. We will devote the rest of this section and the next one to prove the details of this reduction.

Lemma 4.2. *About A_ξ^m and A_ξ:*

(a) For any ξ, A_ξ^m (defined initially on $\mathcal{S}$) admit bounded extensions to $L^2(\mathbb{R})$ uniformly in m and ξ;

(b) A_ξ is bounded on $L^2(\mathbb{R})$ uniformly in ξ.

(c) For $\xi > 0$, $\forall\, f \in L^1(\mathbb{R})$, the series (124) *converges absolutely pointwise and is bounded by a constant multiple of $\xi\ \|f\|_{L^1}$*

Proof. Given f and g in $\mathcal{S}$, we have:

$$\langle A_\xi^m(f)|A_\xi^{m'}(g)\rangle = \sum_{s\in\mathbf{D}_m}\sum_{s'\in\mathbf{D}_{m'}} \langle f|\phi_s\rangle\overline{\langle g|\phi_{s'}\rangle}\langle\phi_s|\phi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi). \tag{126}$$

Suppose that $\langle\phi_s|\phi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)$ is different from zero. Then $\langle\phi_s|\phi_{s'}\rangle \neq 0$ and by Parseval:

$$\begin{aligned}\langle\phi_s|\phi_{s'}\rangle \neq 0 \Rightarrow &\int_{\mathbb{R}} \phi_s(x)\overline{\phi_{s'}(x)}dx \neq 0 \Rightarrow \\ &\int_{\mathbb{R}} \widehat{\phi_s}(\xi)\overline{\widehat{\phi_{s'}}(\xi)}d\xi \neq 0\end{aligned}$$

with $\text{supp}(\widehat{\phi_s}) \subset \frac{1}{5}\omega_{s(1)}$ and $\text{supp}(\widehat{\phi_{s'}}) \subset \frac{1}{5}\omega_{s'(1)}$ (cf. proposition 4.2), the integral above would be 0 if $\frac{1}{5}\omega_{s(1)} \cap \frac{1}{5}\omega_{s'(1)} = \emptyset$, so we must have $\frac{1}{5}\omega_{s(1)} \cap \frac{1}{5}\omega_{s'(1)} \neq \emptyset$, which implies $\omega_{s(1)} \cap \omega_{s'(1)} \neq \emptyset$.

On the other hand, the function $\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)$ is nonzero too, so $\omega_{s(2)} \cap \omega_{s'(2)} \neq \emptyset$. It follows that $\omega_s \cap \omega_{s'} \neq \emptyset$, then either $\omega_s \subset \omega_{s'}$ or $\omega_{s'} \subset \omega_s$ (by proposition 4.1). If $\omega_s \subsetneq \omega_{s'}$ we would have either $\omega_s \subset \omega_{s'(1)}$ or $\omega_s \subset \omega_{s'(2)}$ and then either $\omega_{s(1)} \cap \omega_{s'(1)}$ or $\omega_{s(2)} \cap \omega_{s'(2)}$ would be empty, which we already proved that does not occur. It follows that if $\langle \phi_s|\phi_{s'}\rangle \chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi) \neq 0$ then $\omega_s = \omega_{s'}$, which does not happen if $m \neq m'$, and therefore the expression in (126) is zero.

Let us first prove boundedness for A_ξ^m. We have:

$$\begin{aligned}
\|A_\xi^m(f)\|_{L^2}^2 &= \sum_{s\in\mathbf{D}_m}\sum_{s'\in\mathbf{D}_m} \langle f|\phi_s\rangle\overline{\langle f|\phi_{s'}\rangle}\langle\phi_s|\phi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)\\
&= \sum_{s\in\mathbf{D}_m}\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \langle f|\phi_s\rangle\overline{\langle f|\phi_{s'}\rangle}\langle\phi_s|\phi_{s'}\rangle\chi_{\omega_{s(2)}}(\xi)\chi_{\omega_{s'(2)}}(\xi)\\
&\leq \left[\sum_{\substack{s,s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \left[|\langle f|\phi_s\rangle|\chi_{\omega_{s(2)}}(\xi)|\langle\phi_s|\phi_{s'}\rangle|^{\frac{1}{2}}\right]^2\right]^{\frac{1}{2}} \left[\sum_{\substack{s,s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} \left[\overline{|\langle f|\phi_{s'}\rangle}|\chi_{\omega_{s(2)}}(\xi)|\langle\phi_s|\phi_{s'}\rangle|^{\frac{1}{2}}\right]^2\right]^{\frac{1}{2}}\\
&= \sum_{s\in\mathbf{D}_m}\sum_{\substack{s'\in\mathbf{D}_m\\ \omega_{s'}=\omega_s}} |\langle f|\phi_s\rangle|^2\chi_{\omega_{s(2)}}(\xi)|\langle\phi_s|\phi_{s'}\rangle|\\
&\leq C_1\sum_{s\in\mathbf{D}_m} |\langle f|\phi_s\rangle|^2\chi_{\omega_{s(2)}}(\xi),
\end{aligned} \tag{127}$$

where we used the previous fact proved about s and s', Cauchy-Schwarz inequality and proposition 5.5. To estimate (127) we use

$$\begin{aligned}
|\langle f|\phi_s\rangle| &\leq C_2\int_{\mathbb{R}} |f(y)||I_s|^{-\frac{1}{2}}\left(1+\frac{|y-c(I_s)|}{|I_s|}\right)^{-10}dy\\
&\leq C_3|I_s|^{\frac{1}{2}}\int_{\mathbb{R}}|f(y)|\left(1+\frac{|y-z|}{|I_s|}\right)^{-10}\frac{dy}{|I_s|}\\
&\leq C_4|I_s|^{\frac{1}{2}}M(f)(z),
\end{aligned}$$

$\forall\ z\in I_s$, by lemma 5.1. From this last estimate,

$$|\langle f|\phi_s\rangle|^2 \leq (C_4)^2|I_s|\inf_{z\in I_s} M(f)^2(z) \leq (C_3)^2\int_{I_s} M(f)^2(x)dx. \tag{128}$$

Thus summing (128) over $\mathbf{D}_m$ with $\xi \in \omega_{s(2)}$ gives us

$$\begin{aligned}\sum_{s\in\mathbf{D}_m} |\langle f|\phi_s\rangle|^2\chi_{\omega_{s(2)}}(\xi) &\leq (C_3)^2 \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(\xi)\int_{I_s} M(f)^2(x)dx \\ &\leq (C_3)^2 \int_{\mathbb{R}} M(f)^2(x)dx\end{aligned}$$

and this is finite because Hardy-Littlewood's maximal operator is bounded on $L^2(\mathbb{R})$.

Let us now prove boundedness for $A_\xi = \sum_{m\in\mathbb{Z}} A_\xi^m$. For each $m \in \mathbb{Z}$ fixed, the dyadic tiles of the sum defining A_ξ^m have the form

$$s = [k2^m, (k+1)2^m) \times [l2^{-m}, (l+1)2^{-m}),$$

where $\left(l+\frac{1}{2}\right)2^{-m} \leq \xi < (l+1)2^{-m}$. This way, $l = \lfloor 2^m\xi \rfloor$. Define f_m such that

$$\widehat{f_m} = \widehat{f}\chi_{[2^{-m}\lfloor 2^m\xi\rfloor, 2^{-m}(\lfloor 2^m\xi\rfloor+\frac{1}{2}))}$$

Before we proceed, let us verify that all intervals $[2^{-m}l, 2^{-m}(l+\frac{1}{2}))$ on the definition of $\widehat{f_m}$ above are pairwise disjoint. If $m \neq n$, suppose $[2^{-m}l, 2^{-m}(l+1)) \cap [2^{-n}l, 2^{-n}(l+1)) \neq \emptyset$. As we said above, both intervals contain ξ in their upper half, so one of them is contained in the upper half of the other. Suppose that $[2^{-m}l, 2^{-m}(l+1)) \subset [2^{-n}(l+\frac{1}{2}), 2^{-n}(l+1))$. Then it is clear that $[2^{-m}l, 2^{-m}(l+\frac{1}{2})) \cap [2^{-n}l, 2^{-n}(l+\frac{1}{2})) = \emptyset$. If $[2^{-m}l, 2^{-m}(l+1)) \cap [2^{-n}l, 2^{-n}(l+1)) = \emptyset$, same conclusion follows trivially.

By Parseval's identity and by the fact that $\widehat{\phi_s}$ is supported on the lower half of the dyadic interval ω_s, we have:

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \|A_\xi^m(f)\|_{L^2}^2 &= \sum_{m\in\mathbb{Z}} \|A_\xi^m(f_m)\|_{L^2}^2 \\ &\leq C_5 \sum_{m\in\mathbb{Z}} \|f_m\|_{L^2}^2 \\ &= C_5 \sum_{m\in\mathbb{Z}} \|\widehat{f_m}\|_{L^2}^2 \\ &\leq C_5 \|f\|_{L^2}^2\end{aligned}$$

since the supports of $\widehat{f_m}$ are disjoint for different values of $m \in \mathbb{Z}$. Remember that $\langle A_\xi^m(f)|A_\xi^{m'}(f)\rangle = 0$ if $m \neq m'$. Given $\epsilon > 0$, there is N_0 such that for $M \geq N \geq N_0$ we have:

$$\|\sum_{|N|<m<|M|} A_\xi^m(g)\|_{L^2}^2 = \sum_{|N|<m<|M|} \|A_\xi^m\|_{L^2}^2 < \epsilon^2$$

Thus the series $\sum_{m\in\mathbb{Z}} A_\xi^m(g)$ is Cauchy and converges to some element of $L^2(\mathbb{R})$ which will be denoted by $A_m(g)$. Combining what we did above, we obtain that A_ξ is bounded from $L^2(\mathbb{R})$ to itself.

Let us prove the last assertion now. For $x \in \mathbb{R}$, $\xi > 0$, let $m_0 \in \mathbb{Z}$ be such that $2^{-m_0-1} \leq \xi < 2^{-m_0}$. This implies, in particular, that $\xi \in \omega_{s(2)}$, where $s = [k2^{m_0}, (k+1)2^{m_0}) \times [0, 2^{-m_0})$, $k \in \mathbb{Z}$. It also says that $\xi \notin \omega_{s'(2)}$ if $|\omega_{s'}| > 2^{-m_0}$, so we may restrict the sum appearing in definition (124) to tiles in $\mathbf{D}_m$ with $m \geq m_0$. Using the fact that $|\langle f|\phi_s\rangle| \leq \|f\|_{L^1}\|\phi_s\|_{L^\infty}$, we may estimate this sum as

$$|A_\xi(f)| \leq C\|f\|_{L^1} \sum_{m \geq m_0} \sum_{k \in \mathbb{Z}} 2^{-\frac{m}{2}} \frac{2^{-\frac{m}{2}}}{(1 + 2^{-m}|x - 2^m(k+\frac{1}{2})|^2)}$$

where C is a constant that dominates $\|\phi_s\|_{L^\infty}$ for all s where $|\omega_s| \leq 2^{-m_0}$ (i.e., $|\omega_s| \geq 2^{-m_0}$). Summing first over k and then over $m \geq m_0$, we obtain the desired result. □

4.2 Discretization of the Carleson operator

Let $h \in \mathcal{S}(\mathbb{R})$, $\xi \in \mathbb{R}\backslash\{0\}$, and for $m \in \mathbb{Z}$, y, $\eta \in \mathbb{R}$ and $\lambda \in [0,1]$ we introduce the operators

$$B^m_{\xi,y,\eta,\lambda}(h) = \sum_{s \in \mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\phi_s).$$

Proposition 4.4. *For all $f \in \mathcal{S}(\mathbb{R})$, x, $\xi \in \mathbb{R}$ and $\lambda \in [0,1]$, the function $(y,\eta) \mapsto B^m_{\xi,y,\eta,\lambda}(f)(x)$ is periodic in y with period $2^{m-\lambda}$ and in η with period $2^{-m+\lambda}$.*

Proof. This is a very straightforward verification using invariance properties of translations and modulations. □

Proposition 4.5. *Let $h \in \mathcal{S}(\mathbb{R})$, ξ, y, $\eta \in \mathbb{R}$, $s \in \mathbf{D}_m$ and $\lambda \in [0,1]$. Suppose that $2^{-\lambda}(\xi+\eta) \in \omega_{s(2)}$.*

(a) Assume that $m \leq 0$ and that $2^{-m} \geq 40|\xi|$. There is C (independent from y, η and λ) such that

$$|\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle| = |\langle h|M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\phi_s)\rangle| \leq C2^{\frac{m}{2}}\|\widehat{h}\|_{L^1((-\infty,-\frac{1}{40\cdot 2^m})\cup(\frac{1}{40\cdot 2^m},\infty))}$$

(b) When $|m|$ is large compared to ξ, we have

$$\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))|\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle| \leq C_h \min(1, 2^m),$$

where C_h may depend on h, but not on y, η or λ.

Proof of (a). We claim that under these conditions we have

$$\eta \geq 2^\lambda c(\omega_{s(1)}) + \frac{9}{40}2^{-m}$$

Indeed,

$$\frac{\xi+\eta}{2^\lambda} \in \omega_{s(2)} \Rightarrow \frac{\xi+\eta}{2^\lambda} \geq c(\omega_{s(1)}) + \frac{|\omega_s|}{4} \Rightarrow \xi+\eta \geq 2^\lambda c(\omega_{s(1)}) + 2^\lambda \frac{2^{-m}}{4}$$
$$\Rightarrow \eta \geq 2^\lambda c(\omega_{s(1)}) + 2^\lambda \frac{2^{-m}}{4} - \frac{2^{-m}}{40}$$
$$\Rightarrow \eta \geq 2^\lambda c(\omega_{s(1)}) + \frac{9}{40}2^{-m}$$

This way,

$$\begin{aligned} |\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle| &= |\langle h|M^{-\eta}\tau^{-y}D^{2^{-\lambda}}\phi_s\rangle| = |\langle \widehat{h}|\mathcal{F}(M^{-\eta}\tau^{-y}D^{2^{-\lambda}}\phi_s)\rangle| \\ &= |\langle \widehat{h}|\tau^{-\eta}M^y D^{2^\lambda}\widehat{\phi_s}\rangle| \\ &\leq \int_{\mathbb{R}} |\widehat{h}(z)| \left|\widehat{\phi_s}\left(\frac{z+\eta}{2^\lambda}\right)\right| 2^{-\frac{\lambda}{2}}dz \\ &\leq 2^{\frac{m}{2}} \int_{\mathbb{R}} |\widehat{h}(z)| \left|\widehat{\phi}\left(\frac{\left(\frac{z+\eta}{2^\lambda}\right) - c(\omega_{s(1)})}{|\omega_s|}\right)\right| dz \end{aligned} \tag{129}$$

where we used $\mathcal{F}$ to denote the Fourier transform. By the construction of ϕ, the integrand above is nonzero only if

$$-\frac{1}{10} \leq \frac{\left(\frac{z+\eta}{2^\lambda}\right) - c(\omega_{s(1)})}{|\omega_s|} \leq \frac{1}{10}$$

Observe that the argument of $\widehat{\phi}$ above is positive since $2^{-\lambda}(z+\eta) \in \omega_{s(2)}$, so we have the first inequality. For the second one,

$$\frac{\left(\frac{z+\eta}{2^\lambda}\right) - c(\omega_{s(1)})}{|\omega_s|} \leq \frac{1}{10} \Leftrightarrow z \leq 2^\lambda c(\omega_{s(1)}) + \frac{2^\lambda}{10}|\omega_s| - \eta$$
$$\Leftrightarrow z \leq 2^\lambda c(\omega_{s(1)}) + \frac{2^\lambda}{10}|\omega_s| - 2^\lambda c(\omega_{s(1)}) - \frac{9}{40}2^{-m}$$
$$\Leftrightarrow z \leq \frac{2}{10}|\omega_s| - \frac{9}{40}2^{-m}$$
$$\Leftrightarrow z \leq -\frac{1}{40.2^m}$$

By our previous claim. Back to (129),

$$|\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle| \leq C2^{\frac{m}{2}} \int_{(-\infty,-\frac{1}{40.2^m})} |\widehat{h}(z)|dz \leq C2^{\frac{m}{2}} \|\widehat{h}\|_{L^1((-\infty,-\frac{1}{40\cdot 2^m})\cup(\frac{1}{40\cdot 2^m},\infty))}$$

□

Proof of (b). By Cauchy-Schwarz,

$$|\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle| \leq C\|h\|_{L^2} = C_h$$

where C_h is a constant that depends on h. By item (a),

$$\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))|\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle| \leq C_h 2^{\frac{m}{2}}$$

Then we have the desired result. □

Using proposition 4.5 for $|m|$ large enough,

$$\begin{aligned}
&\left|\sum_{s\in\mathbf{D}_m}\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\langle D^{2^\lambda}\tau^y M^\eta(h)|\phi_s\rangle M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\phi_s)(x)\right| \\
&\leq C_h\min(2^m,1)\sum_{s\in\mathbf{D}_m}\chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))2^{-\frac{m}{2}}\left|\phi\left(\frac{x+y-c(I_s)}{2^{m-\lambda}}\right)\right| \\
&\leq C_h\min(2^{\frac{m}{2}},2^{-\frac{m}{2}})\sum_{k\in\mathbb{Z}}\left|\phi\left(\frac{x+y-(k+\frac{1}{2})2^{m-\lambda}}{2^{m-\lambda}}\right)\right| \\
&\leq C_h\min(2^{\frac{m}{2}},2^{-\frac{m}{2}}),
\end{aligned}$$

since this last sum converges and is limited in x, y, η e λ. It follows that for $h\in\mathcal{S}(\mathbb{R})$ we have

$$\sup_{x\in\mathbb{R}}\sup_{y\in\mathbb{R}}\sup_{\eta\in\mathbb{R}}\sup_{0\leq\lambda\leq 1}|B^m_{\xi,y,\eta,\lambda}(h)(x)| \leq C_h\min(2^{\frac{m}{2}},2^{-\frac{m}{2}}), \tag{130}$$

