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FRANCISCO DE ASSIS BENJAMIM FILHO

A PARTIAL ANSWER TO THE CPE CONJECTURE, DIAMETER ESTIMATES AND MANIFOLDS WITH CONSTANT ENERGY

FORTALEZA

2015

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Thesis submitted to the Post-graduate Program of the Mathematical Department of Universidade Federal do Ceará in partial fulfillment of the necessary requirements for the degree of Ph.D. in Mathematics. Area of expertise: Differential Geometry.

Adviser: Prof. Dr. Abdênago Alves de Barros.

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"Maior do que a tristeza de não haver vencido
é a vergonha de não ter lutado."

(Rui Barbosa)

RESUMO

Esta tese está dividida em quatro partes. Na primeira delas estudaremos pontos críticos do funcional curvatura escalar total restrito ao espaço das métricas de curvatura escalar constante e volume unitário. Provaremos que sob certas condições integrais convenientes os pontos críticos de tal funcional são variedades de Einstein provando assim a conjectura dos pontos críticos neste caso.

Na segunda parte, veremos duas estimativas para o primeiro autovalor do Laplaciano de uma variedade compacta com curvatura de Ricci limitada por baixo por uma constante. As estimativas que obtemos melhoram a estimativa correspondente provada por Li e Yau (1980).

Na terceira parte, estamos interessados em estimar o diâmetro de hipersuperfícies mínimas da esfera. A estimativa que encontramos depende apenas do primeiro autovalor do Laplaciano da hipersuperfície considerada. Para superfícies imersas na esfera de dimensão três, obtemos uma estimativa ligeiramente melhor do que a obtida no caso de dimensão mais alta.

Na última parte, introduzimos o conceito de variedade de energia constante e provamos que a esfera e o toro são as únicas superfícies que têm energia constante. Em dimensão mais alta a situação é bem diferente uma vez que o produto de uma esfera por qualquer variedade compacta tem energia constante. Entretanto, se impusermos uma condição sobre a curvatura de Ricci, é possível caracterizar a esfera também neste caso. Em seguida, aplicamos as informações obtidas ao estudo de hipersuperfícies da esfera provando alguns resultados de rigidez desde que a hipersuperfície tenha energia constante.

Palavras-chave: Funcional curvatura escalar total. Primeiro autovalor do Laplaciano. Diâmetro de hipersuperfícies da esfera. Variedades com energia constante.

ABSTRACT

This thesis is divided into four parts. In the first one we study the critical points of the total scalar curvature functional restricted to the space of metrics with constant scalar curvature and volume one. We shall prove that under certain suitable integral conditions the critical points of such functional are Einstein manifolds proving this way the critical point equation conjecture in this case.

In the second part, we will provide an estimate for the first eigenvalue of the Laplacian of a compact manifolds with Ricci curvature bounded from below by a constant. The estimate we obtain improves the corresponding estimate proved by Li and Yau (1980).

In the third part, we are interested in to estimate the diameter of minimal hypersurfaces of the sphere. The estimate we get depends only on the first eigenvalue of the Laplacian of the considered hypersurface. For immersed surfaces on the three dimensional sphere, we obtain an estimate slightly better than the one obtained in the case of higher dimension.

In the last part, we introduce the concept of manifolds with constant energy and prove that the sphere and the torus are the only compact surfaces that have constant energy. For higher dimension, the situation is very different since the product of the sphere with any compact manifold has constant energy. Nevertheless, if we impose a condition over the Ricci curvature it is possible to characterize the sphere also in this case. After that, we apply the informations obtained to the study of hypersurfaces of the sphere proving some rigidity results provided that the hypersurface has constant energy.
Keywords: Total scalar curvature functional. First eigenvalue of the Laplacian. Diameter of hypersurfaces of the sphere. Manifolds with constant energy.

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1 INTRODUCTION

This thesis is divided into four parts. The first part deals with the critical points of the total scalar curvature functional. Given M^n a compact oriented manifold, denote by $\mathcal{M}$ the set of all smooth Riemannian metrics on M of volume 1. For $g \in \mathcal{M}$ we define the total scalar curvature or the Einstein-Hilbert functional $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{S}(g) = \int_M R_g dM_g \tag{1}$$

where R_g and dM_g stand for, respectively, the scalar curvature and the volume form determined by g and the orientation. It is well known that the critical points of the functional $\mathcal{S}$ are Einstein, see Besse (1987) for more details. By the solution of the Yamabe problem there exists a constant scalar curvature metric in every conformal class of Riemannian metrics on a compact manifold M^n. So, we can define the nonempty set $\mathcal{C} = \{g \in \mathcal{M}; R_g \text{ is constant}\}$.

The Euler-Lagrangian equation of Einstein-Hilbert action (1) on the space $\mathcal{C}$ is given by

$$Ric - \frac{R}{n}g = \nabla^2 f - f\big(Ric - \frac{R}{n-1}g\big). \tag{2}$$

Definition 1. *A CPE metric is a 3-tuple $(M^n,\, g,\, f)$, where (M^n, g) is a compact oriented Riemannian manifold of dimension $n \geq 3$ with constant scalar curvature while f is a smooth potential satisfying equation (2).*

In the middle of 1980's, Besse (1987) proposed the following conjecture:

Conjecture 1. *A CPE metric is always Einstein.*

Conjecture 1 has been proved in many particular cases but its general solution has not been presented yet. Recently, Leandro (2015) has proved Conjecture 1 under a condition on the first eigenfunction, namely:

$$h := |\nabla f|^2 + \frac{R}{n(n-1)}f^2 = \Lambda, \tag{3}$$

where Λ is a constant.

Our first goal on this thesis is to prove some results concerning the CPE conjecture.

Theorem 1. *Let $(M^n,\, g,\, f)$ be a CPE metric. Then M^n is isometric to round sphere and f is a first eigenfunction of the Laplacian, provided that*

1. $\int_M |\nabla f|^4 dM_g = \frac{(n+2)R^2}{3n(n-1)^2}\int_M f^4 dM_g$ *and*
2. $\int_M f^3 dM_g \geq 0.$

As a consequence of the previous theorem we deduce the following result that guarantees the CPE conjecture under a weaker hypothesis than (3).

Corollary 1. *The same conclusion of Theorem 1 is true provided that the function $h =$*

$|\nabla f|^2 + \frac{R}{n(n-1)} f^2$ *is constant along of the flow of* ∇f.

In the second part, we present a new estimate for the first eigenvalue of the Laplacian of a compact Riemannian manifold with Ricci curvature bounded from below as follows:

Theorem 2. *Let* M^m *be a compact Riemannian manifold without boundary. Suppose that the Ricci curvature of* M *satisfies* $Ric_M \geq -R\,(m-1)\,,$ *with* $R \geq 0$. *Then the first eigenvalue* λ *of* M *satisfies*

$$\lambda \;\geq\; b_0\Big(1+\eta\Big),$$

where

$$\begin{aligned}\eta \;=\; & \Big(\frac{2}{m+1}e^{-(1+\sqrt{\mu})} + \frac{2}{(m+1)^2}(1+\sqrt{\mu})e^{-2(1+\sqrt{\mu})} + \frac{4(1+\sqrt{\mu})^3}{3(m+1)^3\mu^{3/2}}e^{-3(1+\sqrt{\mu})} \\ & + \frac{2(3+\mu+3\sqrt{\mu})(1+\sqrt{\mu})^3}{3(m+1)^4\mu^{5/2}}e^{-4(1+\sqrt{\mu})}\Big),\end{aligned}$$

$b_0 := \frac{2}{(m+1)d^2}\Big(1+\sqrt{\mu}\Big)e^{-(1+\sqrt{\mu})}$, $\mu := 1+(n-1)^2 d^2 K^2$ *and* $d = d(M)$ *is the diameter of* M.

In the third part, we are concerned with hypersurfaces of the Euclidean sphere. More precisely, we will use the technique of gradient estimate developed by Li and Yau (2012) to obtain a lower bound for the diameter $d(M)$ of a compact Riemannian manifold M^n which is minimally immersed as a hypersurface of the Euclidean sphere $\mathbb{S}^{n+1}$. More precisely, we shall prove the following results:

Theorem 3. *Let* $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *be a minimal immersion of a compact Riemannian manifold* Σ^n *and let* φ *be an eigenfunction of* M^n, *i.e.* $\Delta\varphi = -\lambda\varphi$. *If the second fundamental form of* M^n *has constant length, and if the maximum of the function* $F = |\nabla u|^2 + (2|A|^2+\lambda)u^2$ *is not a critical point of* u *where* $u = \varphi - a$ *with* a *being a constant (see the discussion below), then we have* $d \geq \frac{\pi}{\sqrt{3\lambda}}$.

For surfaces we can obtain a essentially better estimate for the diameter as we can see in the following result.

Theorem 4. *Let* $\psi : \Sigma^2 \looparrowright \mathbb{S}^3$ *be a minimal immersion of a compact Riemannian surface* Σ^2 *and let* φ *be an eigenfunction of* M^2 *i.e.* $\Delta\varphi = -\lambda\varphi$. *If the second fundamental form of* M^2 *has constant length, and if the maximum of the function* $F = |\nabla u|^2 + (|A|^2+\lambda)u^2$ *is not a critical point of* u *then we have* $d \geq \frac{\pi}{\sqrt{2\lambda}}$.

In the fourth part we will introduce the concept of manifolds with constant energy. We will say that an eigenfunction of the Laplacian f has constant energy if it verifies the condition

$$|\nabla f|^2 + \kappa f^2 = \kappa \tag{4}$$

for some constant κ. In particular, M has constant energy if some eigenfunction of M

has constant energy.

Theorem 5. *Let M^2 be a compact Riemannian surface and let f be an eigenfunction of M for which (4) holds. Then we have either $\lambda = 2\kappa$ or $\lambda = \kappa$. Moreover, if the first case occurs, then M^2 is isometric to a standard sphere $\mathbb{S}^2(r)$, while in the second one M^2 is isometric to a flat torus $\mathbb{T}^2$ foliated by geodesics.*

As a consequence we obtain the following result.

Theorem 6. *Let $\varphi : M^2 \looparrowright \mathbb{S}^3$ be a compact immersion with constant mean curvature into the Euclidean sphere $\mathbb{S}^3$. If $\Delta f + \lambda f = 0$ on M and $|\nabla f|^2 + \kappa f^2 = \kappa$, then either M^2 is isometric to a geodesic sphere or to a Clifford torus.*

In the sequence we focus our attention on the diameter of manifolds with constant energy. More precisely we prove the following result.

Theorem 7. *Let (M^n, g) be a compact Riemannian manifold and let f be a function on M^n such that $\Delta f + \lambda f = 0$ for which (4) holds. Then*

$$d \geq \frac{\pi}{\sqrt{\kappa}}. \tag{5}$$

Moreover, if $Ric \geq (n-1)\kappa g$, then M^n is isometric to a standard sphere $\mathbb{S}^n\big(\frac{1}{\sqrt{\kappa}}\big)$.

This result has the following consequence.

Corollary 2. *Let $\varphi : M^n \to \mathbb{S}^{n+1}$ be a compact minimal immersion into the Euclidean sphere $\mathbb{S}^{n+1}$. If $\Delta f + \lambda f = 0$ and $|\nabla f|^2 + \frac{\lambda}{n} f^2 = \frac{\lambda}{n}$, where λ is the first eigenvalue of the Laplacian of M^n, then $d \geq \pi$.*

Theorem 8. *Let M^n be a compact Riemannian manifold and let f be a function on M^n such that $\Delta f + \lambda f = 0$ for which (4) holds. If*

$$\int_M Ric(\nabla f, \nabla f) dM_g \geq \kappa(n-1) \int_M |\nabla f|^2 dM_g,$$

then M^n is isometric to a standard sphere $\mathbb{S}^n\big(\frac{1}{\sqrt{\kappa}}\big)$.

