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DYNAMIC BAYESIAN STATISTICAL MODELS FOR THE ESTIMATION OF THE ORIGIN-DESTINATION MATRIX

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Tese de Doutorado apresentada ao Programa de Pós-Graduação em Engenharia de Transportes do Departamento de Engenharia de Transportes da Universidade Federal do Ceará como parte dos requisitos para a obtenção do título de Doutor em Engenharia de Transportes. Área de concentração: Planejamento e Operação de Sistemas de Transportes

Orientador: Prof. Dr. Carlos Felipe Grangeiro Loureiro

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Dedicado às mulheres da minha vida:
Minha mãe Rosa, minha irmã Camila, e minha esposa Renata.

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"Whenever a theory appears to you as the only possible one,
take this as a sign that you have neither understood the theory,
nor the problem which it was intended to solve."
Karl Popper

ABSTRACT

In transportation planning, one of the first steps is to estimate the travel demand. A product of the estimation process is the so-called *origin-destination matrix* (OD matrix), whose entries correspond to the number of trips between pairs of zones in a geographic region in a reference time period. Traditionally, the OD matrix has been estimated through direct methods, such as home-based surveys, road-side interviews and license plate automatic recognition. These direct methods require large samples to achieve a target statistical error, which may be technically or economically infeasible. Alternatively, one can use a statistical model to indirectly estimate the OD matrix from observed traffic volumes on links of the transportation network. The first estimation models proposed in the literature assume that traffic volumes in a sequence of days are independent and identically distributed samples of a static probability distribution. Moreover, static estimation models do not allow for variations in mean OD flows or non-constant variability over time. In contrast, day-to-day dynamic models are in theory more capable of capturing underlying changes of system parameters which are only indirectly observed through variations in traffic volumes. Even so, there is still a dearth of statistical models in the literature which account for the day-to-day dynamic evolution of transportation systems. In this thesis, our objective is to assess the potential gains and limitations of day-to-day dynamic models for the estimation of the OD matrix based on link volumes. First, we review the main static and dynamic models available in the literature. We then describe our proposed day-to-day dynamic Bayesian model based on the theory of linear dynamic models. The proposed model is tested by means of computational experiments and compared with a static estimation model and with the generalized least squares (GLS) model. The results show some advantage in favor of dynamic models in informative scenarios, while in non-informative scenarios the performance of the models were equivalent. The experiments also indicate a significant dependence of the estimation errors on the assignment matrices.

Keywords: OD matrix. Estimation. Bayesian statistics.

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS AND ACRONYMS

OD	Origin-destination
GLS	Generalized least squares
ME	Maximum entropy
DLM	Dynamic linear model
MCMC	Markov chain Monte Carlo
SE	Static estimation
DE	Dynamic estimation
MVN	Multivariate normal distribution
MAP	Maximum a posteriori
RMSE	Root mean square error
RRMSE	Relative root mean square error
MAE	Mean absolute error
RMAE	Relative mean absolute error

LIST OF SYMBOLS

$\boldsymbol{x}$	Vector of origin-destination (OD) flows.
$\boldsymbol{y}$	Vector of route flows.
$\boldsymbol{z}$	Vector of traffic volumes on links.
$\boldsymbol{\theta}$	Vector of mean OD flows.
$\boldsymbol{\lambda}$	Vector of mean OD route flows.
$\boldsymbol{p}$	Vector of route choice probabilities.
$\boldsymbol{c}$	Vector of route costs.
$\bar{\boldsymbol{m}}$	Mean vector of the prior distribution in a DLM.
$\bar{\mathbf{C}}$	Covariance matrix of the posterior distribution in a DLM.
$\boldsymbol{m}$	Mean vector of the posterior distribution in a DLM.
$\mathbf{C}$	Covariance matrix of the posterior distribution in a DLM.
$\mathcal{K}_i$	Set of routes for OD pair i.
ξ	Logit scale parameter.
π	Probability of not using a route in the given route choice set.
κ	Coefficient of variation.
τ	Performance (cost) function of a link.
$\boldsymbol{\nu}$	Vector of random errors in the measurement equation of a DLM.
$\boldsymbol{\omega}$	Vector of random errors in the system equation of a DLM.
$\boldsymbol{\Sigma}$	A covariance matrix.
$\mathbf{P}$	Route choice matrix.
$\boldsymbol{\Delta}$	Link-path incidence matrix.
$\mathbf{A}$	Adjustment matrix in the updating equations of a DLM.
$\mathbf{F}$	Assignment matrix/Regression matrix in DLM models.
$\mathbf{W}$	Covariance matrix of the mean OD flows (Evolution matrix).
$\mathbf{V}$	Covariance matrix of the traffic volumes on links.

$\mathbf{I}$	The identity matrix.
$\mathbf{1}$	A vector full of scalars equal to one.
$\mathrm{E}[x]$	The expected value of the random variable x.
$\mathrm{Var}(x)$	The variance of the random variable x.
$\mathrm{Cov}(x, y)$	The covariance between random variables x and y.
$\mathrm{diag}(\boldsymbol{x})$	A matrix whose main diagonal is the vector $\boldsymbol{x}$ and entries off the main diagonal are all zero.
$\mathrm{blockdiag}(\mathbf{X}_i)$	A block-diagonal matrix composed of submatrices $\mathbf{X}_i$
$\lceil x \rceil$	The smallest integer greater than x.
$\mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.
$\mathrm{MN}(x, \boldsymbol{p})$	Multinomial distribution with parameters x and $\boldsymbol{p}$.

CONTENTS

1 INTRODUCTION

1.1 The problem

In transportation planning, one of the first steps is to estimate the travel demand. Generally, the demand is measured in terms of trip flows between zones in a geographic region. The final product of the estimation process is a so-called *origin-destination matrix* (OD matrix, for short), whose entries correspond to the number of trips between pairs of zones in a reference time period.

Traditionally, the OD matrix has been estimated through direct methods, such as home-based surveys, road-side interviews and license plate automatic recognition. These methods collect sample data on the number of trips performed daily, their origins and their destinations. Such data can be compiled and several statistics may be computed, such as the mean, standard deviation and confidence intervals. However, these direct methods require large samples to achieve a target statistical significance, which may be technically or economically infeasible (CASCETTA, 2009).

Another way of estimating the OD matrix is by using trip generation and distribution models. In this approach, social and economic data are used to estimate the number of trips produced and attracted by each zone. In the next step, a gravity-type model is applied in order to distribute the generated trips between zones (ORTÚZAR; WILLUMSEN, 2011). Nevertheless, this approach also has its drawbacks. First, obtaining all the required data demands considerable amounts of resources, with high accompanying costs. Second, these models are in general aimed at long term planning horizons, which limit their use in short term applications, such as traffic management systems and public transit operation.

In the 1970s, researchers started developing alternative mathematical models whose objective was to obtain an OD matrix from indirect data on trip patterns. The main sources of indirect data were traffic volumes observed on links of the transportation network (also called *traffic counts*). The development of traffic monitoring systems opened up the possibility of acquiring data on traffic volumes in an automated way at low costs. In road networks this acquisition takes place by means of sensors installed on the roads, and in transit networks data on traffic of passengers can be acquired by means of electronic ticketing.

The rationale of these alternative models is to estimate OD flows through a mathematical model which relates traffic volumes on links of the transportation network to OD flows between zones. The models are in general of an optimization or statistical nature. The OD matrix so obtained is called a *synthetic* OD matrix, since it is not estimated by direct observation of trips (e.g., by directly sampling OD trips), but as an output of a model which uses indirect data on the travel demand. The usefulness of such a model is evident: the transportation demand patterns in a part or in a whole region may be, in theory, traced to a finer time scale of days or hours, or even in real time. This is a great improvement over household surveys, which are typically carried out once in a decade, a time period during which the demand pattern may have changed considerably.

Since the pioneering work of Robillard (1975), many models have been proposed based on different approaches and assumptions. However, there are several issues related to the problem which have yet to be resolved satisfactorily, both from the theoretical and practical perspectives. Moreover, the literature lacks thorough comparisons and assessments of the performance and properties of the estimates produced by the alternative models. All this has led to a low adoption of the synthetic OD matrix based on link counts as an alternative to the more traditional and costly OD matrix obtained by means of direct estimation.

1.2 Research gap

The first attempts at estimating OD matrices from traffic counts relied on a single sample of volumes. The early data collection procedures involved the manual counting of vehicles in selected points in a transportation network. Due to technological or economical limitations, it was infeasible to take repeated samples of traffic volumes. Since there were no further data on traffic variability, a static and deterministic approach to the problem seemed plausible. The availability of a single sample of volumes provided only a snapshot of the transportation system in a point in time. The so-called reconstruction models sought to estimate mean OD matrices based on a sample of a single day. The validity of static reconstruction models critically hinged on the assumption that observed volumes were representative of a typical day.

More recently, many cities around the world have built traffic control systems, thereby massive data on urban traffic volumes have been collected daily. This opened up the possibility of applying statistical models based on large samples in order to estimate OD matrices and other relevant parameters more accurately. These first estimation models, initially proposed in the 1990's, assume that traffic volumes in a sequence of days are independent and identically distributed samples of a static probability distribution. They use frequentist or Bayesian statistical techniques so as to estimate static quantities, such as mean OD matrices, variances of OD flows, or parameters of the route choice model.

A major weakness of static estimation models is that they do not allow for variations in mean OD flows or heteroscedasticity (i.e., non-constant variability) over time. In the relevant literature, two types of dynamics are often distinguished: *within-day* and *day-to-day* dynamics. Within-day dynamics refer to the temporal variation of the transportation demand for a specified time period within a single day. The extension of the time period under study may be as short as a few minutes or an entire day. Within-day dynamics is often of interest in short-term operational planning, since knowledge of the demand profile is valuable to effective intervention or to designing traffic management policies. In contrast, day-to-day dynamics is related to the variation of the demand for a repeated reference time period (typically the peak hour) over a sequence of days. It is more adequate for mid to long-term planning, when factors such as seasonality, changes in transportation supply and in economic activities are more pronounced. In this thesis, we focus on day-to-day dynamics.

In comparison with static models, day-to-day dynamic models are in theory more capable of capturing underlying changes of system parameters which are only indirectly observed through variations in traffic volumes. They may be more responsive to temporal changes and provide useful information on the dynamic behavior of the transportation system. Despite these promising features, there is still a dearth of statistical models in the literature which account for the day-to-day dynamic evolution of transportation systems.

Moreover, an important issue that we should be aware of when developing models for the estimation of the OD matrix is the occurrence of non-identifiability of the parameters, which refers to the existence of multiple parameter values which fit the data almost equally well. This issue has the implication that, except in very small transportation networks, it is often difficult to estimate OD matrices and other parameters based solely on traffic volume data. Some of the strategies to tackle this problem are: the development of parsimonious models, which are economic in the number of parameters; the use of a prior OD matrix, which can be, for example, an outdated matrix obtained by survey; or the adoption of simplifying and, in some cases, very restrictive assumptions.

In order to contribute to the development and analysis of day-to-day dynamic models for the OD matrix estimation problem, we propose a dynamic model and evaluate its potentials and limitations through computational experiments. Our main hypothesis is that day-to-day dynamic models can produce better estimates of mean OD matrices than static estimation models, since they should be able to account for the evolution of transportation systems over time and make use of the information provided by temporal changes. Our efforts are driven mainly towards attempting to answer the following specific questions:

- Are dynamic models capable of reducing the non-identifiability of mean OD matrices by incorporating the variation of the link volumes over time?
- Can dynamic estimation models produce better estimates of mean OD matrices than static estimation models?
- What is the impact of prior information on estimation errors?
- How do assignment matrices affect estimation errors?

According to the aforementioned research questions, the research objectives of this doctoral thesis are the following:

1.3 General objective

To assess the potential gains and limitations of day-to-day dynamic Bayesian statistical models for the estimation of the OD matrix based on link volumes.

1.4 Specific objectives

- To describe the main static and dynamic OD matrix estimation models currently available in the literature;
- to propose a model for the estimation of the day-to-day dynamic OD matrix based on the theory of dynamic linear models;
- to perform computational experiments in order to evaluate the potential application of dynamic models relative to static models proposed in the literature;
- to assess how assignment matrices affect estimation errors, and
- to assess the impact of prior information on estimation errors.

1.5 Structure of this doctoral thesis

The literature review of our research is compiled in Chapters 2 and 3. We start by describing the traffic assignment models in Chapter 2. As we will see in Chapter 3, the OD matrix estimation problem is the inverse of the traffic assignment problem. Hence, many OD matrix estimation models embed traffic assignment models as part of their solution procedures. We describe the main variables and mathematical relationships involved in modeling transportation networks and the modeling of user behavior through route choice models. The traffic assignment models are presented according to the classification in proportional and equilibrium models, which correspond, in general, to assignment in uncongested and congested networks respectively.

In Chapter 3 we review the main models for OD matrix estimation. The chapter is divided in three sections, corresponding to reconstruction, estimation and dynamic models. In the following Chapter 4, we describe our proposed dynamic Bayesian statistical model for the estimation of the OD matrix and illustrate its application through a small test network from the literature. In Chapter 5 we present and discuss the results of some computational experiments. All experiments are carried out by means of Monte Carlo simulation. We use simulated data in order to evaluate the research questions. The use of synthetic data allows us to have full control of all experimental conditions. We use as test unit a benchmark transportation network from the literature. Finally, in Chapter 6 we draw some conclusive comments and suggest further research directions.

2 TRAFFIC ASSIGNMENT

2.1 Modelling transportation flows on networks

Let $(\mathcal{N}, \mathcal{A})$ be a transportation network, in which $\mathcal{N}$ is a set of *nodes* and $\mathcal{A}$ is a set of *directed links*. Typically, for road networks, the links and nodes correspond to road segments and intersections between road segments, respectively. The transport network connects *zones* of a certain geographic region (e.g., a city), which "produce and attract" trips, so that there is also a set of zones, denoted by $\mathcal{I}$. A *trip* is a movement of a user (person, freight, or vehicle) between an *origin zone* and a *destination zone* (referred to simply as an *OD pair*). All trips enter and exit the network through *centroid nodes*, which are terminal nodes located at zone centroids. Intrazonal trips are not taken into account, since their origin and destination centroid nodes coincide.

We denote by x_i the total flow of trips in an OD pair i for a given time period. In applications, a time period may be, e.g., the morning peak hour or a whole business day. What is traditionally meant as an *OD matrix* is a two-dimensional array whose row indices identify origin zones, column indices identify destination zones, and the entries are the number of trips in an OD pair. As a matter of analytical convenience, in our notation the OD matrix is stretched out as a vector $\boldsymbol{x} \in \mathbb{R}^n_+$, for which $n = |\mathcal{I}|$ is the number of OD pairs. The total count of trips which flow through a link a for a given time period is denoted as the *traffic volume* z_a. The traffic volumes on all links are represented by a vector $\boldsymbol{z} \in \mathbb{R}^m_+$, and $m = |\mathcal{A}|$ is the number of links in the network.