Proposition 4.6. *Let g be integrable in the respective euclidean space below*
(a) If g is periodic on $\mathbb{R}$ with period κ, then

$$\lim_{K\to\infty}\frac{1}{2K}\int_{-K}^{K}g(t)dt = \frac{1}{\kappa}\int_0^\kappa g(t)dt.$$

(b) If g is periodic on $\mathbb{R}^n$ with period $(\kappa_1,\ldots,\kappa_n)$, then

$$\lim_{K_1,\ldots,K_n\to\infty}\frac{2^{-n}}{K_1\ldots K_n}\int_{-K_1}^{K_1}\cdots\int_{-K_n}^{K_n}g(x)dx_1\ldots dx_n = \frac{1}{\kappa_1\ldots\kappa_n}\int_0^{\kappa_1}\cdots\int_0^{\kappa_n}g(x)dx_1\ldots dx_n.$$

Proof of (a). Write $K=n\tau+r$, where $n\in\mathbb{N}$ and $r\in[0,\tau)$. This way,

$$\begin{aligned}
\frac{1}{2K}\int_{-K}^{K}g(t)dt &= \frac{1}{2K}\int_{-n\tau}^{n\tau}g(t)dt + \frac{1}{2K}\int_{[-K,K]\setminus[-n\tau,n\tau]}g(t)dt \\
&= \frac{2n}{2K}\int_0^\tau g(t)dt + \frac{1}{2K}\int_{[-K,K]\setminus[-n\tau,n\tau]}g(t)dt
\end{aligned}$$

The second integral above is bounded by $2\|g\|_{L^1[0,\tau]}$, then it goes to zero when $K\to\infty$. As for the first integral, the quotient $\frac{n}{K}=\frac{n}{n\tau+r}$ goes to $\frac{1}{\tau}$ when n (or K) goes to infinity. Thus we have the desired result. □

Proof of (b). Apply the previous idea multiple times. □

Using proposition 4.6 and the periodicity of $B^m_{\xi,y,\eta,\lambda}(h)$ (cf. proposition 4.4), we conclude that the means

$$\frac{1}{2KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1} B^m_{\xi,y,\eta,\lambda}(h)d\lambda dy d\eta$$

converge pointwise to a function $\Pi^m_\xi(h)$ when K, $L\to\infty$. Note also that

$$M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{\xi+\eta}{2^\lambda}}D^{2^\lambda}\tau^y M^\eta(h) = \sum_{m\in\mathbb{Z}} B^m_{\xi,y,\eta,\lambda}(h)$$

By the Weierstrass M-test, estimate (130) implies uniform convergence for the series $\sum_{m\in\mathbb{Z}} B^m_{\xi,y,\eta,\lambda}(h)$ and then

$$\begin{aligned}
&\lim_{\substack{K\to\infty\\ L\to\infty}}\frac{1}{2KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1} M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{\xi+\eta}{2^\lambda}}D^{2^\lambda}\tau^y M^\eta(h)d\lambda dy d\eta \\
&\quad= \lim_{\substack{K\to\infty\\ L\to\infty}}\frac{1}{2KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1}\sum_{m\in\mathbb{Z}} B^m_{\xi,y,\eta,\lambda}(h)d\lambda dy d\eta \\
&\quad= \sum_{m\in\mathbb{Z}}\lim_{\substack{K\to\infty\\ L\to\infty}}\frac{1}{2KL}\int_{-L}^{L}\int_{-K}^{K}\int_{0}^{1} B^m_{\xi,y,\eta,\lambda}(h)d\lambda dy d\eta \\
&\quad= \sum_{m\in\mathbb{Z}}\Pi^m_\xi(h).
\end{aligned}\tag{131}$$

where the exchange of the infinite sum with the integral above is justified by the uniform convergence of $\sum_{m\in\mathbb{Z}} B^m_{\xi,y,\eta,\lambda}(h)$. We define, therefore, $\Pi_\xi(h) := \sum_{m\in\mathbb{Z}}\Pi^m_\xi(h)$.

Proposition 4.7. *About Π_ξ, we have:*
(a) It is bounded in L^2 with bound independent of ξ:

$$\sup_\xi \|\Pi_\xi\|_{2\to 2} < +\infty$$

(b) $\forall z\in\mathbb{R}$, it holds

$$\tau^{-z}\Pi_\xi\tau^z(h) = \Pi_\xi(h)$$

(c) $\forall \xi,\theta\in\mathbb{R}$, it holds

$$M^{-\theta}\Pi_{\xi+\theta}M^\theta = \Pi_\xi$$

(d) For all integers k we have

$$A_\xi(h) = D^{2^{-k}}A_{2^{-k}\xi}D^{2^k}(h)$$

(e) $M^{-\xi}\Pi_\xi M^\xi$ *commutes with dilations* D^{2^a}, $a \in \mathbb{R}$*:*

$$D^{2^{-a}} M^{-\xi}\Pi_\xi M^\xi D^{2^a} = M^{-\xi}\Pi_\xi M^\xi$$

(f) $M^{-\xi}\Pi_\xi M^\xi$ *is a positive semi-definite operator, nonzero and, if* $\widehat{h}(x) = 0,\ \forall x < 0,$ *then* $M^{-\xi}\Pi_\xi M^\xi(h) = 0$.

Proof of (a). It follows directly from the definition of Π_ξ, lemma 4.2 and Fatou's lemma. □

Proof of (b). One can easily see that $\tau^{-z}M^{-\eta} = 2^{-2\pi i\eta z}M^{-\eta}\tau^{-z}$. Using this we obtain

$$\begin{aligned}
&\sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\langle D^{2^\lambda}\tau^y M^\eta \tau^z(h)|\phi_s\rangle \tau^{-z}M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\phi_s) \\
&\qquad = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\langle h|\tau^{-z}M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\phi_s)\rangle \tau^{-z}M^{-\eta}\tau^{-y}D^{2^{-\lambda}}(\phi_s) \\
&\qquad = \sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\langle h|M^{-\eta}\tau^{-y-z}D^{2^{-\lambda}}(\phi_s)\rangle M^{-\eta}\tau^{-y-z}D^{2^{-\lambda}}(\phi_s)
\end{aligned}$$

Now, recall that $\tau^{-z}\Pi_\xi^m\tau^z(h)$ is equal to the limit of the averages of the preceding expressions over all $(y,\eta,\lambda) \in [-K,K]\times[0,L]\times[0,1]$. In view of the previous identity, this is equal to the limit of the averages of the expressions

$$\sum_{s\in\mathbf{D}_m} \chi_{\omega_{s(2)}}(2^{-\lambda}(\xi+\eta))\langle D^{2^\lambda}\tau^{y'} M^\eta \tau^z(h)|\phi_s\rangle M^{-\eta}\tau^{-y'}D^{2^{-\lambda}}(\phi_s) \tag{132}$$

over all $(y',\eta,\lambda) \in [-K+z, K+z]\times[0,L]\times[0,1]$. Since (132) is periodic in (y',η), it follows that its average over the set $[-K+z,K+z]\times[0,L]\times[0,1]$ is equal to its average over the set $[-K,K]\times[0,L]\times[0,1]$. Taking limits as $K, L\to\infty$, we obtain the identity $\tau^{-y}\Pi_\xi^m\tau^y(h) = \Pi_\xi^m(h)$. Summing over all $m\in\mathbb{Z}$ we have

$$\tau^{-z}\Pi_\xi\tau^z(h) = \Pi_\xi(h)$$

□

Proof of (c). Using averages over the shifted rectangles $[-K,K]\times[\theta, L+\theta]$ and an argument analogous to the previous item we obtain the desired identity. □

Proof of (d). Observe that

$$\begin{aligned}
D^{2^{-k}}A_{2^{-k}\xi}D^{2^k}(h)(x) &= 2^{\frac{k}{2}}A_{2^{-k}\xi}D^{2^k}(h)(2^k x) \\
&= 2^{\frac{k}{2}}\sum_{s\in\mathbf{D}} \chi_{\omega_{s(2)}}(2^{-k}\xi)\langle D^{2^k}h|\phi_s\rangle\phi_s(2^k x) \\
&= 2^{\frac{k}{2}}\sum_{s\in\mathbf{D}} \chi_{\omega_{s(2)}}(2^{-k}\xi)\langle h|D^{2^{-k}}\phi_s\rangle\phi_s(2^k x)
\end{aligned} \tag{133}$$

Take s to be a dyadic tile of order m, i.e., there are $r, s \in \mathbb{Z}$ such that

$$s = [2^m r, 2^m(r+1)) \times [2^{-m}s, 2^{-m}(s+1))$$

Looking at the expression inside the sum in (133) we note that

$$\begin{aligned} \chi_{\omega_{s(2)}}(2^{-k}\xi) = 1 &\Leftrightarrow 2^{-k}\xi \in \left[2^{-m}\left(s+\frac{1}{2}\right), 2^{-m}(s+1)\right) \\ &\Leftrightarrow \xi \in \left[2^{k-m}\left(s+\frac{1}{2}\right), 2^{k-m}(s+1)\right) \end{aligned} \tag{134}$$

And also

$$\begin{aligned} D^{2^{-k}}\phi_s(x) = 2^{\frac{k}{2}}\phi_s(2^k x) &= 2^{\frac{k}{2}}|I_s|^{-\frac{1}{2}}\phi\left(\frac{2^k x - c(I_s)}{|I_s|}\right)e^{2\pi i c(\omega_{s(1)})2^k x} \\ &= \left(\frac{|I_s|}{2^k}\right)^{-\frac{1}{2}}\phi\left(\frac{x - \frac{c(I_s)}{2^k}}{\frac{|I_s|}{2^k}}\right)e^{2\pi i(2^k c(\omega_{s(1)}))x} \\ &= \phi_{\tilde{s}}(x) \end{aligned} \tag{135}$$

where $\tilde{s}$ is the tile of order $m-k$ such that

$$c(I_{\tilde{s}}) = \frac{c(I_s)}{2^k}$$

$$c(\omega_{\tilde{s}(1)}) = 2^k c(\omega_{s(1)})$$

This way,

$$\begin{aligned} 2^{\frac{k}{2}}\sum_{s\in\mathbf{D}}\chi_{\omega_{s(2)}}(2^{-k}\xi)\langle h|D^{2^{-k}}\phi_s\rangle\phi_s(2^k x) &= \sum_{s\in\mathbf{D}}\chi_{\omega_{s(2)}}(2^{-k}\xi)\langle h|D^{2^{-k}}\phi_s\rangle D^{2^{-k}}\phi_s(x) \\ &= \sum_{\tilde{s}\in\mathbf{D}}\chi_{\omega_{\tilde{s}(2)}}(\xi)\langle h|\phi_{\tilde{s}}\rangle\phi_{\tilde{s}}(x) \\ &= A_\xi(h) \end{aligned}$$

Since the correspondence between s and $\tilde{s}$ established above is unique, we conclude that

$$A_\xi(h) = D^{2^{-k}}A_{2^{-k}\xi}D^{2^k}(h)$$

□

Proof of (e). Denote by $[a]$ the integer part of a real number a. Using the identities

$D^b M^\eta = M^{\frac{\eta}{b}} D^b$ and $D^b \tau^z = \tau^{bz} D^b$ we obtain

$$\begin{aligned} & D^{2^{-a}} M^{-(\xi+\eta)} \tau^{-y} D^{2^{-\lambda}} A_{\frac{\xi+\eta}{2^\lambda}} D^{2^\lambda} \tau^y M^{\xi+\eta} D^{2a} \\ &\qquad = M^{-2^a(\xi+\eta)} \tau^{-2^{-a} y} D^{2^{-(a+\lambda)}} A_{\frac{\xi+\eta}{2^\lambda}} D^{2^{a+\lambda}} \tau^{2^{-a} y} M^{2^a(\xi+\eta)} \\ &\qquad = M^{-2^a(\xi+\eta)} \tau^{-y'} D^{2^{-\lambda'}} D^{-[a+\lambda]} A_{\frac{2^a(\xi+\eta)}{2^{\lambda'} 2^{[a+\lambda]}}} D^{[a+\lambda]} D^{2^{\lambda'}} \tau^{y'} M^{2^a(\xi+\eta)} \\ &\qquad = M^{-2^a \xi} M^{-\eta'} \tau^{-y'} D^{2^{-\lambda'}} A_{\frac{2^a \xi + 2^a \eta}{2^{\lambda'}}} D^{2^{\lambda'}} \tau^{y'} M^{\eta'} M^{2^a \xi} \\ &\qquad = M^{-\xi} M^{-\theta} \left(M^{-\eta'} \tau^{-y'} D^{2^{-\lambda'}} A_{\frac{\xi+\theta+\eta'}{2^{\lambda'}}} D^{2^{\lambda'}} \tau^{y'} M^{\eta'} \right) M^\theta M^\xi \end{aligned} \tag{136}$$

where we set $y' = 2^{-a} y$, $\eta' = 2^a \eta$, $\lambda' = a + \lambda - [a+\lambda]$, and $\theta = (2^a - 1)\xi$. The averages of the left-hand side of (136) over all $(y, \eta, \lambda) \in [-K, K] \times [0, L] \times [0, 1]$ converges to the operator $D^{2^{-a}} M^{-\xi} \Pi_\xi M^\xi D^{2^a}$ as $K, L \to \infty$ (look at (131)). On the other hand, this limit is equal to the limit of the averages of the last expression in the right-hand side of (136) over all (y', η', λ') in $[-2^{-a} K, 2^{-a} K] \times [0, 2^a L] \times [0, 1]$, which is

$$M^{-\xi} M^{-\theta} \Pi_{\xi+\theta} M^\theta M^\xi$$

Using (c), we obtain

$$D^{2^{-a}} M^{-\xi} \Pi_\xi M^\xi D^{2^a} = M^{-\xi} M^{-\theta} \Pi_{\xi+\theta} M^\theta M^\xi = M^{-\xi} \Pi_\xi M^\xi$$

which means that the operator $M^{-\xi} \Pi^\xi M^\xi$ commutes with dilations. □

Proof of (f). The second assertion is a consequence of

$$\langle D^{2^\lambda} \tau^y M^\eta M^\xi(h) | \phi_s \rangle = \langle M^\xi(h) | M^{-\eta} \tau^{-y} D^{-2^\lambda}(\phi_s) \rangle = \langle (M^\xi(h))^\wedge | (M^{-\eta} \tau^{-y} D^{-2^\lambda}(\phi_s))^\wedge \rangle = 0$$

since the Fourier transform of $\tau^{-z} M^{-\eta} \tau^{-y} D^{-2^\lambda}(\phi_s)$ is supported in the set $(-\infty, 2^\lambda c(\omega_{s(1)}) - \eta + \frac{2^\lambda}{10} |\omega_s|)$, which is disjoint from the interval (ξ, ∞) whenever $2^{-\lambda}(\xi + \eta) \in \omega_{s(2)}$. Now we want to prove that

$$\langle \Pi_\xi(h) | h \rangle \geq 0 \tag{137}$$

This follows from the fact that (137) is equal to

$$\lim_{\substack{K \to \infty \\ L \to \infty}} \frac{1}{2KL} \int_0^L \int_{-K}^K \int_0^1 \sum_{s \in \mathbf{D}} \chi_{\omega_{s(2)}} \left(\frac{\xi + \eta}{2^\lambda} \right) |\langle D^{2^\lambda} \tau^y M^\eta(h) | \phi_s \rangle|^2 d\lambda dy d\eta \tag{138}$$

This also implies that Π_ξ is not the zero operator. Indeed, observe that

$$\sum_{s \in \mathbf{D}_0} \chi_{\omega_{s(2)}} \left(\frac{\xi + \eta}{2^\lambda} \right) |\langle D^{2^\lambda} \tau^y M^\eta(h) | \phi_s \rangle|^2 = \langle h | B^0_{\xi, y, \eta, \lambda}(h) \rangle$$

is periodic with period $(2^{-\lambda}, 2^{\lambda})$ in (y, η), so by proposition (4.6) the limit (138) is greater than or equal to

$$\int_0^{2^\lambda} \int_0^{2^{-\lambda}} \int_0^1 \sum_{s \in \mathbf{D}_0} \chi_{\omega_{s(2)}} \left(\frac{\xi + \eta}{2^\lambda} \right) |\langle D^{2^\lambda} \tau^y M^\eta(h) | \phi_s \rangle|^2 d\lambda dy d\eta$$

(observe that the sum above is over $\mathbf{D}_0$, not $\mathbf{D}$). Since we can always find a Schwartz function h and a tile s such that $\langle D^{2^\lambda} \tau^y M^\eta(h) | \phi_s \rangle$ is nonzero for (y, η, λ) near $(0, 0, 0)$, the expression in (138) is strictly positive for this h, so $\langle \Pi_\xi(h) | h \rangle$ is strictly positive. This way we proved that the operators $M^{-\xi} \Pi_\xi M^\xi$ are nonzero for every ξ. $\square$

Define the operator P by:

$$P(f) = (\widehat{f} \chi_{(-\infty,0)})^\vee$$

Proposition 4.8. *Up to a constant multiple, P is the unique bounded operator on $L^2(\mathbb{R})$ that has the following properties:*
(a) Commutes with translations.
(b) Commutes with dilations.
(c) Vanishes when applied to functions whose Fourier transforms are supported on the positive axis.