2 PRELIMINARIES

Here we will collect some definitions and results. Given two operators $S, T : \mathcal{H} \to \mathcal{H}$ defined over a finite dimensional Hilbert space $\mathcal{H}$ the Hilbert-Schmidt inner product is defined according to

$$\langle S, T\rangle = \mathrm{tr}\big(\mathrm{ST}^*\big), \tag{6}$$

where tr and $*$ denote, respectively, the trace and the adjoint operation. The Hilbert-Schmidt norm for a $(0, 2)$-tensor in a Riemannian manifold (M^n, g) is the one induced from this inner product. If we set $T_{ij} = T(\partial_i, \partial_j)$, $S_{ij} = S(\partial_i, \partial_j)$ where ∂_i is the coordinate basis of a coordinate system and if we denote by (g_{ij}) and g^{ij} the matrix of the metric and the inverse of the metric, respectively then we get

$$\langle S, T\rangle = \sum_{ijkl} g^{ik} g^{jl} S_{ij} T_{kl}.$$

In particular, for an orthonormal basis $\{e_1, ..., e_n\}$ after identifying the $(0, 2)$-tensor with its correponding $(1, 1)$-tensor i.e. $S(e_i, e_j) = \langle Se_i, e_j\rangle$ we can obtain

$$\begin{aligned}\langle S, T\rangle &= \sum_{ij} S_{ij}T_{ij} = \sum_{ij}\langle Se_i, e_j\rangle\langle Te_i, e_j\rangle \\ &= \sum_{i}\langle Se_i, \sum_{j}\langle Te_i, e_j\rangle = \sum_{i}\langle Se_i, Te_i\rangle.\end{aligned}$$

For example, if $S = \mu g$ then $\langle S, T\rangle = \mu \cdot tr(T)$.

Moreover, if I denotes the identity operator on $\mathcal{H}$ of dimension n the traceless of an operator T is given by

$$\mathring{T} = T - \frac{\mathrm{trT}}{n} I. \tag{7}$$

In particular, the norm of $\mathring{T}$ is given by

$$|\, \mathring{T} \,|^2 = |\, T \,|^2 - \frac{(\mathrm{trT})^2}{n}. \tag{8}$$

Now we present a lemma that was proved by Barros and Gomes (2013) which is a very useful tool for computing the divergence of tensors.

Lema 2.1. *Let T be a symmetric $(0, 2)$-tensor on a Riemannian manifold (M^n, g) and φ a smooth function on M^n. Then*

$$div(T(\varphi(Z))) = \varphi(divT)(Z) + \varphi\langle\nabla Z, T\rangle + T(\nabla\varphi, Z). \tag{9}$$

A smooth vector field X on (M^n, g) is conformal if there is a function $\psi \in C^\infty(M)$, such that

$$\mathcal{L}_X g = 2\psi g,$$

where $\mathcal{L}$ is the Lie derivative. The function ψ is the *conformal factor* of X. This condition is equivalent to say that X satisfies the equation

$$\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 2\psi\langle Y, Z\rangle,$$

for any vector fields $Y, Z \in \mathfrak{X}(M)$. In particular, we have

$$\psi = \frac{1}{n} div\, X.$$

An interesting particular case of a conformal vector field X occurs when it satisfies the following condition

$$\nabla_Y X = \psi Y,$$

for every $Y \in \mathfrak{X}(M)$, then we say that X is *closed*. The expression closed comes from the fact that its dual 1-form ω is closed, in fact

$$\begin{aligned} d\omega(Y, Z) &= Y\langle X, Z\rangle - Z\langle X, Z\rangle - \langle X, [Y, Z]\rangle \\ &= \psi\langle Y, Z\rangle - \psi\langle Y, Z\rangle + \langle X, [Y, Z]\rangle - \langle X, [Y, Z]\rangle \;=\; 0. \end{aligned}$$

A gradient conformal vector field $X = \nabla\varphi$ is a closed vector field, in fact

$$\langle \nabla_Y X, Z\rangle = \nabla^2\varphi(Y, Z) = \langle \psi Y, Z\rangle,$$

thus $\nabla_Y X = \psi Y$ for all $Y \in \mathfrak{X}(M)$.

It is known that if X is a closed conformal vector field with conformal factor ψ then $Ric(X) = -(n-1)\nabla\psi$. For a proof see, for instance, Proposition 2.6 of Silva Filho (2013).

Hereafter, we collect some well known facts that will be used on this work. The first one is the following theorem, due to Obata (1962).

Theorem 2.1. *(Obata 1962) In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a nonconstant function ϕ with $\nabla^2\phi = -c^2\phi g$, it is necessary and sufficient that the manifold be isometric with a sphere $\mathbb{S}^n(c)$ of radius $1/c$ in the $(n+1)-$Euclidean space.*

Theorem 2.2 (Cheng (1975))**.** *Let M be a complete n-dimensional Riemannian manifold with Ricci curvature bounded from below by $Ric \geq (n-1)K > 0$. If the diameter d of M satisfies*

$$d = \frac{\pi}{\sqrt{K}},$$

then M is isometric to the standard sphere of radius $1/\sqrt{K}$.

Now we have the following generalized Hadamard theorem.

Theorem 2.3 (Frankel ((1966))). *Let M^{n+1} be a complete connected manifold with positive Ricci curvature. Let N and P be immersed minimal hypersurfaces of M^{n+1}, each immersed as a closed subset, and let N be compact. Then N and P must intersect.*

3 CRITICAL POINT EQUATION CONJECTURE

Our goal in this section is to study the space of metrics with constant scalar curvature satisfying the critical point equation on compact manifolds, for simplicity, CPE metrics. It has been conjectured that every CPE metric must be Einstein. Here, we prove this conjecture under some suitable integral conditions inherited from the standard sphere. The content of this section was taken from the paper Filho (2015).

3.1 Critical point metrics of the total scalar curvature functional

An important and interesting problem in differential geometry is to find Riemannian metrics on a given manifold that provides constant curvature. One of the tools for understanding this problem is the study of the critical points of the total scalar curvature functional. More precisely, let M^n be a compact oriented manifold and $\mathcal{M}$ the set of smooth Riemannian structures on M^n of volume 1. Given a metric $g \in \mathcal{M}$, we define the total scalar curvature functional $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{S}(g) = \int_M R_g dM_g, \tag{10}$$

where R_g and dM_g stand, respectively, for the scalar curvature and the volume form of metric g. We highlight that the critical points of the functional $\mathcal{S}$ are Einstein, for more details see Besse (1987, chap. 4).

The linearization of the total scalar curvature operator is given by the following expression, see Besse (1987, chap. 4),

$$\mathcal{L}_g(h) = -\Delta_g(trh) + div_g(div_g h)) - g(h, Ric_g), \tag{11}$$

and the formal L^2-adjoint, $\mathcal{L}_g^*$, of $\mathcal{L}_g$ is given by

$$\mathcal{L}_g^*(f) = \nabla_g^2 f - (\Delta_g f)g - fRic_g, \tag{12}$$

where, Δ_g, tr, div_g and $\nabla_g^2 f$ stand for the Laplacian, the trace, the divergence operator and the Hessian with respect to the metric g. The Einstein-Hilbert functional restricted to a given conformal class is just the Yamabe functional, whose critical points are constant scalar curvature metrics in that class. It is well-known that there is a constant scalar curvature metric in every conformal class of Riemannian metrics on a compact manifold M^n. So, it is natural to consider the set

$$\mathcal{C} = \{g \in \mathcal{M};\ R_g \text{ is constant}\}.$$

Koiso (1979), showed that, under generic conditions, $\mathcal{C}$ is an infinite dimensional manifold. It has been conjectured that the critical points of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{C}$ are Einstein; for more details see Besse (1987, p. 128).

Formally the Euler-Lagrangian equation of Hilbert-Einstein action on the space $\mathcal{C}$ is given by

$$\mathcal{L}_g^*(f) = \nabla_g^2 f - (\Delta_g f)g - fRic_g = \mathring{Ric}, \tag{13}$$

i.e.

$$Ric - \frac{R}{n}g = \nabla^2 f - f\big(Ric - \frac{R}{n-1}g\big), \tag{14}$$

where f is a smooth function defined on M^n and Ric, R and $\nabla^2 f$ stand, respectively, for the Ricci tensor, the scalar curvature and the Hessian form on M^n. Moreover, computing the trace in (14) we obtain

$$\Delta f + \frac{R}{n-1}f = 0. \tag{15}$$

In particular, R lies on the spectrum of M^n, then it must be positive.

Definition 3.1. *A CPE metric is a 3-tuple $(M^n,\ g,\ f)$, where (M^n, g) is a compact oriented Riemannian manifold of dimension $n \geq 3$ with constant scalar curvature while f is a smooth potential satisfying equation (14).*

We point out that Einstein metrics are recovered when $f = 0$. Moreover, the existence of a non constant solution is only known in the round sphere for height function. The conjecture proposed in Besse (1987) in 1980's may be restated in terms of CPE, see also Qing and Yuan (2013) and Hwang (2003). More precisely, the authors proposed the following conjecture.

Conjecture 3.1. *A CPE metric is always Einstein.*

It should be pointed out that the CPE conjecture combined with Obata's Theorem 2.1, allows us to deduce that a CPE metric with non-constant potential function is isometric to a round sphere metric and the potential f is a height function on the sphere.

Although Conjecture 3.1 has not been proved yet, many partial answer are known. For instance, Lafontaine (1983) proved the CPE Conjecture under locally conformally flat assumption. Hwang (2000) was able to obtain the conjecture provided $f \geq -1$. In 2010 Chang, Hwang and Yun (2010) showed that the conjecture is true for three-dimensional manifolds with null second homology group such that $Ker\,\mathcal{L}_g^* \neq 0$. Barros and Ribeiro Jr (2014) showed that the conjecture is also true for 4-dimensional half conformally flat manifolds. A metric is half conformally flat if it is selfdual or antiselfdual i.e. if $W^- = 0$ or $W^+ = 0$, respectively, where, W is the Weyl-tensor. For more details see Besse (1987). Note that half-conformally flat condition is weaker than locally conformally flat condition in dimension 4. Qing and Yuan (2013) obtained a positive answer for Bach-flat manifolds in any dimension. Chang et al. (2014) proved that the conjecture is true if the manifold has harmonic curvature. Barros et al. (2015) have proved the conjecture for 4-dimensional manifolds with harmonic tensor W^+. Recently, Leandro (2015) has proved CPE conjecture under an assumption on the first eigenfunction. His result reads as follows.

Theorem 3.1. *Let $(M^n,\, g,\, f)$ be a CPE metric. Then M^n is Einstein if and only if*

$$|\nabla f|^2 + \frac{R}{n(n-1)} f^2 = \Lambda, \tag{16}$$

where Λ is a constant.

Since we expect that a CPE metric is isometric to the canonical metric of the standard sphere $\mathbb{S}^n$ and f must be a first eigenfunction of the Laplacian, we will seek for properties of the eigenfunction of the Euclidean sphere. It is easy to check that the next integral formulae are true on $\mathbb{S}^n$ for a height function:

$$\int_M f^3 dM_g = 0 \tag{17}$$

and

$$\int_M |\nabla f|^4 dM_g = \frac{(n+2)R^2}{3n(n-1)^2} \int_M f^4 dM_g. \tag{18}$$

In general, it suffices that

$$|\nabla f|^2 + \kappa f^2 = \tau, \tag{19}$$

where κ and τ are constants, to get the same formulae, see Lemma 3.1 below.

On the other hand, those formulae are not sufficient to recuperate the sphere. In fact, it suffices to take $\mathbb{S}^n \times \mathbb{S}^n$ and a suitable function on this product, see Section 6.

In the next lemma we collect some properties of a manifold supporting an eigenfunction verifying a relation like (19). On this work, we will use only properties 1, 2 and 4 but we include another ones for the sake of completeness.

Lema 3.1. *Let M^n be a compact Riemannian manifold and let f be an eigenfunction of M, i.e. $\Delta f = -\lambda f$. Suppose that the following equation holds:*

$$|\nabla f|^2 + \kappa f^2 = \kappa. \tag{20}$$

Then we have the following assertions:

1. $\int_M f^{m+2} dM_g = \left(\frac{\kappa(m+1)}{\lambda+(m+1)\kappa}\right) \int_M f^m dM_g,$ *for any* $m \in \mathbb{N} \cup \{0\}$.
2. *In particular,* $\int_M f^{2m+1} dM_g = 0,$ *for any* $m \in \mathbb{N}$.
3. *In particular,*

$$\int_M f^{2m} dM_g = vol(M) \prod_{j=1}^{m} \frac{(2j-1)\kappa}{\lambda + (2j-1)\kappa}.$$

Moreover, if $\kappa = \frac{\lambda}{n}$ *then*

$$\int_M f^{2m} dM_g = vol(M) \prod_{j=1}^{m} \frac{2j-1}{n+2j-1},$$

for any $m \in \mathbb{N}$.

4. $\int_M |\nabla f|^4 dM_g = \frac{\lambda(\lambda+2\kappa)}{3} \int_M f^4 dM_g$.
5. *We have the integral*

$$\int_M f^{2m} \overset{\circ}{\nabla^2} f(\nabla f, \nabla f) dM_g = 0, \tag{21}$$

for any $m \in \mathbb{N} \cup \{0\}$.