For a given OD pair i, there is a set of *routes* connecting its origin and destination zones. A *route* is a simple path between a pair of nodes. In general, we consider only a reduced set of routes, since the total number of routes may be prohibitively large in real scale networks. (See Bekhor, Ben-Akiva and Ramming (2006) for an evaluation of route choice set generation algorithms). Let $\mathcal{K}_i$ be a set of routes associated with OD pair i. For a route $k \in \mathcal{K}_i$, we define y_{ik} as the flow of trips through route k in the reference time period. Let also $\boldsymbol{y} \in \mathbb{R}^r_+$ be the vector whose components are y_{ik}, and $r = \sum_{i \in \mathcal{I}} |\mathcal{K}_i|$ is the number of routes over all OD pairs.

Flows in OD pairs, route flows and traffic volumes are all related by flow conservation equations (2.1) and (2.2), given below:

$$\sum_{k \in \mathcal{K}_i} y_{ik} = x_i \qquad \forall i \in \mathcal{I} \tag{2.1}$$

$$z_a = \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}_i} \delta_{ak} y_{ik} \qquad \forall a \in \mathcal{A} \tag{2.2}$$

In equation (2.2), δ_{ak} takes the value 1 if link a is part of route k, and 0 otherwise. It synthesizes information on the topology of the network. Conveniently, we will refer to equation (2.2) in matrix notation, in which $\mathbf{\Delta} = [\delta_{ak}]_{m \times r}$ is called the *link-path incidence matrix*:

$$\boldsymbol{z} = \mathbf{\Delta}\boldsymbol{y} \tag{2.3}$$

It is also worth defining the route choice fraction $p_{ik} = y_{ik}/x_i$, which gives the proportion of users in OD pair i that chooses to follow route k. Notice that it is possible to establish a direct relationship between OD flows and traffic volumes through the route fractions. Let $\mathbf{P} = [p_{ik}]_{r \times n}$ be a *route choice matrix*, then:

$$\boldsymbol{z} = \mathbf{\Delta}\mathbf{P}\boldsymbol{x} \tag{2.4}$$

The route choice matrix $\mathbf{P}$ is commonly specified by means of a route choice model, which estimates the probability that a user chooses a route as a function of the travel time (or generalized cost) associated with the route. Deterministic route choice models assume that the users have perfect knowledge of costs and always choose the route with minimum cost. In contrast, stochastic models assume that *perceived* costs of the users are different from actual costs, so that they may choose routes which do not have minimum costs. The probabilities are estimated by means of discrete choice models, among which the multinomial logit and probit are the most used (CASCETTA, 2009).

The term $\mathbf{F} = \mathbf{\Delta}\mathbf{P}$ is called the *assignment matrix* (CASCETTA; NGUYEN, 1988), through which the relationship between $\boldsymbol{x}$ and $\boldsymbol{z}$ may be directly expressed by:

$$\boldsymbol{z} = \mathbf{F}\boldsymbol{x} \tag{2.5}$$

As we will see in the forthcoming chapters, the assignment matrix plays a key role both in traffic assignment as in the OD matrix estimation problem.

2.2 Route choice models

Route choice models try to capture the choice behavior of users on which route to follow when going from an origin to a destination. Each route has a cost associated to it, and it is assumed that a user chooses the one route with minimum cost (rational behavior). Most of the occasions, costs will be measured in terms of travel time spent from origin to destination by following the chosen route. Other sources of costs may include, e.g., the gas consumption, toll fees or transit fares. In reality, all those may be aggregated in a *generalized cost* measure.

It is assumed that the cost associated with a particular route depends linearly on the costs associated with the links that make up the route. Let c_k be the cost associated with a route $k \in \mathcal{K}_i$ for an OD pair i, and let τ_a be the cost associated with a link a. Then:

$$c_k = \sum_{a \in \mathcal{A}} \delta_{ak} \tau_a + c_k^{NA} \quad k \in \mathcal{K}_i \tag{2.6}$$

In equation 2.6, the term c_k^{NA} corresponds to some fixed cost incurred when using the route, e.g., a toll fee. Without loss of generalization, we assume it to be zero henceforth. Let $p_k = y_k/x_i$ be the fraction of trips in od pair i that flows through route $k \in \mathcal{K}_i$, with $\sum_{i \in \mathcal{K}_i} p_k = 1$, and p_k is a function of the route costs c_j for all $j \in \mathcal{K}_i$ for OD pair i. There are two classes of models to determine the value of p_k: Deterministic and stochastic (CASCETTA, 2009).

In deterministic models, it is assumed that the user knows with certainty the costs associated to each route. Let $c^* = \min_{k \in \mathcal{K}_i}(c_k)$ be the minimum cost among all routes in OD pair i. Thus:

$$p_k = \begin{cases} 1 & \text{if } c_k = c^* \\ 0 & \text{if } c_k > c^* \end{cases} \quad \forall k \in \mathcal{K}_i \tag{2.7}$$

In other words, the determination of p_k by means of equation (2.7) implies that all trips in OD pair i will flow through the minimum cost route. In case there are multiple routes with minimum cost, p_k is undefined, since users will show no preference for a particular route. A procedure to define a unique value for p_k based on entropy maximization is presented in (LARSSON et al., 2001).

On the other hand, in stochastic models, it is assumed that users do not have full knowledge of the actual costs associated with links and routes. In this way, the user chooses the route of minimum *perceived cost*, which is modelled as a random variable, whose probability distribution captures the variability in perception among users. The route cost c_k is then given by the following expression:

$$c_k = \sum_{a} \delta_{ak} \tau_a + c_k^{NA} + \epsilon_k \tag{2.8}$$

The term $\sum_a \delta_k c_a + c_k^{NA}$ is called *systematic cost* or *actual cost* associated with the route, and the term ϵ_k corresponds to *random error*, which represents the variability in the perceptions of users with respect to actual cost. The fraction p_k in stochastic models may be regarded as the probability of a user choosing route k, which corresponds to the probability of k being a route with minimum cost:

$$p_k = \text{Prob}(c_k \leq c_l, \quad \forall l \neq k) \tag{2.9}$$

The determination of p_k in stochastic models depends on the probability distribution of the error ϵ_k. Some of the models widely applied in the literature are the *probit*, which assumes a normal distribution, and the *logit*, which assumes a Gumbel distribution (BEN-AKIVA; LERMAN, 1985). In the case of the probit model, p_k must be estimated through Monte Carlo simulation, while for the logit model, there is an analytical solution given in equation (2.10):

$$p_k = \frac{\exp(-c_k/\xi)}{\sum_{l \in \mathcal{K}_i} \exp(-c_l/\xi)} \tag{2.10}$$

In equation (2.10), ξ is a scaling factor proportional to the variance of the error term ϵ_k.

An important issue in route choice models is the generation of the *route choice set*. Ideally, the models should consider exhaustively all available routes, but the number of routes in real networks is prohibitively large, which poses computational challenges in the development of efficient algorithms. This difficulty has been addressed in two directions: one consider the generation of a reduced choice set, by means of some criterion or procedure which samples meaningful routes from the point of view of the user (BEKHOR; BEN-AKIVA; RAMMING, 2006); the other consider the development of algorithms which do not require the enumeration of routes, such as Dial's Algorithm (DIAL, 1971) and the algorithms proposed by Akamatsu (1996) and Bell (1995).

2.3 Proportional assignment

The earlier approaches to traffic assignment considered that congestion effects on costs were negligible, so that costs could be treated as constants independent of traffic link flows. The methods based on this assumption were called *proportional traffic assignment*, since scaling the OD matrix by a constant would result in link flows scaled by the same constant. Proportional assignment procedures are generally classified in *deterministic* and *stochastic*, according to the corresponding route choice model adopted (See section 2.2).

A simple deterministic assignment method is the *all-or-nothing* (SHEFFI, 1985). It consists in identifying a minimum cost route for each OD pair, and assigning all the flow in the OD pair to this route. Afterwards, the link flows are determined by simple application of the flow conservation equation (2.2). The most computationally expensive step in the application of the all-or-nothing is in the determination of the minimum cost routes. A minimum cost route may be obtained by means of the application of one of the several available algorithms for finding minimum paths in a graph. In practice, Dijkstra's algorithm is very convenient, by virtue of its computational efficiency and the fact that it can generate multiple routes at a time (AHUJA; MAGNANTI; ORLIN, 1993). The main drawback of the all-or-nothing procedure is that it is not much realistic. In real situations, the trips are distributed among multiple routes between each OD pair, instead of only one. Notwithstanding this limitation, the all-or-nothing procedure is still useful in intermediate steps in more sophisticated assignment procedures, as in equilibrium assignment.

In contrast, stochastic assignment procedures assign flows to multiple routes for an OD pair. They are based on stochastic route choice models, as the multinomial logit or probit. In summary, the procedure consists in computing the probabilities of each route, and then applying flow conservation equations (2.1) and (2.2) in order to determine route and link flows. The main issue in applying stochastic assignment is the generation of the route choice set. For a given OD pair, the number of possible routes in a real scale network can be overwhelmingly large, so that the enumeration of all possible routes may be infeasible, as already pointed out in Section 2.2.

As stated in the beginning of this Section, proportional assignment is more suitable for networks where congestion effects are negligible. In the next section we present a class of non-proportional assignment procedures, equilibrium methods, which take congestion into account.

2.4 Equilibrium assignment

From a practical point of view, proportional assignment models are useful only for networks with low to moderate congestion levels. Nevertheless, as flows increase in the network, so does congestion, and costs are significantly affected by it. Users respond to changes in costs by changing their routes, trying to minimize their private costs. By its turn, the route choices made by users determine traffic flows on routes and links, affecting back the costs of routes. This feedback loop makes the whole transportation network behave as a dynamic system. *Ceteris paribus*, the transportation network may eventually reach an equilibrium state in the long run, in which the flows remain constant over time. We refer to equilibrium traffic assignment as the determination of traffic flows for a network in an equilibrium state for a given OD matrix.

One of the first definitions of traffic equilibrium is known as *Wardrop first principle*, whose statement is: in *Wardrop equilibrium*, “The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.” (WARDROP, 1952, p. 345).

In mathematical terms, in a Wardrop equilibrium, for any two routes k and l:

$$c_l > c_k \implies y_l = 0 \tag{2.11}$$

When a transportation network is at Wardrop equilibrium, no user has incentive to unilaterally change route because all other routes have cost equal to or greater than the route currently used. Devarajan (1981) shows that this is equivalent to the definition of Nash equilibrium in non-cooperative games. Although a Wardrop equilibrium is reached as users try to optimize their particular costs, it is not necessarily optimal to the user, as there may be traffic distributions that produce lower costs for all users, as is demonstrated by the so-called *Braess paradox* (BRAESS; NAGURNEY; WAKOLBINGER, 2005). There may also be a traffic distribution which is socially optimal, in which the sum of costs of all users is minimum (this is known in the literature as *Wardrop second principle*).

Equilibrium assignment relies on link *performance functions* in order to evaluate the effect of congestion on links. In general, as traffic volumes increase, so do the time spent on links, the consumption of gas by vehicles, and other proxies of cost. The dependence of costs on link flows is modelled by the performance functions, among which the BPR function is very popular in the transportation theory and practice, given below (CASCETTA, 2009):

$$\tau_a(z_a) = \tau_{a0}\left[1 + \alpha\left(\frac{z_a}{z_a^{\max}}\right)^{\beta}\right] \tag{2.12}$$

Where $\tau_a(z_a)$ is the cost in link a when there is a flow of z_a. The cost in "free flow" is denoted as τ_{a0}, i.e., the cost the user incurs if no other user is using the link, and $z_a^{\max}$ is the capacity of the link. α and β are parameters of the function (typical values adopted in the literature are 0.15 and 4, respectively). Note that the capacity of the link is not treated as a hard constraint, i.e., actual volumes are allowed to be greater than the theoretical maximum. It is also worth noting that BPR function (equation (2.12)) has the following properties: it is continuous; it is strictly increasing, which means cost always increases with flow; and it is separable, which means the link costs depend solely on the flow on that link.

Smith (1979) has shown that a Wardropian equilibrium point satisfies the following variational inequalities with respect to link costs and flows:

$$\boldsymbol{\tau}(\boldsymbol{z}^*)^T(\boldsymbol{z} - \boldsymbol{z}^*) \geq 0 \tag{2.13}$$

In equation 2.13, $\boldsymbol{\tau}(\boldsymbol{z})$ is the vector of link costs, $\boldsymbol{z}$ is the vector of link flows, and $\boldsymbol{z}^*$ is the vector of equilibrium link flows. The satisfaction of inequalities (2.13) are also shown to be sufficient for the existence, uniqueness and stability of a Wardropian equilibrium point if the cost functions are continuous and strictly increasing. (BPR function, equation (2.12), satisfies both conditions).

It can be shown (SHEFFI, 1985) that the variational inequalities given by (2.13) correspond to first order optimality conditions of the following mathematical program proposed by Beckmann, McGuire and Winsten (1956):

$$\min \quad \sum_{a \in \mathcal{A}} \int_0^{z_a} \tau_a(v)dv \tag{2.14}$$

s.t.

$$\sum_{k \in \mathcal{K}_i} y_k = x_i \qquad \forall i \in I \tag{2.15}$$

$$z_a = \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}_i} \delta_{ak} y_k \qquad \forall a \in \mathcal{A} \tag{2.16}$$

$$y_k \geq 0 \qquad \forall k \in \mathcal{K}_i \quad \forall i \in \mathcal{I} \tag{2.17}$$

The objective function in (2.14) in known as *Beckmann's transformation*, and has no physical meaning. The constraints (2.15) and (2.16) are the flow conservation equations and (2.17) are non-negativity constraints on route flows. The mathematical program given by equations (2.14) to (2.17) is convex, and may be solved efficiently and without route enumeration by means of the Frank-Wolfe algorithm (LEBLANC; MORLOK; PIERSKALLA, 1975).

The definition of Wardrop equilibrium is often referred to as *deterministic user equilibrium*, since a basic assumption is that the user knows with certainty the costs associated to each route (See section 2.2). Daganzo and Sheffi (1977, p. 255) proposed the notion of *stochastic user equilibrium* as an extension of the Wardrop first principle to the case when users do not have full knowledge of the costs associated to routes, stated as: “In stochastic user equilibrium, no user *believes* he can improve his travel time by unilaterally changing routes.”