Proof. Using proposition (5.4), (a) implies the existence of $m \in L^\infty(\mathbb{R}^n)$ such that $\widehat{Tf}(\xi) = m(\xi)\hat{f}$. We also have (using $\mathcal{F}$ to denote the Fourier transform):

$$\begin{aligned} \mathcal{F} D_\delta f(\xi) &= \int_{-\infty}^{\infty} f(\delta x) e^{-2\pi i x \xi} dx \\ &= \delta^{-1} \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi / \delta} dx \\ &= \delta^{-1} (D_{\delta^{-1}} \mathcal{F} f)(\xi) \end{aligned} \tag{139}$$

Let us write symbolically $\mathcal{F}T = m\mathcal{F}$, where by m we mean the operator of multiplication by the L^∞ function m. If $g = (\chi_{[0,\infty)} \widehat{f})^\vee$, we have $\widehat{g} = \chi_{[0,\infty)} \widehat{f}$ and supp $\widehat{g} \subset [0, \infty)$. Using hypothesis (b),

$$Tg = 0 \Rightarrow T \left(f - (\chi_{(-\infty,0)} \widehat{f})^\vee \right) = 0 \Rightarrow Tf = T \left[(\chi_{(-\infty,0)} \widehat{f})^\vee \right]$$

This way,

$$\widehat{Tf}(\xi) = T[(\chi_{(-\infty,0)} \widehat{f})^\vee]^\wedge(\xi) = m(\xi)[(\chi_{(\infty,0)} \widehat{f})^\vee]^\wedge(\xi) = \chi_{(-\infty,0)}(\xi) m(\xi) \widehat{f}(\xi) \tag{140}$$

On the other hand, by (139),

$$D_\delta m = D_\delta \mathcal{F} T \mathcal{F}^{-1} = \delta^{-1} \mathcal{F} D_{\delta^{-1}} T \mathcal{F}^{-1} = \delta^{-1} \mathcal{F} T D_{\delta^{-1}} \mathcal{F}^{-1} = \delta^{-1} \delta \mathcal{F} T \mathcal{F}^{-1} D_\delta = m D_\delta$$

So

$$D_\delta \circ m \circ f(x) = m \circ D_\delta \circ f(x) \Rightarrow (m \circ f)(\delta x) = m(x)(D_\delta f)(x) \Rightarrow$$
$$m(\delta x) f(\delta x) = m(x) f(\delta x), \quad \forall f \in L^2 \quad \therefore \quad m(\delta x) = m(x)$$

i.e., m is constant. By (140),

$$\widehat{Tf}(\xi) = c\chi_{(-\infty,0)}(\xi)\widehat{f}(\xi) \Rightarrow Tf = cP(f)$$

□

Propositions 4.7 and 4.8 imply:

$$M^{-\xi}\Pi_\xi M^\xi = c_\xi P(f), \quad \exists c_\xi \in \mathbb{R}$$

Proposition 4.7 (b) gives us:

$$M^{-\xi-\theta}\Pi_{\xi+\theta}M^{\xi+\theta} = M^{-\xi}\Pi_\xi M^\xi \Rightarrow c_\xi = c_{\xi+\theta}$$

So the unilateral Carleson operator can be written as:

$$\mathcal{C}(f) = \frac{1}{|c|}\sup_{\xi>0}|\Pi_\xi(f)| \tag{141}$$

4.3 Linearization of a maximal dyadic sum

We start by observing that for a fixed $f \in \mathcal{S}(\mathbb{R})$, the function

$$(x,\xi) \mapsto \int_{-\infty}^{\xi} \widehat{f}(y)e^{2\pi ixy}dy$$

defined on $\mathbb{R} \times \mathbb{R}^+$ is continuous in both variables. This way, we can restrict the range of $N > 0$ in (119) to $N \in \mathbb{Q}^+$. We can go one step further:

Claim 4.3. *If we prove* (120) *to any finite* $Q_0 \subset \mathbb{Q}^+$ *instead of* $\mathbb{Q}^+$ *with bound* C *independent of* Q_0 *and* f*,* (120) *holds.*

Proof. From the observation above:

$$\mathcal{C}_1(f)(x) = \sup_{N \in \mathbb{Q}^+}\left|\int_{-\infty}^{N} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi\right|$$

Given an enumeration $\{q_1, q_2, \ldots, q_n, \ldots\}$ of $\mathbb{Q}^+$, define $Q_n := \{q_j; j \leq n\}$. If the supreme above is attained at a rational $q < \infty$, the following is obviously true:

$$\mathcal{C}_{Q_n}(f)(x) = \sup_{N \in Q_n} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \nearrow \sup_{N \in \mathbb{Q}^+} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| = \mathcal{C}_1(f)(x) \tag{142}$$

If the supreme is attained at infinity, we can take a monotone sequence of rationals r_j such that

$$\left| \int_{-\infty}^{r_j} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \to \mathcal{C}_1(f)(x)$$

But $r_j = q_{i_j}$, so:

$$\left| \int_{-\infty}^{r_j} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \leq \sup_{N \in Q_{i_j}} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \leq \mathcal{C}_1(f)(x)$$

Then (142) is also true in this case. This way, by Fatou's lemma for L^p-weak:

$$\|\mathcal{C}_1(f)\|_{2,\infty} = \left\| \lim_{n\to\infty} \mathcal{C}_{Q_n}(f) \right\|_{2,\infty} \leq \liminf_{n\to\infty} \|\mathcal{C}_{Q_n}(f)\|_{2,\infty} \leq C\|f\|_2$$

by hypothesis. □

Take now $Q_0 \subset \mathbb{Q}$. For each $\xi_0 \in Q_0$ we have that

$$\left\{ x \in \mathbb{R}; \max_{\xi \in Q_0} \left| \int_{-\infty}^{\xi} \widehat{f}(y) e^{2\pi i x y} dy \right| = \left| \int_{-\infty}^{\xi_0} \widehat{f}(y) e^{2\pi i x y} dy \right| \right\}$$

is closed and hence measurable. Therefore, we may select a measurable real-valued function $N_f : \mathbb{R} \to Q_0$ such that for all $x \in \mathbb{R}$ we have:

$$\sup_{\xi \in Q_0} \left| \int_{-\infty}^{\xi} \widehat{f}(y) e^{2\pi i x y} dy \right| = \left| \int_{-\infty}^{N_f(x)} \widehat{f}(y) e^{2\pi i x y} dy \right|$$

This measurable function N_f motivates the following approach to our problem: consider the operator

$$f \mapsto \left| \int_{-\infty}^{N(x)} \widehat{f}(y) e^{2\pi i x y} dy \right|$$

where $N : \mathbb{R} \to Q_0$ is a general measurable function. If we prove a $(2,2)$-weak estimate for this operator applied to functions of $\mathcal{S}(\mathbb{R})$ with bounds independent of N, then for a given $f \in \mathcal{S}(\mathbb{R})$ we choose $N = N_f$ and obtain $(2,2)$-weak boundedness of

$$f \mapsto \sup_{\xi \in Q_0} \left| \int_{-\infty}^{\xi} \widehat{f}(y) e^{2\pi i x y} dy \right|$$

Therefore, $(2,2)$-weak boundedness for $\mathcal{C}_1$ will follow by our previous argument.

Fix $N : \mathbb{R} \to Q_0$ measurable assuming at most a finite number of rational values. Define on $\mathcal{S}(\mathbb{R})$ the operator:

$$\mathcal{D}_N(f)(x) = A_{N(x)}(f)(x) = \sum_{s\in\mathbf{D}} (\chi_{\omega_{s(2)}} \circ N)(x)\langle f|\phi_s\rangle\phi_s(x).$$

we have already seen that this sum converges absolutely.

Claim 4.4. *It suffices to show that*

$$\|\mathcal{D}_N(f)\|_{2,\infty} \le C\|f\|_2, \forall f \in L^2 \tag{143}$$

Proof. Taking $\xi = N(x)$ in the definition of Π_ξ,

$$\Pi_{N(x)}(f)(x) = \lim_{K,L\to\infty} \frac{1}{2KL} \int_{[-L,L]\times[-K,K]\times[0,1]} M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{N(x)+\eta}{2^\lambda}}D^{2^\lambda}\tau^y M^\eta(f)d\lambda dy d\eta$$

This way,

$$|\Pi_{N(x)}(f)(x)| \le \liminf_{K,L\to\infty} \frac{1}{2KL} \int_{[-L,L]\times[-K,K]\times[0,1]} |M^{-\eta}\tau^{-y}D^{2^{-\lambda}}A_{\frac{N(x)+\eta}{2^\lambda}}D^{2^\lambda}\tau^y M^\eta(f)|d\lambda dy d\eta$$

We now apply $L^{2,\infty}$ norms on both sides, use Fatou's lemma for L^p-weak and use the fact that translations, modulations and L^2-dilation are isometries on L^2. Using (143), we are done. □

For each fixed measurable function $N : \mathbb{R} \to \mathbb{R}_+$, we define the *N-linearization* of the Carleson operator *associated to the finite set of tiles* $\mathbf{P}$ as the linear operator:

$$\mathcal{D}_{N,\mathbf{P}}(f)(x) = \sum_{s\in\mathbf{P}} (\chi_{\omega_{s(2)}} \circ N)(x)\langle f|\phi_s\rangle\phi_s(x)$$

By monotone convergence, to obtain the limitation in (143) it is enough to obtain:

$$\|\mathcal{D}_{N,\mathbf{P}}(f)\|_{2,\infty} \le C\|f\|_2, \tag{144}$$

where C does not depend on $f, \mathbf{P}, N$.
By duality in Lorentz spaces, it suffices to show that for all $f \in \mathcal{S}(\mathbb{R})$ we have

$$\left|\int_{\mathbb{R}} D_{N,\mathbf{P}}(f)(x)g(x)dx\right| = \left|\sum_{s\in\mathbf{P}} \langle(\chi_{\omega_{s(2)}} \circ N)\phi_s, g\rangle\langle f|\phi_s\rangle\right| \le C\|g\|_{L^{2,1}}\|f\|_2$$

We can restrict ourselves even more (see 1.4.7 in GRAFAKOS (2014a)). It suffices to show that for all measurable E with $|E| < \infty$ we have

$$\left|\int_E D_{N,\mathbf{P}}(f)(x)dx\right| = \left|\sum_{s\in\mathbf{P}}\langle(\chi_{\omega_{s(2)}}\circ N)\phi_s,\chi_E\rangle\langle f|\phi_s\rangle\right| \leq C|E|^{\frac{1}{2}}\|f\|_2$$

In particular, this estimate holds if it holds the following:

$$\sum_{s\in\mathbf{P}}|\langle(\chi_{\omega_{s(2)}}\circ N)\phi_s|\chi_E\rangle||\langle f|\phi_s\rangle| \leq C|E|^{1/2}\|f\|_2, \tag{145}$$

With C again independent from $f, \mathbf{P}, N$. Let us now work to obtain this last inequality.

4.4 The main argument

In this section, let us define a partial order on the set of all dyadic tiles. From now on, we will call them *tiles*.

Definition 4.3. *Define the partial order $<$ on $\mathbf{D}$ by*

$$s < s' \Longleftrightarrow I_s \subseteq I_{s'} \quad \text{e} \quad \omega_{s'} \subseteq \omega_s.$$

By proposition 4.1,

$$s \cap s' \neq \emptyset \Rightarrow s < s' \text{ or } s' < s.$$

As a consequence, if $\mathbf{P}$ is a finite set of tiles, all maximal elements of $\mathbf{P}$ with respect to $<$ are pairwise disjoint.

Definition 4.4. *A finite set of tiles $\mathbf{P}$ is a* tree *if there is a tile $t \in \mathbf{P}$ such that all $s \in \mathbf{P}$ satisfies $s < t$. We will call t the* top *of $\mathbf{P}$ and denote it by $t = \text{top}(\mathbf{P})$. It is immediate that the top of a tree is unique.*

Note that a finite set of tiles $\mathbf{P}$ can be written as union of trees which tops are maximal elements. Indeed, consider all maximal elements of $\mathbf{P}$ under $<$. Then every non-maximal element $s \in \mathbf{P}$ satisfies $s < t$ for some maximal $t \in \mathbf{P}$, therefore it belongs to some tree with t as its top. Observe also that tiles can be written as union of two semi-tiles $I_s \times \omega_{s(1)}$ and $I_s \times \omega_{s(2)}$.

Definition 4.5. *A tree $\mathbf{T}$ is a* 1-tree *if*

$$\omega_{\text{top}(\mathbf{T})(1)} \subseteq \omega_{s(1)}$$

for all $s \in \mathbf{T}$. *A tree* $\mathbf{T}'$ *is a* 2-tree *if for all* $s \in \mathbf{T}'$ *we have*

$$\omega_{\mathrm{top}(\mathbf{T}')(2)} \subseteq \omega_{s(2)}$$

Claim 4.5. *About* 1-trees *and* 2-trees*,*

(a) Every tree $\mathbf{T}$ *can be written as union of a* 1-tree *and a* 2-tree *such that their intersection is precisely the top of* $\mathbf{T}$*.*

(b) If $\mathbf{T}$ *is a* 1-tree*, then* $\omega_{\mathrm{top}(\mathbf{T})(2)} \cap \omega_{s(2)} = \emptyset$ *for all* $s \in \boldsymbol{T}$ *different from the top. An analogous statement holds for* 2-trees*.*

Proof of (a). We verify this by using a simple algorithm. For each $s \in \mathbf{T}$ different from the top, if $\omega_{\mathrm{top}(\mathbf{T})(1)} \subseteq \omega_{s(1)}$ add s to the set $\mathbf{A}$ (initially empty). If $\omega_{\mathrm{top}(\mathbf{T}')(2)} \subseteq \omega_{s(2)}$, add s to the set $\mathbf{B}$ (also initially empty). Observe that we don't add the same tile to both sets $\mathbf{A}$ and $\mathbf{B}$. Finally, when all tiles but the top were selected to some set, we put the top in both sets and have $\mathbf{T} = \mathbf{A} \cup \mathbf{B}$ with the desired property. □

Proof of (b). Immediate from the definition. □

We introduce now the concepts of *mass* and *energy.*

Definition 4.6. *Let* $N \colon \mathbb{R} \to \mathbb{R}^+$ *be a measurable function,* $s \in \mathbf{D}$ *and* E *be a measurable set with* $|E| < \infty$*. Let*

$$\mathcal{M}(E;\{s\}) = \frac{1}{|E|} \sup_{\substack{u \in \mathbf{D} \\ s<u}} \int_{E \cap N^{-1}[\omega_u]} \frac{|I_u|^{-1} dx}{(1 + \frac{|x - c(I_u)|}{|I_u|})^{10}}$$

We call $\mathcal{M}(E;\{s\})$ the *mass* of E with respect to $\{s\}$. Given $\mathbf{P} \subset \mathbf{D}$, the *mass of* E *with respect to* $\mathbf{P}$ is defined by

$$\mathcal{M}(E;\mathbf{P}) = \sup_{s \in \mathbf{P}} \mathcal{M}(E;\{s\}).$$

Observe that the mass of E with respect to any set of tiles is at most:

$$\frac{1}{|E|} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{10}} \le \frac{1}{|E|}.$$

Definition 4.7. *Given* $\mathbf{P} \subset \mathbf{D}$ *finite and a function* $f \in L^2(\mathbb{R})$*, define*

$$\mathcal{E}(f;\mathbf{P}) = \frac{1}{\|f\|_{L^2}} \sup_{\mathbf{T}} \left(\frac{1}{|I_{\mathrm{top}(\mathbf{T})}|} \sum_{s \in \mathbf{T}} |\langle f | \phi_s \rangle|^2 \right)^{\frac{1}{2}}$$

where the supreme is taken over all 2-trees $\mathbf{T} \subset \mathbf{P}$*. We call* $\mathcal{E}(f;\mathbf{P})$ *the* energy *of* f *with respect to* $\mathbf{P}$*.*

We will state now tree lemmas which will be proved in the three subsequent subsections.

Lemma 4.3. *There exists a constant* C_1 *such that for any measurable function* $N\colon \mathbb{R} \to \mathbb{R}^+$*, for all measurable* $E \subset \mathbb{R}$ *with finite measure and for all finite set of tiles* $\mathbf{P}$ *there is a subset* $\mathbf{P}'$ *of* $\mathbf{P}$ *such that*

$$\mathcal{M}(E; \mathbf{P} \backslash \mathbf{P}') \leq \frac{1}{4} \mathcal{M}(E; \mathbf{P})$$

and $\mathbf{P}'$ *is a union of trees* $\mathbf{T}_j$ *satisfying*

$$\sum_j |I_{\mathrm{top}(\mathbf{T}_j)}| \leq \frac{C_1}{\mathcal{M}(E; \mathbf{P})} \tag{146}$$

Lemma 4.4. *There exists a constant* C_2 *such that for all finite set of tiles* $\mathbf{P}$ *and all functions* $f \in L^2(\mathbb{R})$ *there is a subset* $\mathbf{P}''$ *of* $\mathbf{P}$ *such that*

$$\mathcal{E}(f; \mathbf{P} \backslash \mathbf{P}'') \leq \frac{1}{2} \mathcal{E}(f; \mathbf{P})$$

and $\mathbf{P}''$ *is a union of trees* $\mathbf{T}_j$ *satisfying*

$$\sum_j |I_{\mathrm{top}(\mathbf{T}_j)}| \leq \frac{C_2}{\mathcal{E}(f; \mathbf{P})^2} \tag{147}$$

Lemma 4.5. *There exists a finite constant* C_3 *such that for all trees* $\mathbf{T}$*, all functions* $f \in L^2(\mathbb{R})$*, for any measurable function* $N\colon \mathbb{R} \to \mathbb{R}^+$ *and for all measurable subsets* E *we have*

$$\sum_{s \in \mathbf{T}} |\langle f | \phi_s \rangle \langle \chi_{E \cap N^{-1}[\omega_{s(2)}]} | \phi_s \rangle| \leq C_3 |I_{\mathrm{top}(\mathbf{T})}| \mathcal{E}(f; \mathbf{T}) \mathcal{M}(E; \mathbf{T}) \|f\|_{L^2} |E|. \tag{148}$$

In the rest of this section, we conclude the proof of theorem 4.1 assuming lemmas 4.3, 4.4 and 4.5.