6. *We have the integral*

$$\int_M f^{2m+1} \overset{\circ}{\nabla^2} f(\nabla f, \nabla f) dM_g = vol(M) \Big(\frac{\lambda - \kappa n}{n}\Big)\Big(\frac{\lambda}{2m+3}\Big) \prod_{j=1}^{m+2} \frac{(2j-1)\kappa}{\lambda + (2j-1)\kappa}, \tag{22}$$

for any $m \in \mathbb{N} \cup \{0\}$.

7. *If* $f(t) := (f \circ \gamma)(t)$, *then* $f(t) = \cos(t)$ *along any integral curve of* ∇f *such that* $\gamma(0) = p$ *where* p *is a point where* f *attains its maximum.*

Equation (20) seems to be more restrictive than (19) but we shall see on Section 6 that there is no loss of generality with that assumption.

Equation (17) is second item 2 with $m = 1$ and (18) is equivalent to item 4 with $R = n(n-1)$, $\lambda = n$ and $\kappa = 1$.

Proof. To prove the first assertion, multiply (20) by f^m and use that $\int_M f^m |\nabla f|^2 dM_g = \frac{\lambda}{m+1} \int_M f^{m+2} dM_g$. Items 2 and 3 follow using 1 and induction. To prove 4 we multiply (20) by $|\nabla f|^2$, integrate and use item 1 together with the facts that $\lambda \int_M f^2 dM_g = \int_M |\nabla f|^2 dM_g$ and $\int_M f^2 |\nabla f|^2 dM_g = \frac{\lambda}{3} \int_M f^4 dM_g$, see (26) below. Equation (20) implies that

$$\nabla^2 f(\nabla f, \nabla f) = -\kappa f |\nabla f|^2. \tag{23}$$

Hence, $f^{2m} \overset{\circ}{\nabla^2} f(\nabla f, \nabla f) = \Big(\frac{\lambda - \kappa n}{n}\Big) f^{2m+1} |\nabla f|^2$, which gives

$$\begin{aligned} \int_M f^{2m} \overset{\circ}{\nabla^2} f(\nabla f, \nabla f) dM_g &= \Big(\frac{\lambda - \kappa n}{n}\Big) \int_M f^{2m+1} |\nabla f|^2 dM_g \\ &= \Big(\frac{\lambda - \kappa n}{n}\Big)\Big(\frac{\lambda}{2m+2}\Big) \int_M f^{2m+3} dM_g = 0, \end{aligned}$$

where the last equality follows from item 2. Analogously, we have

$$\begin{aligned}\int_M f^{2m+1}\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g &= \Big(\frac{\lambda-\kappa n}{n}\Big)\Big(\frac{\lambda}{2m+3}\Big)\int_M f^{2m+4}dM_g\\ &= vol(M)\Big(\frac{\lambda-\kappa n}{n}\Big)\Big(\frac{\lambda}{2m+3}\Big)\prod_{j=1}^{m+2}\frac{(2j-1)\kappa}{\lambda+(2j-1)\kappa}\end{aligned}$$

where the last equality follows from item 3.

To prove item 7, we note that, up to normalization of the metric, we can assume $\kappa=1$. Taking an integral curve γ of ∇f, we get, in virtue of (23), that

$$\frac{d^2(f\circ\gamma)}{dt^2}(t) = -(f\circ\gamma)(t). \tag{24}$$

Therefore, $f(t) = A\cos(t)+B\sin(t)$ and since f attains its maximum at p, we get $A=(f\circ\gamma)(0)=1$. Moreover, $(f\circ\gamma)'(0)=0$ gives $B=0$ and the result follows. □

Now, we consider a smooth function f defined on a Riemannian manifold M^n such that $\Delta f=-\lambda f$. Then we have

$$\frac{1}{m}\Delta f^m = -\lambda f^m + (m-1)f^{m-2}|\nabla f|^2. \tag{25}$$

Whence, for M^n compact, we immediately obtain

$$\int_M f^m dM_g = \frac{m-1}{\lambda}\int_M f^{m-2}|\nabla f|^2 dM_g. \tag{26}$$

Moreover, multiplying identity (25) by $|\nabla f|^2$ and integrating we also derive

$$(m-1)\int_M f^{m-2}|\nabla f|^4 dM_g = \frac{(n+2)}{n(m+1)}\lambda^2\int_M f^{m+2}dM_g - 2\int_M f^{m-1}\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g. \tag{27}$$

On the other hand, for any smooth function u on M^n the Bochner formula in tensorial language says

$$div\nabla^2 u = Ric(\nabla u,\cdot)+\nabla\Delta u. \tag{28}$$

Whence, we have

$$div(\mathring{\nabla}^2 u) = Ric(\nabla u,\cdot)+\frac{n-1}{n}\nabla\Delta u. \tag{29}$$

Moreover, when $\Delta u=-\frac{R}{n-1}u$ we obtain

$$div(\mathring{\nabla}^2 u) = \mathring{Ric}(\nabla u,\cdot). \tag{30}$$

From this and from the fact that $\langle\psi g,\mathring{\nabla}^2 f\rangle=0$, for any smooth function ψ, we can apply Lemma (2.1) with $T=\mathring{\nabla}^2 f$ and $Z=\nabla f$ to conclude that for any smooth function φ on

M,

$$div\big(\varphi \overset{\circ}{\nabla^2} f(\nabla f)\big) = \varphi\big(\overset{\circ}{Ric}(\nabla f, \nabla f) + |\overset{\circ}{\nabla^2} f|^2\big) + \overset{\circ}{\nabla^2} f(\nabla\varphi, \nabla f). \tag{31}$$

Since $\nabla|\nabla f|^2 = 2\nabla^2 f(\nabla f)$, letting $h = |\nabla f|^2 + \frac{R}{n(n-1)} f^2$, we deduce

$$\overset{\circ}{\nabla^2} f(\nabla f) = \frac{1}{2}\nabla h. \tag{32}$$

Therefore, choosing $\varphi = 1$ in (31) and using (32) we obtain the next lemma.

Lema 3.2. *Let (M^n, g) be a Riemannian manifold and f a smooth function on M such that $\Delta f + \frac{R}{n-1} f = 0$. Then we have:*

1. $\frac{1}{2}\Delta h = \overset{\circ}{Ric}(\nabla f, \nabla f) + |\overset{\circ}{\nabla^2} f|^2$.
2. $\frac{1}{2}\langle \nabla\varphi, \nabla h\rangle = \overset{\circ}{\nabla^2} f(\nabla\varphi, \nabla f)$.

As a consequence of Lemma 3.2 we get the following result.

Theorem 3.2. *Let $(M^n,\, g,\, f)$ be a CPE metric. Suppose that*

$$\int_M Ric(\nabla f, \nabla f) dM_g \geq \frac{R}{n}\int_M |\nabla f|^2 dM_g,$$

then M^n is isometric to a round sphere $\mathbb{S}^n$.

Proof. Using the quoted lemma we obtain that $\overset{\circ}{\nabla^2} f = 0$, which is equivalent to Obata's result. □

More generally, we have the following result.

Theorem 3.3. *Let M be a compact Riemannian manifold and let f be an eigenfunction associated to λ, i.e. $\Delta f = -\lambda f$. Then*

$$\int_M |\overset{\circ}{\nabla^2} f|^2 dM_g + \int_M \Big(Ric(\nabla f, \nabla f) - \frac{\lambda}{n}(n-1)|\nabla f|^2\Big) dM_g = 0. \tag{33}$$

As a consequence, we obtain that:

1. *If the tensor $\mathcal{T} = Ric - \frac{\lambda}{n}(n-1)g \geq 0$, or more generally, if $\int_M \mathcal{T}(\nabla f, \nabla f) dM_g \geq 0$ then M is isometric to the Euclidean sphere.*
2. *If ∇f is a conformal vector field then M^n is isometric to the Euclidean sphere.*

Note that if $(M^n,\, g,\, f)$ is a CPE metric then $\lambda = \frac{R}{n-1}$, and thus

$$\mathcal{T}(\nabla f, \nabla f) = Ric(\nabla f, \nabla f) - \frac{R}{n}|\nabla f|^2.$$

The condition $\int_M \mathcal{T}(\nabla f, \nabla f) dM_g \geq 0$ can then be seen as a generalization of inequality $\int_M Ric(\nabla f, \nabla f) dM_g \geq \frac{R}{n}\int_M |\nabla f|^2 dM_g$, used in Theorem 3.2.

Proof. Define the function $h = |\nabla f|^2 + \kappa f^2$ where κ is any nonzero constant. Then

Computing the Laplacian of the function h, we have

$$\begin{aligned}\frac{1}{2}\Delta h &= Ric(\nabla f,\nabla f)+\langle\Delta\nabla f,\nabla f\rangle+|\nabla^2 f|^2-\frac{(\Delta f)^2}{n}+\frac{(\Delta f)^2}{n}+\kappa(f\Delta f+|\nabla f|^2)\\ &= Ric(\nabla f,\nabla f)-\lambda|\nabla f|^2+|\mathring{\nabla}^2 f|^2+\frac{\lambda^2 f^2}{n}+\kappa(-\lambda f^2+|\nabla f|^2)\\ &= Ric(\nabla f,\nabla f)+(\kappa-\lambda)|\nabla f|^2+(\frac{\lambda^2}{n}-\kappa\lambda)f^2+|\mathring{\nabla}^2 f|^2\end{aligned}$$

Integrating over M and using that $\int_M f^2=\frac{\int_M|\nabla f|^2}{\lambda}$ we get

$$\int_M|\mathring{\nabla}^2 f|^2dM_g+\int_M(Ric(\nabla f,\nabla f)-\frac{\lambda}{n}(n-1)|\nabla f|^2)dM_g=0. \quad (34)$$

If $\int_M\mathcal{T}(\nabla f,\nabla f)dM_g\geq 0$ then equation (33) combined with Obata's theorem gives item 1. To show 2, we recall that if ∇f is a conformal vector field then

$$Ric(\nabla f,\nabla f)=-(n-1)\langle\nabla f,\nabla(\frac{\Delta f}{n})\rangle=\frac{\lambda}{n}(n-1)|\nabla f|^2.$$

Therefore, $\int_M Ric(\nabla f,\nabla f)dM_g=\int_M\frac{\lambda}{n}(n-1)|\nabla f|^2dM_g$ and the result follows. □

Proceeding, taking into account the compactness of M^n, we use the second item of Lemma 3.2 and the fact that $\int_M f|\nabla f|^2dM_g=\frac{R}{2(n-1)}\int_M f^3dM_g$ to deduce

$$\int_M\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g=\frac{(n+2)R^2}{4n(n-1)^2}\int_M f^3dM_g. \quad (35)$$

Moreover, we consider again M^n compact and choosing $\varphi=f^m$ in (31) we have

$$\int_M f^m\big(\mathring{Ric}(\nabla f,\nabla f)+|\mathring{\nabla}^2 f|^2\big)dM_g=-m\int_M f^{m-1}\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g. \quad (36)$$

Lema 3.3. *Let* $(M^n,\,g,\,f)$ *be a* CPE metric. Then we have:

1. $(f+1)\mathring{Ric}=\mathring{\nabla}^2 f$. In particular, $(M^n,\,g,\,f)$ is Einstein if and only if ∇f is a conformal vector field.
2. $\int_M f^m\langle\mathring{Ric},\mathring{\nabla}^2 f\rangle dM_g=-m\int_M f^{m-1}\mathring{Ric}(\nabla f,\nabla f)dM_g$.
3. $\int_M\big(f+1\big)|\mathring{\nabla}^2 f|^2dM_g=-2\int_M\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g$.
4. $\int_M f^m\langle\mathring{Ric},\mathring{\nabla}^2 f\rangle dM_g=\sum_{i=1}^m(-1)^{i+1}\int_M f^{m-i}|\mathring{\nabla}^2 f|^2dM_g$.

Proof. Taking into account that $\Delta f=-\frac{R}{n-1}f$ we may write equation (14) according to

$$\begin{aligned}\mathring{Ric} &= \mathring{\nabla}^2 f-\big(Ric-\frac{R}{n-1}g+\frac{R}{n(n-1)}g\big)f\\ &= \mathring{\nabla}^2 f-f\mathring{Ric},\end{aligned}$$

which gives the first part of 1. Since $\frac{1}{2}\mathcal{L}_{\nabla f}g=\nabla^2 f$, we complete the proof of assertion 1.

Now, since R is a constant we have by the second contracted Bianchi identity that $div(\mathring{Ric}) = 0$. Hence, we apply Lemma 2.1 with $T = \mathring{Ric}$, $\varphi = f^m$ and $Z = \nabla f$ to get

$$div(f^m \mathring{Ric}(\nabla f)) = f^m \langle \mathring{Ric}, \mathring{\nabla}^2 f\rangle + m f^{m-1} \mathring{Ric}(\nabla f, \nabla f),$$

which after integration yields the second item.