The stochastic user equilibrium concept has been proposed as more realistic than the deterministic user equilibrium, since it takes explicitly into account the different perception of users on costs, and it is a generalization of deterministic equilibrium. Nonetheless, stochastic route choice models require the calibration of parameters so that the model fits the observed behavior of users. Fisk (1980) proposed the following mathematical programming model, whose solution is a stochastic equilibrium point according to a logit route choice model:

$$\min \quad \sum_{a\in\mathcal{A}} \int_0^{z_a} \tau_a(v)dv + \xi \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}_i} y_k \ln y_k \tag{2.18}$$

s.t.

$$\sum_{k\in\mathcal{K}_i} y_k = x_i \qquad \forall i \in I \tag{2.19}$$

$$z_a = \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}_i} \delta_{ak} y_k \qquad \forall a \in \mathcal{A} \tag{2.20}$$

$$y_k \geq 0 \qquad \forall k \in \mathcal{K}_i \quad \forall i \in \mathcal{I} \tag{2.21}$$

The solution of the model (2.18)-(2.21) may be obtained by the iterated application of Dial's Algorithm and use of the *method of successive averages* (MSA) (CHEN; ALFA, 1991; POWELL; SHEFFI, 1982). It should be pointed out that, when $\xi = 0$ the model for stochastic user equilibrium reduces to the model for deterministic user equilibrium given by (2.14)-(2.17).

More recently, Watling (2002a), Watling (2002b) has proposed a new formulation for the stochastic user equilibrium, which he called a *general* stochastic user equilibrium with stochastic flows. According to the author, the definition of stochastic user equilibrium proposed by Daganzo and Sheffi (1977) assumes that OD flows are deterministic, and there is randomness only in users perception of route costs. Then, he proposes a more general definition of stochastic user equilibrium which account for stochastic OD flows. Let $\boldsymbol{p}_i$ be a vector of route choice probabilities for an OD pair i, whose OD flow is a random variable with a specified probability distribution (e.g., binomial, Poisson, beta-binomial). Given a realized OD flow x_i and $\boldsymbol{p}_i$, the route flows are random variables with multinomial probability distribution, i.e., $\boldsymbol{y}_i \sim \text{MN}(x_i, \boldsymbol{p}_i)$. The random route flows will induce random route costs, with a corresponding probability distribution. If we define $c_k(\boldsymbol{p}_i)$ as the expected cost of route k given route choice probabilities $\boldsymbol{p}_i$, then the probability of route k being chosen is $\phi_k(\boldsymbol{p}_i) = \text{Prob}\{c_k(\boldsymbol{p}_i) + \epsilon_k < c_l(\boldsymbol{p}_i) + \epsilon_l\} \quad \forall l \neq k$. If we denote $\boldsymbol{\phi}(\boldsymbol{p})$ as the vector of *output* route choice probabilities for the vector of *input* route choice probabilities $\boldsymbol{p}$, then the network is in general stochastic user equilibrium when the following fixed point condition is verified (NAKAYAMA; WATLING, 2014):

$$\boldsymbol{p} = \boldsymbol{\phi}(\boldsymbol{p}) \tag{2.22}$$

Watling (2002a) has also shown that his proposed notion of general stochastic equilibrium reduces to the traditional notion proposed by Daganzo and Sheffi (1977) if the performance functions on links are linear.

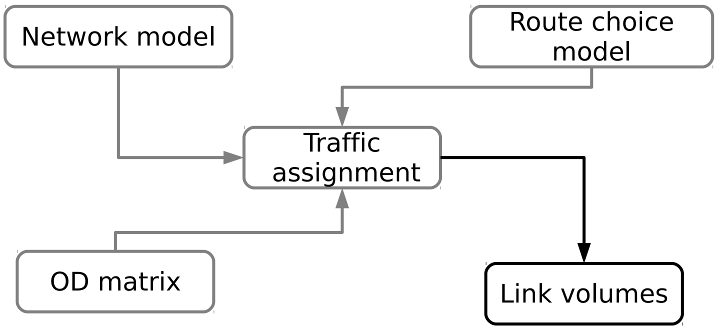
With all the background theory on traffic assignment presented, we are finally ready to discuss in Chapter 3 the models for the estimation of OD matrix based on traffic counts.

3 ORIGIN-DESTINATION MATRIX ESTIMATION

3.1 Problem description

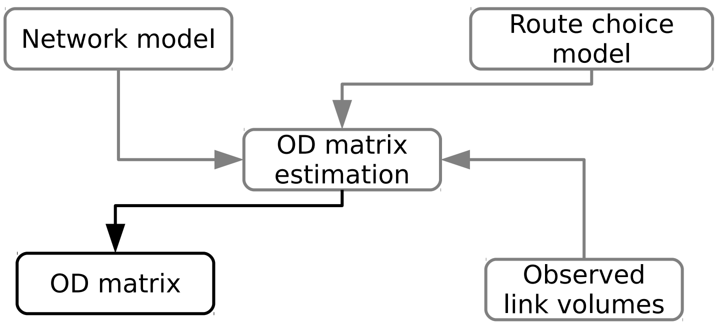
The OD matrix estimation problem may be defined as the inverse of the traffic assignment problem: given a set of observed link volumes observed in a reference time period, to determine the corresponding OD matrix which generated those volumes. Figures 1 and 2 show schematic representations of the traffic assignment problem (described in Chapter 2) and of the OD matrix estimation problem, respectively, where we can see the close relationship between the two problems.

Figure 1 – The traffic assignment problem



Source: Cascetta (2009)

Figure 2 – The OD matrix estimation problem



Source: Cascetta (2009)

The idea of estimating the OD matrix from observed link volumes came out from the observation that, as link volumes are functions of the OD matrix through flow conservation equations (2.1) and (2.2), observed link volumes could be in theory used to estimate the corresponding OD matrix. One straight approach would be to try to directly invert the mapping given by equation (2.5), if we knew the assignment matrix. The main hurdle is that this mapping is generally non-invertible, since there may be many OD matrices corresponding to the same observed link volumes. This is known in the literature as the *underspecification problem* (WILLUMSEN, 1981) (also known as non-identifiability problem). For example, if we treat the assignment matrix as independent of the OD matrix, equation (2.5) results in a linear system. In most practical settings, the number of OD pairs is much greater than the number of observed links, so that a solution of the linear system will not be unique. This problem is even more severe in practice as the number of independent observed links is commonly only a fraction of all links.

In order to overcome the underspecification problem, most models use a *prior* OD matrix, which could be an outdated matrix obtained by past household surveys, a recent sample matrix, or a modelled matrix produced by a trip generation and distribution model. Such a prior matrix provides additional information on the OD flows, thus mitigating the underspecification. They are in general used as a target or seed matrix. When it is used as a target matrix, the model outputs as estimate an OD matrix consistent with observed link volumes and least distant from the target matrix. In contrast, a seed OD matrix is intended to be a starting solution, so that the estimated OD matrix may not bear any resemblance to the seed matrix.

A natural concern when dealing with transportation networks is the level of congestion. In uncongested networks, the route choice matrix $\mathbf{P}$ may be determined exogenously, since it is assumed that the level of congestion does not influence the choices of the users. On the other hand, in congested networks the choices of the users are significantly influenced by the level of congestion, so that the route choice matrix is a function of the OD matrix. OD matrix estimation models for congested networks generally take a *bilevel* form, in which an embedded equilibrium traffic assignment model iteratively estimates the route choice matrix.

Currently in the literature, a distinction between *reconstruction* and *estimation* is made, as pointed out by Spiess (1987), Lo, Zhang and Lam (1996), Hazelton (2000), Timms (2001) and Carvalho (2014). We define as reconstruction of the OD matrix the attempt to recover the "exact" OD matrix which produced an observed vector of link volumes in a given time period. Reconstruction models do not take into account the variability of OD flows and link volumes. On the other hand, we refer to estimation of the OD matrix when we intend to estimate the mean OD flows or other parameters of a "population" of OD matrices. Thus, estimation models assume that OD flows and link volumes are stochastic and try to statistically estimate their mean values.

In Sections 3.2 and 3.3 we review reconstruction and estimation models, respectively.

3.2 Models for the reconstruction of the OD matrix

The first models proposed to the estimation of the OD matrix were optimization models with the aim of reconstructing the OD matrix given the observation of link volumes on selected links in a single time period. They try to overcome the underspecification problem by choosing an OD matrix which is consistent with the observed volumes and, at the same time, is optimal according to some objetive-function. The main models in this category are the maximum entropy model (ME) and the generalized least squares model (GLS), which we review in Sections 3.2.1 and 3.2.2, respectively. Another noteworthy one is the Bayesian model of Maher (1983) (Section 3.2.3). These models are based on the assumption that the network is uncongested and that the assignment matrix is known. In Section 3.2.4 we describe the bilevel approch, which allows these models to be applied to congested networks.

3.2.1 Maximum entropy

Van Zuylen and Willumsen (1980) proposed the use of the principle of entropy maximization, from Statistical Mechanics, so as to uniquely determine an OD matrix. This principle had already been applied in Transportation Planning by Wilson (1967). The idea is to make an analogy between trips in the network and particles in a gas. For a given trip pattern, it is possible to identify *macro*, *meso* and *microstates* of the network. For example, a macrostate is identified with the total number of trips in the network. For a given macrostate, there are many mesostates, which are identified with each different OD matrix. And for a given mesostate, there are many microstates, each identified with a particular labeling of individual trips. Assuming that all microstates are equally likely, the more microstates are associated with an OD matrix $\boldsymbol{x}$, the more likely it is. The objective of the ME model is then to obtain the most likely OD matrix subject to the constraint that observed volumes are reproduced exactly, as given below (WILLUMSEN, 1981):

$$\max \quad -\sum_{i\in\mathcal{I}}(x_i \ln x_i - x_i) \tag{3.1}$$

s.t.

$$\mathbf{F}\boldsymbol{x} = \hat{\boldsymbol{z}} \tag{3.2}$$

$$\boldsymbol{x} \geq \mathbf{0} \tag{3.3}$$

In which $\hat{\boldsymbol{z}}$ is the vector of observed link volumes. In the ME model, it is assumed that the route choice matrix $\mathbf{P}$ is exogenously determined, so that the assignment matrix $\mathbf{F} = \boldsymbol{\Delta}\mathbf{P}$ is constant (and $\boldsymbol{\Delta}$ includes only rows corresponding to observed links). As the objective function (3.1) is strictly concave, and constraints (3.2) and (3.3) are convex, the optimal solution is unique. By forming the Lagrangian function and applying first order conditions for a maximum, we obtain the following set of nonlinear equations:

$$x_i^{\text{ME}} = \prod_{a\in\mathcal{A}'} b_a^{f_{ai}} \quad \forall i \in \mathcal{I} \tag{3.4}$$

In which b_a are balancing factors, f_{ai} are the corresponding elements from the assignment matrix $\mathbf{F}$ and $\mathcal{A}'$ is the set of observed links. The equations (3.4) may be solved by an iterative algorithm described by Van Zuylen and Willumsen (1980), though it has no convergence guarantees. Alternatively, an ME model may be directly solved by using one of the standard nonlinear programming methods (See Nocedal and Wright (2006), Boyd and Vandenberghe (2004)). The model also accommodates the possibility of using a prior OD matrix $\hat{\boldsymbol{x}}$. In this case, it can be shown that the solution to the ME model is:

$$x_i^{\mathrm{ME}} = \hat{x}_i \prod_{a \in \mathcal{A}'} b_a^{f_{ai}} \quad \forall i \in \mathcal{I} \tag{3.5}$$

In equation (3.5), it can be seen that the reconstructed OD matrix will be an expansion of the prior OD matrix. In particular, $x_i^{\mathrm{ME}} = 0$ if the corresponding prior OD flow $\hat{x}_i = 0$. In case the prior OD matrix is a sample matrix obtained by means of a household survey, many prior OD flows will be zero due the relative small size of the sample, so that equation (3.5) cannot provide a better estimate of the OD flows. A further difficulty which may arise when applying an ME model is the presence of inconsistencies in the linear constraints given by equation (3.2). Due to observation errors, the observed link volumes may not satisfy all equations, and some correction procedure may have to be applied (VAN ZUYLEN; BRANSTON, 1982).

3.2.2 Generalized least squares

Cascetta (1984) proposed a model based on the generalized least squares (GLS) method. The GLS extends the simple least squares method used in linear regression, and allows for observations with heteroscedasticity and correlation. The GLS model is more general than the ME model, in the sense that it takes into account the presence of errors in both a prior OD matrix obtained by survey and the observed link volumes.

Let vector $\hat{\boldsymbol{x}}$ be a prior estimate of the OD matrix, obtained by direct sampling such as household surveys or obtained by a trip generation and distribution model. Bear in mind that we are trying to reconstruct the OD matrix $\boldsymbol{x}$, which we regard as an unknown constant. In the process of trying to estimate $\boldsymbol{x}$, there will be errors from sampling or from the inadequacy of the model. Thus, the prior estimate $\hat{\boldsymbol{x}}$ may be expressed as:

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\epsilon} \tag{3.6}$$

In equation (3.6), $\boldsymbol{\epsilon}$ is a random error vector with zero mean and covariance matrix $\boldsymbol{\Sigma}_\epsilon$. The GLS model also considers the fact that observed link volumes may not comply with flow conservation in nodes due to observation errors. Let $\boldsymbol{z} = \mathbf{F}\boldsymbol{x}$ be the actual link volumes, in which $\mathbf{F} = \boldsymbol{\Delta}\mathbf{P}$ is assumed known. The vector $\hat{\boldsymbol{z}}$ of observed volumes is then given by:

$$\hat{\boldsymbol{z}} = \mathbf{F}\boldsymbol{x} + \boldsymbol{\zeta} \tag{3.7}$$

Where $\boldsymbol{\zeta}$ is a random error term with zero mean and covariance matrix $\mathbf{\Sigma}_\zeta$. If we assume that the prior matrix $\hat{\boldsymbol{x}}$ and the observed link volumes $\hat{\boldsymbol{z}}$ are independent, then the GLS estimator $\boldsymbol{x}^{\mathrm{GLS}}$ is the solution of the following quadratic program:

$$\boldsymbol{x}^{\mathrm{GLS}} = \underset{\boldsymbol{x}\geq\mathbf{0}}{\operatorname{argmin}} \left\{ (\boldsymbol{x}-\hat{\boldsymbol{x}})^{\mathsf{T}}\mathbf{\Sigma}_\epsilon^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}}) + (\mathbf{F}\boldsymbol{x}-\hat{\boldsymbol{z}})^{\mathsf{T}}\mathbf{\Sigma}_\zeta^{-1}(\mathbf{F}\boldsymbol{x}-\hat{\boldsymbol{z}}) \right\} \tag{3.8}$$

If we assume that $x_i > 0 \quad \forall i \in \mathcal{I}$, a unique solution may be obtained explicitly by applying first orders conditions for an optimum, since the objective-function is strictly convex, resulting in the following expression:

$$\boldsymbol{x}^{\mathrm{GLS}} = (\mathbf{\Sigma}_\epsilon^{-1} + \mathbf{F}^{\mathsf{T}}\mathbf{\Sigma}_\zeta^{-1}\mathbf{F})^{-1}(\mathbf{\Sigma}_\epsilon^{-1}\hat{\boldsymbol{x}} + \mathbf{F}^{\mathsf{T}}\mathbf{\Sigma}_\zeta^{-1}\hat{\boldsymbol{z}}) \tag{3.9}$$

In case any of the non-negativity constraints is active, the solution may be obtained by gradient methods as the one proposed by Bell (1991). In equation (3.8), it can be seen that GLS is a kind of weighted least squares, in which the weights are given by the inverse of the covariance matrices of the errors. This is, the less the covariance of a term is, the more weight is put on it. Another interpretation is that the GLS model tries to find an OD matrix which is consistent with observed link volumes and the least distant from the target prior OD matrix $\hat{\boldsymbol{x}}$. The covariance matrices $\mathbf{\Sigma}_\epsilon$ and $\mathbf{\Sigma}_\zeta$ are obtained according to the properties of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{z}}$, respectively. If we do not have data to estimate them, we must use identity matrices and the GLS model becomes a simple least squares model.