Given a finite set of tiles $\mathbf{P}$, E measurable with finite measure, $N\colon \mathbb{R} \to \mathbb{R}^+$ measurable and $f \in L^2(\mathbb{R})$, take $n_0 \in \mathbb{Z}$ big enough such that

$$\mathcal{E}(f; \mathbf{P}) \leq 2^{n_0},$$

$$\mathcal{M}(E; \mathbf{P}) \leq 2^{n_0}.$$

We shall construct inductively a sequence of pairwise disjoint sets

$$\mathbf{P}_{n_0}, \mathbf{P}_{n_0 - 1}, \mathbf{P}_{n_0 - 2}, \mathbf{P}_{n_0 - 3}, \ldots$$

such that

$$\bigcup_{j=-\infty}^{n_0} \mathbf{P}_j = \mathbf{P} \tag{149}$$

and such that the following conditions are satisfied:

1. $\mathcal{E}(f; \mathbf{P}_j) \leq 2^{j+1}$ for all $j \leq n_0$;
2. $\mathcal{M}(E; \mathbf{P}_j) \leq 2^{2j+2}$ for all $j \leq n_0$;
3. $\mathcal{E}\left(f; \mathbf{P}\backslash(\mathbf{P}_{n_0} \cup \ldots \cup \mathbf{P}_j)\right) \leq 2^j$ for all $j \leq n_0$;
4. $\mathcal{M}\left(E; \mathbf{P}\backslash(\mathbf{P}_{n_0} \cup \ldots \cup \mathbf{P}_j)\right) \leq 2^{2j}$ for all $j \leq n_0$;
5. $\mathbf{P}_j$ is a union of trees $\mathbf{T}_{jk}$ such that for all $j \leq n_0$ we have

$$\sum_k |I_{\text{top}(\mathbf{T}_{jk})}| \leq C_0 2^{-2j},$$

where $C_0 = C_1 + C_2$ and C_1 and C_2 are the constants obtained in lemmas 4.3 and 4.4, respectively.

Assume for a moment that we already constructed the sequence $\{\mathbf{P}_j\}_{j \leq n_0}$ with the properties above. To obtain (145), we use 1, 2 and 5, the fact that the mass of any set of tiles is bounded by $|E|^{-1}$ and lemma 4.5 to obtain

$$\begin{aligned}
\sum_{s \in \mathbf{P}} |\langle f|\phi_s\rangle\langle \chi_{E \cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle| &= \sum_j \sum_{s \in \mathbf{P}_j} |\langle f|\phi_s\rangle\langle \chi_{E \cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle| \\
&\leq \sum_j \sum_k \sum_{s \in \mathbf{T}_{jk}} |\langle f|\phi_s\rangle\langle \chi_{E \cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle| \\
&\leq C_3 \sum_j \sum_k |I_{\text{top}(\mathbf{T}_{jk})}| \mathcal{E}(f; \mathbf{T}_{jk}) \mathcal{M}(E; \mathbf{T}_{jk}) \|f\|_{L^2} |E| \\
&\leq C_3 \sum_j \sum_k |I_{\text{top}(\mathbf{T}_{jk})}| 2^{j+1} \min(|E|^{-1}, 2^{2j+2}) \|f\|_{L^2} |E| \\
&\leq C_3 \sum_j C_0 2^{-2j} 2^{j+1} \min(|E|^{-1}, 2^{2j+2}) \|f\|_{L^2} |E| \\
&= C_3 \sum_j C_0 2^{-j+1} \min(|E|^{-1}, 2^{2j+2}) \|f\|_{L^2} |E|
\end{aligned}$$

We have to estimate this last sum:

$$\begin{cases} |E|^{-1} < 2^{2j+2} \Leftrightarrow \log_2 |E|^{-1} < 2j+2 \Leftrightarrow j > \frac{-2+\log_2 |E|^{-1}}{2} := k_0 \\ |E|^{-1} \geq 2^{2j+2} \Leftrightarrow \log_2 |E|^{-1} \geq 2j+2 \Leftrightarrow j \leq \frac{-2+\log_2 |E|^{-1}}{2} := k_0 \end{cases}$$

Let us break this sum in two pieces:

$$\begin{aligned}\sum_j 2^{-j+1}\min(|E|^{-1},2^{2j+2}) &= \sum_{j=-\infty}^{\lfloor k_0\rfloor} 2^{-j+1}\min(|E|^{-1},2^{2j+2})\\ &+ \sum_{j=\lfloor k_0\rfloor+1}^{n_0} 2^{-j+1}\min(|E|^{-1},2^{2j+2})\\ &= I + II\end{aligned}$$

For I:

$$\sum_{j=-\infty}^{\lfloor k_0\rfloor} 2^{-j+1}\min(|E|^{-1},2^{2j+2}) = \sum_{j=-\infty}^{\lfloor k_0\rfloor} 2^{j+3} = 8\sum_{j=-\lfloor k_0\rfloor} 2^{-j} = 16.2^{\lfloor k_0\rfloor} \le 16.2^{k_0} = 8.|E|^{-\frac{1}{2}}$$

For II:

$$\sum_{j=\lfloor k_0\rfloor+1}^{n_0} 2^{-j+1}\min(|E|^{-1},2^{2j+2}) \le 2|E|^{-1}\sum_{j=\lfloor k_0\rfloor+1}^{\infty} 2^{-j} \le 4|E|^{-1}2^{-k_0} = 8|E|^{-\frac{1}{2}}$$

Summing $I + II$:

$$\sum_j 2^{-j+1}\min(|E|^{-1},2^{2j+2}) \le 16|E|^{-\frac{1}{2}}$$

Then:

$$\begin{aligned}\sum_{s\in\mathbf{P}} |\langle f|\phi_s\rangle\langle\chi_{E\cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle| &\le \dots\\ &\le C_3\sum_j C_0 2^{-j+1}\min(|E|^{-1},2^{2j+2})\|f\|_{L^2}|E|\\ &\le C_3C_0\|f\|_{L^2}|E|16|E|^{-\frac{1}{2}}\\ &= C|E|^{\frac{1}{2}}\|f\|_{L^2}\end{aligned}$$

This proves estimate (145).

We shall now construct a sequence $\{\mathbf{P}_j\}_{j\le n_0}$ satisfying conditions 1 to 5 above. We will proceed by induction. Start with $j = n_0$ putting $\mathbf{P}_{n_0} = \emptyset$. This way, 1, 2 and 5 are immediately satisfied and

$$\begin{aligned}\mathcal{E}(f;\mathbf{P}\backslash\mathbf{P}_{n_0}) &= \mathcal{E}(f;\mathbf{P}) \le 2^{n_0};\\ \mathcal{M}(E;\mathbf{P}\backslash\mathbf{P}_{n_0}) &= \mathcal{M}(E;\mathbf{P}) \le 2^{n_0}.\end{aligned}$$

therefore 3 and 4 are also satisfied for $\mathbf{P}_{n_0}$.

Suppose we already constructed pairwise disjoint sets $\mathbf{P}_{n_0}$, $\mathbf{P}_{n_0-1}$, ..., $\mathbf{P}_n$ for some $n < n_0$ such that $1-5$ hold for all $j \in \{n_0, n_0-1, \dots, n\}$. We will construct a set of tiles $\mathbf{P}_{n-1}$ disjoint from $\mathbf{P}_j$ with $j \leq n$ such that $1-5$ hold for it.

Let us define an auxiliary set $\mathbf{P}'_{n-1}$. If $\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) \leq 2^{2(n-1)}$, put $\mathbf{P}'_{n-1}=\emptyset$. If $\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) > 2^{2(n-1)}$, use lemma 4.3 to find a subset $\mathbf{P}'_{n-1}$ of $\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)$ such that

$$\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}'_{n-1})\right) \leq \frac{1}{4}\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) \leq \frac{2^{2n}}{4} = 2^{2(n-1)}$$

(using the induction hypothesis 4 for $j=n$) and $\mathbf{P}'_{n-1}$ is a union of trees $\mathbf{T}'_k$ satisfying

$$\sum_k |I_{\mathrm{top}(\mathbf{T}'_k)}| \leq C_1\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right)^{-1} \leq C_1 2^{-2(n-1)}. \tag{150}$$

This way, if $\mathcal{E}\left(f;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) \leq 2^{n-1}$, put $\mathbf{P}''_{n-1}=\emptyset$. Otherwise, use lemma 4.4 to obtain a subset $\mathbf{P}''_{n-1}$ of $\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)$ such that

$$\mathcal{E}\left(f;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}''_{n-1})\right) \leq \frac{1}{2}\mathcal{E}\left(f;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) \leq \frac{1}{2}2^n = 2^{n-1}$$

(using the induction hypothesis 3 for $j=n$) and $\mathbf{P}''_{n-1}$ is a union of trees $\mathbf{T}''_k$ satisfying

$$\sum_k |I_{\mathrm{top}(\mathbf{T}''_k)}| \leq C_2\mathcal{E}\left(f;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right)^{-2} \leq C_2 2^{-2(n-1)}. \tag{151}$$

In any case, we conclude:

$$\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}'_{n-1})\right) \leq 2^{2(n-1)} \tag{152}$$

$$\mathcal{E}\left(f;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}''_{n-1})\right) \leq 2^{n-1} \tag{153}$$

Define $\mathbf{P}_{n-1}=\mathbf{P}'_{n-1}\cup\mathbf{P}''_{n-1}$ and let us verify properties $1-5$. As $\mathbf{P}_{n-1}\subset\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)$, we have:

$$\mathcal{E}(f;\mathbf{P}_{n-1}) \leq \mathcal{E}\left(f;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) \leq 2^n = 2^{(n-1)+1},$$

where the last inequality is a consequence of the induction hypothesis 3 for $j=n$; this way, 1 holds for $j=n-1$. Likewise,

$$\mathcal{M}(E;\mathbf{P}_{n-1}) \leq \mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n)\right) \leq 2^{2n} = 2^{2(n-1)+2},$$

where we used the induction hypothesis 4 for $j=n$, therefore 2 holds for $j=n-1$.

To prove 3 for $j=n-1$, note that $\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}_{n-1})\subset\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}''_{n-1})$ and this last one has energy at most 2^{n-1} by estimate 153. To prove 4 with $j=n-1$, note that $\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}_{n-1})\subset\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_n\cup\mathbf{P}'_{n-1})$, which has mass at most $2^{2(n-1)}$ by 152. Finally, if we add (150) to (151), we obtain 5 for $j=n-1$

with $C_0 = C_1 + C_2$.

Choose $j \in \mathbb{Z}$ with $0 < 2^{2j} < \min_{s\in\mathbf{P}} \mathcal{M}(E;\{s\})$. Therefore:

$$\mathcal{M}\left(E;\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_j)\right) \leq 2^{2j} < \min_{s\in\mathbf{P}} \mathcal{M}(E;\{s\})$$

which implies $\mathbf{P}\backslash(\mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_j) = \emptyset$, i.e., $\mathbf{P} = \mathbf{P}_{n_0}\cup\ldots\cup\mathbf{P}_j$ and (149) holds. It also follows that there exists n_1 such that, for all $n \leq n_1$, $\mathbf{P}_n = \emptyset$. This concludes the construction of $\{\mathbf{P}_j\}_{j\leq n_0}$.

4.5 Proof of lemma 4.3

Let μ to be the mass of a set of tiles $\mathbf{P}$. Define:

$$\mathbf{P}' = \left\{s \in \mathbf{P} : \mathcal{M}(E;\{s\}) > \frac{\mu}{4}\right\}$$

This is our natural candidate to be the set of "fat" tiles. By definition, $\mathcal{M}\left(E;\mathbf{P}\backslash\mathbf{P}'\right) \leq \frac{\mu}{4}$, i.e., $\mathbf{P}\backslash\mathbf{P}'$ is "light". We now show that $\mathbf{P}'$ can be written as a union of trees such that (146) holds. It follows by definition that for each $s \in \mathbf{P}'$ there is $u(s) \in \mathbf{D}$ with $u(s) > s$ and

$$\frac{1}{|E|}\int_{E\cap N^{-1}[\omega_{u(s)}]} \frac{|I_{u(s)}|^{-1}dx}{(1+\frac{|x-c(I_{u(s)})|}{|I_{u(s)}|})^{10}} > \frac{\mu}{4} \tag{154}$$

Let $\mathbf{U} = \{u(s) : s \in \mathbf{P}'\}$ and $\mathbf{U}_{max} \subset \mathbf{U}$ be the subset containing all maximal elements of $\mathbf{U}$ according to $<$. Now define $\mathbf{P}'_{max}$ to be the set of all maximal elements of $\mathbf{P}'$. As we have already said, the tiles of $\mathbf{P}'$ can be grouped in trees $\mathbf{T}_j = \{s \in \mathbf{P}' : s < t_j\}$ with tops $t_j \in \mathbf{P}'_{max}$. If $t_j < u$ e $t_{j'} < u$ for some $u \in \mathbf{U}$, then $\omega_u \subset \omega_{t_j} \cap \omega_{t_{j'}}$, and as t_j and $t_{j'}$ are disjoint, it follows that I_{t_j} and $I_{t_{j'}}$ are disjoint subsets of I_u. Enumerate the elements of $\mathbf{U}_{max}$ from u_1 to u_n. Starting with u_1, let A_1 be the set of all t_j such that $t_j < u_1$. Let A_2 be the set of all $t_j \notin A_1$ such that $t_j < u_2$. Repeating this algorithm until there is no top of tree left, we have construct disjoint sets A_k's. From this,

$$\sum_j |I_{t_j}| = \sum_{i=1}^{n}\sum_{t_j\in A_i} |I_{t_j}| \leq \sum_{i=1}^{n} |I_{u_i}| \leq \sum_{u\in\mathbf{U}_{max}} |I_u|$$

So it suffices to prove:

$$\sum_{u\in\mathbf{U}_{max}} |I_u| \leq C\mu^{-1} \tag{155}$$

for some constant C. For $u \in \mathbf{U}_{max}$ we can rewrite (154) as

$$\frac{1}{|E|}\sum_{k=0}^{\infty}\int_{E\cap N^{-1}[\omega_u]\cap(2^kI_u\backslash 2^{k-1}I_u)}\frac{|I_u|^{-1}dx}{(1+\frac{|x-c(I_u)|}{|I_u|})^{10}} > \frac{\mu}{8}\sum_{k=0}^{\infty}2^{-k}$$

with the convention $2^{-1}I_u = \emptyset$. It follows that for all $u \in \mathbf{U}_{max}$ there is an integer $k \geq 0$ such that

$$|E|\frac{\mu}{8}|I_u|2^{-k} < \int_{E\cap N^{-1}[\omega_u]\cap(2^kI_u\backslash 2^{k-1}I_u)}\frac{dx}{(1+\frac{|x-c(I_u)|}{|I_u|})^{10}} \leq \frac{|E\cap N^{-1}[\omega_u]\cap 2^kI_u|}{(\frac{4}{5})^{10}(1+2^{k-2})^{10}}$$

Therefore $u \in \mathbf{U}_k$ where

$$\mathbf{U}_k = \{u \in \mathbf{U}_{max} : |I_u| \leq 8\cdot 5^{10}2^{-9k}\mu^{-1}|E|^{-1}|E\cap N^{-1}[\omega_u]\cap 2^kI_u|\}$$

So

$$\mathbf{U}_{max} = \bigcup_{k=0}^{\infty}\mathbf{U}_k$$

Observe that in order to prove (155) it suffices to prove that

$$\sum_{u\in\mathbf{U}_k}|I_u| \leq C2^{-8k}\mu^{-1}, \qquad k\geq 0 \tag{156}$$

Take $v_0 \in \mathbf{U}_k$ such that $|I_{v_0}|$ is the largest possible among elements of $\mathbf{U}_k$. Select now $v_1 \in \mathbf{U}_k\backslash\{v_0\}$ such that $(2^kI_{v_1})\times\omega_{v_1}$ is disjoint from $(2^kI_{v_0})\times\omega_{v_0}$ and $|I_{v_1}|$ is the largest possible. By induction, at the j^{th} step we choose $v_j \in \mathbf{U}_k\backslash\{v_0,\ldots,v_{j-1}\}$ such that $(2^kI_{v_j})\times\omega_{v_j}$ is disjoint from all enlarged rectangles of tiles previously selected and $|I_{v_j}|$ is the largest possible. This algorithm will finish after a finite number of steps (because we will have exhausted $\mathbf{U}_k$ or because we won't be able to select tiles using this criterion). We will call $\mathbf{V}_k$ the set of these tiles selected from $\mathbf{U}_k$.