To prove the third item, note that by Lemma 3.2 we have

$$\begin{aligned}\int_M \mathring{\nabla}^2 f(\nabla f, \nabla f) dM_g &= -\int_M f\Big(\mathring{Ric}(\nabla f, \nabla f) + |\mathring{\nabla}^2 f|^2\Big) dM_g \\ &= \frac{1}{2}\int_M f^2 \langle \mathring{Ric}, \mathring{\nabla}^2 f\rangle dM_g - \int_M f|\mathring{\nabla}^2 f|^2 dM_g \\ &= -\frac{1}{2}\int_M (f+1)|\mathring{\nabla}^2 f|^2 dM_g,\end{aligned}$$

where we have used the first and the second items.

We now treat of 4. Indeed, note that

$$\begin{aligned}\int_M f^m \langle \mathring{Ric}, \mathring{\nabla}^2 f\rangle dM_g &= \int_M f^{m-1}\langle \mathring{\nabla}^2 f - \mathring{Ric}, \mathring{\nabla}^2 f\rangle dM_g \\ &= \int_M f^{m-1}|\mathring{\nabla}^2 f|^2 dM_g - \int_M f^{m-1}\langle \mathring{Ric}, \mathring{\nabla}^2 f\rangle dM_g\end{aligned}$$

and use induction to complete the proof. □

Now we are ready to prove our first results. We will prove the CPE conjecture under some integral assumptions.

Theorem 3.4. *Let $(M^n,\ g,\ f)$ be a CPE metric. Then M^n is isometric to round sphere and f is a first eigenfunction of the Laplacian, provided that*

1. $\int_M |\nabla f|^4 dM_g = \frac{(n+2)R^2}{3n(n-1)^2}\int_M f^4 dM_g$ *and*
2. $\int_M f^3 dM_g \geq 0.$

Proof. Choosing $m = 2$ and $\lambda = R/(n-1)$ in Equation (27) we have

$$\int_M |\nabla f|^4 dM_g = \frac{(n+2)R^2}{3n(n-1)^2}\int_M f^4 dM_g - 2\int_M f\mathring{\nabla}^2 f(\nabla f, \nabla f) dM_g. \qquad (37)$$

Hence, under our first assumption, we derive $\int_M f\mathring{\nabla}^2 f(\nabla f, \nabla f) dM_g = 0$. Now, we use (36), items 2, 4 and 3 of Lemma 3.3 and (35) to obtain

$$\begin{aligned}\int_M f^2 |\mathring{\nabla}^2 f|^2 dM_g &= -\int_M f^2 \mathring{Ric}(\nabla f, \nabla f) dM_g \\ &= \frac{1}{3}\int_M f^3 \langle \mathring{Ric}, \mathring{\nabla}^2 f\rangle dM_g \\ &= \frac{1}{3}\int_M f^2 |\mathring{\nabla}^2 f|^2 dM_g - \frac{1}{3}\int_M f|\mathring{\nabla}^2 f|^2 dM_g + \frac{1}{3}\int_M |\mathring{\nabla}^2 f|^2 dM_g,\end{aligned}$$

which gives

$$\int_M f^2|\overset{\circ}{\nabla}{}^2 f|^2 dM_g = \frac{1}{2}\int_M (1-f)|\overset{\circ}{\nabla}{}^2 f|^2 dM_g. \tag{38}$$

Therefore, we deduce

$$\begin{aligned}\int_M (f+1)^2|\overset{\circ}{\nabla}{}^2 f|^2 dM_g &= \frac{3}{2}\int_M (f+1)|\overset{\circ}{\nabla}{}^2 f|^2 dM_g\\ &= -\frac{3}{4}\frac{(n+2)R^2}{n(n-1)^2}\int_M f^3 dM_g,\end{aligned}$$

i.e.

$$\int_M (f+1)^2|\overset{\circ}{\nabla}{}^2 f|^2 dM_g = -\frac{3}{4}\frac{(n+2)R^2}{n(n-1)^2}\int_M f^3 dM_g. \tag{39}$$

Since we are supposing that $\int_M f^3 dM_g \geq 0$, (39) gives $\int_M f^3 dM_g = 0$. Therefore, since $f^{-1}(-1)$ has measure zero we conclude from (39) that $\overset{\circ}{\nabla}{}^2 f = 0$, i.e. ∇f is a conformal vector field. Thus it suffices to use Lemma 3.3 to finish the proof of the theorem. □

As an immediate consequence of Theorem 3.4 we obtain the next corollary under an assumption that is weaker than (3).

Corollary 3.1. *The same conclusion of Theorem 3.4 is true provided that the function* $h = |\nabla f|^2 + \frac{R}{n(n-1)}f^2$ *is constant along of the flow of* ∇f.

That is, the function h does not need to be a constant, it suffices that its restriction to any integral curve of the field ∇f is constant.

Proof. For the proof of Corollary 3.1 it suffices to notice that $2\overset{\circ}{\nabla}{}^2 f(\nabla f, \nabla f) = \langle \nabla h, \nabla f\rangle$. Since we are supposing $\langle \nabla h, \nabla f\rangle = 0$ we use identities (35) and (37), respectively, to arrive at $\int_M f^3 dM_g = 0$ and $\int_M |\nabla f|^4 dM_g = \frac{(n+2)R^2}{3n(n-1)^2}\int_M f^4 dM_g$. Thus we are in position to use Theorem 3.4 to finish the proof of the corollary. □

4 LI-YAU INEQUALITY

In this section M will always be a closed Riemannian manifold, i.e. compact without boundary. We will be interested in to obtain improvements of an estimate of the first eigenvalue of the Laplacian of M obtained by Li and Yau (1980). The content of this section was taken from the preprint Barros and Filho (2014a).

In view of the importance of the Poincaré inequality in analysis on manifolds it is much desirable to get optimal quantitative estimates of the first eigenvalue λ of a compact Riemannian manifold M. The task of estimating the first eigenvalue starts when Lichnerowicz (1958) proved that if the Ricci curvature of M^n satisfies

$$Ric \geq (n-1)K > 0 \tag{40}$$

then

$$\lambda \geq nK. \tag{41}$$

Four years later, Obata (1962) showed that Lichnerowicz inequality is indeed sharp in the sense that the equality occurs on 41 if and only if M is isometric to the Euclidean sphere $\mathbb{S}^n$ with constant scalar curvature K.

Later on, Yau (1975) proved that λ has a lower bound in terms of the lower bound of the Ricci curvature, the diameter and the volume. In the same paper he conjectured that there should have a lower bound for λ depending only on d and K and emphasized that in view of known examples the dependence of the diameter and the lower bound of the Ricci curvature could not be dropped.

If we make K converge to zero in (40) then the estimate (41) gives no information and thus it is necessary to find another kind of estimate, possibly depending on more geometric quantities. In this direction, Li and Yau (1980) showed that if $Ric \geq 0$, then

$$\lambda \geq \frac{\pi^2}{(1+a)d^2},$$

with $0 \leq a < 1$, and they conjectured that

$$\lambda \geq \frac{\pi^2}{d^2}.$$

Li-Yau's conjecture was proved by Zhong and Yang (1984).

Considering the lower bound of the Ricci curvature to be a negative constant, i.e. $Ric \geq -(n-1)K$ with $K > 0$, it is possible to obtain an estimate from below for λ. There are two distinguished results concerning to this problem, the first one is due to Li and Yau (2012). They proved that

$$\lambda \geq \frac{2}{(n+1)d^2}\big(1+\sqrt{\mu}\big)e^{-(1+\sqrt{\mu})}, \tag{42}$$

where d is the diameter of M and $\mu := 1+(n-1)^2 d^2 K$. While the second is due to Yang (1990) and it reads as follows:

$$\lambda \geq \frac{\pi^2}{d^2}e^{-(n-1)d\sqrt{K}}. \tag{43}$$

Using the elementary inequality $\sqrt{1+x^2} \leq 1+x$, if $x \geq 0$, we get

$$\sqrt{1+(n-1)^2d^2K} \leq 1+(n-1)d\sqrt{K},$$

from this, we obtain

$$e^{-(1+\sqrt{1+(n-1)^2d^2K}} \geq e^{-(2+(n-1)d\sqrt{K})}.$$

Whence we can obtain from estimate (42) the following

$$\lambda \geq \frac{2}{(n+1)e^2d^2}\big(1+\sqrt{1+(n-1)^2d^2K}\big)e^{-(n-1)d\sqrt{K}}. \tag{44}$$

In particular, if $\frac{2}{(n+1)e^2}\big(1+\sqrt{1+(n-1)^2d^2K}\big) \geq \pi^2$, in other words if

$$d^2K \geq (n+1)\frac{(\pi e)^2}{(n-1)^2}\Big(\frac{(n+1)(\pi e)^2}{4}-1\Big),$$

then we can deduce that the estimate obtained by Li and Yau is better than the one obtained by Yang.

4.1 Improvement of Li-Yau inequality

In this part we shall obtain an improvement of Li-Yau estimate (42) according to the following theorem .

Theorem 4.1. *Let M^n be a closed Riemannian manifold. Suppose that the Ricci curvature of M satisfies $Ric_M \geq -K\,(n-1)$, with $K \geq 0$. Then the first eigenvalue λ of M satisfies*

$$\lambda \;\geq\; b_0\Big(1+\eta\Big),$$

where

$$\begin{aligned}\eta \;=\; & \Big(\frac{2}{n+1}e^{-(1+\sqrt{\mu})}+\frac{2}{(n+1)^2}(1+\sqrt{\mu})e^{-2(1+\sqrt{\mu})}+\frac{4(1+\sqrt{\mu})^3}{3(n+1)^3\mu^{3/2}}e^{-3(1+\sqrt{\mu})}\\ & +\frac{2(3+\mu+3\sqrt{\mu})(1+\sqrt{\mu})^3}{3(n+1)^4\mu^{5/2}}e^{-4(1+\sqrt{\mu})}\Big)\end{aligned}$$

and $b_0 := \frac{2}{(n+1)d^2}\Big(1+\sqrt{\mu}\Big)e^{-(1+\sqrt{\mu})}$.

The techniques of gradient estimates used by Li and Yau (2012) as well as Yang (1990) are supported in the well known Bochner technique. Roughly speaking, the technique is to get an upper bound for the norm of the gradient of an eigenfunction and then integrating the resulting inequality along a minimizing geodesic joining specific points of the manifold. The main difficult is that to obtain such an upper bound it is necessary to choose a suitable "test functon" and this may be nontrivial. Let us illustrate the idea of the technique by proving the theorems of Lichnerowicz-Obata and Li-Yau. We follow closely the approach of Li (2012).

Example 4.1 (Lichnerowicz (1958) and Obata (1962))**.** *Let* M *be an* n *dimensional closed Riemannian manifold. Suppose that the Ricci curvature of* M *is bounded from below by*

$$Ric \geq (n-1)K$$

for some constant $K > 0$*, then the first eigenvalue of the Laplacian of* M *must satisfy* $\lambda \geq nK$*. Moreover, equality holds if and only if* M *is isometric to a standard sphere of radius* $1/\sqrt{K}$*.*

Indeed, let u *be a nonconstant eigenfunction such that* $\Delta f = -\lambda f$*. Consider the function*

$$F = |\nabla u|^2 + cu^2, \tag{45}$$

where the constant c *will be chosen. By the Bochner formula, we have*

$$\begin{aligned}\frac{1}{2}\Delta F &= Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle + |\nabla^2 u|^2 + c(u\Delta u + |\nabla u|^2)\\ &\geq (n-1)K|\nabla u|^2 - \lambda|\nabla u|^2 + \frac{\lambda^2 u^2}{n} + c(-\lambda u^2 + |\nabla u|^2)\\ &= (c+(n-1)K-\lambda)|\nabla u|^2 + \frac{\lambda}{n}(\lambda - nc)u^2.\end{aligned}$$

We would like to find a constant c *such that* F *is a subharmonic function for* $\lambda \leq nK$*. If we choose* $c = \lambda/n$*, we will get*

$$\Delta F \geq 2(n-1)\Big(K - \frac{\lambda}{n}\Big)|\nabla u|^2. \tag{46}$$

Suppose that $\lambda \leq nK$*, then* F *is a subharmonic function. Since* M *is closed, the maximum principle implies that* F *must be constant on all of* M *and the above inequalities are, in fact, equalities. In particular, we have equality in (46) and thus* $\lambda = nK$*. Moreover*

$$|\nabla u|^2 + \frac{\lambda}{n}u^2 = \frac{\lambda}{n}|u|^2_\infty,$$

where $|u|_\infty = \sup_M u$*. Normalizing* u *such that* $|u|_\infty = 1$ *and noting that at the maximum* *and minimum of u the gradient vanishes, we conclude that u is symmetric, i.e.* $\min u = -1 = -\max u$ *and*

$$\frac{|\nabla u|}{\sqrt{1-u^2}} = \sqrt{K}.$$

Integrating along a minimal geodesic joining the minimum and the maximum of u we have

$$\begin{aligned} d\sqrt{K} &\geq \int_\gamma \frac{|\nabla u|}{\sqrt{1-u^2}} \\ &\geq \int_0^1 \frac{du}{\sqrt{1-u^2}} \\ &= \pi, \end{aligned}$$

where d is the diameter of M. But, the Bonnet-Meyer theorem gives the reversed inequality $d\sqrt{K} \leq \pi$, thus we get $d\sqrt{K} = \pi$. By Cheng's result, Theorem 2.2, M is isometric to the Euclidean sphere.