An advantage of the GLS model when a prior OD matrix is available is that, unlike the ME model, it can produce estimated OD flows greater than zero even if the corresponding prior OD flows are zero in the prior OD matrix (see equation (3.5)). On the other hand, the GLS model cannot be applied without a prior OD matrix. In case such a prior OD matrix is not available, the term related to $\hat{\boldsymbol{x}}$ in equation (3.8) is dropped, and only the term related to $\hat{\boldsymbol{z}}$ remains. However, the solution to the resulting model is not unique, due to the underspecification problem. If we want a unique solution, we must *regularize* the problem in some way. This is precisely what the term related to the prior OD matrix $\hat{\boldsymbol{x}}$ does. Spiess (1990), Doblas and Benitez (2005) also develop GLS models which circumvent the underspecification problem by not allowing the estimated matrix to be much different from a prior matrix.

3.2.3 Maher's Bayesian model

In a Bayesian model for the reconstruction of the OD matrix $\boldsymbol{x}$, we must specify the prior distribution $p(\boldsymbol{x})$ of the real OD matrix and the likelihood function $\boldsymbol{p}(\hat{\boldsymbol{z}}|\boldsymbol{x})$ of the observed link volumes (See the Appendix B for a basic review on Bayesian inference). The prior distribution may be specified from subjective knowledge from a practitioner or expert in the field of transportation systems. If a prior OD matrix is available, it may be included in the model as mean values of the prior distribution, for example.

Maher (1983) proposed one of the first Bayesian models to reconstruct the OD matrix. The author assumes a multivariate normal distribution as prior (See A), with *hyperparameters* mean $\boldsymbol{\mu}_0$ and covariance matrix $\boldsymbol{\Sigma}_0$, and as likelihood also a MVN distribution with mean $\boldsymbol{\mu}_z = \mathbf{F}\boldsymbol{x}$ and covariance matrix $\boldsymbol{\Sigma}_z$. The author also assumes that the network is uncongested and that the assignment matrix is known. It can be shown that the posterior is also MVN, i.e., $p(\boldsymbol{x}|\hat{\boldsymbol{z}}) = \mathrm{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$, where the posterior parameters mean $\boldsymbol{\mu}_1$ and covariance matrix $\boldsymbol{\Sigma}_1$ are given by the following expressions:

$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\Sigma}_0\mathbf{F}^\intercal(\boldsymbol{\Sigma}_z + \mathbf{F}\boldsymbol{\Sigma}_0\mathbf{F}^\intercal)^{-1}(\hat{\boldsymbol{z}} - \mathbf{F}\boldsymbol{\mu}_0) \tag{3.10}$$

$$\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_0\mathbf{F}^\intercal(\boldsymbol{\Sigma}_z + \mathbf{F}\boldsymbol{\Sigma}_0\mathbf{F}^\intercal)^{-1}\mathbf{F}\boldsymbol{\Sigma}_0 \tag{3.11}$$

Then, we can take the posterior mean $\boldsymbol{\mu}_1$ as an estimator for the OD matrix. It should be emphasized that equations (3.10) and (3.11) do not assure non-negativity, which may be a problem in OD pairs with low OD flows. Alternatively one may obtain the posterior maximum by maximizing the posterior density $p(\boldsymbol{x}|\hat{\boldsymbol{z}})$ subject to non-negativity constraints.

3.2.4 Equilibrium-based models and the bilevel approach

Both ME and GLS models assume that the assignment matrix $\mathbf{F}$ was determined exogenously. This is suitable for uncongested networks, in which the assignment matrix is independent of the OD matrix. Nonetheless, in congested networks we cannot isolate the estimation of the OD matrix from the estimation of the assignment matrix, since the level of congestion influences the choices of the users.

One of the first models for the estimation of OD matrices in congested networks was proposed by Nguyen (1977), from an adaptation of Beckmann's model for deterministic equilibrium assignment (Section 2.4), given below (YANG et al., 1992):

$$\min \quad \sum_a \int_0^{z_a} \tau_a(v)dv - \sum_{i \in I} c_i^* x_i \tag{3.12}$$

s.t.

$$\sum_{k \in \mathcal{K}_i} y_k = x_i \qquad \forall i \in \mathcal{I} \tag{3.13}$$

$$z_a = \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}_i} \delta_{ak} y_k \tag{3.14}$$

$$y_k \geq 0 \qquad \forall k \in \mathcal{K}_i \quad \forall i \in \mathcal{I} \tag{3.15}$$

In which the term c_i^* corresponds to the minimum route cost in OD pair i. Nguyen's model assumes that the network is in deterministic user equilibrium and that all links are observed. In practice, one does not know the minimum route cost for an OD pair, so that Nguyen's model is has not been considered in practice. Another limitation of Nguyen's model is that the objective function (3.12) is not strictly convex in the OD flows vector $\boldsymbol{x}$, so that it suffers from the underspecification problem. In order to ensure a unique solution, LeBlanc and Farhangian (1982) proposed an extension to Nguyen's model which produces an OD matrix of minimum Euclidian distance to a target prior OD matrix, while Turnquist and Gur (1979) proposed the use of a target prior OD matrix as an initial solution in the algorithm. This strategy is similar to the one adopted in the GLS model.

An alternative to Nguyen's model are the bilevel models, which bear their origins in the Stackelberg game framework from Game Theory (FISK, 1984; FISK, 1988; FISK, 1989). In bilevel models, there is an *upper level*, in which the OD matrix is estimated using a route choice matrix estimated by an equilibrium traffic assignment in the *lower level* (SHEFFI, 1985; BELL; IIDA, 1997). A general bilevel model, proposed by Yang et al. (1992), is given below:

$$\min \quad \rho g_1(\boldsymbol{x}, \hat{\boldsymbol{x}}) + g_2(\boldsymbol{z}, \hat{\boldsymbol{z}}) \quad \text{(upper level)} \tag{3.16}$$

$$\text{s.t.}$$

$$\boldsymbol{z} = \boldsymbol{z}^{\text{eq}}(\boldsymbol{x}) \quad \text{(lower level)} \tag{3.17}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{3.18}$$

At the upper level, one must obtain an OD matrix $\boldsymbol{x}$ which minimizes the objective function (3.16). The model is multiobjective, since the objective-function is a linear combination of two functions, g_1 and g_2 (BRENNINGER-GÖTHE; JÖRNSTEN, 1989). g_1 is a function of the OD matrix $\boldsymbol{x}$ and a prior OD matrix $\hat{\boldsymbol{x}}$, while g_2 is a function of the link volumes $\boldsymbol{z}$ resulting from the traffic assignment of the OD matrix $\boldsymbol{x}$ and observed link volumes $\hat{\boldsymbol{z}}$. $\rho > 0$ is a weighting factor which balances the importance of functions g_1 and g_2. Notice that the ME and GLS models are special cases of the general bilevel model (CASCETTA; NGUYEN, 1988).

At the lower level, constraint (3.17) means that the vector of link volumes $\boldsymbol{z} = \boldsymbol{z}^{\text{eq}}(\boldsymbol{x})$ corresponds to a network in equilibrium for a given OD matrix $\boldsymbol{x}$. The equilibrium volumes are obtained by assigning the OD matrix reconstructed at the upper level by means of a deterministic or stochastic user equilibrium assignment (MAHER; ZHANG; VAN VLIET, 2001). The whole procedure is iterative, since the process of reconstructing the OD matrix and assigning it to the network must be repeated until convergence.

Bilevel models are difficult to solve, since they are not convex and may have many local optima (FLORIAN; CHEN, 1995). Moreover, evaluating constraint (3.17) in the lower level is a computationally intensive task, since traffic assignment is in general obtained by the solution of an optimization model. Yang (1995) proposed the following heuristic algorithm to solve the general bilevel model:

Initial Step: Make $t := 0$ and get an initial assignment matrix $\mathbf{F}^{(t)} = \mathbf{\Delta P}^{(t)}$.

Step 1 (upper level): Get $\boldsymbol{x}^{(t+1)}$ by solving the upper level using a linear approximation $\boldsymbol{z}(\boldsymbol{x}) = \mathbf{F}^{(t)}\boldsymbol{x}$ in the objective-function;

Step 2 (lower level): Determine a new assignment matrix $\mathbf{F}^{(t+1)} = \mathbf{\Delta P}(\boldsymbol{x}^{(t+1)})$ by means of an equilibrium assignment of $\boldsymbol{x}^{(t+1)}$;

Step 3: If a stopping criterion is met, stop. Otherwise, make $t := t + 1$ and return to step 1.

If convergence is reached, the estimated OD matrix, the link volumes and the assignment matrix should all correspond to an equilibrium state of the network. Other algorithms proposed in the literature include the ones by Codina and Barceló (2004) and Codina, García and Marín (2006), which are based on conjugate directions; and the one proposed by Lundgren and Peterson (2008), which is based on the projected gradient. They may be computationally more efficient than Yang's heuristics since they use gradient information.

A different direction is taken by Cascetta and Postorino (2001), who proposed a formulation of the bilevel model as a compound fixed point problem. That is, as in the case of bilevel programming models, where there is an optimization problem within another optimization problem, in the compound fixed point problem there is a fixed point problem within another fixed point problem. The upper level is given by the following equation:

$$\boldsymbol{x}^* \quad = \quad \text{argmin}_{\boldsymbol{x} \geq \mathbf{0}} \quad \rho g_1(\boldsymbol{x}, \hat{\boldsymbol{x}}) + g_2(\mathbf{F}(\boldsymbol{z}^*(\boldsymbol{x}^*))\boldsymbol{x}, \hat{\boldsymbol{z}}) \tag{3.19}$$

Where the vector of modeled link flows $\boldsymbol{z}^*(\boldsymbol{x})$ is the result of a stochastic traffic assignment of OD matrix $\boldsymbol{x}$ formulated as a fixed point problem in the lower level:

$$\boldsymbol{z}^*(\boldsymbol{x}) = \mathbf{F}(\boldsymbol{z}^*(\boldsymbol{x}))\boldsymbol{x} \tag{3.20}$$

Finally, the iterative algorithm to solve the compound fixed point problem given by equations (3.19) and (3.20) is similar to Yang's heuristic to solve the general bilevel model:

Initial step: Take $t = 0$. Get an initial solution $\boldsymbol{x}^{(0)}$;

Step 1: Obtain the assignment matrix $\mathbf{F}(\boldsymbol{x}^{(t)})$ by means of a stochastic equilibrium assignment (solution of the fixed point problem at the lower level).

Step 2: Get a linear approximation of the assignment map by doing:

$$\boldsymbol{z}(\boldsymbol{x}) = \mathbf{F}(\boldsymbol{x}^{(t)})\boldsymbol{x} \tag{3.21}$$

Step 3: Get $\boldsymbol{x}^{(t+1)}$ by means of the solution of the optimization model below:

$$\min \quad \rho g_1(\boldsymbol{x}, \hat{\boldsymbol{x}}) + g_2(\boldsymbol{x}, \hat{\boldsymbol{z}}) \tag{3.22}$$

$$\text{s.t.}$$

$$\boldsymbol{z} = \mathbf{F}(\boldsymbol{x}^{(t)})\boldsymbol{x} \tag{3.23}$$

$$\boldsymbol{x} \geq \mathbf{0} \tag{3.24}$$

Step 4: If the convergence criterion is met, stop. Otherwise, make $t := t + 1$ and return to step 1.

In the next Section 3.3 we review the main models for the estimation of the mean OD matrix.

3.3 Models for the estimation of the mean static OD matrix

Unlike reconstruction models, reviewed in Section 3.2, estimation models assume that OD flows are stochastic and follow some specified probability distribution. The aim of these models is to estimate parameters of "population" of OD matrices, such as mean OD matrices, variances and covariances and other quantities of interest.

In the following sections, we classify the models according to the estimation method: maximum likelihood, moment-based or Bayesian.