For each $u \in \mathbf{U}_k$ there is $v \in \mathbf{V}_k$ with $|I_u| \leq |I_v|$ such that:

$$[(2^kI_u)\times\omega_u]\cap[(2^kI_v)\times\omega_v] \neq \emptyset$$

Indeed, if $u \in \mathbf{V}_k$ take $v = u$. If $u \in \mathbf{U}_k\backslash\mathbf{V}_k$, there is such v because otherwise u would have been selected previously. We choose such v and say that it is *associated* to u. Note that if u and u' are associated to the same v, as they are disjoint and as $\omega_u\cap\omega_{u'}\supset\omega_v$, then $I_u\cap I_{u'} = \emptyset$. This way, tiles $u \in \mathbf{U}_k$ associated to a $v \in \mathbf{V}_k$ fixed have I_u's disjoint ans satisfy

$$I_u \subseteq 2^{k+2}I_v$$

From this,

$$\sum_{\substack{u \in \mathbf{U}_k \\ u \text{ associated to } v}} |I_u| \leq |2^{k+2} I_v| = 2^{k+2} |I_v|$$

Finally,

$$\begin{aligned}
\sum_{u \in \mathbf{U}_k} |I_u| &\leq \sum_{v \in \mathbf{V}_k} \sum_{u \text{ associated to } v} |I_u| \\
&\leq 2^{k+2} \sum_{v \in \mathbf{V}_k} |I_v| \\
&\leq 2^{k+5} 5^{10} \mu^{-1} |E|^{-1} 2^{-9k} \sum_{v \in \mathbf{V}_k} |E \cap N^{-1}[\omega_v] \cap 2^k I_v| \\
&\leq 32 \cdot 5^{10} \mu^{-1} 2^{-8k}
\end{aligned}$$

since the enlarged rectangles $(2^k I_v) \times \omega_v$ of the selected tiles v are disjoint and then $E \cap N^{-1}[\omega_v] \cap 2^k I_v$ also are. This proves (156) and lemma 4.3.

4.6 Proof of lemma 4.4

Let $g \in L^2(\mathbb{R})$ and $\mathbf{T}'$ a 2-tree. Define the following quantity:

$$\Delta(g; \mathbf{T}') = \frac{1}{\|g\|_{L^2}} \left(\frac{1}{|I_{\text{top}(\mathbf{T}')}|} \sum_{s \in \mathbf{T}'} |\langle g | \phi_s \rangle|^2 \right)^{\frac{1}{2}}$$

We now describe an algorithm to construct a set $\mathbf{P}''$. Start with $\tilde{\mathbf{T}}$ being the set of all 2-trees contained in $\mathbf{P}$ and perform the following:

1. Take the subset $\overline{\mathbf{T}}$ of $\tilde{\mathbf{T}}$ that satisfies

$$\Delta(g; \mathbf{T}') \geq \frac{1}{2} \mathcal{E}(g; \mathbf{P}), \text{ for all } \mathbf{T}' \in \overline{\mathbf{T}} \tag{157}$$

2. Select $\mathbf{T}'_1$ from $\overline{\mathbf{T}}$ such that $c(\omega_{\text{top}(\mathbf{T}'_1)})$ is the smallest possible.
3. Let $\mathbf{T}_1$ be the set of all $s \in \mathbf{P}$ such that $s < \text{top}(\mathbf{T}'_1)$.

Now redefine $\tilde{\mathbf{T}}$ to be the set of all 2-trees contained in $\mathbf{P} \backslash \mathbf{T}_1$ and run the algorithm again, obtaining a tree $\mathbf{T}_2$. At the j^{th} iteration we will have constructed $\mathbf{T}_1$, $\mathbf{T}_2$, ..., $\mathbf{T}_{j-1}$ and run the algorithm again for $\tilde{\mathbf{T}} = \mathbf{P} \backslash (\mathbf{T}_1 \cup \mathbf{T}_2 \cup \ldots \cup \mathbf{T}_{j-1})$. We keep doing this till there is no 2-tree left to pass step 1 above. This way we construct a finite sequence of pairwise disjoint 2-trees $\mathbf{T}'_1$, $\mathbf{T}'_2$, ..., $\mathbf{T}'_n$ and a set of pairwise disjoint trees $\mathbf{T}_1$, $\mathbf{T}_2$, ..., $\mathbf{T}_n$ such that $\mathbf{T}'_j \subseteq \mathbf{T}_j$, $\text{top}(\mathbf{T}_j) = \text{top}(\mathbf{T}'_j)$ and $\mathbf{T}'_j$ satisfy the condition of step 1 above. Define

$$\mathbf{P}'' = \bigcup_j \mathbf{T}_j$$

Observe that the process we just performed eliminates all trees trees with high energy. In other words,

$$\mathcal{E}(g; \mathbf{P} \backslash \mathbf{P}'') \leq \frac{1}{2}\mathcal{E}(g; \mathbf{P})$$

$\mathbf{P}''$ is our candidate to be the set with high energy that satisfies the conditions of the statement of this lemma. We now prove (147). Using (157) for each tree $\mathbf{T}_j$ and summing,

$$\begin{aligned}\frac{1}{4}\mathcal{E}(g;\mathbf{P})^2 \sum_j |I_{\text{top}(\mathbf{T}_j)}| &\leq \frac{1}{\|g\|_{L^2}^2} \sum_j \sum_{s \in \mathbf{T}'_j} |\langle g|\phi_s\rangle|^2 \\ &= \frac{1}{\|g\|_{L^2}^2} \sum_j \sum_{s \in \mathbf{T}'_j} \langle g|\phi_s\rangle \overline{\langle g|\phi_s\rangle} \\ &= \frac{1}{\|g\|_{L^2}} \left\langle \frac{g}{\|g\|_{L^2}} \Big| \sum_j \sum_{s \in \mathbf{T}'_j} \langle g|\phi_s\rangle \phi_s \right\rangle \\ &\leq \frac{1}{\|g\|_{L^2}} \left\| \sum_j \sum_{s \in \mathbf{T}'_j} \langle \phi_s|g\rangle \phi_s \right\|_{L^2}\end{aligned} \tag{158}$$

Set $\mathbf{U} = \cup_j \mathbf{T}'_j$. We claim that it suffices to prove that

$$\frac{1}{\|g\|_{L^2}} \left\| \sum_j \sum_{s \in \mathbf{T}'_j} \langle \phi_s|g\rangle \phi_s \right\|_{L^2} \leq C \left(\mathcal{E}(g;\mathbf{P})^2 \sum_j |I_{\text{top}(\mathbf{T}_j)}| \right)^{\frac{1}{2}} \tag{159}$$

Indeed, using (158) we easily get (147) by transitivity. If we are given ω_s and $\omega_{s'}$ we have four possibilities:

1. $\omega_s = \omega_{s'}$
2. $\omega_s \subsetneq \omega_{s'}$
3. $\omega_{s'} \subsetneq \omega_s$
4. $\omega_s \cap \omega_{s'} = \emptyset$

As we have already said earlier, the last case gives us $\langle \phi_s|\phi_{s'}\rangle = 0$ by Parseval and by the disjointness of the supports of $\widehat{\phi_s}$ and $\widehat{\phi_{s'}}$. This way,

$$\left\| \sum_j \sum_{s \in \mathbf{T}'_j} \langle \phi_s|g\rangle \phi_s \right\|_{L^2}^2 \leq \underbrace{\sum_{\substack{s,u \in \mathbf{U} \\ \omega_s = \omega_u}} |\langle \phi_s|g\rangle \langle \phi_u|g\rangle \langle \phi_s|\phi_u\rangle|}_{\textbf{(I)}} + 2 \underbrace{\sum_{\substack{s,u \in \mathbf{U} \\ \omega_s \subsetneq \omega_u}} |\langle \phi_s|g\rangle \langle \phi_u|g\rangle \langle \phi_s|\phi_u\rangle|}_{\textbf{(II)}} \tag{160}$$

Our two jobs now are to estimate **(I)** and **(II)**. For **(I)**:

$$
\begin{aligned}
&\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s=\omega_u}} |\langle\phi_s|g\rangle||\langle\phi_s|\phi_u\rangle|^{\frac{1}{2}}|\langle\phi_u|g\rangle||\langle\phi_s|\phi_u\rangle|^{\frac{1}{2}} \\
&\qquad \le \left(\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s=\omega_u}} |\langle\phi_s|g\rangle|^2|\langle\phi_s|\phi_u\rangle|\right)^{\frac{1}{2}} \left(\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s=\omega_u}} |\langle\phi_u|g\rangle|^2|\langle\phi_s|\phi_u\rangle|\right)^{\frac{1}{2}} \\
&\qquad = \sum_{s\in\mathbf{U}} |\langle g|\phi_s\rangle|^2 \sum_{\substack{u\in\mathbf{U}\\ \omega_u=\omega_s}} |\langle\phi_s|\phi_u\rangle| \\
&\qquad \le C' \sum_{s\in\mathbf{U}} |\langle g|\phi_s\rangle|^2 \\
&\qquad = C' \sum_j \sum_{s\in\mathbf{T}'_j} |\langle g|\phi_s\rangle|^2 \\
&\qquad = C' \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}||I_{\mathrm{top}(\mathbf{T}_j)}|^{-1} \sum_{s\in\mathbf{T}'_j} |\langle g|\phi_s\rangle|^2 \\
&\qquad \le C' \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|\mathcal{E}(g;\mathbf{P})^2\|g\|_{L^2}^2
\end{aligned} \tag{161}
$$

where we used Cauchy-Schwarz in the first inequality and the almost orthogonality proposition 5.5 in the second. Before we proceed, let us make an useful observation: if $\omega_s \subsetneq \omega_u$ and $\langle\phi_s|\phi_u\rangle \neq 0$, then by the same argument used in the beginning of the proof of lemma 4.2 we have $\omega_s \subset \omega_{u(1)}$. For **(II)**:

$$
\begin{aligned}
&\sum_{\substack{s,u\in\mathbf{U}\\ \omega_s\subsetneq\omega_u}} |\langle\phi_s|g\rangle\langle\phi_u|g\rangle\langle\phi_s|\phi_u\rangle| \\
&\qquad = \sum_j \sum_{s\in\mathbf{T}'_j} |\langle\phi_s|g\rangle| \sum_{\substack{u\in\mathbf{U}\\ \omega_s\subsetneq\omega_u}} |\langle\phi_u|g\rangle\langle\phi_s|\phi_u\rangle| \\
&\qquad \le \sum_j \left(\sum_{s\in\mathbf{T}'_j} |\langle\phi_s|g\rangle|^2\right)^{\frac{1}{2}} \left[\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subsetneq\omega_u}} |\langle\phi_u|g\rangle\langle\phi_s|\phi_u\rangle|\right)^2\right]^{\frac{1}{2}} \\
&\qquad \le \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|^{\frac{1}{2}}\Delta(g;\mathbf{T}'_j)\|g\|_{L^2} \left[\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subsetneq\omega_u}} |\langle\phi_u|g\rangle\langle\phi_s|\phi_u\rangle|\right)^2\right]^{\frac{1}{2}} \\
&\qquad \le \mathcal{E}(g;\mathbf{P})\|g\|_{L^2} \sum_j |I_{\mathrm{top}(\mathbf{T}_j)}|^{\frac{1}{2}} \left[\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subsetneq\omega_{u(1)}}} |\langle\phi_u|g\rangle\langle\phi_s|\phi_u\rangle|\right)^2\right]^{\frac{1}{2}}
\end{aligned} \tag{162}
$$

where we used Cauchy-Schwarz in the first inequality, the definition of Δ in the second and the definition of $\mathcal{E}$ and our previous useful observation in the third. Since $\{s\} \subset \mathbf{P}$ is a 2-tree, we have:

$$\mathcal{E}(g;\{u\}) = \frac{1}{\|g\|_{L^2}}\left(\frac{|\langle g,|\phi_u\rangle|^2}{|I_u|}\right)^{\frac{1}{2}} = \frac{1}{\|g\|_{L^2}}\frac{|\langle g|\phi_u\rangle|}{|I_u|^{\frac{1}{2}}} \leq \mathcal{E}(g;\mathbf{P})$$

so the last inequality gives us:

$$|\langle g|\phi_u\rangle| \leq \|g\|_{L^2}|I_u|^{\frac{1}{2}}\mathcal{E}(g;\mathbf{P}) \tag{163}$$

Using (163) to estimate the term inside the brackets in the last inequality of (162),

$$\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}} |\langle\phi_u|g\rangle\langle\phi_s|\phi_u\rangle|\right)^2 \leq \mathcal{E}(g;\mathbf{P})^2\|g\|^2_{L^2}\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}} |I_u|^{\frac{1}{2}}|\langle\phi_s|\phi_u\rangle|\right)^2 \tag{164}$$

Our next step is to establish the following lemma:

Lemma 4.6. (Strong disjointness) *Let $\mathbf{T}_j$ and $\mathbf{T}'_j$ be as before.*
(a) Let $s \in \mathbf{T}'_j$ and $u \in \mathbf{T}'_k$. If $\omega_s \subseteq \omega_{u(1)}$, then $I_u \cap I_{\text{top}(\mathbf{T}_j)} = \emptyset$.
(b) If $u \in \mathbf{T}'_k$ and $v \in \mathbf{T}'_l$ are different tiles and satisfy $\omega_s \subseteq \omega_{u(1)} \cap \omega_{v(1)}$ for some fixed $s \in \mathbf{T}'_j$, then $I_u \cap I_v = \emptyset$.

Proof. Let us first prove (a). Note that $s \in \mathbf{T}'_j$ implies $\omega_{\text{top}(\mathbf{T}'_j)} \subset \omega_{s(2)} \subset \omega_{u(1)}$. On the other hand, $u \in \mathbf{T}'_k$ implies $\omega_{\text{top}(\mathbf{T}'_k)} \subset \omega_{u(2)}$, so $\omega_{\text{top}(\mathbf{T}'_j)} < \omega_{\text{top}(\mathbf{T}'_k)}$, which means that $\mathbf{T}'_j$ was selected before $\mathbf{T}'_k$ in our initial algorithm. This way, $j < k$. Remember that $I_{\text{top}(\mathbf{T}_j)} = I_{\text{top}(\mathbf{T}'_j)}$, so

$$|I_{\text{top}(\mathbf{T}'_j)}| = \frac{1}{|\omega_{\text{top}(\mathbf{T}'_j)}|} \geq \frac{1}{|\omega_s|} \geq \frac{1}{|\omega_{u(1)}|} = \frac{2}{|\omega_u|} = 2|I_u|$$

if $I_u \cap I_{\text{top}(\mathbf{T}_j)} \neq \emptyset$, then $I_u \subset I_{\text{top}(\mathbf{T}_j)}$. Also, $\omega_{\text{top}(\mathbf{T}'_j)} \subset \omega_s \subset \omega_u$. This way $u < \text{top}(\mathbf{T}'_j)$, which implies $u \in \mathbf{T}_j$ since this tree was selected first. On the other hand, $\mathbf{T}_j \cap \mathbf{T}'_k = \emptyset$, contradiction. Therefore it follows that $I_u \cap I_{\text{top}(\mathbf{T}_j)} = \emptyset$. Assume now the conditions of (b). $\emptyset \neq \omega_s \subseteq \omega_{u(1)} \cap \omega_{v(1)}$ implies one of the three below:

1. $\omega_u \subset \omega_{v(1)}$. In this case, (a) gives us $I_v \cap I_{\text{top}(\mathbf{T}'_k)} = \emptyset$. As $I_u \subset I_{\text{top}(\mathbf{T}'_k)}$, we conclude that $I_u \cap I_v = \emptyset$.
2. $\omega_v \subset \omega_{u(1)}$. Same argument used in the previous item applies here to conclude that $I_u \cap I_v = \emptyset$.
3. $\omega_u = \omega_v$, which implies $|I_u| = |I_v|$. As u and v are different tiles, we must have $I_u \cap I_v = \emptyset$.

□

We claim that it suffices to prove that

$$\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}}|I_u|^{\frac{1}{2}}|\langle\phi_s|\phi_u\rangle|\right)^2\leq C|I_{\mathrm{top}(\mathbf{T}_j)}| \tag{165}$$

Indeed, using 163, 162 and 165 we have:

$$\begin{aligned}\mathbf{(II)}&\leq\mathcal{E}(g;\mathbf{P})\|g\|_{L^2}\sum_j|I_{\mathrm{top}(\mathbf{T}_j)}|^{\frac{1}{2}}\mathcal{E}(g;\mathbf{P})\|g\|_{L^2}\left[\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}}|I_u|^{\frac{1}{2}}|\langle\phi_s|\phi_u\rangle|\right)^2\right]^{\frac{1}{2}}\\&\leq C\mathcal{E}(g;\mathbf{P})^2\|g\|_{L^2}^2\sum_j|I_{\mathrm{top}(\mathbf{T}_j)}|\end{aligned}$$

Adding our estimates for **(I)** and **(II)** in (160) we conclude (159). Let us prove 165. By lemma 4.6, different tiles u that appear as parameters in 165 have associated disjoint intervals I_u contained in $(I_{\mathrm{top}(\mathbf{T}_j)})^c$. Let $t_j=\mathrm{top}(\mathbf{T}_j)$. Using proposition 5.5 we get:

$$\begin{aligned}\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}}|I_u|^{\frac{1}{2}}|\langle\phi_s|\phi_u\rangle|\right)^2&\leq C\sum_{s\in\mathbf{T}'_j}\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}}|I_u|^{\frac{1}{2}}\left(\frac{|I_s|}{|I_u|}\right)^{\frac{1}{2}}\int_{I_u}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}}\right)^2\\&\leq C\sum_{s\in\mathbf{T}'_j}|I_s|\left(\sum_{\substack{u\in\mathbf{U}\\ \omega_s\subset\omega_{u(1)}}}\int_{I_u}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}}\right)^2\\&\leq C\sum_{s\in\mathbf{T}'_j}|I_s|\left(\int_{(I_{t_j})^c}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}}\right)^2\\&\leq C\sum_{s\in\mathbf{T}'_j}|I_s|\underbrace{\int_{(I_{t_j})^c}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}}}_{\mathbf{(III)}}\end{aligned}$$

Since **(III)** <1 as one can see by making a change of variables and comparing it to $\int_{\mathbb{R}}(1+|y|)^{-20}dx<1$. For each $k\geq 0$ the intervals I_s with $s\in\mathbf{T}'_j$ and $|I_s|=2^{-k}|I_{t_j}|$ are pairwise disjoint and contained in I_{t_j}. This way,

$$\begin{aligned}\sum_{s\in\mathbf{T}'_j}|I_s|\int_{(I_{t_j})^c}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}} &= \sum_{k=0}^{\infty}\frac{2^k}{|I_{t_j}|}\sum_{\substack{s\in\mathbf{T}'_j\\|I_s|=2^{-k}|I_{t_j}|}}|I_s|\int_{(I_{t_j})^c}\frac{dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}}\\
&\le C\sum_{k=0}^{\infty}\frac{2^k}{|I_{t_j}|}\sum_{\substack{s\in\mathbf{T}'_j\\|I_s|=2^{-k}|I_{t_j}|}}\int_{I_s}\int_{(I_{t_j})^c}\frac{dxdy}{\left(1+\frac{|x-y|}{|I_s|}\right)^{20}}\\
&\le C\sum_{k=0}^{\infty}\frac{2^k}{|I_{t_j}|}\int_{I_{t_j}}\int_{(I_{t_j})^c}\frac{dxdy}{\left(1+\frac{|x-y|}{|I_s|}\right)^{20}}\\
&\le C'\sum_{k=0}^{\infty}\frac{2^k}{|I_{t_j}|}2^{-2k}|I_{t_j}|^2\\
&= C''|I_{t_j}|\end{aligned}$$

where we used lemma 5.5. This finishes the proof of this lemma.