Example 4.2 (Li and Yau (1980))**.** *Let M be an n dimensional closed Riemannian manifold. Suppose that the Ricci curvature of M is bounded from below by*

$$Ric \geq 0.$$

Then the first eigenvalue satisfies

$$\lambda \geq \frac{\pi^2}{(1+a)d^2}$$

where d is the diameter of M.

In fact, suppose that φ is a first eigenfunction of M. By multiplying φ by a constant it is possible to choose φ so that

$$a - 1 = \inf_M \varphi, \quad a + 1 = \sup_M \varphi,$$

where $0 \leq a < 1$ is the median of φ.

Let $u = \varphi - a$, u satisfies $\Delta u = -\lambda(u + a)$. Let $F = |\nabla u|^2 + cu^2$. Suppose $x_0 \in M$ is the point where F achieves its maximum. In case $\nabla u(x_0) \neq 0$ we can choose coordinates so that $u_1(x_0) = |\nabla u(x_0)|$. Since in the maximum point, the gradient of F vanishes we have $u_{11}(x_0) = -cu(x_0)$ at x_0. Proceeding as before, we obtain that at x_0,

$$\begin{aligned} 0 &\geq \frac{1}{2}\Delta F \\ &= Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u \rangle + |\nabla^2 u|^2 + c(u\Delta u + |\nabla u|^2) \\ &\geq c^2u^2 - \lambda|\nabla u|^2 + c|\nabla u|^2 - c\lambda u(u + a) \\ &= (c - \lambda)(|\nabla u|^2 + cu^2) - ac\lambda u, \end{aligned}$$

where we have used the fact that $|\nabla^2 u|^2 \geq u_{11}^2 = c^2u^2$. *Now we choose* $c = \lambda(1+a)$ *to get* $0 \geq a\lambda F(x_0) - ac\lambda$, *which gives the gradient estimate*

$$|\nabla u(x)|^2 \leq \lambda(1+a)(1-u^2(x)). \tag{47}$$

This estimate is trivial if $\nabla u(x_0) = 0$, *then the hypothesis of* $\nabla u(x_0) \neq 0$ *is not necessary. Let* γ *be the minimizing geodesic joining the points where* $u = -1$ *and* $u = 1$. *Integrating as before, we obtain*

$$d\sqrt{\lambda(1+a)} \geq \sqrt{\lambda(1+a)}\int_\gamma ds \geq \int_\gamma \frac{|\nabla u|^2}{\sqrt{1-u^2}} \geq \int_{-1}^{1} = \pi.$$

More generally, we can adapt the proof of Theorem 1 of Schoen and Yau (1994, p. 110) to get the following

Example 4.3. *Let* M *be a closed Riemannian manifold with* $Ric \geq (n-1)K \geq 0$, *then for any* $\epsilon \geq 0$,

$$\lambda \geq \frac{1}{\left(1+\frac{(1-k)}{(1+k)(1+\epsilon)^2}\right)}\left(\frac{(2\sin^{-1}(\frac{1}{1+\epsilon}))^2}{d^2} + K(n-1)\Big(1-\frac{1}{(1+\epsilon)^2}\Big)\right). \tag{48}$$

To prove formula (48) we assume $-1 \leq k = \inf u < \sup u = 1$, *where* $0 < k \leq 1$ *and define* $\tilde{u} = \frac{2u-(1-k)}{1+k}$. *Then*

$$\begin{cases} \Delta u &= -\lambda(\tilde{u}+a), \quad a = \frac{1-k}{1=k}, \quad 0 \leq a < 1, \\ \max \tilde{u} &= 1, \\ \min \tilde{u} &= -1. \end{cases}$$

For small values of $\epsilon > 0$ *let* $v = \frac{\tilde{u}}{1+\epsilon}$, *then*

$$\begin{cases} \Delta v &= -\lambda(v+a-\epsilon), \quad a_\epsilon = \frac{a}{1+\epsilon}, \\ \max \tilde{u} &= \frac{1}{1+\epsilon}, \\ \min \tilde{u} &= -\frac{1}{1+\epsilon}. \end{cases}$$

Now we consider the function $F(x) = \frac{|\nabla v|^2}{1-v^2}$. *After applying Bochner formula in the expression* $F(1-v^2) = |\nabla v|^2$, *we can take a referential such that at the point where* F *attains its maximum we have* $u_{11}^2 = \frac{|\nabla v|^4 v^2}{(1-v^2)^2}$. *In a similar way to Example 4.2 we conclude that*

$$\begin{aligned} \frac{|\nabla v|^2}{1-v^2} &\leq \lambda(1+va_\epsilon) - K(n-1)(1-v^2) \\ &\leq \lambda\Big(1+\frac{1-k}{(1+k)(1+\epsilon)^2}\Big) - K(n-1)\Big(1-\frac{1}{(1+\epsilon)^2}\Big). \end{aligned}$$

Now we integrate this inequality along the minimizing geodesic joining the points where $v = \pm\frac{1}{1+\epsilon}$ to get (48).

In the following table, we collect some values for the right hand side of (48) according to the value of ϵ.

Table 1: Lower bounds for λ

$\frac{1}{1+\epsilon}$	ϵ	lower bound for λ
$\frac{1}{2}$	1	$\frac{2(k+1)}{(k+3)}\left(\frac{\pi^2}{9d^2} + \frac{3(n-1)K}{4}\right)$
$\frac{\sqrt{2}}{2}$	$\sqrt{2}-1$	$\left(\frac{\sqrt{2}(k+1)}{\sqrt{2}(k+1)+(1-k)}\right)\left(\frac{\pi^2}{4d^2} + \frac{(n-1)K}{2}\right)$
$\frac{\sqrt{3}}{2}$	$\frac{2-\sqrt{3}}{\sqrt{3}}$	$\left(\frac{2(k+1)}{2(k+1)-\sqrt{3}(k+1)}\right)\left(\frac{4\pi^2}{9d^2} + \frac{(n-1)K}{4}\right)$
1	0	$\left(\frac{k+1}{2}\right)\frac{\pi^2}{d^2}$

Source: Own author.

Analyzing the above table we note that if $\epsilon \to 0$ then we get $\lambda \geq \frac{\pi^2}{2}$, since $0 \leq a < 1$, and we obtain again the result of Example 4.2.

Comparing Examples 4.1, 4.2 and 4.3 it is possible to see that the choice of the test function F was affected by the change of the lower bound of the Ricci curvature. The more negative we allow the Ricci curvature be the more complicated the function F becomes. We will see more situations where this happens in the proof of Theorem 4.1 and in Section 5.

For the proof of Theorem 4.1 we shall follow the approach adopted by Li and Yau. The fundamental ingredient is the next lemma.

Lema 4.1 (Lemma 5.6, Li and Yau (2012))**.** *Let M be a complete n-dimensional Riemannian manifold whose Ricci curvature of M is bounded from below by $Ric \geq -(n-1)K$ for some constant $K > 0$. Let u be a function defined on M satisfying the equation*

$$\Delta u = -\lambda u, \tag{49}$$

in such way that $\min u = -1$. *Letting $Q = |\nabla \log(a+u)|^2$, $v = \log(a+u)$ and $a > 1$, then we have*

$$\Delta Q - \frac{n}{2(n-1)}|\nabla Q|^2 Q^{-1} + \langle \nabla v, \nabla Q\rangle Q^{-1}\Big(\frac{2(n-2)}{n-1}Q - \frac{2}{n-1}\frac{\lambda u}{a+u}\Big) \geq$$
$$2(n-1)Q^2 + \Big(\frac{4}{n-1}\frac{\lambda u}{a+u} - 2\frac{\lambda a}{a+u} - 2(n-1)K\Big)Q + \frac{2}{n-1}\Big(\frac{\lambda u}{a+u}\Big)^2.$$

Now we are in position to prove Theorem 4.1.

Proof. From estimate (42) we know that $\lambda \geq b_0$, where $b_0 = \frac{2}{(n+1)d^2}\Big(1 + \sqrt{\mu}\Big)e^{-(1+\sqrt{\mu})}$. Now we shall argue as in the proof presented by Li and Yau (2012) for estimate (42). Let

u be a nonconstant eigenfunction satisfying

$$\Delta u = -\lambda u,$$

and $\min u = -1$ and $\max u \leq 1$. Let us consider the function

$$v = \log(a+u),$$

for some constant $a > 1$ and

$$Q = |\nabla v|^2.$$

Take a point x_0 where Q attains its maximum. By using Lemma 4.1 and the maximum principle we deduce

$$\begin{aligned} Q(x) \leq Q(x_0) &\leq (n+1)\frac{\lambda a}{a+u} + (n-1)^2 K - 2\lambda \\ &\leq (n+1)\frac{\lambda a}{a-1} + (n-1)^2 K - 2b_0 \end{aligned}$$

for all $x \in M$.

In the proof of Theorem 5.7 Li (2012) throw away the term -2λ in the above inequality. Instead of doing that, we use the estimate (42) to produce another estimate.

In fact, integrating $Q^{1/2} = |\nabla \log(a+u)|$ along a minimizing geodesic γ from $u = -1$ to $u = \max u = u_\infty$ we have

$$\begin{aligned} \log\big(\frac{a}{a-1}\big) &\leq \log\big(\frac{a+u_\infty}{a-1}\big) \\ &= \int_{s_0}^{s_1} \frac{d}{ds}\log(a+u(\gamma(s))ds \\ &\leq \int_\gamma \sqrt{Q} \leq \Big((n+1)\frac{\lambda a}{a-1} + (n-1)^2 K - 2b_0\Big)^{\frac{1}{2}} d \end{aligned}$$

Letting $t = \frac{a-1}{a}$, the last inequality becomes

$$\big(\log(\frac{1}{t})\big)^2 \leq \Big((n+1)\frac{\lambda}{t} + (n-1)^2 K - 2b_0\Big)d^2.$$

Whence we deduce

$$(n+1)\frac{\lambda}{t} \geq \frac{1}{d^2}\big(\log(\frac{1}{t})\big)^2 - (n-1)^2 K + 2b_0,$$

i.e.

$$(n+1)\lambda \geq t\Big(\frac{1}{d^2}\big(\log(\frac{1}{t})\big)^2 - (n-1)^2 K\Big) + 2b_0 t. \tag{50}$$

Define the function $f : (0, 1) \to \mathbb{R}$ by

$$f(t) = t\Big(\frac{1}{d^2}\big(\log(\frac{1}{t})\big)^2 - (n-1)^2 K\Big) + 2b_0 t. \tag{51}$$

From here, it follows that

$$f'(t) = -(n-1)^2 K + \frac{(\log t)^2}{d^2} + \frac{2\log t}{d^2} + 2b_0, \tag{52}$$

whose roots are given by

$$t_1 = e^{-(1+\sqrt{\mu - 2b_0 d^2})}$$

and

$$t_2 = e^{-(1-\sqrt{\mu - 2b_0 d^2})}.$$

Analyzing the second derivative of f we conclude that t_1 is the global maximum of f. Besides,

$$f(t_1) \;\; = \;\; \frac{2}{d^2}(1 + \sqrt{\mu - 2b_0 d^2})e^{-(1+\sqrt{\mu - 2b_0 d^2})}.$$

Thus,

$$\lambda \;\; \geq \;\; \frac{2}{(n+1)d^2}(1 + \sqrt{\mu - 2b_0 d^2})e^{-(1+\sqrt{\mu - 2b_0 d^2})} =: b_1$$

Therefore, if we set $y_0 := 2b_0 d^2$ we get

$$\lambda \geq \frac{2}{(n+1)d^2}\left(1 + \sqrt{\mu - y_0}\right) e^{-(1+\sqrt{\mu - y_0})} =: b_1. \tag{53}$$