3.3.1 Maximum likelihood

Let the OD flow x_i in OD pair $i \in \mathcal{I}$ be a random variable. Many authors (SPIESS, 1987; VARDI, 1996; TEBALDI; WEST, 1998; HAZELTON, 2000) assume that OD flows follow a Poisson distribution with parameter $\theta_i = \mathrm{E}[x_i]$. We denote by $\boldsymbol{\theta} \in \mathbb{R}^n$ the vector of mean OD flows for all OD pairs. If we assume that the OD flows are independent, it can be shown (HAZELTON, 2000) that route flows y_{ik}, $\forall k \in \mathcal{K}_i, i \in \mathcal{I}$ are also independent random variables which follow Poisson distributions with expected value $\mathrm{E}[y_{ik}] = p_{ik}\theta_i$, in which p_{ik} is the probability that a trip occurs in route $k \in \mathcal{K}_i$. Moreover, as traffic volumes on links are sums of route flows according to the flow conservation equation (2.2), they also marginally follow Poisson distributions. Nevertheless, since some links share the same route flows, link volumes are not independent random variables, so that their covariances will be different from zero. Their joint distribution should be some complicated form of multivariate Poisson, which may be well approximated by a multivariate normal distribution if link volumes are far from zero (HAZELTON, 2000; VARDI, 1996). From equation (2.4), we see that the mean vector and covariance matrix of the link volumes vector $\boldsymbol{z} \in \mathbb{R}^m$ are given respectively by $\boldsymbol{\mu} = \mathbf{\Delta P}\boldsymbol{\theta}$ and $\mathbf{\Sigma} = \mathbf{\Delta \Theta \Delta}^\mathsf{T}$, in which $\mathbf{\Theta} = \operatorname{diag}(\mathbf{P}\boldsymbol{\theta})$. Making $\mathbf{F} = \mathbf{\Delta P}$, the joint probability density function of the link volumes is given by:

$$f(\boldsymbol{z}|\boldsymbol{\theta}) = (2\pi)^{-m/2}|\mathbf{\Sigma}|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{z} - \mathbf{F}\boldsymbol{\theta})^\mathsf{T}\mathbf{\Sigma}^{-1}(\boldsymbol{z} - \mathbf{F}\boldsymbol{\theta})\right\} \tag{3.25}$$

Where $|.|$ denotes the determinant of a matrix. It should be noted in the density function defined by (3.25) that the matrix $\mathbf{\Delta}$ should be formed only by independent rows, otherwise the covariance matrix $\mathbf{\Sigma}$ will be singular. This means that only the rows corresponding to non-redundant observed links should be included in $\mathbf{\Delta}$. Another important issue related to (3.25) is how to treat the route choice matrix $\mathbf{P}$. If the network is uncongested, $\mathbf{P}$ can be estimated independently from the estimation of the mean OD matrix, by means of a route choice model. Another approach would be to treat the route choice matrix as a parameter of the model, so that in theory it could be jointly estimated with the mean OD matrix, irrespective of the network being congested or not.

Given a sample of size N of traffic volumes vectors $\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \ldots, \boldsymbol{z}^{(N)}$, each observed in different days during the same reference time period and assumed independent, we define the *likelihood function* as the following:

$$\mathcal{L}(\boldsymbol{\theta}) = \prod_{j=1}^{N} f(\boldsymbol{z}^{(j)}|\boldsymbol{\theta}) \tag{3.26}$$

We can define as a maximum likelihood estimate of the mean OD matrix the maximizer of equation (3.26). It is though computationally more convenient to maximize the *log-likelihood* function, given by $\ell(\boldsymbol{\theta}) = \log \mathcal{L}(\boldsymbol{\theta})$, which in our present case takes the following form:

$$\ell(\boldsymbol{\theta}) = -\frac{N}{2}\log|\mathbf{\Sigma}| - \frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{z}^{(j)} - \mathbf{F}\boldsymbol{\theta})^\mathsf{T}\mathbf{\Sigma}^{-1}(\boldsymbol{z}^{(j)} - \mathbf{F}\boldsymbol{\theta}) + c \tag{3.27}$$

In which $c = -(Nm/2)\log(2\pi)$ is constant with respect to $\boldsymbol{\theta}$. Let $\boldsymbol{\theta}^* = \operatorname{argmax}_{\boldsymbol{\theta}\geq\mathbf{0}}\, \ell(\boldsymbol{\theta})$ be a maximizer of the log-likelihood function. As the log-likelihood may not be strictly concave, due to the existence of multiple $\boldsymbol{\theta}$ with the same likelihood value (the underspecification problem), it may have multiple maximizers. Some ways of circumventing this problem is to resort to a prior OD matrix as a target matrix in a way similar to the GLS model described in section 3.2.2, or advance to a Bayesian approach as described in section 3.3.3.

In case the sample of observed volume vectors is large, Hazelton (2000) proposes to substitute the population covariance matrix by the sample covariance matrix $\mathbf{S}$ in the log-likelihood, resulting in the following approximation to the log-likelihood that is computationally more tractable:

$$\tilde{\ell}(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{z}^{(j)} - \mathbf{F}\boldsymbol{\theta})^{\mathsf{T}}\mathbf{S}^{-1}(\boldsymbol{z}^{(j)} - \mathbf{F}\boldsymbol{\theta}) \tag{3.28}$$

Maximizing equation (3.28) can be interpreted as finding an estimate $\hat{\boldsymbol{\theta}}$ for which the sum of the weighted quadratic distance ($\mathbf{S}^{-1}$ is the matrix of weights) between the expected volume vector $\boldsymbol{\mu} = \mathbf{F}\boldsymbol{\theta}$ and the observed volume vectors $\boldsymbol{z}^{(j)}, j = 1, 2, \dots N$ is minimal. However, once more $\tilde{\ell}(\boldsymbol{\theta})$ is not stricly concave if $\boldsymbol{\Delta}$ has more columns than rows, which occurs if there are more OD pairs than observed links and we may have to resort to a prior OD matrix as a target matrix in a way similar to the GLS model described in section 3.2.2.

3.3.2 Moment-based models

The method of moments is a classical technique in point estimation. Its basic idea is to solve the equations obtained by equating population and sample moments. Let $\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \dots, \boldsymbol{z}^{(N)}$ be a sample of traffic volumes vectors, each observed in different days during the same reference time period and assumed independent. Their sample mean $\bar{\boldsymbol{z}}$ and sample covariance matrix $\mathbf{S}$ are given respectively by:

$$\bar{\boldsymbol{z}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{z}^{(j)} \tag{3.29}$$

$$\mathbf{S} = \frac{1}{N-1}\sum_{j=1}^{N}(\boldsymbol{z}^{(j)} - \bar{\boldsymbol{z}})(\boldsymbol{z}^{(j)} - \bar{\boldsymbol{z}})^{\mathsf{T}} \tag{3.30}$$

In theory, an estimate of the mean OD flows $\boldsymbol{\theta}$ may be obtained by solving the linear system of equations resulting from equating population and sample moments for $\boldsymbol{\theta} \geq \mathbf{0}$. A problem is that this linear system is often inconsistent, as pointed out by Vardi (1996). The inconsistencies arise from the sampling errors. To overcome this problem, Hazelton (2003) proposed to set an optimization model whose objective-function is the minimization of the distances between theoretical and sample moments:

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \geq \mathbf{0}}{\operatorname{argmin}}\{\|\mathbf{F}\boldsymbol{\theta} - \bar{\boldsymbol{z}}\| + \rho\|\operatorname{vec}(\boldsymbol{\Sigma}) - \operatorname{vec}(\mathbf{S})\|\} \tag{3.31}$$

In which vec(.) is the vector concatenation of the columns of a matrix, $\|.\|$ denotes a suitable distance measure (a popular one is the Euclidian distance), $\rho \geq 0$ is a weighting factor, $\mathbf{F} = \mathbf{\Delta P}$, $\mathbf{\Sigma} = \mathbf{\Delta \Theta \Delta}^{\mathsf{T}}$ and $\mathbf{\Theta} = \text{diag}(\mathbf{P}\boldsymbol{\theta})$. It is worth noting that the method of moments is not dependent on the Poisson assumption, and may be applied to more general models in which only a relationship between the mean vector and the covariances matrix is assumed. See Hazelton (2003) for more details.

3.3.3 Bayesian inference

The models based on Bayesian inference define, in addition to the likelihood of the data, prior and posterior probability distributions for the OD matrix. We have already presented a Bayesian model for reconstruction of the OD matrix, Maher's model, in Section 3.2.3. See Appendix B for a short review of the main concepts of Bayesian inference.

The main model in this class was proposed by Tebaldi and West (1998). Following Vardi (1996), they assume that the OD flows follow independent Poisson distributions, whose mean value θ_i for OD pair i follows a gamma prior probability distribution. They also assume that only one route is available for each OD pair, so that the route choice matrix $\mathbf{P}$ equals the identity matrix. Let $\boldsymbol{z}$ be a vector of observed link volumes, with the following likelihood function:

$$\text{p}(\boldsymbol{z}|\boldsymbol{\theta}) = \sum_{\boldsymbol{x} \in \mathcal{X}} \text{p}(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{\theta}) \tag{3.32}$$

In which $\mathcal{X} = \{\boldsymbol{x} : \mathbf{\Delta}\boldsymbol{x} = \boldsymbol{z}\}$ is the set of OD flow vectors consistent with the observed link volume vector $\boldsymbol{z}$. The problem with the likelihood function in (3.32) is that the evaluation of this sum requires the enumeration of all vectors $\boldsymbol{x} \in \mathcal{X}$, which is computationally infeasible for even moderately-sized networks. Their solution strategy is to evaluate not the marginal posterior $\text{p}(\boldsymbol{\theta}|\boldsymbol{z})$, but the joint posterior distribution $\text{p}(\boldsymbol{x}, \boldsymbol{\theta}|\boldsymbol{z})$, by noting that:

$$\text{p}(\boldsymbol{x}, \boldsymbol{\theta}|\boldsymbol{z}) \propto \text{p}(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{\theta})\text{p}(\boldsymbol{x}, \boldsymbol{\theta}) \tag{3.33}$$

If we further assume that $\boldsymbol{z}$ is conditionally independent of $\boldsymbol{\theta}$ given $\boldsymbol{x}$, and by noting that $\text{p}(\boldsymbol{x}, \boldsymbol{\theta}) = \text{p}(\boldsymbol{x}|\boldsymbol{\theta})\text{p}(\boldsymbol{\theta})$, we have:

$$\text{p}(\boldsymbol{x}, \boldsymbol{\theta}|\boldsymbol{z}) \propto \text{p}(\boldsymbol{z}|\boldsymbol{x})\text{p}(\boldsymbol{x}|\boldsymbol{\theta})\text{p}(\boldsymbol{\theta}) \tag{3.34}$$

In addition, they assume there are no observation errors in link volumes, so that $\text{p}(\boldsymbol{z}|\boldsymbol{x}) = \text{I}(\mathbf{\Delta}\boldsymbol{x} = \boldsymbol{z})$ and I denotes the *indicator function*, $\text{I}(A) = 1$ if A is true and 0 otherwise. In order to sample from the joint posterior given by 3.34, Tebaldi and West proposed a Gibbs sampler (GEMAN; GEMAN, 1984), which iteratively samples from the conditional distributions $\text{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z})$ and $\text{p}(\theta_i|\boldsymbol{\theta}_{[-i]}, \boldsymbol{x}, \boldsymbol{z})$, in which $\boldsymbol{x}_{[-i]}$ and $\boldsymbol{\theta}_{[-i]}$

denote the corresponding vectors with component i excluded. This sequence of conditional samples is a Markov chain, which converges (mixes) in the long run to the desired joint posterior distribution $\mathrm{p}(\boldsymbol{x}, \boldsymbol{\theta}|\boldsymbol{z})$. We can draw samples from the corresponding marginal posteriors $\mathrm{p}(\boldsymbol{x}|\boldsymbol{z})$ and $\mathrm{p}(\boldsymbol{\theta}|\boldsymbol{z})$ by taking samples from $\boldsymbol{x}$ and $\boldsymbol{\theta}$ in isolation.

Given gamma priors on θ_i, sampling from $\mathrm{p}(\theta_i|\boldsymbol{\theta}_{[-i]}, \boldsymbol{x}, \boldsymbol{z})$ is easy by virtue of conjugacy, so that $\mathrm{p}(\theta_i|\boldsymbol{\theta}_{[-i]}, \boldsymbol{x}, \boldsymbol{z}) = \mathrm{Ga}(\alpha_i + x_i, \beta_i + 1)$, in which α_i and β_i are the parameters of the gamma prior distribution of θ_i. With regards to the posterior $\mathrm{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z})$, there is no conjugate distribution, so that we have to resort to a scheme called *Metropolis-within-Gibbs*. This scheme is used to sample from the posterior $\mathrm{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z})$ by means of the *Metropolis-Hastings* algorithm (HASTINGS, 1970).

In order to sample from $\mathrm{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z})$, we should also notice that the assumption of error-free observed volumes implies that $\mathrm{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z})$ has positive support only for values of the vector $\boldsymbol{x}$ which satisfy the constraints $\boldsymbol{\Delta x} = \boldsymbol{z}$ and non-negativity $\boldsymbol{x} \geq \boldsymbol{0}$. They then proposed to partition the link-route incidence matrix $\boldsymbol{\Delta} = [\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2]$, thereby $\boldsymbol{\Delta}_1$ is a nonsingular $m \times m$ matrix and $\boldsymbol{\Delta}_2$ is an $m \times (n - m)$ matrix, with a corresponding partition $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2]^\mathsf{T}$ and m and n are respectively the number of observed independent links and the number of OD pairs. Thus, we can write the dependent subvector $\boldsymbol{x}_1$ as a linear combination of the independent subvector $\boldsymbol{x}_2$:

$$\boldsymbol{x}_1 = \boldsymbol{\Delta}_1^{-1}(\boldsymbol{z} - \boldsymbol{\Delta}_2\boldsymbol{x}_2) \tag{3.35}$$

Equation (3.35) implies that we do not have to sample conditionally all components from $\boldsymbol{x}$. Once $\mathrm{p}(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{z}) = \mathrm{p}(\boldsymbol{x}_1|\boldsymbol{x}_2, \boldsymbol{\theta}, \boldsymbol{z})\mathrm{p}(\boldsymbol{x}_2|\boldsymbol{\theta}, \boldsymbol{z})$ and $\mathrm{p}(\boldsymbol{x}_1|\boldsymbol{x}_2, \boldsymbol{\theta}, \boldsymbol{z}) = \mathrm{I}(\boldsymbol{x}_1 = \boldsymbol{\Delta}_1^{-1}(\boldsymbol{z} - \boldsymbol{\Delta}_2\boldsymbol{x}_2))$, we need to sample only the components from $\boldsymbol{x}_2$. Finally, the conditional distribution $\mathrm{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z})$ is given by:

$$\mathrm{p}(x_i|\boldsymbol{x}_{[-i]}, \boldsymbol{\theta}, \boldsymbol{z}) \propto \frac{\theta_i}{x_i!} \prod_{k \in I_1} \frac{\theta_k}{x_k!} \qquad i \in I_2 \tag{3.36}$$

In equation (3.36), I_1 and I_2 denote, respectively, the sets of indexes of the subvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. A major caveat in sampling from the conditional distribution in (3.36) is identifying its support. For x_i conditional on the other values $\boldsymbol{x}_{[-i]}$ and $i \in I_2$, Tebaldi and West proposed to start by setting the lower bound $x_i^{\mathrm{LB}} = 0$ and the upper bound $x_i^{\mathrm{UB}} \leq \min_{a \in \mathcal{A}(i)}\{z_a - \sum_{j \in I_2, j \neq i} \delta_{aj}x_j\}$, where $\mathcal{A}(i)$ denotes the set of links affected by OD pair i, and then testing for non-negativity of $\boldsymbol{x}_1$. In case $\boldsymbol{x}_1 \geq 0$ is not true, either the lower bound $x_i^{\mathrm{LB}} = 0$ is incremented or the upper bound x_i^{UB} is decremented and non-negativity of $\boldsymbol{x}_1$ tested. The procedure is repeated until the condition $\boldsymbol{x}_1 \geq 0$ is satisfied, and the lower and upper bounds are fixed. For real-scale networks, this trial-and-error procedure can be computationally prohibitive, as pointed out by Parry and Hazelton (2013).