4.7 Proof of lemma 4.5

Proving this lemma for $\|g\|_{L^2}=1$ and replacing g by $g/\|g\|_{L^2}$ proves it for a general $g\in L^2(\mathbb{R})$, so we will concentrate ourselves in this case. Fix a function g with L^2 norm 1, a tree $\mathbf{T}$, a measurable function $N:\mathbb{R}\to\mathbb{R}^+$ and a measurable set E with finite measure.

Define $\mathcal{J}'$ to be the set of all dyadic intervals J such that $3J$ does not contain any I_s, $s\in\mathbf{T}$. By taking dyadic intervals small enough, we can always find one interval in $\mathcal{J}'$ that contains a $x\in\mathbb{R}$. Define $\mathcal{J}$ to be the set of all maximal subsets of $\mathcal{J}'$ (under inclusion). This way, $\mathcal{J}$ is a partition of $\mathbb{R}$.

Observe that $\langle g|\phi_s\rangle\langle\chi_{E\cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle$ is a complex number, so we can multiply it by some unitary vector $\epsilon_s\in\mathbb{C}$ such that the result is its absolute value (we are just rotating the original complex number). Precisely, there exists $\epsilon_s\in\mathbb{C}$ with $|\epsilon_s|=1$ such that

$$|\langle g|\phi_s\rangle\langle\chi_{E\cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle| = \epsilon_s\langle g|\phi_s\rangle\langle\chi_{E\cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle$$

We can rewrite the left-hand side of (148) as

$$\begin{aligned}\sum_{s\in\mathbf{T}}\epsilon_s\langle g|\phi_s\rangle\langle\chi_{E\cap N^{-1}[\omega_{s(2)}]}|\phi_s\rangle &\le \left\|\sum_{s\in\mathbf{T}}\epsilon_s\langle g|\phi_s\rangle\chi_{E\cap N^{-1}[\omega_{s(2)}]}\phi_s\right\|_{L^1(\mathbb{R})}\\
&= \sum_{J\in\mathcal{J}}\left\|\sum_{s\in\mathbf{T}}\epsilon_s\langle g|\phi_s\rangle\chi_{E\cap N^{-1}[\omega_{s(2)}]}\phi_s\right\|_{L^1(\mathcal{J})}\\
&\le \Sigma_1+\Sigma_2\end{aligned}$$

where

$$\Sigma_1 = \sum_{J\in\mathcal{J}} \left\| \sum_{\substack{s\in\mathbf{T}\\|I_s|\leq 2|J|}} \epsilon_s \langle g|\phi_s\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\phi_s \right\|_{L^1(\mathcal{J})} \tag{166}$$

$$\Sigma_2 = \sum_{J\in\mathcal{J}} \left\| \sum_{\substack{s\in\mathbf{T}\\|I_s|>2|J|}} \epsilon_s \langle g|\phi_s\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}\phi_s \right\|_{L^1(\mathcal{J})} \tag{167}$$

Let us start by estimating Σ_1. For every $s \in \mathbf{T}$, $\{s\}$ is a 2-tree contained in $\mathbf{T}$. By the definition of energy,

$$|\langle g|\phi_s\rangle| \leq |I_s|^{\frac{1}{2}}\mathcal{E}(g;\mathbf{T}) \tag{168}$$

Using this, we obtain

$$\begin{aligned}
\Sigma_1 &\leq \sum_{J\in\mathcal{J}} \sum_{\substack{s\in\mathbf{T}\\|I_s|\leq 2|J|}} \mathcal{E}(g;\mathbf{T}) \int_{J\cap E\cap N^{-1}[\omega_{s(2)}]} |I_s|^{\frac{1}{2}}|\phi_s(x)|dx \\
&\leq C\sum_{J\in\mathcal{J}} \sum_{\substack{s\in\mathbf{T}\\|I_s|\leq 2|J|}} \mathcal{E}(g;\mathbf{T})|I_s| \int_{J\cap E\cap N^{-1}[\omega_{s(2)}]} \frac{|I_s|^{-1}}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{20}}dx \\
&\leq C\sum_{J\in\mathcal{J}} \sum_{\substack{s\in\mathbf{T}\\|I_s|\leq 2|J|}} \mathcal{E}(g;\mathbf{T})|E|\mathcal{M}(E;\mathbf{T})|I_s| \sup_{x\in J} \frac{1}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{10}} \\
&\leq C\mathcal{E}(g;\mathbf{T})|E|\mathcal{M}(E;\mathbf{T}) \sum_{J\in\mathcal{J}} \sum_{k=-\infty}^{\log_2 2|J|} 2^k \sum_{\substack{s\in\mathbf{T}\\|I_s|=2^k}} \frac{1}{\left(1+\frac{\mathrm{dist}(J,I_s)}{2^k}\right)^5} \frac{1}{\left(1+\frac{\mathrm{dist}(J,I_s)}{2^k}\right)^5}
\end{aligned}$$

Remember that all I_s with $s \in \mathbf{T}$ and $|I_s| = 2^k$ are pairwise disjoint and contained in $I_{\mathrm{top}(\mathbf{T})}$. Therefore, $2^{-k}\mathrm{dist}(J, I_s) \geq |I_{\mathrm{top}(\mathbf{T})}|^{-1}\mathrm{dist}(J, I_{\mathrm{top}(\mathbf{T})})$ and we have the estimate

$$\left(1+\frac{\mathrm{dist}(J,I_s)}{2^k}\right)^{-5} \leq \left(1+\frac{\mathrm{dist}(J,I_{\mathrm{top}(\mathbf{T})})}{|I_{\mathrm{top}(\mathbf{T})}|}\right)^{-5}$$

Moreover, the sum

$$\sum_{\substack{s\in\mathbf{T}\\|I_s|=2^k}} \frac{1}{\left(1+\frac{\mathrm{dist}(J,I_s)}{2^k}\right)^5}$$

is controlled by a finite constant, since for every $m \in \mathbb{Z}^+$ there exist at most two tiles $s \in \mathbf{T}$ with $|I_s| = 2^k$ such that I_s are not contained in $3J$ and $m2^k \leq \mathrm{dist}(J, I_s) \leq (m+1)2^k$.

Therefore, we obtain

$$\begin{aligned}
\Sigma_1 &= C\mathcal{E}(g;\mathbf{T})|E|\mathcal{M}(E;\mathbf{T})\sum_{J\in\mathcal{J}}\sum_{k=-\infty}^{\log_2 2|J|}\frac{2^k}{\left(1+\frac{\text{dist}(J,I_{\text{top}(\mathbf{T})})}{|I_{\text{top}(\mathbf{T})}|}\right)^5} \\
&\leq C\mathcal{E}(g;\mathbf{T})|E|\mathcal{M}(E;\mathbf{T})\sum_{J\in\mathcal{J}}\frac{|J|}{\left(1+\frac{\text{dist}(J,I_{\text{top}(\mathbf{T})})}{|I_{\text{top}(\mathbf{T})}|}\right)^5} \\
&\leq^{(*)} C\mathcal{E}(g;\mathbf{T})|E|\mathcal{M}(E;\mathbf{T})\sum_{J\in\mathcal{J}}\int_J\frac{1}{\left(1+\frac{|x-c(I_{\text{top}(\mathbf{T})})|}{|I_{\text{top}(\mathbf{T})}|}\right)^5}dx \\
&\leq C|I_{\text{top}(\mathbf{T})}|\mathcal{E}(g;\mathbf{T})|E|\mathcal{M}(E;\mathbf{T})
\end{aligned} \tag{169}$$

since $\mathcal{J}$ is a partition of $\mathbb{R}$ and by a change of variables. For this part to be complete, we still must verify the penultimate inequality $(*)$ above.

Since J and $I_{\text{top}(\mathbf{T})}$ are dyadic intervals, we have three cases:

1. $J \cap I_{\text{top}(\mathbf{T})} = \emptyset$
2. $J \subset I_{\text{top}(\mathbf{T})}$
3. $I_{\text{top}(\mathbf{T})} \subset J$

Actually, we just have the first two cases. The third does not happen because $3J$ does not contain $I_{\text{top}(\mathbf{T})}$. In the first case we have $|J| \leq \text{dist}(J, I_{\text{top}(\mathbf{T})})$ since $3J$ does not contain $I_{\text{top}(\mathbf{T})}$. In the second case we obviously have $|J| \leq |I_{\text{top}(\mathbf{T})}|$. Thus in both cases we have $|J| \leq \text{dist}(J, I_{\text{top}(\mathbf{T})}) + |I_{\text{top}(\mathbf{T})}|$. This way, for any $x \in J$ we have

$$\begin{aligned}
|x - c(I_{\text{top}(\mathbf{T})})| &\leq |J| + \text{dist}(J, I_{\text{top}(\mathbf{T})}) + \frac{|I_{\text{top}(\mathbf{T})}|}{2} \\
&\leq 2\text{dist}(J, I_{\text{top}(\mathbf{T})}) + \frac{3|I_{\text{top}(\mathbf{T})}|}{2}
\end{aligned}$$

Therefore,

$$\int_J \frac{dx}{\left(1+\frac{|x-c(I_{\text{top}(\mathbf{T})})|}{|I_{\text{top}(\mathbf{T})}|}\right)^5} \geq \frac{|J|}{\left(\frac{5}{2}+\frac{2\text{dist}(J,I_{\text{top}(\mathbf{T})})}{|I_{\text{top}(\mathbf{T})}|}\right)^5} \geq \frac{(\frac{2}{5})^5|J|}{\left(1+\frac{\text{dist}(J,I_{\text{top}(\mathbf{T})})}{|I_{\text{top}(\mathbf{T})}|}\right)^5}$$

This verification completes the estimate for Σ_1.

Let us now focus on Σ_2. Assume without loss of generality that for all J appearing in (167), the set of s in $\mathbf{T}$ with $2|J| < |I_s|$ is nonempty. Thus, if J appears in this sum we have $2|J| < |I_s| < |I_{\text{top}(\mathbf{T})}|$. We claim that J is contained in $3I_{\text{top}(\mathbf{T})}$. Indeed, since J is maximal, if it is not contained in $3I_{\text{top}(\mathbf{T})}$ it has size larger than $\frac{1}{2}|I_{\text{top}(\mathbf{T})}|$.

Let $\mathbf{T}_2$ be the 2-tree of all $s \in \mathbf{T}$ such that $\omega_{\text{top}(\mathbf{T})(2)} \subset \omega_{s(2)}$. By defining

$\mathbf{T}_1 = \mathbf{T} \backslash \mathbf{T}_2$ we see that $\mathbf{T}_1$ is a 1-tree minus its top. Now set

$$F_{1J} = \sum_{\substack{s \in \mathbf{T}_1 \\ |I_s| > 2|J|}} \epsilon_s \langle g | \phi_s \rangle \chi_{E \cap N^{-1}[\omega_{s(2)}]} \phi_s$$

$$F_{2J} = \sum_{\substack{s \in \mathbf{T}_2 \\ |I_s| > 2|J|}} \epsilon_s \langle g | \phi_s \rangle \chi_{E \cap N^{-1}[\omega_{s(2)}]} \phi_s$$

We obviously have

$$\Sigma_2 \leq \sum_{J \in \mathcal{J}} \|F_{1J}\|_{L^1(J)} + \sum_{J \in \mathcal{J}} \|F_{2J}\|_{L^1(J)} = \Sigma_{21} + \Sigma_{22}$$

To finish the proof, we will deal with both Σ_{21} and Σ_{22}. If the tiles s and s' that appear in the definition of F_{1J} have different scales, then the sets $\omega_{s(2)}$ and $\omega_{s'(2)}$ are disjoint and thus so are the sets $E \cap N^{-1}[\omega_{s(2)}]$ and $E \cap N^{-1}[\omega_{s'(2)}]$. Set

$$G_J = J \cap \bigcup_{\substack{s \in \mathbf{T} \\ |I_s| > 2|J|}} E \cap N^{-1}[\omega_{s(2)}]$$

Then F_{1J} is supported in the set G_J and we have

$$\begin{aligned}
\|F_{1J}\|_{L^1(J)} &\leq \|F_{1J}\|_{L^\infty(J)} |G_J| \\
&\leq \left\| \sum_{k > \log_2 2|J|} \sum_{\substack{s \in \mathbf{T}_1 \\ |I_s| = 2^k}} \epsilon_s \langle g | \phi_s \rangle \chi_{E \cap N^{-1}[\omega_{s(2)}]} \phi_s \right\|_{L^\infty(J)} |G_J| \\
&\leq \sup_{k > \log_2 2|J|} \left\| \sum_{\substack{s \in \mathbf{T}_1 \\ |I_s| = 2^k}} \epsilon_s \langle g | \phi_s \rangle \chi_{E \cap N^{-1}[\omega_{s(2)}]} \phi_s \right\|_{L^\infty(J)} |G_J| \\
&\leq \sup_{k > \log_2 2|J|} \sup_{x \in J} \sum_{\substack{s \in \mathbf{T}_1 \\ |I_s| = 2^k}} \mathcal{E}(g; \mathbf{T}) 2^{\frac{k}{2}} \frac{2^{\frac{-k}{2}}}{\left(1 + \frac{|x - c(I_s)|}{2^k}\right)^{10}} |G_J| \\
&\leq C \mathcal{E}(g; \mathbf{T}) |G_J|
\end{aligned}$$

where we used (168) and the fact that all the I_s that appear in the sum are disjoint.

Claim 4.6. *For all $J \in \mathcal{J}$ we have*

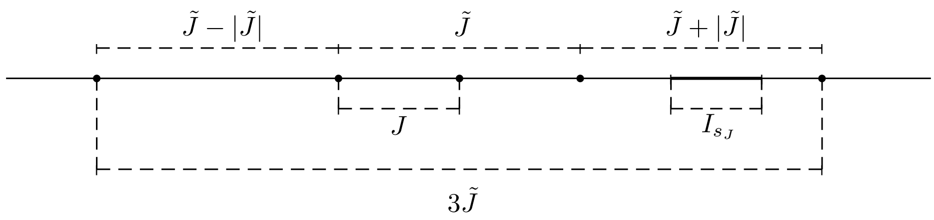
$$|G_J| \leq C |E| \mathcal{M}(E; \mathbf{T}) |J| \tag{170}$$

Once we prove (170), sum over all J that appear in the definition of F_{1J} and observe that all of these intervals are pairwise disjoint and contained in $3I_{\mathrm{top}(\mathbf{T})}$. This implies the desired estimate for Σ_{21}.

Proof of claim 4.6. Consider the unique dyadic interval $\tilde{J}$ of length $2|J|$ that contains J. By the maximality of $\mathcal{J}$, $3\tilde{J}$ contains I_{s_J}, where $s_J \in \mathbf{T}$ is a tile. We have two cases:

1. I_{s_J} is either $(\tilde{J} - |\tilde{J}|) \cup \tilde{J}$ or $\tilde{J} \cup (\tilde{J} + |\tilde{J}|)$.
2. I_{s_J} is contained in one of the two dyadic intervals $\tilde{J} - |\tilde{J}|$, $\tilde{J} + |\tilde{J}|$.