In order to show that b_1 is greater than b_0 it suffices to prove that the function $\varphi : [0, \mu] \to [0, \infty)$,

$$\varphi(y) := (1 + \sqrt{\mu - y})e^{-(1+\sqrt{\mu - y})},$$

is increasing. To see this, just note that

$$\varphi'(y) = \frac{e^{-(1+\sqrt{\mu - y})}}{2} > 0.$$

The inequality $\varphi(y_0) > \varphi(0)$ gives that $b_1 > b_0$. In order to analyze how much b_1 is greater than b_0, we calculate

$$\begin{array}{rclrcl}
\varphi(y) & = & (\sqrt{\mu-y}+1)\, e^{-\sqrt{\mu-y}-1}, & \varphi(0) & = & \left(\sqrt{\mu}+1\right) e^{-\sqrt{\mu}-1},\\
\varphi^{(1)}(y) & = & \frac{e^{-\sqrt{\mu-y}-1}}{2}, & \varphi^{(1)}(0) & = & \frac{e^{-\sqrt{\mu}-1}}{2},\\
\varphi^{(2)}(y) & = & \frac{\sqrt{\mu-y}e^{-\sqrt{\mu-y}-1}}{4(\mu-y)}, & \varphi^{(2)}(0) & = & \frac{e^{-\sqrt{\mu}-1}}{4\sqrt{\mu}},\\
\varphi^{(3)}(y) & = & \frac{(\sqrt{\mu-y}+1)e^{-\sqrt{\mu-y}-1}}{8\sqrt{\mu-y}(\mu-y)}, & \varphi^{(3)}(0) & = & \frac{\left(\sqrt{\mu}+1\right)e^{-\sqrt{\mu}-1}}{8\mu^{\frac{3}{2}}},\\
\varphi^{(4)}(y) & = & \frac{(\mu-y+3\sqrt{\mu-y}+3)e^{-\sqrt{\mu-y}-1}}{16\sqrt{\mu-y}(\mu-y)^2}, & \varphi^{(4)}(0) & = & \frac{\left(\mu+3\sqrt{\mu}+3\right)e^{-\sqrt{\mu}-1}}{16\mu^{\frac{5}{2}}},\\
\varphi^{(5)}(y) & = & \frac{(\sqrt{\mu-y}(\mu-y+15)+6(\mu-y)+15)e^{-\sqrt{\mu-y}-1}}{32\sqrt{\mu-y}(\mu-y)^3}, & \varphi^{(5)}(0) & = & \frac{\left(\mu^{\frac{3}{2}}+6\mu+15\sqrt{\mu}+15\right)e^{-\sqrt{\mu}-1}}{32\mu^{\frac{7}{2}}}.
\end{array}$$

This way, $\varphi^{(i)} > 0$ for $0 \le i \le 5$, i.e. $\varphi^{(i)}$ is increasing for $0 \le i \le 4$. In particular, $\varphi^{(4)}$ is increasing. By Taylor's formula, there exists $c_{y_0} \in (0, y_0)$ such that

$$\begin{aligned}
\varphi(y_0) &= \varphi(0) + \varphi^{(1)}(0)y_0 + \frac{\varphi^{(2)}(0)}{2}y_0^2 + \frac{\varphi^{(3)}(0)}{3!}y_0^3 + \frac{\varphi^{(4)}(c_{y_0})}{4!}y_0^4\\
&> \varphi(0) + \varphi^{(1)}(0)y_0 + \frac{\varphi^{(2)}(0)}{2}y_0^2 + \frac{\varphi^{(3)}(0)}{3!}y_0^3 + \frac{\varphi^{(4)}(0)}{4!}y_0^4.
\end{aligned}$$

From this we get

$$\begin{aligned}
\varphi(y_0) &> (1+\sqrt{\mu})e^{-(1+\sqrt{\mu})}\Big(1 + \frac{2}{n+1}e^{-(1+\sqrt{\mu})} + \frac{2(1+\sqrt{\mu})}{(n+1)^2}e^{-2(1+\sqrt{\mu})}\\
&\quad + \frac{4(1+\sqrt{\mu})^3}{3(n+1)^3\mu^{3/2}}e^{-3(1+\sqrt{\mu})} + \frac{2(3+\mu+3\sqrt{\mu})(1+\sqrt{\mu})^3}{3(n+1)^4\mu^{5/2}}e^{-4(1+\sqrt{\mu})}\Big).
\end{aligned}$$

Since $b_0 = \frac{2}{(n+1)d^2}\varphi(0)$ and $b_1 = \frac{2}{(n+1)d^2}\varphi(y_0)$, the previous inequality implies that

$$\lambda > b_1 > b_0\Big(1+\eta\Big),$$

where

$$\begin{aligned}
\eta &= \Big(\frac{2}{n+1}e^{-(1+\sqrt{\mu})} + \frac{2}{(n+1)^2}(1+\sqrt{\mu})e^{-2(1+\sqrt{\mu})} + \frac{4(1+\sqrt{\mu})^3}{3(n+1)^3\mu^{3/2}}e^{-3(1+\sqrt{\mu})}\\
&\quad + \frac{2(3+\mu+3\sqrt{\mu})(1+\sqrt{\mu})^3}{3(n+1)^4\mu^{5/2}}e^{-4(1+\sqrt{\mu})}\Big).
\end{aligned}$$

□

4.2 Another improvement of Li-Yau inequality

In this section, we provide, for manifolds with small diameter, another improvement of inequality (42) according to the following

Theorem 4.2. *Let M^n be a compact Riemannian manifold with $Ric \ge -(n-1)$. Then* *we have*

$$\lambda \geq \frac{\pi^2}{2d^2} - \frac{n-1}{2}. \tag{54}$$

Note that in (54), we need to assume that d is relatively small in order to get a positive lower bound for λ. More precisely, it is necessary that $d^2 \leq \frac{\pi^2}{n-1}$.

Proof. Define the function $F = |\nabla u|^2 + cu^2$. Proceeding as before we have

$$|\nabla u|^2 \leq \frac{c\lambda}{c - \lambda + 1 - n} - cu^2.$$

Choosing $c - \lambda + 1 - n = \lambda$ i.e. $c = 2\lambda + n - 1$ we get the gradient estimate

$$\frac{|\nabla u|}{\sqrt{1-u^2}} \leq \sqrt{2\lambda + n - 1}. \tag{55}$$

Integrating from $u = -1$ to $u = 1$ we get the estimate (54).

To show that (54) is better than (42) for d small, it suffices to show that

$$\pi^2 \geq (n-1)d^2 + \frac{4}{n+1}(1 + \sqrt{1+(n-1)d^2})e^{-(1+\sqrt{1+(n-1)d^2})}. \tag{56}$$

But, (56) holds for $d = 0$, so it holds for d small.

In the following table we exhibit the maximal diameter $d_{\max}$ for which (54) is better than (42) in the interval $(0, d_{\max})$, for some values of n.

Table 2: Maximal diameter

n	d_{max}
2	3.1290
3	2.2148
4	1.8183
5	1.5676
6	1.4048
7	1.2806
8	1.1854
9	1.1094
10	1.0460
100	0.3157
1000	0.0993

Source: Own author.

To get this table we define the function $F : (1,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$F(n,d) := \pi^2 - (n-1)d^2 - \frac{4}{n+1}(1 + \sqrt{1+(n-1)d^2})e^{-(1+\sqrt{1+(n-1)d^2})},$$

and for each fixed value of n we obtain a function $F_n(d) := F(n,d)$. The number d_{max}

is the first positive root of F_n. For example, if we take $n = 2$, then the estimate (54) is better than (42) for d in the interval $(0, 3.1290]$. $\square$

5 LOWER BOUND FOR THE DIAMETER OF COMPACT MINIMAL HYPERSURFACES OF THE EUCLIDEAN SPHERE

Our goal in this section is to investigate the diameter of a compact minimally immersed hypersurface of the Euclidean sphere. The content of this section was taken from the preprint Barros and Filho (2014b).

Given a compact Riemannian manifold M^n without boundary let us consider $\psi : M^n \looparrowright \mathbb{S}^{n+p}$ a minimal immersion. One fundamental identity for such immersion is the celebrated result of Takahashi (1966), which asserts that $\Delta\psi + n\psi = 0$, where Δ stands for the Laplacian of M^n with respect to the induced metric. Takahashi's result immediately gives that the first eigenvalue of such an immersion is less than or equal to n.

The simplest example of such a submanifold is a totally geodesic sphere $\mathbb{S}^n$ that has the first eigenvalue equal to n. Furthermore, $\mathbb{S}^n$ has another important propriety: its diameter is π. The two dimensional Clifford torus embedded into $\mathbb{S}^3$ also has the same diameter. Whence it arises a natural question concerning the diameter of this class of submanifolds: is there a lower bound for the diameter of such submanifolds?

In this section we give a positive answer to this question. Our main ingredients rely in the techniques introduced by Li and Yau (1982) and Li (2012) to estimate eigenvalues of the Laplacian of general compact Riemannian manifolds. Now we announce our first result.

Theorem 5.1. *Let $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a minimal immersion of a compact Riemannian manifold Σ^n and let φ be an eigenfunction of M^n, i.e. $\Delta\varphi = -\lambda\varphi$. If the second fundamental form of M^n has constant length, and if the maximum of the function $F = |\nabla u|^2 + (2|A|^2 + \lambda)u^2$ is not a critical point of u, where $u = \varphi - a$, with a being a constant, then we have $d \geq \frac{\pi}{\sqrt{3\lambda}}$.*

Proof. Let φ be an eigenfunction of a compact Riemannian manifold M^n. We can suppose that $\min\varphi = a - 1$ and $\max\varphi = a + 1$, where $0 \leq a < 1$. If we take $u := \varphi - a$ then $\min u = -1$, $\max u = 1$ and $\Delta u = -\lambda(u + a)$.

Let us consider $\psi : \Sigma^n \to \mathbb{S}^{n+1}$ a minimal isometric immersion of a compact Riemannian manifold Σ^n with second fundamental form A. Gauss equation

$$Ric(\nabla u, \nabla u) = (n-1)|\nabla u|^2 - |A\nabla u|^2 \tag{57}$$

implies

$$Ric(\nabla u, \nabla u) \geq (n - 1 - |A|^2)|\nabla u|^2. \tag{58}$$

Now let $F = |\nabla u|^2 + cu^2$ where c is a constant to be chosen. Suppose that its maximal value is attained at x_0 and x_0 is not a critical point of u. Since $\nabla u(x_0) \neq 0$, we may choose a referential near x_0 such that $|\nabla u|e_1 = \nabla u$. From that we deduce $u_{11} = -cu$, which gives $|\nabla^2 u| \geq c^2u^2$. Without loss of generality, we can suppose that $|A|^2 \geq n$, otherwise the result of Chern et al. (1970) implies that M^n is totally geodesic. Hence at x_0 we have, by the maximum principle, $\Delta F \geq 0$, i.e.

$$\begin{aligned} 0 &\geq Ric(\nabla u, \nabla u) + \langle \nabla\Delta u, \nabla u\rangle + |\nabla^2 u|^2 + c(u\Delta u + |\nabla u|^2) \\ &= (n-1-|A|^2)|\nabla u|^2 - \lambda|\nabla u|^2 + |\nabla^2 u|^2 - c\lambda u^2 - c\lambda ua + c|\nabla u|^2 \\ &= (c-\lambda+n-1-|A|^2)|\nabla u|^2 + c^2u^2 - c\lambda u^2 - c\lambda ua \\ &= (c-\lambda+n-1-|A|^2)(|\nabla u|^2 + c^2u^2) + cu^2(|A|^2-(n-1)) - c\lambda ua \\ &\geq (c-\lambda+n-1-|A|^2)(|\nabla u|^2 + cu^2) - c\lambda. \end{aligned}$$

where we have used the fact that $au \leq 1$. Therefore, choosing $c - \lambda + n - 1 - |A|^2 > 0$ we have

$$\frac{c\lambda}{\left(c-\lambda+n-1-|A|^2\right)} \geq |\nabla u|^2 + cu^2.$$

Hence, we arrive at

$$|\nabla u|^2 \leq \frac{c\lambda}{\left(c-\lambda+n-1-|A|^2\right)} - cu^2. \tag{59}$$

Next, letting $c = 2|A|^2 + \lambda$ we have $c \geq 2n + \lambda \geq 3\lambda$, where we have used that $\lambda \leq n$. Using this choice of c in (59), we have

$$|\nabla u|^2 \leq \lambda \cdot \frac{2|A|^2+\lambda}{\left(|A|^2+n-1\right)} - 3\lambda u^2. \tag{60}$$

Since

$$\begin{aligned} \frac{2|A|^2+\lambda}{\left(|A|^2+n-1\right)} &\leq \frac{2|A|^2+|A|^2}{|A|^2} \\ &= 3, \end{aligned}$$

we deduce our main inequality

$$|\nabla u|^2 \leq 3\lambda\left(1-u^2\right). \tag{61}$$

Therefore we have the next lemma.