After identifying the support of the posterior (3.36), we should use a proposal distribution $q(x_i)$ to sample candidate values for x_i. The authors suggest using uniform or Poisson proposals. Given a candidate value x_i^*, the candidate is accepted with probability given by the following ratio r:

$$r = \min\left(1, \frac{p(x_i^*)q(x_i)}{p(x_i)q(x_i^*)}\right) \tag{3.37}$$

Finally, the description of the Gibbs sampler is the following:

1. (Initialization) Set starting values for the OD flow vector $\boldsymbol{x}^{(0)}$. This can be accomplished by finding a non-negative solution to the underdetermined system of equations $\boldsymbol{\Delta x} = \boldsymbol{z}$
2. (Step 1) Draw independent samples for the mean values θ_i from the posterior $p(\theta_i|\boldsymbol{\theta}_{[-i]}, \boldsymbol{x}, \boldsymbol{z})$, for $i \in \mathcal{I}$;
3. (Step 2) For each i, sample a candidate x_i^* from a proposal distribution q(.) and accept it with probability given by $\min\left(1, \frac{p(x_i^*)q(x_i)}{p(x_i)q(x_i^*)}\right)$.
4. Repeat steps 1 and 2 until convergence.

In the following Section 3.4 we review dynamic models for estimation of the OD matrix.

3.4 Models for the estimation of the dynamic OD matrix

All approaches discussed up to this point in this review modeled the transportation system as *static* and in equilibrium. By static we mean that the parameters of the system do not vary over time. However, in reality the system is *dynamic*, since social-economic factors and infrastructure do change over time, affecting the system parameters. In order to predict future behavior of the system or assess the impact of intervention in the system over time, we should be able to model the dynamic variation of the demand and other relevant parameters.

The types of dynamic models for the OD matrix may be classified in two broad classes, according to the time scale of the model: *within-day* and *day-to-day* models. Within-day models consider the time variation of the demand for a specified time period within a single day. The extension of the time period under study may be as short as a few minutes or the whole day. In contrast, day-to-day models are often concerned with the variation of the demand for a repeated reference time period (typically the peak hour) over a sequence of days.

In this work we are concerned with day-to-day dynamic models, since our main interest is in mid to long-term planning horizon, while the within-day models are more useful for short-term operational decisions. We refer to the works of Willumsen (1984), Cremer and Keller (1987), Cascetta (1993), Ashok and Ben-Akiva (2002) for further reading on within-day dynamic models. The development of models for the estimation of the OD matrix in day-to-day dynamic settings is very recent, with the main paper published by Hazelton (2008). According to his words:

> Previous work on inference for OD matrices from link count data can be split broadly into two types. First there is static matrix estimation, where it is most often assumed that a single set of link counts is available, typically augmented by highly relevant prior information such as an outdated OD matrix. [...] Second, there is dynamic matrix estimation, based on a sequence of consecutive traffic counts taken at say 5–15 min intervals. [...] We look at OD matrix estimation based on a sequence of daily link counts. This is different to the within-day dynamic matrix estimation problem mentioned above since we assume that all trips are completed within a single observational period. [...] Neither is the problem of estimating daily OD matrices simply equivalent to a sequence of static estimation problems. While in principle we could apply existing static matrix estimators, we would lose a great deal of information in terms of likely similarities between OD matrices from one day to the next. (HAZELTON, 2008, p.542)

In the day-to-day dynamic OD matrix estimation problem, we want to estimate a sequence of unobserved mean OD matrices $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_T$ given a sample of link volume vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T$ observed in a sequence of $t = 1, 2, \ldots, T$ consecutive days on some links of the network. In the referred paper, Hazelton assumes that the OD flows are independent and follow Poisson distributions. The independence assumption implies that the covariance between OD pairs is zero, while the Poisson assumption implies that the mean OD flows and their variances are equal, i.e., given the OD flow x_{ti} in OD pair i at time t, $\mathrm{E}[x_{ti}] = \mathrm{Var}(x_{ti}) = \theta_{ti}$. Furthermore, both assumptions have the implication that the OD route flows y_{tk} at time t will also be Poisson with mean $\lambda_{tk} = p_{tk}\theta_{ti}$ for route $k \in \mathcal{K}_i$, and p_{tk} is the probability of choosing route k. Thus, inference can be made directly on the mean route flows λ_{tk} and the mean OD flows in each OD pair i may be estimated by summing up the estimated mean route flows in each route in OD pair i.

A problem we face when assuming Poisson OD flows is that the joint probability distribution of the link volumes will have a complicated form, which is not tractable, as already discussed in Section 3.3.1. By using a normal approximation, the conditional distribution of the link volumes $\boldsymbol{z}_t$ given the mean OD route flows $\boldsymbol{\lambda_t}$ is given by $\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\lambda_t}) = \mathrm{N}(\boldsymbol{\Delta\lambda_t}, \phi\boldsymbol{\Delta\Lambda}_t\boldsymbol{\Delta}^{\mathsf{T}})$, in which $\boldsymbol{\Lambda}_t = \mathrm{diag}(\boldsymbol{\lambda_t})$ and $\phi > 0$ is a scale factor which adjusts for link volumes which are not compliant with the Poisson assumption.

Hazelton further considers parsimonious parametrizations of the mean OD route flows. The idea is to represent the mean OD route flows $\boldsymbol{\lambda}_t$ as a function of a vector of parameters $\boldsymbol{\beta}$ which does not change with time. A possible parsimonious model is to represent the vector $\boldsymbol{\lambda}_t$ as a linear model $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + t\boldsymbol{\delta}$, with a vector of parameters $\boldsymbol{\beta} = (\boldsymbol{\lambda}_0, \boldsymbol{\delta}, \phi)$, so that we make inferences on the initial vector $\boldsymbol{\lambda}_0$, the time increment vector $\boldsymbol{\delta}$ and the link volume scale factor ϕ. Another possibility is a weekday-weekend model, in which $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0$ in weekdays and $\boldsymbol{\lambda}_t = \gamma\boldsymbol{\lambda}_0$ for weekends, and γ is a demand adjustment factor, so that $\boldsymbol{\beta} = (\boldsymbol{\lambda}_0, \gamma, \phi)$ is the vector of parameters on which we make inference. Assuming that link volumes are independent over time, the likelihood function of the observed link volumes will be given by the following expression:

$$\mathrm{L}(\boldsymbol{\beta}) = \prod_{t=1}^{T} (2\pi)^{-T/2} |\boldsymbol{\Sigma}_t(\boldsymbol{\beta})|^{-1/2} \exp\left\{ -\frac{1}{2} (\boldsymbol{z}_t - \boldsymbol{\Delta}\boldsymbol{\lambda}_t(\boldsymbol{\beta}))^{\mathsf{T}} \boldsymbol{\Sigma}_t(\boldsymbol{\beta})^{-1} (\boldsymbol{z}_t - \boldsymbol{\Delta}\boldsymbol{\lambda}_t(\boldsymbol{\beta})) \right\} \tag{3.38}$$

Where $\boldsymbol{\Sigma}_t(\boldsymbol{\beta}) = \phi\boldsymbol{\Delta}\boldsymbol{\Lambda}_t\boldsymbol{\Delta}^{\mathsf{T}}$, which is a function of $\boldsymbol{\beta}$ since $\boldsymbol{\Lambda}_t = \mathrm{diag}(\boldsymbol{\lambda}_t(\boldsymbol{\beta}))$. Inference on $\boldsymbol{\beta}$ may be performed by maximizing equation (3.38), so that we obtain a maximum likelihood estimator. Alternatively, we obtain a Bayesian estimator by specifying a prior distribution on $\boldsymbol{\beta}$ and using the maximum a posteriori or mean of the posterior distribution.

A limitation of this latter model is that it assumes independence of link volumes, which may not correspond to reality. In fact, it is expected that link volumes exhibit correlation over time due the nature of the decision process of users, who dynamically adapt to congestion conditions. In a recent paper, Parry and Hazelton (2013) make an attempt at modeling this time dependence of the OD route flows according to an n-step Markovian process with transition kernel given by $\mathrm{p}(\boldsymbol{y}_t | \boldsymbol{z}_{t-1}, \boldsymbol{z}_{t-2}, \ldots, \boldsymbol{z}_{t-n}, \boldsymbol{\beta})$, in which $\boldsymbol{\beta}$ is a vector of parameters of interest. They propose a Gibbs sampler in order to iteratively sample from the conditionals $\mathrm{p}(\boldsymbol{\beta}|\mathbf{Y}, \mathbf{Z})$ and $\mathrm{p}(\mathbf{Y}|\boldsymbol{\beta}, \mathbf{Z})$, where $\mathbf{Y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_T)$ and $\mathbf{Z} = (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T)$. Nevertheless, a major drawback in their model is that it relies on sampling route flows, which is a daunting computational task for which no efficient procedure was until this moment proposed in the literature. Moreover, their model demands observations in all links of the network, preventing its use in large networks where one often observes only a subset of all links.

In the following Chapter 4, we describe our proposed dynamic model for the estimation of OD matrices in day-to-day dynamic settings.

4 A BAYESIAN DYNAMIC LINEAR MODEL FOR THE DAY-TO-DAY ESTIMATION OF THE OD MATRIX

In this chapter, we describe our proposed dynamic model for the day-to-day estimation of the OD matrix. Unlike the dynamic model of Hazelton (2008) described in Section 3.4, and static estimation models such as the ones from Vardi (1996) and Hazelton (2000), which assume independent Poisson OD flows, we model OD flows as random variables following multivariate normal distributions. By assuming normal/Gaussian OD flows, we can benefit from a more flexible covariance structure. For example, in contrast to the independent Poisson assumption, we may allow OD flows to be correlated and variances not be equal to mean OD flows. Moreover, the multivariate normal distribution is analytically convenient for Bayesian inference, since it is amenable to conjugacy.

For each OD pair, we model OD route flows also as Gaussian variables with a multinomial-like covariance structure, defined by the route choice set and the route choice probabilities. Accordingly, the link volumes are also Gaussian with a mean vector which is a function of the mean OD flows and whose covariance matrix reflects the variability of the OD flows, of the route choices and of the measurements of volumes on links. We cast all these variables in a dynamic linear model, which allows us to model the time dependence and evolution of the OD flows.

In addition, unlike the model of Hazelton (2008), the dynamic linear formulation of our model allows *online* estimation, i.e., we can update the estimate of the OD matrix as soon as a new vector of link volumes is observed. This is in contrast to *batch* estimation, in which we must gather a sample of link volume vectors and then apply the model to the batch of observations.

We describe the mathematical formulation and updating equations of our proposed model in the following Sections 4.1 and 4.2. In Section 4.3, we propose a method to estimate the route choice probability within our proposed dynamic model. In Section 4.4, we consider the static equilibrium-based case. Finally, in Section 4.5 we illustrate the application of the model to a small network.

4.1 Mathematical formulation of the model

We model the day-to-day dynamics of origin-destination flows with basis on the theory of dynamic linear models (DLM), which are Markovian state space models (WEST; HARRISON, 1997; SÄRKKÄ, 2013) (See Appendix C for a review of dynamic linear models). In the context of the estimation of the OD matrix in a day-to-day setting, we define $\boldsymbol{\theta}_t = (\theta_{t1}, \theta_{t2}, \ldots, \theta_{tn})^\mathsf{T}$ as the mean OD flow vector, in which θ_{tj} is the mean OD flow between OD pair $j \in \mathcal{J}$ at time t, and $n = |J|$ is the number of OD pairs. We define $\boldsymbol{z}_t = (z_{t1}, z_{t2}, \ldots, z_{tm})^\mathsf{T}$ as the vector of observed volumes in a subset of links in the network at time t, in which z_{ti} is the observed volume on link $i \in \mathcal{I}$ and $m = |\mathcal{I}|$ is the number of observed links.

A simple model for the temporal variation of the mean OD flow vector is based on the assumption that, in the short term, mean OD flows are *locally constant*. In other words, at time t mean OD flows should be equal to the previous OD flows at time $t-1$ but shifted by some stochastic error:

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t \tag{4.1}$$

In which $\boldsymbol{\omega}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{W}_t)$ and the covariances matrix $\mathbf{W}_t$ can be construed as an *evolution matrix*. It measures the uncertainty of the modeller with regards to the variability of OD flows over the time step from $t-1$ to t. Moreover, the vector of observed volumes $\boldsymbol{z}_t$ is related to the current mean OD vector $\boldsymbol{\theta}_t$ through an *observation model*, given below:

$$\boldsymbol{z}_t = \mathbf{F}_t \boldsymbol{\theta}_t + \boldsymbol{\nu}_t \tag{4.2}$$

In (4.2), $\mathbf{F}_t = \boldsymbol{\Delta}\mathbf{P}_t$ is an assignment matrix, $\boldsymbol{\Delta}$ is the link-path incidence matrix and $\mathbf{P}_t$ a route choice matrix at time t. The observation error at time t is given by $\boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{V}_t)$. It represents the variability of the observed volumes around the mean expected volumes given by $\mathrm{E}[\boldsymbol{z}_t] = \mathbf{F}_t \boldsymbol{\theta}_t$.

We should give special attention to the specification of the covariance matrix $\mathbf{V}_t$ of the observed volumes. Due to the network structure, link volumes are correlated. This correlation structure has to be represented in the covariances matrix $\mathbf{V}_t$. We can identify three sources of variability affecting link volumes: the generation of OD flows; the route choice process; and the counting of volumes on the links.