Figure 4: J and $\tilde{J}$



In the first case let $u_J = s_J$. This way, $|u_J| = 2|\tilde{J}|$. In the second case, whichever of these two dyadic intervals that contains I_{s_J} is contained in $I_{\mathrm{top}(\mathbf{T})}$, since it intersects is and has smaller length than it. This way, there exists a tile $u_J \in \mathbf{D}$ with $|I_{u_J}| = |\tilde{J}|$ such that $I_{s_J} \subseteq I_{u_J} \subseteq I_{\mathrm{top}(\mathbf{T})}$ and $\omega_{\mathrm{top}(\mathbf{T})} \subseteq \omega_{u_J} \subseteq \omega_{s_J}$. In both cases we have a tile u_J satisfying $s_J < u_J < \mathrm{top}(\mathbf{T})$ with $|\omega_{u_J}|$ being either $\frac{1}{4|J|}$ or $\frac{1}{2|J|}$. Then for any $s \in \mathbf{T}$ with $|I_s| > 2|J|$ we have $|\omega_s| \leq |\omega_{u_J}|$. Since both ω_s and ω_{u_J} contain $\omega_{\mathrm{top}(\mathbf{T})}$, they must intersect, so $\omega_s \subseteq \omega_{u_J}$. We conclude that any $s \in \mathbf{T}$ with $|I_s| > 2|J|$ must satisfy $N^{-1}[\omega_s] \subseteq N^{-1}[\omega_{u_J}]$. This (together with the definition of G_J) implies

$$G_J \subseteq J \cap E \cap N^{-1}[\omega_{u_J}] \tag{171}$$

Therefore

$$\begin{aligned}
|E|\mathcal{E}(M; \mathbf{T}) &= \sup_{s \in \mathbf{T}} \sup_{\substack{u \in \mathbf{D} \\ s < u}} \int_{E \cap N^{-1}[\omega_u]} \frac{|I_u|^{-1}}{\left(1 + \frac{|x - c(I_u)|}{|I_u|}\right)^{10}} dx \\
&\geq \int_{J \cap E \cap N^{-1}[\omega_{u_J}]} \frac{|I_{u_J}|^{-1}}{\left(1 + \frac{|x - c(I_{u_J})|}{|I_{u_J}|}\right)^{10}} dx \\
&\geq c|I_{u_J}|^{-1} |J \cap E \cap N^{-1}[\omega_{u_J}]| \\
&\geq c|I_{u_J}|^{-1} |G_J|
\end{aligned}$$

where we used (171) and the fact that for $x \in J$ we have

$$|x - c(I_{u_J})| \leq 4|J| \leq 2|I_{u_J}|$$

It follows that

$$|G_J| \leq \frac{1}{c}|E|\mathcal{M}(E; \mathbf{T})|I_{u_J}| \leq \frac{4}{c}|E|\mathcal{M}(E; \mathbf{T})|J|$$

and this is exactly what we wanted in (4.6). $\square$

Now we turn our attention to the estimate for $\Sigma_{22} = \sum_{J\in\mathcal{J}} \|F_{2J}\|_{L^1(J)}$. Since $\mathbf{T}_2$ is a 2-tree, all intervals $\omega_{s(2)}$ with $s \in \mathbf{T}_2$ are nested. Therefore, for each $x \in J$ for which $F_{2J}(x)$ is nonzero, there exists a largest dyadic interval ω_{u_x} and a smallest dyadic interval ω_{v_x} (for some $u_x, v_x \in \mathbf{T}_2 \cap \{s : |I_s| \geq 4|J|\}$) such that for $s \in \mathbf{T}_2 \cap \{s : |I_s| \geq 4|J|\}$ we have $N(x) \in \omega_{s(2)}$ if and only if $\omega_{v_x} \subseteq \omega_s \subseteq \omega_{u_x}$. Then we have:

$$\begin{aligned} F_{2J}(x) &= \sum_{\substack{s\in\mathbf{T}_2 \\ |I_s|>2|J|}} \epsilon_s \langle g|\phi_s\rangle \chi_{E\cap N^{-1}[\omega_{s(2)}]}(x)\phi_s(x) \\ &= \chi_E(x) \sum_{\substack{s\in\mathbf{T}_2 \\ |\omega_{v_x}|\leq|\omega_s|\leq|\omega_{u_x}|}} \epsilon_s\langle g|\phi_s\rangle\phi_s(x) \end{aligned}$$

Now pick a Schwartz function ψ whose Fourier transform $\widehat{\psi}(t)$ is supported in $|t| \leq \frac{1}{2} + \frac{1}{100}$ and that it is equal to 1 in $|t| \leq \frac{1}{2}$.

Claim 4.7. *For all $z \in \mathbb{R}$, if $|\omega_{v_x}| \leq |\omega_s| \leq |\omega_{u_x}|$, then*

$$\left(\phi_s * \left\{ \frac{M^{c(\omega_{u_x})}D^{|\omega_{u_x}|^{-1}}(\psi)}{|\omega_{u_x}|^{-\frac{1}{2}}} - \frac{M^{c(\omega_{v_x(2)})}D^{|\omega_{v_x(2)}|^{-1}}(\psi)}{|\omega_{v_x(2)}|^{-\frac{1}{2}}} \right\}\right)(z) = \phi_s(z) \tag{172}$$

Proof. Computing the Fourier transform (in z) of the function inside the curly brackets (using the relations in lemma 2.1) we get

$$\widehat{\psi}\left(\frac{\xi - c(\omega_{u_x})}{|\omega_{u_x}|}\right) - \widehat{\psi}\left(\frac{\xi - c(\omega_{v_x(2)})}{|\omega_{v_x(2)}|}\right)$$

which is equal to 1 in the support of $\widehat{\phi_s}$ for all $s \in \mathbf{T}_2$ that satisfy $|\omega_{v_x}| \leq |\omega_s| \leq |\omega_{u_x}|$ but vanishes on $\omega_{v_x(2)}$. □

Taking $z = x$ in yields

$$\begin{aligned} F_{2J}(x) = &\sum_{\substack{s\in\mathbf{T}_2 \\ |\omega_{v_x}|\leq|\omega_s|\leq|\omega_{u_x}|}} \epsilon_s\langle g|\phi_s\rangle\phi_s(x)\chi_E(x) \\ = &\left[\sum_{s\in\mathbf{T}_2} \epsilon_s\langle g|\phi_s\rangle\phi_s\right] * \left\{ \frac{M^{c(\omega_{u_x})}D^{|\omega_{u_x}|^{-1}}(\psi)}{|\omega_{u_x}|^{-\frac{1}{2}}} - \frac{M^{c(\omega_{v_x(2)})}D^{|\omega_{v_x(2)}|^{-1}}(\psi)}{|\omega_{v_x(2)}|^{-\frac{1}{2}}} \right\}(x)\chi_E(x) \end{aligned}$$

Since all s that appear in the definition of F_{2J} satisfy $|\omega_s| \leq (4|J|)^{-1}$, we have:

$$\begin{aligned} |F_{2J}(x)| &\leq 2\chi_E(x) \sup_{\delta>|\omega_{u_x}|^{-1}} \int_{\mathbb{R}} \left|\sum_{s\in\mathbf{T}_2} \epsilon_s\langle g|\phi_s\rangle\phi_s(z)\right| \frac{1}{\delta}\left|\psi\left(\frac{x-z}{\delta}\right)\right| dz \\ &\leq C \sup_{\delta>4|J|} \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} \left|\sum_{s\in\mathbf{T}_2} \epsilon_s\langle g|\phi_s\rangle\phi_s(z)\right| dz \end{aligned}$$

Where we used lemma 5.2 in the last inequality. **Observe that the maximal function above satisfies the property**

$$\sup_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|h(z)|dz\leq 2\inf_{x\in J}\sup_{\delta>2|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|h(z)|dz$$

Using this we obtain

$$\begin{aligned}
\Sigma_{22} &\leq \sum_{J\in\mathcal{J}}\|F_{2J}\|_{L^1(J)}\leq\sum_{J\in\mathcal{J}}\|F_{2J}\|_\infty|G_J| \\
&\leq C\sum_{\substack{J\in\mathcal{J}\\ J\subset 3I_{\mathrm{top}(\mathbf{T})}}}|E|\mathcal{M}(E;\mathbf{T})|J|\sup_{x\in J}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\left|\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s(z)\right|dz \\
&\leq 2C|E|\mathcal{M}(E;\mathbf{T})\sum_{\substack{J\in\mathcal{J}\\ J\subset 3I_{\mathrm{top}(\mathbf{T})}}}\int_J\sup_{\delta>2|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\left|\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s(z)\right|dzdx \\
&\leq C|E|\mathcal{M}(E;\mathbf{T})\left\|M\left(\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s\right)\right\|_{L^1(3I_{\mathrm{top}(\mathbf{T})})}
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator. Using that M is $(2,2)$-strong and Cauchy-Schwarz inequality,

$$\begin{aligned}
\left\|M\left(\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s\right)\right\|_{L^1(3I_{\mathrm{top}(\mathbf{T})})} &= \int_{\mathbb{R}}\chi_{3I_{\mathrm{top}(\mathbf{T})}}(z)\left|M\left(\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s\right)(z)\right|dz \\
&\leq C|I_{\mathrm{top}(\mathbf{T})}|^{\frac{1}{2}}\left\|\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s\right\|_{L^2(\mathbb{R})}
\end{aligned}$$

By lemma 5.4 we deduce

$$\left\|\sum_{s\in\mathbf{T}_2}\epsilon_s\langle g|\phi_s\rangle\phi_s\right\|_{L^2(\mathbb{R})}\leq C\left(\sum_{s\in\mathbf{T}_2}|\epsilon_s\langle g|\phi_s\rangle\phi_s|^2\right)^{\frac{1}{2}}\leq C'|I_{\mathrm{top}(\mathbf{T})}|^{\frac{1}{2}}\mathcal{E}(g;\mathbf{T})$$

where we used (159) to justify the second inequality above. We conclude that

$$\Sigma_{22}\leq C|E|\mathcal{M}(E;\mathbf{T})|I_{\mathrm{top}(\mathbf{T})}|\mathcal{E}(g;\mathbf{T})$$

which is exactly what we needed to prove. This completes the proof of this lemma.

5 USEFUL RESULTS

Proposition 5.1. *Fix $c_0 > 0$ and a function $\phi \in \mathcal{S}(\mathbb{R})$ such that its Fourier transform is supported in $[-\frac{3}{8}, \frac{3}{8}]$ and satisfies*

$$\sum_{l\in\mathbb{Z}} \left|\hat{\phi}\left(t+\frac{l}{2}\right)\right|^2 = c_0$$

for all $t \in \mathbb{R}$. Fix $m \in \mathbb{Z}$ and define

$$\phi_s(x) = 2^{-\frac{m}{2}}\phi(2^{-m}x-k)e^{2\pi i 2^{-m}x\frac{l}{2}}$$

where $s = [k2^m, (k+1)2^m) \times [l2^{-m}, (l+1)2^{-m})$ is a tile in $\mathbf{D}_m$. Prove that for all $f \in \mathcal{S}(\mathbb{R})$ we have:
(a)

$$\sum_{s\in\mathbf{D}_m} \langle f|\phi_s\rangle \, \phi_s = c_0 f$$

(b)

$$\|f\|_2^2 = \frac{\|\phi\|_2^2}{c_0^2}\sum_{s\in\mathbf{D}_m} |\langle f|\phi_s\rangle|^2$$

Proof of (a). Using the Poisson summation formula,

$$\begin{aligned}
\sum_{s\in\mathbf{D}_m} \phi_s(x)\overline{\hat{\phi}_s(y)} &= \sum_{s\in\mathbf{D}_m} \phi(2^{-m}x-k)\overline{\hat{\phi}\left(2^m y-\frac{l}{2}\right)}e^{2\pi i(2^{-m}x\frac{l}{2}+2^m yk-\frac{l}{2}k)} \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \phi(2^{-m}x-k)\overline{\widehat{\phi}\left(2^m y-\frac{l}{2}\right)}e^{2\pi i(2^{-m}x\frac{l}{2}+2^m yk-\frac{l}{2}k)} \\
&= \sum_{l\in\mathbb{Z}} \overline{\widehat{\phi}\left(2^m y-\frac{l}{2}\right)}e^{2\pi i(2^{-m}x\frac{l}{2})}\left(\sum_{k\in\mathbb{Z}} \phi(2^{-m}x-k)e^{2\pi i(\frac{l}{2}-2^m y)(-k)}\right) \\
&= \sum_{l\in\mathbb{Z}} \overline{\widehat{\phi}\left(2^m y-\frac{l}{2}\right)}e^{2\pi i(2^{-m}x\frac{l}{2})}\left(\sum_{k\in\mathbb{Z}} \tau_{-2^{-m}x}\phi(-k)e^{2\pi i(\frac{l}{2}-2^m y)(-k)}\right) \\
&= \sum_{l\in\mathbb{Z}} \overline{\widehat{\phi}\left(2^m y-\frac{l}{2}\right)}e^{2\pi i(2^{-m}x\frac{l}{2})}\left(\sum_{k\in\mathbb{Z}} \widehat{\tau_{-2^{-m}x}\phi}\left(2^m y-\frac{l}{2}+k\right)\right) \\
&= \sum_{l\in\mathbb{Z}} \overline{\widehat{\phi}\left(2^m y-\frac{l}{2}\right)}e^{2\pi i(2^{-m}x\frac{l}{2})}\widehat{\tau_{-2^{-m}x}\phi}\left(2^m y-\frac{l}{2}\right) \\
&= \sum_{l\in\mathbb{Z}} \overline{\widehat{\phi}\left(2^m y-\frac{l}{2}\right)}\widehat{\phi}\left(2^m y-\frac{l}{2}\right)e^{2\pi i xy} \\
&= c_0 e^{2\pi i xy}
\end{aligned}$$

Multiplying both sides of the equation obtained by $\widehat{f}$ and integrating on y, using Parseval and Fourier inversion, we conclude this item. ☐

Proof of (b). Take L^2 norms on both sides of the previous item, use the orthogonality of the functions ϕ_s and the fact that $\|\phi_s\|_2 = \|\phi\|_2$ for all s and we are done. ☐

Proposition 5.2.

$$\int_{-\infty}^{\infty} \frac{sin(bx)}{x} dx = \pi \mathrm{sgn}(b)$$

Proof. Let I be the integral above. If $b > 0$ we make the change of variables $bx = y$ and get

$$\frac{I}{2} = \int_0^{\infty} \frac{sin(bx)}{x} dx = \int_0^{\infty} \frac{sin(y)}{y} dy = \int_0^{\infty} \mathcal{L}(sin(y))(s) ds = \int_0^{\infty} \frac{ds}{1+s^2} = \frac{\pi}{2}$$

where $\mathcal{L}$ is the Laplace transform. If $b < 0$, observe that $sin(bx) = -sin(-bx)$ and proceed the same way. In any case we get

$$I = \pi \mathrm{sgn}(b)$$

☐

Proposition 5.3.

$$\lim_{\epsilon \to 0} \int_{\epsilon < |t| < \frac{1}{\epsilon}} \frac{e^{-2\pi i t(\xi - \eta)}}{t} dt = -i\pi \mathrm{sgn}(\xi - \eta)$$

Proof.

$$\begin{aligned}\lim_{\epsilon \to 0} \int_{\epsilon < |t| < \frac{1}{\epsilon}} \frac{e^{-2\pi i t(\xi - \eta)}}{t} dt &= -i \mathrm{sgn}(\xi - \eta) \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2\pi} < |t| < \frac{1}{2\pi\epsilon}} \frac{sin(|\xi - \eta| t)}{t} dt \\ &= -i\pi \mathrm{sgn}(\xi - \eta)\end{aligned}$$

by proposition 5.2. ☐

Lemma 5.1. *Let $k > 0$ be a function on $[0, \infty)$ that is continuous except at a finite number of points. Suppose that $K(x) = k(|x|)$ is an integrable function on $\mathbb{R}^n$ and satisfies*

$$|x| \leq |y| \Rightarrow K(x) \geq K(y)$$

In other words, k is decreasing. Define K_ϵ as

$$K_\epsilon(x) = \frac{1}{\epsilon^n} K\left(\frac{x}{\epsilon}\right)$$

Then the following estimate is true:

$$\sup_{\epsilon > 0} \left(|f| * K_\epsilon\right)(x) \leq \|K\|_{L^1} M(f)(x)$$

for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. See theorem 2.1.10 in GRAFAKOS (2014a). □

Lemma 5.2. *Let* $K(x) = (1+|x|)^{-n-\delta}$ *be defined on* $\mathbb{R}^n$. *There exists a constant* $C_{n,\delta}$ *such that for all* $\epsilon_0 > 0$ *we have the estimate*

$$\sup_{\epsilon>\epsilon_0}(|f| * K_\epsilon)(x) \leq C_{n,\delta} \sup_{\epsilon>\epsilon_0} \frac{1}{\epsilon^n} \int_{|y-x|\leq\epsilon} |f(y)|dy$$

for all f *locally integrable on* $\mathbb{R}^n$.

Proof. Apply a minor and convenient modification to the proof of lemma 5.1. □

Lemma 5.3. *Let* (X,μ) *be a* σ*-finite measure space and let* $0 < p < \infty$. *For all measurable functions* g_n *on* X *we have:*

$$\| \liminf_{n\to\infty} |g_n| \|_{L^{p,\infty}} \leq C_p \liminf_{n\to\infty} \|g_n\|_{L^{p,\infty}}$$

Proof. See GRAFAKOS (2014a), 1.1.12, page 14. □

Lemma 5.4. *Let* ϕ_s *be as in definition 4.1 and let* $\mathbf{T}_2$ *be a 2-tree. There is a constant* C *such that for all sequences of complex scalars* $\{\lambda_s\}_{s\in\mathbf{T}_2}$ *we have*

$$\left\| \sum_{s\in\mathbf{T}_2} \lambda_s \phi_s \right\|_{L^2(\mathbb{R})} \leq C \left(\sum_{s\in\mathbf{T}_2} |\lambda_s|^2 \right)^{\frac{1}{2}}$$

Sketch of the proof. Define $\mathcal{G}_m = \{s \in \mathbf{T}_2 : |I_s| = 2^m\}$. Then for $s \in \mathcal{G}_m$ and $s' \in \mathcal{G}_{m'}$, the functions ϕ_s and $\phi_{s'}$ are orthogonal to each other, and it suffices to obtain the corresponding estimate when the summation is restricted to a given $\mathcal{G}_m$. But for $s \in \mathcal{G}_m$, the intervals I_s are disjoint and we may use the idea of the proof of lemma 4.2. Use that $\sum_{u:\omega_u=\omega_s} |\langle \phi_s | \phi_u \rangle| \leq C$ for every fixed s. □

Proposition 5.4. *We have the following characterizations of operators in an Euclidean space with certain properties of symmetry:*

(a) Let T *be a bounded linear operator mapping* $L^1(\mathbb{R}^n)$ *to itself. Then a necessary and sufficient condition that* T *commutes with translations is that there exists a measure* $\mu \in \mathcal{B}(\mathbb{R}^n)$ *so that* $Tf = f * \mu$, *for all* $f \in L^1(\mathbb{R}^n)$. *One has then* $\|T\| = \|\mu\|$.