Lema 5.1. *Let $\psi : \Sigma \to \mathbb{S}^{n+1}$ be a minimal isometric immersion of a compact Riemann surface Σ^n and suppose that $|A|^2$ is a constant. If u is given according to the above choice, then we have*

$$\frac{|\nabla u|}{\sqrt{1-u^2}} \leq \sqrt{3\lambda}. \tag{62}$$

Integrating inequality (62) along the minimizing geodesic joining the points where $u = -1$ and $u = 1$ we obtain the claimed diameter estimate of Theorem 5.1. □

For surfaces we can obtain a slightly better estimate for the diameter as we can see in the following result.

Theorem 5.2. *Let $\psi : \Sigma^2 \looparrowright \mathbb{S}^3$ be a minimal immersion of a compact Riemannian surface Σ^2 and let φ be an eigenfunction of M^2 i.e. $\Delta\varphi = -\lambda\varphi$. If the second fundamental form of M^2 has constant length, and if the maximum of the function $F = |\nabla u|^2 + (|A|^2 + \lambda)u^2$ is not a critical point of u then we have $d \geq \frac{\pi}{\sqrt{2\lambda}}$.*

We should point out that Choi and Wang (1983) have proved that $\lambda \geq n/2$ for embedded hyersurfaces of $\mathbb{S}^{n+1}$ and it is not hard to see that a minimal hypersurface of $\mathbb{S}^{n+1}$ has diameter $d \geq \pi/2$. Indeed, to see that we just need to apply Theorem 2.3 to conclude that Σ^n must intersect every equator, and then the assertion follows. The relevance of Theorems (5.1) and (5.2) is on the absence of a similar result to the one of Choi-Wang for immersed hypersurfaces. In other words, one does not know whether λ converges to zero or not in the immersed case.

Proof. The proof is similar to the proof of Theorem 5.1.

Firstly, we consider $\psi : \Sigma^2 \to \mathbb{S}^3$ a minimal isometric immersion of a compact Riemannian surface Σ^2 with second fundamental form A. Now we choose an eigenfunction φ of the Laplacian as before. So, Gauss equation becomes

$$Ric(\nabla u, \nabla u) = \big(1 - |Ae_1|^2\big)|\nabla u|^2. \tag{63}$$

Moreover, we also have

$$|A\nabla u|^2 = |\nabla u|^2 |Ae_1|^2. \tag{64}$$

On the other hand, we can choose a referential $\{v_1, v_2\}$ which diagonalizes A at x_0. Suppose that $Av_i = k_i v_i$ for each $i = 1, 2$. Letting $e_1 = \sum_{i=1}^{2} a_{i1} v_i$ we have

$$Ae_1 = k_1\big(a_{11}v_1 - a_{21}v_2\big). \tag{65}$$

Whence we obtain

$$|Ae_1|^2 = k_1^2 = \frac{1}{2}|A|^2. \tag{66}$$

Therefore, we can combine (64) with (66) to infer that at x_0

$$|A\nabla u|^2 = \frac{1}{2}|A|^2|\nabla u|^2. \tag{67}$$

Since $\nabla u(x_0) \neq 0$, we may choose a referential near x_0 such that $|\nabla u|e_1 = \nabla u$.

As before, we deduce

$$\begin{aligned} 0 &\geq \left(\frac{2-|A|^2}{2}\right)|\nabla u|^2 - \lambda|\nabla u|^2 + |\nabla^2 u|^2 - c\lambda u^2 - c\lambda u a + c|\nabla u|^2 \\ &= \left(c - \lambda + \left(\frac{2-|A|^2}{2}\right)\right)\left(|\nabla u|^2 + cu^2\right) + cu^2\Big(\frac{|A|^2}{2} - 1\Big) - c\lambda u a \\ &\geq \left(c - \lambda + \left(\frac{2-|A|^2}{2}\right)\right)\left(|\nabla u|^2 + cu^2\right) - c\lambda u a \\ &\geq \left(c - \lambda + \left(\frac{2-|A|^2}{2}\right)\right)\left(|\nabla u|^2 + cu^2\right) - c\lambda, \end{aligned}$$

Therefore, if $c - \lambda + 1 - \frac{|A|^2}{2} > 0$ we have

$$\frac{c\lambda}{\big(c - \lambda + 1 - \frac{|A|^2}{2}\big)} \geq |\nabla u|^2 + cu^2$$

From that we deduce

$$|\nabla u|^2 \leq \frac{c\lambda}{\big(c - \lambda + 1 - \frac{|A|^2}{2}\big)} - cu^2 \tag{68}$$

Next, letting $c = |A|^2 + \lambda$ we have $c \geq 2 + \lambda \geq 2\lambda$. Using this choice in (68) we arrive at

$$|\nabla u|^2 \leq \lambda \frac{|A|^2 + \lambda}{\big(\frac{|A|^2}{2} + 1\big)} - 2\lambda u^2. \tag{69}$$

Whence we deduce the following inequality

$$|\nabla u|^2 \leq 2\lambda\big(1 - u^2\big). \tag{70}$$

Therefore we have the following lemma.

Lema 5.2. *Let $\psi : \Sigma \to \mathbb{S}^3$ be a minimal isometric immersion of a compact Riemann surface Σ^2 and suppose that $|A|^2$ is a constant. If u is given according to the above choice, then we have*

$$\frac{|\nabla u|}{\sqrt{1-u^2}} \leq \sqrt{2\lambda}. \tag{71}$$

Now we proceed as before to conclude the proof of the theorem. □

6 COMPACT MANIFOLDS WITH CONSTANT ENERGY

Now, we introduce the concept of manifold with constant energy in order to characterize the Euclidean sphere and the Clifford torus. We also investigate the problem of estimating from below the diameter of such manifolds and get some informations in the case of a hypersurface of the sphere which has constant energy. The content of this section was taken from the preprint Barros and Filho (2015).

We recall that Obata's theorem asserts that a compact Riemannian manifold carrying a smooth function satisfying $\nabla^2 f + \kappa f I = 0$ must be isometric to a standard sphere $\mathbb{S}^n(\rho)$, for some $\rho > 0$, moreover f is a first eigenfunction of $\mathbb{S}^n(\rho)$. Since gradient of $|\nabla f|^2$ is $2\nabla^2 f(\nabla f)$ we conclude that Obata's condition implies $|\nabla f|^2 + \kappa f^2$ constant. In the opposite direction, it is natural to ask if this condition is sufficient to characterize the sphere. More precisely, we are interested in eigenfunction f of the Laplacian on compact Riemannian manifold M^n verifying

$$|\nabla f|^2 + \kappa f^2 = \tau, \tag{72}$$

for some real constants κ and τ. Following Castañeda (2007), we will say that an eigenfunction f has constant energy if it verifies the condition $|\nabla f|^2 + \kappa f^2 = \kappa$, for some constant κ. We will say that M has constant energy if some eigenfunction of M has constant energy. Next we consider the Riemannian product $M^{n+k} = \mathbb{S}^n(\sqrt{n/\lambda}) \times N^k$ with its product metric $\sigma = g_{\mathbb{S}^n} + g_{N^k}$, where $\mathbb{S}^n(\sqrt{n/\lambda})$ is the sphere of radius $\sqrt{n/\lambda}$ and N^k is any compact k-dimensional Riemannian manifold. Let f be any first eigenfunction of $\mathbb{S}^n(\sqrt{n/\lambda})$ which satisfies $|\nabla^{\mathbb{S}^n} f|^2 + \frac{\lambda}{n} f^2 = \frac{\lambda}{n}$. Defining $F : M^{n+k} \to \mathbb{R}$ by $F(p,q) = f(p)$ we immediately have $\Delta_M F = \Delta_{\mathbb{S}^n} f = -\lambda f$ and

$$|\nabla^M F|^2 + \frac{\lambda}{n} F^2 = |\nabla^{\mathbb{S}^n} f|^2 + \frac{\lambda}{n} f^2 = \frac{\lambda}{n}.$$

This shows that the class of manifolds satisfying (72) is big enough. But, restricting our attention to surfaces, we will give a positive answer, see Theorem 6.1, by showing that the sphere and a flat torus are the unique Riemannian surfaces carrying such a structure. As a consequence of this theorem we deduce that a sphere as well as a Clifford torus are the unique surfaces in $\mathbb{S}^3$ with constant mean curvature and constant energy, see Theorem 6.2. On the other hand, considering constant mean curvature immersions into the Euclidean sphere $\mathbb{S}^{n+1}$, we can ask if there are another examples up to geodesic spheres or Clifford tori with constant energy.

Finally, we present a lower bound for the diameter of a compact Riemannian manifold with constant energy, see Theorem 6.3, which allow us to conclude that π is a lower bound for the diameter of this class of Riemannian manifold M^n minimally immersed into the Euclidean sphere $\mathbb{S}^{n+1}$.

6.1 Riemannian manifolds with $|\nabla f|^2 + \kappa f^2 = \tau$

We notice that if (72) holds, then since at the maximum point of f, its gradient vanishes τ must be of the form $\tau = \kappa |f|^2_\infty$, so we have $|\nabla f|^2 + \kappa f^2 = \kappa |f|^2_\infty$ and the same property holds at the infimum, this gives that f is symmetric, i.e. $\max f = -\min f$. We can further suppose that $|f|_\infty = 1$, (otherwise, we work with $g := f/|f|_\infty$). Thus, we obtain that $\max f = -\min f = 1$, from this, we have

$$|\nabla f|^2 + \kappa f^2 = \kappa. \tag{73}$$

In particular, this last identity tell us that $\cos^{-1}(f)$ is well defined and has $|\nabla(1/\sqrt{\kappa})\cos^{-1} f)| = 1$ away from the points where $f = \pm 1$. That is, $(1/\sqrt{\kappa})\cos^{-1} f$ is a distance function to the points where $f = \pm 1$.

6.2 Level sets of f

In this part, we present some properties of the level sets $f^{-1}(c)$, where c is a regular value of f. We start remembering the next result which corresponds to Proposition 2.14 of Barbosa et al. (1991).

Proposition 6.1. *Let $\mathfrak{F}$ be a codimension one C^3-foliation of a Riemannian manifold M^{n+1} and let N be a unit vector field normal to the leaves of $\mathfrak{F}$ in some open set U of M. Then, on U we have*

$$divN = -nH.$$

where H is the mean curvature in the direction of N.

In particular, if $\Sigma = f^{-1}(c)$, where c is a regular value of f, we can use the above proposition with $N = \frac{\nabla f}{|\nabla f|}$. First we observe that for any $Y \in TM$

$$Y\big(|\nabla f|^{-1}\big) = -|\nabla f|^{-3}\langle \nabla_{\nabla f}\nabla f, Y\rangle. \tag{74}$$

Since $X = \nabla_{\left(\frac{\nabla f}{|\nabla f|}\right)}\left(\frac{\nabla f}{|\nabla f|}\right)$, in this case we have

$$X = \frac{1}{|\nabla f|^2}\nabla_{\nabla f}\nabla f - \frac{1}{|\nabla f|^4}\langle \nabla_{\nabla f}\nabla f, \nabla f\rangle \nabla f.$$

In particular, if $|\nabla f|^2 + \kappa f^2 = \kappa$, we have $\nabla_{\nabla f}\nabla f = -\kappa f \nabla f$, whence we deduce that $X = 0$.

On the other hand we also have

$$div\Big(\frac{1}{|\nabla f|}\nabla f\Big) = -\frac{\lambda}{|\nabla f|}f - \frac{1}{|\nabla f|^3}\langle \nabla_{\nabla f}\nabla f, \nabla f\rangle. \tag{75}$$

Indeed, since $div\big(\frac{1}{|\nabla f|}\nabla f\big) = \frac{1}{|\nabla f|}div(\nabla f) + \langle\nabla(\frac{1}{|\nabla f|}), \nabla f\rangle$ it suffices to use (74).

In particular, if $|\nabla f|^2 + \kappa f^2 = \kappa$, we have

$$div\big(\frac{1}{|\nabla f|}\nabla f\big) = \frac{1}{|\nabla f|}(\kappa - \lambda)f. \tag{76}$$

Whence we obtain the next lemma.