Let us define $\boldsymbol{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tn})^\intercal$ as the vector of actual OD flows at time t. The conditional probability density function of $\boldsymbol{x}_t$ is given by:

$$\mathrm{p}(\boldsymbol{x}_t | \boldsymbol{\theta}_t) = \mathrm{N}(\boldsymbol{\theta}_t, \boldsymbol{\Sigma}_{tx}) \tag{4.3}$$

Where $\boldsymbol{\Sigma}_{tx}$ is the covariance matrix of the actual OD flows, which can account for correlations among OD flows or simply be a diagonal matrix in case OD flows are deemed independent. Given a realized vector $\boldsymbol{x}_t$, for each OD pair j we have a vector of route flows $\boldsymbol{y}_{tj} = (y_{tj1}, y_{tj2}, \ldots, y_{tj\mathrm{n}(j)})$, in which $\mathrm{n}(j) = |\mathcal{K}_j|$ is the size of the route set $\mathcal{K}_j$ of OD pair j. We assume that $\boldsymbol{y}_{tj} \sim \mathrm{MN}(\lceil x_{tj} \rceil, \boldsymbol{p}_{tj})$, in which $\boldsymbol{p}_{tj} = (p_{tj1}, p_{tj2}, \ldots, p_{tj\mathrm{n}(j)})^\intercal$ is the vector of route choice probabilities of OD pair j. Notice that, according to the properties of the multinomial distribution, we must have $\sum_{k \in \mathcal{K}_j} p_{tjk} = 1$ and $\sum_{k \in \mathcal{K}_j} y_{tjk} = \lceil x_{tj} \rceil$, and the mean and covariances are respectively given by:

$$\mathrm{E}[y_{tjk}] = \lceil x_{tj} \rceil p_{tjk} \tag{4.4}$$

$$\mathrm{Cov}(y_{tjk}, y_{tjl}) = \begin{cases} \lceil x_{tj} \rceil p_{tjk}(1 - p_{tjk}) & \text{if} \quad k = l \\ -\lceil x_{tj} \rceil p_{tjk} p_{tjl} & \text{if} \quad k \neq l \end{cases} \tag{4.5}$$

In order to work within the framework of Gaussian DLMs, we consider a normal approximation to the multinomial, given by:

$$\mathrm{p}(\boldsymbol{y}_{tj} | x_{tj}, \boldsymbol{p}_{tj}) \approx \mathrm{N}(x_{tj}\boldsymbol{p}_{tj}, \boldsymbol{\Sigma}_{tyj}) \tag{4.6}$$

Where the covariance matrix of the route flows for OD pair j at time t is given by:

$$\boldsymbol{\Sigma}_{tyj} = x_{tj}(\mathrm{diag}(\boldsymbol{p}_{tj}) - \boldsymbol{p}_{tj}\boldsymbol{p}_{tj}^{\mathsf{T}}) \tag{4.7}$$

The approximation given by expression (4.6) will be good provided that route flows are large. Notice also that, as long as $\sum_{k \in \mathcal{K}_j} p_{tjk} = 1$, the covariance matrix $\boldsymbol{\Sigma}_{tyj}$ will be singular and the corresponding multivariate normal distribution will be degenerate. In order to avoid this, we assume there is a positive probability of none of the routes in the route set being chosen, so that $\sum_{k \in \mathcal{K}_j} p_{tjk} \leq 1$, assuring that $\boldsymbol{\Sigma}_{tyj}$ is non-singular.

The conditional distribution of the vector of route flows $\boldsymbol{y}_t = (\boldsymbol{y}_{t1}, \boldsymbol{y}_{t2}, \ldots, \boldsymbol{y}_{tn})^{\mathsf{T}}$ will be multivariate normal (since we defined normal distributions for all subvectors $\boldsymbol{y}_{tj}$):

$$\mathrm{p}(\boldsymbol{y}_t | \boldsymbol{x}_t) = \mathrm{N}(\mathbf{P}_t \boldsymbol{x}_t, \boldsymbol{\Sigma}_{ty}) \tag{4.8}$$

Where $\mathbf{P}_t$ and $\boldsymbol{\Sigma}_{ty}$ are *block-diagonal* matrices represented, respectively, by:

$$\mathbf{P}_t = \begin{bmatrix} \boldsymbol{p}_{t1} & & & \\ & \boldsymbol{p}_{t2} & & \\ & & \ddots & \\ & & & \boldsymbol{p}_{tn} \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{ty} = \begin{bmatrix} \boldsymbol{\Sigma}_{ty1} & & & \\ & \boldsymbol{\Sigma}_{ty2} & & \\ & & \ddots & \\ & & & \boldsymbol{\Sigma}_{tyn} \end{bmatrix}$$

Since we assume that $\mathbf{P}_t$ is known, we omit its explicit dependence in (4.8) as well as in subsequent conditional densities. Notice also from (4.7) that the covariance matrix

$\mathbf{\Sigma}_{ty}$ is dependent on the OD flow vector $\boldsymbol{x}_t$. In order to avoid such dependence, which may impose some “analytic hurdles”, we define an *approximate* covariance matrix $\hat{\mathbf{\Sigma}}_{ty}$ calculated on an approximate OD flow vector $\hat{\boldsymbol{x}}_t$.

The next step is to obtain the conditional distribution of route flows given mean OD flows $\boldsymbol{\theta}_t$:

$$\begin{aligned} \mathrm{p}(\boldsymbol{y}_t|\boldsymbol{\theta}_t) &= \int \mathrm{p}(\boldsymbol{y}_t, \boldsymbol{x}_t|\boldsymbol{\theta}_t)\mathrm{d}\boldsymbol{x}_t \\ &= \int \mathrm{p}(\boldsymbol{y}_t|\boldsymbol{x}_t)\mathrm{p}(\boldsymbol{x}_t|\boldsymbol{\theta}_t)\mathrm{d}\boldsymbol{x}_t \end{aligned} \tag{4.9}$$

Where (4.9) results from the conditional independence of $\boldsymbol{y}_t$ from $\boldsymbol{\theta}_t$, given $\boldsymbol{x}_t$ and $\mathbf{P}_t$. From (4.3) and (4.8), and from equations (A.2) and (A.3) in Appendix A, the marginal density (4.9) is multivariate normal:

$$\mathrm{p}(\boldsymbol{y}_t|\boldsymbol{\theta}_t) = \mathrm{N}(\mathbf{P}_t\boldsymbol{\theta}_t, \mathbf{P}_t\mathbf{\Sigma}_{tx}\mathbf{P}^{\mathsf{T}} + \hat{\mathbf{\Sigma}}_{ty}) \tag{4.10}$$

In order to complete the specification of our model, we must obtain the conditional distribution of the observed volumes given the mean OD flows $\boldsymbol{\theta}_t$. First we notice that the conditional distribution of observed volumes $\boldsymbol{z}_t$, given route flows $\boldsymbol{y}_t$ is:

$$\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{y}_t) = \mathrm{N}(\mathbf{\Delta}\boldsymbol{y}_t, \mathbf{\Sigma}_{tz}) \tag{4.11}$$

In which $\mathbf{\Sigma}_{zt}$ is the covariance matrix of the errors originated when observing the volumes on links, and $\mathbf{\Delta}$ is the link-path incidence matrix. Then we have:

$$\begin{aligned} \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t) &= \int \mathrm{p}(\boldsymbol{z}_t, \boldsymbol{y}_t|\boldsymbol{\theta}_t)\mathrm{d}\boldsymbol{y}_t \\ &= \int \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{y}_t)\mathrm{p}(\boldsymbol{y}_t|\boldsymbol{\theta}_t)\mathrm{d}\boldsymbol{y}_t \end{aligned} \tag{4.12}$$

From (4.10), (4.11) and from equations (A.2) and (A.3), we have:

$$\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t) = \mathrm{N}(\mathbf{\Delta}\mathbf{P}_t\boldsymbol{\theta}_t, \mathbf{V}_t) \tag{4.13}$$

In which the covariance matrix $\mathbf{V}_t$ is given by:

$$\begin{aligned} \mathbf{V}_t &= \mathbf{\Delta}(\mathbf{P}\mathbf{\Sigma}_{tx}\mathbf{P}_t^{\mathsf{T}} + \hat{\mathbf{\Sigma}}_{ty})\mathbf{\Delta}^{\mathsf{T}} + \mathbf{\Sigma}_{tz} \\ &= \mathbf{F}_t\mathbf{\Sigma}_{tx}\mathbf{F}_t^{\mathsf{T}} + \mathbf{\Delta}\hat{\mathbf{\Sigma}}_{ty}\mathbf{\Delta}^{\mathsf{T}} + \mathbf{\Sigma}_{tz} \end{aligned} \tag{4.14}$$

Finally, we are able to state fully our dynamic linear model for the day-to-day variation of the OD matrix:

Dynamic model:

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t \qquad \boldsymbol{\omega}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{W}_t) \tag{4.15}$$

Observational model:

$$\boldsymbol{z}_t = \mathbf{F}_t\boldsymbol{\theta}_t + \boldsymbol{\nu}_t \qquad \boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{V}_t) \tag{4.16}$$

Observational covariance matrix:

$$\mathbf{V}_t = \mathbf{F}_t\boldsymbol{\Sigma}_{tx}\mathbf{F}_t^{\mathsf{T}} + \boldsymbol{\Delta}\hat{\boldsymbol{\Sigma}}_{ty}\boldsymbol{\Delta}^{\mathsf{T}} + \boldsymbol{\Sigma}_{tz} \tag{4.17}$$

And $\mathbf{W}_t$ is determined according to the knowledge of the analyst regarding how the OD flows vary over the days. For example, if the OD flows are independent and vary only slightly from time $t-1$ to t, one may set $\mathbf{W}_t$ as a diagonal matrix with small variances for the OD flows.

4.2 Updating equations for the formulated model

At a time t, let $D_{t-1} = \{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{t-1}\}$ and $\mathbf{F}_t$ is the (known) assignment matrix. The updating equations of our model are the following (See Appendix C):

Posterior distribution of $\boldsymbol{\theta}_t$ at time $t-1$:

$$\mathrm{p}(\boldsymbol{\theta}_{t-1}|D_{t-1}) = \mathrm{N}(\boldsymbol{m}_{t-1}, \mathbf{C}_{t-1})$$

Prior distribution of $\boldsymbol{\theta}_t$ at time t:

$$\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1}) = \mathrm{N}(\bar{\boldsymbol{m}}_t, \bar{\mathbf{C}}_t)$$

Where

$$\bar{\boldsymbol{m}}_t = \boldsymbol{m}_{t-1}$$
$$\bar{\mathbf{C}}_t = \mathbf{C}_{t-1} + \mathbf{W}_t$$

One-step forecast of link volumes $\boldsymbol{z}_t$:

$$\mathrm{p}(\boldsymbol{z}_t|D_{t-1}) = \mathrm{N}(\boldsymbol{f}_t, \mathbf{Q}_t)$$

And

$$\boldsymbol{f}_t = \mathbf{F}_t\bar{\boldsymbol{m}}_t$$
$$\mathbf{Q}_t = \mathbf{F}_t\bar{\mathbf{C}}_t\mathbf{F}_t^{\mathsf{T}} + \mathbf{V}_t$$

It is worth noting that in computing $\mathbf{V}_t$, the covariance matrix of route flows $\hat{\boldsymbol{\Sigma}}_{ty}$ is computed based on the predicted value of mean OD flows, i.e., $\hat{\boldsymbol{\Sigma}}_{ty} = \mathrm{blockdiag}_{j\in\mathcal{J}}\{\hat{\boldsymbol{\Sigma}}_{tyj}\}$ in which $\hat{\boldsymbol{\Sigma}}_{tyj} = \bar{\mathrm{m}}_{\mathrm{tj}}(\mathrm{diag}(\mathrm{p}_{\mathrm{tj}}) - \boldsymbol{p}_{tj}\boldsymbol{p}_{tj}^{\mathsf{T}})$.

Compute the prediction error:

$$\boldsymbol{e}_t = \boldsymbol{z}_t - \boldsymbol{f}_t$$

And $\mathbf{A}_t$ is an adjustment matrix:

$$\mathbf{A}_t = \bar{\mathbf{C}}_t\mathbf{F}_t^{\mathsf{T}}\mathbf{Q}_t^{-1} \tag{4.18}$$

Finally, the posterior distribution of $\boldsymbol{\theta}_t$ given $D_t = D_{t-1} \cup \{\boldsymbol{z}_t\}$:

$$\mathrm{p}(\boldsymbol{\theta}_t|D_t) = \mathrm{N}(\boldsymbol{m}_t, \mathbf{C}_t)$$

With posterior mean and covariance matrix:

$$\boldsymbol{m}_t = \bar{\boldsymbol{m}}_t + \mathbf{A}_t\boldsymbol{e}_t \tag{4.19}$$
$$\mathbf{C}_t = \bar{\mathbf{C}}_t - \mathbf{A}_t\mathbf{Q}_t\mathbf{A}_t^{\mathsf{T}} \tag{4.20}$$

In equations (4.19) and (4.20), we see that the adjustment matrix $\mathbf{A}_t$ controls how the parameters from the posterior distribution are modified according to the new observation $\boldsymbol{z}_t$. In particular, we see from equation (4.18) that the adjustment matrix is a function of the prior covariance matrix $\bar{\mathbf{C}}_t$ and of the inverse $\mathbf{Q}_t^{-1}$ of the covariance matrix of the forecast distribution of the link volumes, so that the adjustment matrix gives more or less weight on the observed link volumes according to their uncertainty relative to the uncertainty on OD flows.

4.3 Congestion and the estimation of route choice probabilities

In the preceding Sections 4.1 and 4.2, we developed the base dynamic linear model for the problem of estimation of the OD matrix in a day-to-day dynamic setting. In our exposition, we assumed that the route choice probabilities, represented in the form of a route choice matrix $\mathbf{P}_t$, are known for all time periods. In practice, these probabilities have to be estimated.

We consider the estimation of the route choice matrix by an exogenous route choice model, which has been the practice in most literature on OD matrix estimation. In general, these models are based on the theoretical body of *discrete choice theory* (also referred to as *random utility theory*), briefly reviewed in Section 2.2.

In the uncongested case, we may plausibly assume that the route choice matrix does not depend on time, at least for the the planning horizon under consideration, in order that $\mathbf{P}_t = \mathbf{P}$ for all t. Then, we can estimate $\mathbf{P}$ by applying, for example, a logit or probit model based on free flow times (for further details, refer to Section 2.2).

The problem gets a new layer of complexity in congested networks. In this case, in contrast to uncongested networks, the levels of traffic volumes on links have non-negligible deteriorating impact on route travel times. In response to congestion, rational users change routes in order to minimize their travel times. This means that route choice probabilities will change with time as a function of users expected travel times.

Moreover, unlike static models, in dynamic settings the network may not be in an equilibrium state. As described in Section 2.4, in stochastic equilibrium, route choice probabilities (i.e., expected route choice proportions) are constant functions of mean OD flows, corresponding to stationary probability distributions reached in the long run. If mean OD flows vary with time, the transportation network will likely not have enough time to converge to an equilibrium state.