(b) Let T *be a bounded linear operator mapping* $L^2(\mathbb{R}^n)$ *to itself. Then a necessary and sufficient condition that* T *commutes with translations is that there exists a bounded measurable function* $m(y)$ *(called multiplier) such that* $\widehat{Tf}(y) = m(y)\hat{f}(y)$, *for all* $L^2(\mathbb{R}^n)$. *One has then* $\|T\| = \|f\|_\infty$.

Proof of (a). Define

$$\phi(x) = \begin{cases} c\, e^{|x|^2/(|x|^2-1)} & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1 \end{cases}$$

where c is chosen such that $\int_{\mathbb{R}^n}\phi(x)dx = 1$. Define also $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$. We have that $\|T\phi_\epsilon\|_1$ is bounded as $\epsilon \to 0$, so there is a sequence $\{\epsilon_k\}$ and a Radon measure μ such that $T\phi_{\epsilon_k} \to \mu$ weakly in L^1. Since T is continuous, linear and commutes with translation,

$$\begin{aligned} f*\mu(x) &= \lim_{k\to\infty}\int_{\mathbb{R}^n} f(y)\, T\phi_{\epsilon_k}(x-y)\, dy \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} f(y)\, T(\phi_{\epsilon_k}(x-y))\, dy \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} T(f(y)\, \phi_{\epsilon_k}(x-y))\, dy \\ &= \lim_{k\to\infty} T\left(\int_{\mathbb{R}^n} f(y)\, \phi_{\epsilon_k}(x-y)\, dy\right) \\ &= T\left(\lim_{k\to\infty}\int_{\mathbb{R}^n} f(y)\, \phi_{\epsilon_k}(x-y)\, dy\right) \\ &= Tf(x) \end{aligned}$$

The other direction is trivial. □

Proof of (b). Let $\psi(x) = e^{-\pi x^2}$ so that $\hat{\psi}(\xi) = \psi(\xi)$.

$$\begin{aligned} \psi(\xi)(Tf)^\wedge(\xi) &= (\psi * Tf)^\wedge(\xi) \\ &= \left(\int_{\mathbb{R}^n}\psi(x-y)\, Tf(y)\, dy\right)^\wedge(\xi) \\ &= \left(\int_{\mathbb{R}^n}\psi(y)\, Tf(x-y)\, dy\right)^\wedge(\xi) \\ &= \left(\int_{\mathbb{R}^n}T^*\psi(y)\, f(x-y)\, dy\right)^\wedge(\xi) \\ &= (T^*\psi)^\wedge(\xi)\hat{f}(\xi) \end{aligned}$$

Let $m(\xi) = (T^*\psi)^\wedge(\xi)/\psi(\xi)$, then we have

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$$

and therefore, $\|m\|_{L^\infty} = \|T\|_{L^2}$. The other direction is also trivial. □

Proposition 5.5. *For ϕ_s as in definition 4.1 we have:*
(a)

$$\sum_{\substack{s'\in\mathbf{D}_m \\ \omega_{s'}=\omega_s}} |\langle\phi_s|\phi_{s'}\rangle| \leq C \sum_{\substack{s'\in\mathbf{D}_m \\ \omega_{s'}=\omega_s}} \left(1+\frac{dist(I_s, I_{s'})}{2^m}\right)^{-10} \leq C_1$$

(b) If $|I_u| < |I_s|$, then for M big enough:

$$|\langle \phi_s | \phi_u \rangle| \leq C_M \left(\frac{|I_s|}{|I_u|} \right)^{\frac{1}{2}} \int_{I_u} \frac{|I_s|^{-1} dx}{\left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^M}$$

Proof of (a). Let Φ be the set of all ϕ_s in the sense of definition 4.1 where s is a dyadic rectangle of scale m and $\rho : \Phi \mapsto \mathbf{D}_m$ given by $\rho(\phi_s) = I_s \times \omega_s$, which we already know to be injective since different dyadic tiles of same scale are disjoint. In the proof of 4.2 we proved (25). (26) follows from the definition of ϕ_s and (27) is obvious since $|I_s||\omega_s| = |I_{s'}||\omega_{s'}| = 1$ for all $s, s' \in \mathbf{D}_m$. Remember that we are now considering only tiles for which $\omega_{s'} = \omega_s$, so $c(I_s) \neq c(I_{s'})$ and $2|I_s| \geq |I_{s'}|$. This shows that ρ is a phase plane representation and we are in the same setting of lemma 3.4. Following its proof (and using the same conventions) we have:

$$\begin{aligned} |\langle \phi_s, \phi_{s'} \rangle| &\leq C|I_s|^{\frac{1}{2}}|I_{s'}|^{-\frac{1}{2}} \left(1 + \frac{|c - c(I_{s'})|}{|I_{s'}|}\right)^{-10} + C|I_{s'}|^{\frac{1}{2}}|I_s|^{-\frac{1}{2}} \left(1 + \frac{|c - c(I_s)|}{|I_s|}\right)^{-10} \\ &\leq \tilde{C} \left(1 + \frac{dist(I_s, I_{s'})}{2^m}\right)^{-10} \end{aligned}$$

Summing over all tiles $s, s' \in \mathbf{D}_m$ such that $\omega_{s'} = \omega_s$ we obtain (a). □

Proof of (b). Just use the identity

$$\langle \phi_s | \phi_u \rangle \leq C_M \frac{\min\left(\frac{|I_s|}{|I_u|}, \frac{|I_u|}{|I_s|}\right)^{\frac{1}{2}}}{\left(1 + \frac{|c(I_s) - c(I_u)|}{\max(|I_s|, |I_u|)}\right)^M}$$

and the fact that

$$\left| \frac{|x - c(I_s)|}{|I_s|} - \frac{|c(I_u) - c(I_s)|}{|I_s|} \right| \leq \frac{1}{2}$$

for all $x \in I_u$. □

Lemma 5.5. *There is a constant $C > 0$ such that for any interval J and any $b > 0$ we have:*

$$\int_J \int_{J^c} \frac{dxdy}{\left(1 + \frac{|x-y|}{b|J|}\right)^{20}} = Cb^2|J|^2$$

Proof. If $J = (\alpha, \beta)$ and we make the change of variables

$$x \to x - \alpha$$

$$y \to y - \alpha$$

then the integral above becomes

$$I = \int_{(0,|J|)} \int_{(0,|J|)^c} \frac{dxdy}{\left(1 + \frac{|x-y|}{b|J|}\right)^{20}}$$

Now we proceed with the change

$$x \to \frac{x}{b|J|}$$
$$y \to \frac{y}{b|J|}$$

and we get

$$I = b^2|J|^2 \int_{\left(0,\frac{1}{b}\right)} \int_{\left(0,\frac{1}{b}\right)^c} \frac{dxdy}{(1 + |x-y|)^{20}} = b^2|J|^2 \int_{\left(0,\frac{1}{b}\right)^c} \int_{\left(0,\frac{1}{b}\right)} \frac{dxdy}{(1 + |x-y|)^{20}}$$

We only need to show that this integral is dominated by a constant C. Indeed,

$$I_1(x) = \int_{\frac{1}{b}}^{\infty} \frac{dy}{(1+y-x)^{20}} = \int_{\frac{1}{b}-x}^{\infty} \frac{dt}{(1+t)^{20}} = \frac{1}{19\left(1+\frac{1}{b}-x\right)^{19}}$$

$$I = \int_0^{\frac{1}{b}} I_1(x)dx = \int_0^{\frac{1}{b}} \frac{dx}{19\left(1+\frac{1}{b}-x\right)^{20}} = \frac{1}{19}\int_1^{1+\frac{1}{b}} \frac{dz}{z^{19}} = \frac{1}{19}\left(\frac{1}{18} - \frac{1}{18\left(1+\frac{1}{b}\right)^{18}}\right) = C$$

□

6 CONCLUSION

We conclude by discussing the proof of theorem 1.4 as a consequence of the stronger version of Carleson's theorem. It is known that the Carleson operator is not only (2, 2)-weak, but also (p, p)-strong for all $1 < p < \infty$. For $f \in C^\infty(\mathbb{T}^1)$ and $R > 0$ we have

$$f * D_R(x) = \sum_{\substack{k\in\mathbb{Z}\\|k|\leq\lfloor R\rfloor}} \widehat{f}(k)e^{2\pi ikx}$$

where D_R is the Dirichlet kernel

$$D_R(x) = \sum_{\substack{k\in\mathbb{Z}\\|k|\leq\lfloor R\rfloor}} e^{2\pi ikx}$$

We have the following theorem:

Theorem 6.1. *There is a constant C such that for all $f \in C^\infty(\mathbb{T}^1)$ we have*

$$\left\| \sup_{N\in\mathbb{Z}^+} |f * D_N| \right\|_{L^2(\mathbb{T}^1)} \leq C\|f\|_{L^2(\mathbb{T}^1)} \tag{173}$$

By an argument similar to the given in claims 4.1 and 4.2 we see that theorem 6.1 implies theorem 1.4.

Definition 6.1. *Given $1 \leq p < \infty$ we denote my $\mathcal{M}_p(\mathbb{R}^n)$ the set of all bounded functions m on $\mathbb{R}^n$ such that the operator defined on $\mathcal{S}(\mathbb{R}^n)$ by*

$$T_m(f) = (m\widehat{f})^\vee$$

is bounded from $L^p(\mathbb{R}^n)$ to itself. The norm of m in $\mathcal{M}_p(\mathbb{R}^n)$ is denoted by

$$\|m\|_{\mathcal{M}_p} := \|T_m\|_{L^p\to L^p}$$

Definition 6.2. *Given $1 \leq p < \infty$ we denote my $\mathcal{M}_p(\mathbb{Z}^n)$ the set of all bounded sequences $\{a_m\}_{m\in\mathbb{Z}^n}$ such that the operator defined on $C^\infty(\mathbb{T}^n)$ by*

$$T(f)(x) = \sum_{m\in\mathbb{Z}^n} a_m\widehat{f}(m)e^{2\pi im\cdot x}$$

is bounded from $L^p(\mathbb{Z}^n)$ to itself. The norm of $\{a_m\}_{m\in\mathbb{Z}^n}$ in $\mathcal{M}_p(\mathbb{Z}^n)$ is denoted by

$$\|\{a_m\}_m\|_{\mathcal{M}_p} := \|Tf\|_{L^p\to L^p}$$

Definition 6.3. *Let $t_0 \in \mathbb{R}^n$. A bounded function b on $\mathbb{R}^n$ is called* regulated at t_0 *if*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \int_{|t| \leq \epsilon} (b(t_0 - t) - b(t_0))dt = 0$$

b is called regulated *if it is regulated at every $t_0 \in \mathbb{R}^n$.*

Let b be a bounded function defined on $\mathbb{R}^n$. For $R > 0$ we introduce the multiplier operators

$$S_{b,R}(f)(x) = \sum_{m \in \mathbb{Z}^n} b(m/R)\widehat{F}(m)e^{2\pi imx}, \quad f \in C^\infty(\mathbb{T}^n)$$
$$T_{b,R}(f)(x) = \int_{\mathbb{R}^n} b(\xi/R)\widehat{f}(\xi)e^{2\pi i\xi x}dx, \quad f \in C_c^\infty(\mathbb{R}^n)$$

We introduce the maximal operators

$$\begin{aligned} M_b(f)(x) &= \sup_{R>0} |S_{b,R}(f)(x)|, \\ N_b(f)(z) &= \sup_{R>0} |T_{b,R}(f)(x)|, \end{aligned} \tag{174}$$

And we have the following theorem:

Theorem 6.2. *Let b be a function defined on $\mathbb{R}^n$, regulated and integrable on any cube. Assume also that $t \mapsto b(\xi/t)$ has only countably many discontinuities on $\mathbb{R}^+$. Let $1 < p < \infty$ and $C_p < \infty$ and suppose that $b \in \mathcal{M}_p(\mathbb{R}^n)$. Let M_b and N_b as in* (174). *Then the following are equivalent:*

$$\|M_b(f)\|_{L^p(\mathbb{T}^n)} \leq C_p\|b\|_{\mathcal{M}_p}\|f\|_{L^p(\mathbb{T}^n)}, \quad f \in C^\infty(\mathbb{T}^n)$$
$$\|N_b(f)\|_{L^p(\mathbb{R}^n)} \leq C_p\|b\|_{\mathcal{M}_p}\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_c^\infty(\mathbb{R}^n)$$

Proof. See GRAFAKOS (2014a) page 282. □

Proof of theorem 6.1. Consider the following function defined on $\mathbb{R}$:

$$b(x) = \begin{cases} 1 & \text{when } |x| < 1, \\ \frac{1}{2} & \text{when } |x| = 1, \\ 0 & \text{when } |x| > 1. \end{cases}$$

then b is bounded, integrable over any interval and regulated. Also, given any $x \in \mathbb{R}$, the function $t \mapsto b(x/t)$ is discontinuous only for $t \in \{x, -x\}$. Let $S_{b,R}$ be as above with b being the function we just defined.

Claim 6.1. *The inequality*

$$\left\|\sup_{R>0}|S_{b,R}(f)|\right\|_{L^p} \leq C_p'\|f\|_{L^p}$$

implies theorem 6.1

Proof of claim 6.1. We have:

$$S_{b,R}(f)(x) = \begin{cases} \displaystyle\sum_{m\leq\lfloor R\rfloor} \widehat{f}(m)e^{2\pi imx} & \text{if } R \notin \mathbb{Z}^+, \\ D_{R-1} * f(x) + \dfrac{\widehat{f}(R)e^{2\pi ixR} + \widehat{f}(-R)e^{-2\pi ixR}}{2} & \text{if } R \in \mathbb{Z}^+, \end{cases} \tag{175}$$

Since $\sup_{R>0}|\widehat{f}(\pm R)| \leq \|f\|_{L^1} \leq \|f\|_{L^p}$, it follows that if the above holds, then theorem 6.1 holds with constant $C_p' = C_p + 1$ □

To use theorem 6.2 we only need to prove that $b \in \mathcal{M}_p$. To this end we invoke the following:

Theorem 6.3. *Suppose that b is a bounded function on $\mathbb{R}^n$, integrable over any cube. Suppose that the sequences $\{b(m/R)\}_{m\in\mathbb{Z}^n}$ are in $\mathcal{M}_p(\mathbb{Z}^n)$ uniformly in $R > 0$ for some $1 < p < \infty$ for some $1 < p < \infty$. Then $b \in \mathcal{M}_p$ and we have*

$$\|b\|_{\mathcal{M}_p(\mathbb{R}^n)} \leq \sup_{R>0}\|\{b(m/R)\}_{m\in\mathbb{Z}^n}\|_{\mathcal{M}_p(\mathbb{Z}^n)}$$

Proof. See GRAFAKOS (2014a) page 279. □

Observe that

$$\sup_{R>0}\|f * D_R\|_{L^p(\mathbb{T}^1)} = \sup_{N\in\mathbb{Z}^+}\|f * D_N\|_{L^p(\mathbb{T}^1)} \leq C_p''\|f\|_{L^p(\mathbb{T}^1)}$$

where the last estimate follows from the L^p convergence theory of Fourier series. The equality is due to the fact that $D_N = D_{N+\epsilon}$ if $0 < \epsilon < 1$. Also,

$$\|b(\cdot/R)\|_{\mathcal{M}_p(\mathbb{Z})} = \|f * D_R\|_{L^p(\mathbb{T}^1)} \quad \text{if } R \notin \mathbb{Z}^+$$
$$\|b(\cdot/R)\|_{\mathcal{M}_p(\mathbb{Z})} \leq \|f * D_{R-1}\|_{L^p(\mathbb{T}^1)} + \|f\|_{L^p(\mathbb{T}^1)} \quad \text{if } R \in \mathbb{Z}^+$$

Then $\sup_{R>0}\|b(\cdot/R)\|_{\mathcal{M}_p(\mathbb{Z})} < \infty$. By theorem 6.3, b lies in $\mathcal{M}_p$. We can now use theorem 6.2 in the particular case $p = 2$ and $n = 1$. The $L^2(\mathbb{R})$ boundedness of the maximal operator

$$N_{b,R}(f)(x) = \sup_{R>0}\left|\int_{-R}^{R}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi\right| \tag{176}$$

is equivalent to that of

$$M_{b,R}(f)(x) = \sup_{R>0} \left| \sum_{m\in\mathbb{Z}} \widehat{f}(m) e^{2\pi imx} b\left(\frac{m}{R}\right) \right| = \sup_{R>0} |S_{b,R}(f)(x)| \tag{177}$$

But (176) is the Carleson operator, which we said in the beginning of this section that is (p, p)-strong for $1 < p < \infty$. In particular, it is $(2, 2)$-strong. In view of (175) and of the fact that $\sup_{R>0} |\widehat{f}(\pm R)| \leq \|f\|_{L^p}$, the L^p boundedness of (177) is equivalent to the conclusion of theorem 6.1. This completes the proof. $\square$

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