Lema 6.1. *If $|\nabla f|^2 + \kappa f^2 = \kappa$, then any $c \in (-1, 1)$ is a regular value of f. Hence the mean curvature of $\Sigma_c = f^{-1}(c)$ is given by $(n-1)H = \frac{(\lambda-\kappa)}{\sqrt{\kappa(1-c^2)}}c$. Therefore, up to a set of measure zero, M^n is foliated by hypersurfaces Σ_c^{n-1} of constant mean curvature. Moreover, $f^{-1}(0)$ is always a minimal hypersurface.*

Proof. Since $|\nabla f|^2 = \kappa(1-f^2)$ any $c \in (-1, 1)$ is a regular value of f and from Proposition 6.1 we know that $(n-1)H = -div\big(\frac{1}{|\nabla f|}\nabla f\big)$. Now it suffices to use (76) and $|\nabla f| = \sqrt{\kappa(1-f^2)}$ to conclude the proof of the lemma. □

6.3 Surfaces with constant energy

In this section we will characterize surfaces with constant energy. The next theorem characterizes a standard sphere and a flat torus as the unique compact surfaces carrying property (73).

Theorem 6.1. *Let M^2 be a compact Riemannian surface and let f be an eigenfunction of M for which (73) holds. Then we have either $\lambda = 2\kappa$ or $\lambda = \kappa$. Moreover, if the first case occurs, then M^2 is isometric to a standard sphere $\mathbb{S}^2(r)$, while in the second one M^2 is isometric to a flat torus $\mathbb{T}^2$ foliated by geodesics.*

Proof. If $\nabla f(p) \neq 0$, then we can choose a frame so that $|\nabla f|e_1 = \nabla f$. Equation (73) then gives that $f_{11} = \langle\nabla_{e_1}\nabla f, e_1\rangle = -\kappa f$ and similarly, $f_{12} = f_{21} = 0$. Using the condition $\Delta f = -\lambda f$ we also obtain $f_{22} = (\kappa - \lambda)f$. Therefore,

$$\nabla^2 f = f\begin{pmatrix} -\kappa & 0 \\ 0 & \kappa - \lambda \end{pmatrix}, \tag{77}$$

which implies,

$$|\nabla^2 f|^2(p) = f^2(p)\Big(2\kappa^2 - 2\kappa\lambda + \lambda^2\Big). \tag{78}$$

If p is a point where $\nabla f(p) = 0$ we choose a sequence of points $p_n \in M$ such that $p_n \to p$ and $\nabla f(p_n) \neq 0$. Then, since (78) holds at p_n, by continuity, it must hold at p and therefore on all of M.

By one hand (73) gives, at any point p of M, $\frac{1}{2}\Delta|\nabla f|^2 = -\frac{\kappa}{2}\Delta f^2 = \kappa\lambda f^2 - \kappa|\nabla f|^2$. On the other hand, Bochner formula combined with (78) gives

$$\frac{1}{2}\Delta|\nabla f|^2 = Ric(\nabla f, \nabla f) - \lambda|\nabla f|^2 + \big(2\kappa^2 - 2\kappa\lambda + \lambda^2\big)f^2(p). \tag{79}$$

We compare these two informations to arrive at

$$Ric(\nabla f, \nabla f) = (\lambda - \kappa)|\nabla f|^2 - (\kappa - \lambda)(2\kappa - \lambda)f^2. \tag{80}$$

If we evaluate (80) at a critical point we obtain either $\lambda = 2\kappa$ or $\lambda = \kappa$.

We can rewrite (77) as

$$\nabla^2 f + \kappa f I = f \begin{pmatrix} 0 & 0 \\ 0 & 2\kappa - \lambda \end{pmatrix},$$

at every point p where $\nabla f \neq 0$. Whence

$$\int_M |\nabla^2 f + \kappa f I|^2 dM_g = (\lambda - 2\kappa)^2 \int_M f^2 dM_g, \tag{81}$$

from where we conclude that M^2 is isometric to $\mathbb{S}^2$ if and only if $\lambda = 2\kappa$. On the other hand, if $\lambda = \kappa$ we deduce that the Gaussian curvature K of M^2 is null, since $Ric(\nabla f, \nabla f) = K|\nabla f|^2$. Therefore we conclude that M^2 is isometric to a flat torus. Moreover, according to Lemma 6.1 the mean curvature is null, but in dimension two this mean curvature coincides with the geodesic curvature, and this completes the proof of the theorem. □

As a consequence of Theorem 6.1 we obtain the following result.

Theorem 6.2. *Let $\varphi : M^2 \looparrowright \mathbb{S}^3$ be a compact immersion with constant mean curvature into the Euclidean sphere $\mathbb{S}^3$. If $\Delta f + \lambda f = 0$ on M and $|\nabla f|^2 + \kappa f^2 = \kappa$, then M^2 is isometric to either a geodesic sphere or a Clifford torus.*

Proof. According to Theorem 6.1 either $2\kappa = \lambda$ or $\lambda = \kappa$. In the former case we have a geodesic sphere, while in the second one we have a flat torus. But, by a result due to Hoffman (1972), a flat torus of $\mathbb{S}^3$ with constant mean curvature is isometric to a Clifford torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ for some $r \in (0,1)$ and this completes the proof of our theorem. □

6.4 Ricci curvature and the Einstein case

First we notice that since $\Delta f = -\lambda f$ and $\Delta|\nabla f|^2 = -\kappa \Delta f^2 = -2\kappa(f\Delta f + |\nabla f|^2)$ we use Bochner formula to arrive at

$$Ric(\nabla f, \nabla f) = (\lambda - \kappa)|\nabla f|^2 + \lambda\big(\kappa - \frac{\lambda}{n}\big)f^2 - |\mathring{\nabla}^2 f|^2, \tag{82}$$

where $\mathring{\nabla}^2 f$ stands for the traceless of the Hessian of f.

We observe that we can use (82) to derive the next inequality

$$\kappa \geq \frac{\lambda}{n}. \tag{83}$$

In fact, applying this formula at point p where f achieves its maximum we obtain $|\nabla^2 f|^2 = \lambda\kappa f^2$. On the other hand $|\nabla^2 f|^2 \geq \frac{\lambda^2}{n} f^2$ will gives the desired result.

We assume now that M is an Einstein manifold with constant energy. We obtained an estimate of κ in terms of the scalar curvature R and the dimension.

Lema 6.2. *Let M^n be a compact Einstein Riemannian manifold and let f be a function on M^n such that $\Delta f + \lambda f = 0$ for which (73) holds. Then*

$$\kappa \geq \frac{R}{n^2}.$$

Proof. Supposing that $Ric = \frac{R}{n} g$, we have by Bochner's formula,

$$\frac{1}{2}\Delta|\nabla f|^2 = \Big(\frac{R}{n} - \lambda\Big)|\nabla f|^2 + |\nabla^2 f|^2.$$

Because of Schur's Lemma and Theorem 6.1 we can suppose $n \geq 3$, which gives that R is a constant. Integrating and using (83), we get

$$0 \leq \int_M |\nabla^2 f|^2 dM_g = \Big(\lambda - \frac{R}{n}\Big)\int_M |\nabla f|^2 dM_g \tag{84}$$

$$\leq \Big(n\kappa - \frac{R}{n}\Big)\int_M |\nabla f|^2 dM_g. \tag{85}$$

Whence, it follows that $\kappa \geq \frac{R}{n^2}$. □

6.5 Estimate of diameter

For a compact Riemannian manifold M^n let us denote its diameter by d. The aim of the next result is to present a lower bound for d in the class of manifolds with constant energy. We recall that a unit standard sphere $\mathbb{S}^n$ has diameter π. Then a natural question is to achieve a lower bound for $d = diam(M)$ among this class of Riemannian manifolds. Our first result gives a lower bound which depends only on κ.

Theorem 6.3. *Let (M^n, g) be a compact Riemannian manifold and let f be a function on M^n such that $\Delta f + \lambda f = 0$ for which (73) holds. Then*

$$d \geq \frac{\pi}{\sqrt{\kappa}}. \tag{86}$$

Moreover, if $Ric \geq (n-1)\kappa g$, then M^n is isometric to a standard sphere $\mathbb{S}^n\big(\frac{1}{\sqrt{\kappa}}\big)$.

Proof. We follow the technique developed by Li and Yau 2012 to provide gradient estimate. To do that we can write (73) as follows

$$\frac{|\nabla f|}{\sqrt{1-f^2}} = \sqrt{\kappa}. \tag{87}$$

Now we integrate (87) over a minimizing geodesic γ joining the points where $f=-1$ and $f=1$ to get

$$\begin{aligned} d\sqrt{\kappa} &\geq \int_\gamma \frac{|\nabla f|}{\sqrt{1-f^2}} \\ &\geq \int_{-1}^{1} \frac{df}{\sqrt{1-f^2}} \\ &= \pi, \end{aligned}$$

which gives the pinching of the diameter. Under the hypothesis of the *Ric*, arguing as in the proof of Myers' theorem, we arrive at $d \leq \frac{\pi}{\sqrt{\kappa}}$. Whence we obtain equality for the diameter which enables us to use the result of Cheng (1975) to complete the proof of the theorem. □

We notice that we can improve the part of the rigidity of the sphere by requiring a weaker condition on the Ricci tensor.

Theorem 6.4. *Let M^n be a compact Riemannian manifold and let f be a function on M^n such that $\Delta f + \lambda f = 0$ for which (73) holds. If $\int_M Ric(\nabla f, \nabla f) \geq \kappa(n-1)\int_M |\nabla f|^2$, then M^n is isometric to a standard sphere $\mathbb{S}^n\big(\frac{1}{\sqrt{\kappa}}\big)$.*

Proof. On integrating identity (82) we deduce

$$\int_M Ric(\nabla f, \nabla f) + \int_M |\mathring{\nabla}^2 f|^2 = \frac{\lambda}{n}(n-1)\int_M |\nabla f|^2 \leq \kappa(n-1)\int_M |\nabla f|^2, \tag{88}$$

where we used (83) on the last part. Now it suffices to use our assumption jointly with Obata's theorem to conclude the proof of the theorem. □

Proceeding we recall that given a minimal immersion $\psi : M^n \looparrowright \mathbb{S}^{n+1}$ of a compact Riemannian manifold M^n without boundary into the Euclidean sphere $\mathbb{S}^{n+1}$ the Laplacian of M^n in the induced metric satisfies $\Delta\psi + n\psi = 0$. As we have seen in the previous section, the diameter of this immersion must be bigger than $\pi/2$. Therefore, we can use Theorem 6.3 to show that a sphere attains the minimal value of the diameter among minimal hypersurfaces of $\mathbb{S}^{n+1}$ with constant energy.

Corollary 6.1. *Let $\varphi : M^n \to \mathbb{S}^{n+1}$ be a compact minimal immersion into the Euclidean sphere $\mathbb{S}^{n+1}$. If $\Delta f + \lambda f = 0$ and $|\nabla f|^2 + \frac{\lambda}{n} f^2 = \frac{\lambda}{n}$, where λ is the first eigenvalue of the Laplacian of M^n, then $d \geq \pi$.*

Proof. Taking into account that $\Delta\varphi + n\varphi = 0$ we deduce $\lambda \leq n$. Now it suffices to apply Theorem 6.3 to conclude the proof of the corollary. $\square$

7 CONCLUSION

In this work we obtained a partial answer to the critical point equation conjecture under some integral assumptions, Theorem 3.4. It would be nice if one were able to prove that theorem with just one of its hypotheses.

In the sequence, we have obtained some estimates for the first eigenvalue of a compact manifold that improve the corresponding inequality of Li-Yau.

Concerning the diameter of compact minimal hypersurfaces of the Euclidean sphere we got two estimates, one for manifolds of dimension bigger than two and the other for surfaces slightly better than the previous one. In both cases, we have assumed the hypothesis that the maximum of the function F (see Theorems 5.1 and 5.2), is not a critical point of u. We believe that this hypothesis is not necessary.

In the last part, after introducing the concept of manifold with constant energy, we proved that the sphere and the flat torus are the only compact surfaces with constant energy. In higher dimensions though, simple examples show that this is not true in general, but it is if we assume that the Ricci tensor is bigger than the one of the corresponding sphere. Since surface are Einstein manifolds, it seems to be natural to try to generalize Theorem 6.1 under Einstein assumption. Even in dimension four, this may be an interesting problem.

Our goals on this work were reached since we have given a contribution to the important CPE conjecture. We have also improved some known eigenvalue estimates and obtained diameter estimates for hypersurfaces of the sphere. Finally, with the concept of manifold with constant energy at our disposal we have characterized the sphere and the flat torus in dimension two.

The questions raised above may be useful to someone who wants to work on the mentioned subjects.

To finish, we highlight that this thesis gave rise to four scientific articles one of which has been published, namely Filho (2015).

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