Our strategy is then to estimate the route choice matrix at each time t. Let $\boldsymbol{f}_{t-1} = \mathbf{F}_{t-1}\bar{\mathbf{m}}_{t-1}$ be the forecast traffic volumes at time $t-1$. We define the vector $\boldsymbol{u}_{t-1}$ of *forecast* route choice travel times as:

$$\boldsymbol{u}_{t-1} = \boldsymbol{\Delta}^\mathsf{T}\boldsymbol{\tau}(\boldsymbol{f}_{t-1}) \tag{4.21}$$

In which $\boldsymbol{\tau}(.)$ is the link cost vector, given by performance functions (e.g., BPR functions, see Section 2.4), and we assume *additive* route costs as standard in the literature. We can estimate the route choice probabilities at time t taking into account the forecast costs at the previous time:

$$\hat{\boldsymbol{p}}_t = \boldsymbol{p}(\boldsymbol{u}_{t-1}) \tag{4.22}$$

And the function $\boldsymbol{p}(.)$ is computed by means of a route choice model. In practice, it is plausible to consider that users decide which route to follow based on *average* route travel times, which they average out from past experiences. Then, at a time t, we recursively define as estimate of users average route travel time the weighted average of forecast travel times and estimated average travel times at time $t-1$:

$$\tilde{\boldsymbol{u}}_t = \alpha \boldsymbol{\Delta}^{\mathsf{T}} \boldsymbol{\tau}(\boldsymbol{f}_{t-1}) + (1-\alpha)\tilde{\boldsymbol{u}}_{t-1} \tag{4.23}$$

In (4.23), $0 \leq \alpha \leq 1$ is the weight that users put on recently experienced travel times, with $\alpha = 0$ resulting in constant route travel times equal to the initial travel times, $\tilde{\boldsymbol{u}}_t = \tilde{\boldsymbol{u}}_{t-1}, \ldots, \tilde{\boldsymbol{u}}_0$, and $\alpha = 1$ resulting in users totally ignoring past experiences. $\tilde{u}_0$ may be estimated, for example, from free-flow route travel times. Finally, route choice probabilities may be estimated by the following expression:

$$\hat{\boldsymbol{p}}_t = \boldsymbol{p}(\tilde{\boldsymbol{u}}_t) \tag{4.24}$$

4.4 The static equilibrium-based case

If we assume the premise that the mean OD matrix is constant over the considered time horizon, then the evolution covariance matrix from time t to time $t+1$ is zero, $\mathbf{W}_t = \mathbf{W} = \mathbf{0}$. Thus according to equation (4.1) $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} = \boldsymbol{\theta}$, so that we do not need a dynamic model. In addition, in the static case it is typically assumed that the network is in equilibrium, so that route choice probabilities also do not vary over time. Then, we have the following observational model, which relates the mean OD matrix $\boldsymbol{\theta}$ to the observed link volumes:

$$\boldsymbol{z}_t = \mathbf{F}\boldsymbol{\theta} + \boldsymbol{\nu} \tag{4.25}$$

In which $\mathbf{F} = \boldsymbol{\Delta}\mathbf{P}$ is the constant assignment matrix corresponding to an equilibrium state, and $\boldsymbol{\nu} \sim \mathrm{N}(\mathbf{0}, \mathbf{V})$ are homoscedastic errors (do not vary over time) with covariance matrix (according to equation (4.14)):

$$\mathbf{V} = \mathbf{F}\boldsymbol{\Sigma}_x\mathbf{F}^{\mathsf{T}} + \boldsymbol{\Delta}\boldsymbol{\Sigma}_y\boldsymbol{\Delta}^{\mathsf{T}} + \boldsymbol{\Sigma}_z \tag{4.26}$$

In which $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $\boldsymbol{\Sigma}_z$ are the covariance matrices of OD flows, route flows and counting errors, respectively.

Given a prior probability distribution $\mathrm{p}(\boldsymbol{\theta})$ and observed volume vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T$, the posterior distribution $\mathrm{p}(\boldsymbol{\theta}|\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T)$ is given by Bayes theorem:

$$\mathrm{p}(\boldsymbol{\theta}|\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T) \propto \mathrm{p}(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T|\boldsymbol{\theta})\mathrm{p}(\boldsymbol{\theta}) \tag{4.27}$$

Assuming conditional independence of $\boldsymbol{z_t}$ given $\boldsymbol{\theta}$, the likelihood in expression (4.27) may be written as:

$$\mathrm{p}(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T|\boldsymbol{\theta}) = \prod_{t=1}^{T} \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}) \tag{4.28}$$

Further, notice that $\mathrm{p}(\boldsymbol{\theta}|\boldsymbol{z}_t, \boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}_1) \propto \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta})\mathrm{p}(\boldsymbol{\theta}|\boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}_1)$, once more due to conditional independence, so that we can obtain the posterior $\mathrm{p}(\boldsymbol{\theta}|\boldsymbol{z}_t, \boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}_1)$ by iterating through $t = 1, 2, \ldots, T$ and updating the prior to posterior distributions as in the dynamic case. The following are the updating equations for the static case:

Posterior distribution of $\boldsymbol{\theta}_t$ at time $t-1$, equals to prior distribution at time t:

$$\mathrm{p}(\boldsymbol{\theta}_{t-1}|D_{t-1}) = \mathrm{N}(\boldsymbol{m}_{t-1}, \mathbf{C}_{t-1})$$

One-step forecast of link volumes $\boldsymbol{z}_t$:

$$\mathrm{p}(\boldsymbol{z}_t|D_{t-1}) = \mathrm{N}(\boldsymbol{f}_t, \mathbf{Q}_t)$$

Where

$$\begin{aligned} \boldsymbol{f}_t &= \mathbf{F}_t \boldsymbol{m}_{t-1} \\ \mathbf{Q}_t &= \mathbf{F}_t \mathbf{C}_{t-1} \mathbf{F}_t^{\mathsf{T}} + \mathbf{V}_t \end{aligned}$$

Compute the prediction error:

$$\boldsymbol{e}_t = \boldsymbol{z}_t - \boldsymbol{f}_t$$

And the adjustment matrix:

$$\mathbf{A}_t = \mathbf{C}_{t-1} \mathbf{F}_t^{\mathsf{T}} \mathbf{Q}_t^{-1}$$

Finally, the posterior distribution of $\boldsymbol{\theta}_t$ given $D_t = D_{t-1} \cup \{\boldsymbol{z}_t\}$:

$$\mathrm{p}(\boldsymbol{\theta}_t|D_t) = \mathrm{N}(\boldsymbol{m}_t, \mathbf{C}_t)$$

$$\boldsymbol{m}_t = \boldsymbol{m}_{t-1} + \mathbf{A}_t \boldsymbol{e}_t \tag{4.29}$$

$$\mathbf{C}_t = \mathbf{C}_{t-1} - \mathbf{A}_t \mathbf{Q}_t \mathbf{A}_t^\top \tag{4.30}$$

Moreover, in the static case we assume that the network is in equilibrium. For a given route choice matrix $\mathbf{P}$ (the equilibrium probabilities are deterministic), we define the posterior estimate $\boldsymbol{m}$ of $\boldsymbol{\theta}$ as the expected value of the posterior distribution, $\boldsymbol{m} = \mathrm{E}[\mathrm{p}(\boldsymbol{\theta}|\mathbf{P}, \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T)]$. It turns out that the estimate $\boldsymbol{m}$ must be consistent with the route choice probabilities. Then we set up the following fixed point problem to find $\boldsymbol{m}^*$:

$$\boldsymbol{m}^* = \mathrm{E}[\mathrm{p}(\boldsymbol{\theta}|\mathbf{P}(\boldsymbol{m}^*), \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T)] \tag{4.31}$$

The estimate $\boldsymbol{m}^*$ in (4.31) may be found by a fixed point iteration:

- Initial step ($k = 0$): Start with an initial route choice matrix $\mathbf{P}^{(0)}$ (which can be obtained based on free flow route travel times), and prior distribution $\mathrm{p}(\boldsymbol{\theta})$ based on prior knowledge.
- While a stopping criteria is not met, repeat:
  1. Obtain $\boldsymbol{m}^{(k)} = \mathrm{E}[\mathrm{p}(\boldsymbol{\theta}|\mathbf{P}^{(k)}, \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T)]$ by recursively applying the updating equations (4.29) and (4.30).
  2. Calculate new route choice matrix $\mathbf{P}^{(k+1)}(\boldsymbol{m}^{(k)})$ based on stochastic traffic assignment of $\boldsymbol{m}^{(k)}$. Make $k := k + 1$.

An alternative method is to obtain the *maximum a posteriori* (MAP) by maximizing the logarithm of expression (4.27), with likelihood $\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}) = \mathrm{N}(\boldsymbol{\theta}, \mathbf{V})$ and prior $\mathrm{p}(\boldsymbol{\theta}) = \mathrm{N}(\boldsymbol{m}_0, \mathbf{C}_0)$:

$$\ell(\boldsymbol{\theta}) = -\frac{T}{2}\log|\mathbf{V}| - \frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{z}_t - \mathbf{F}\boldsymbol{\theta})^\top \mathbf{V}^{-1}(\boldsymbol{z}_t - \mathbf{F}\boldsymbol{\theta}) - \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{m}_0)^\top \mathbf{C}_0^{-1}(\boldsymbol{\theta} - \boldsymbol{m}_0) \tag{4.32}$$

The main difficulty in maximizing equation (4.32) is that both the covariance matrix $\mathbf{V}$ and the assignment matrix $\mathbf{F}$ are functions of mean OD flows $\boldsymbol{\theta}$. A relaxation we can take with regards to $\mathbf{V}$ is to substitute it for the sample covariance matrix $\hat{\mathbf{V}}$, given by:

$$\hat{\mathbf{V}} = \frac{1}{T-1}\sum_{t=1}^{T}(\boldsymbol{z}_t - \bar{\boldsymbol{z}})(\boldsymbol{z}_t - \bar{\boldsymbol{z}})^\top \tag{4.33}$$

In which $\bar{\boldsymbol{z}}$ is the vector of sample mean observed volumes. Finally, we set up a bilevel model in order to cope with the dependence between the assignment matrix $\mathbf{F}$ and the vector of mean flows $\boldsymbol{\theta}$:

$$\min \quad \frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{z}_t - \mathbf{F}\boldsymbol{\theta})^{\mathsf{T}}\hat{\mathbf{V}}^{-1}(\boldsymbol{z}_t - \mathbf{F}\boldsymbol{\theta}) + \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{m}_0)^{\mathsf{T}}\mathbf{C}_0^{-1}(\boldsymbol{\theta} - \boldsymbol{m}_0) \tag{4.34}$$

s.t.

$$\mathbf{F} = \mathbf{F}(\boldsymbol{\theta}) \tag{4.35}$$

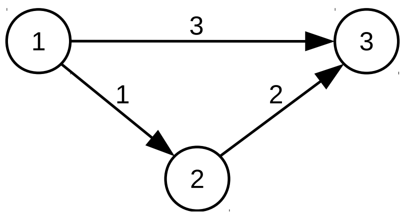
$$\boldsymbol{\theta} \geq \mathbf{0} \tag{4.36}$$

Where $\mathbf{F}(\boldsymbol{\theta})$ is calculated by means of the assignment of the mean OD flow vector $\boldsymbol{\theta}$. It is noteworthy that the model given by equations (4.34)-(4.36) is essentially the GLS model of Hazelton (2000) (see equation (3.28)) with an added term corresponding to the prior OD matrix and cast in a bilevel form. It can be solved by iterating between the upper and lower levels, as described in Section 3.2.4.

4.5 An illustrative example

In order to illustrate the application of our proposed DLM to the estimation of day-to-day OD flows, we simulate dynamic OD flows in a small network from the paper of Hazelton (2000), which is shown in the Figure 3 below:

Figure 3 – Network used in the illustrative example



Source: Hazelton (2000)

There are three OD pairs: (1,2), (1,3) and (2,3). OD pairs (1,2) and (2,3) have only one route each, and OD pair (1,3) has two available routes: The first through links 1 and 2, and the second directly through link 3. Thus, the corresponding link-path incidence matrix is the following:

$$\boldsymbol{\Delta} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We assume that all links have the same length of one unit and capacities respectively of 100.0, 100.0 and 50.0. We adopt BPR-like performance functions, with the traditional values $\alpha = 0.15$ and $\beta = 4.0$ for all three links. Free-flow times are equal to link lengths.

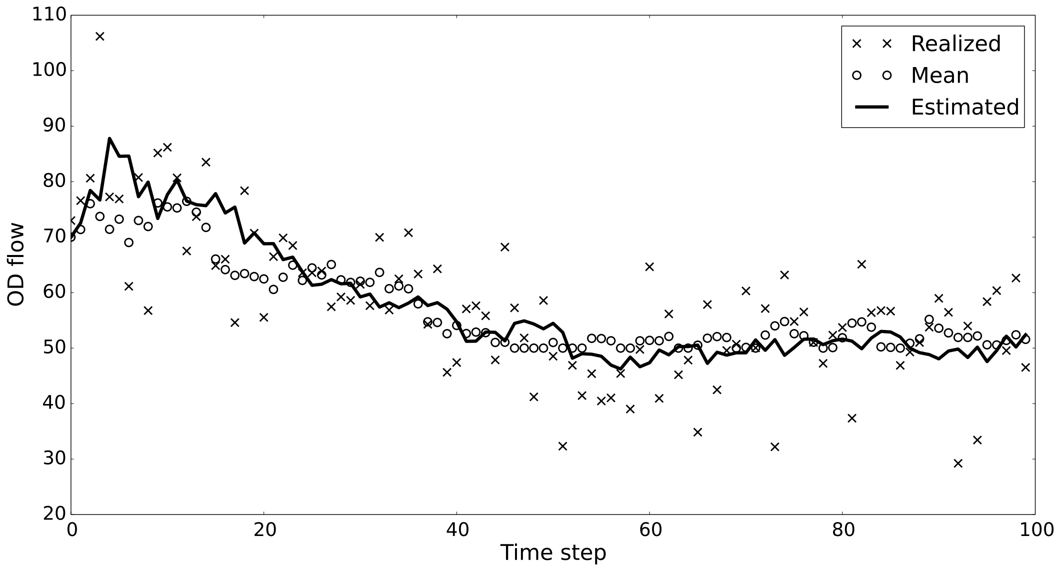
We will simulate T time periods (e.g., days). The simulation is performed according to the following steps for each time period $t = 1, \ldots, T$, starting on initial mean OD flow vector $\boldsymbol{\theta}_0$, average route costs $\tilde{\boldsymbol{u}}_0$ based on free-flow times on links, and specified parameters α (weight of current route costs) and ξ (logit scale):

1. Draw a new mean OD flow vector $\boldsymbol{\theta}_t \sim \mathrm{N}(\boldsymbol{\theta}_{t-1}, \mathbf{W}_t)$ (equation (4.1));
2. draw OD flows $\boldsymbol{x}_t \sim \mathrm{N}(\boldsymbol{\theta}_t, \boldsymbol{\Sigma}_{tx})$;
3. compute route choice probabilities by using a logit route choice model based on current average route costs: $\boldsymbol{p}_{tj} = \mathrm{logit}(\tilde{\boldsymbol{u}}_{tj}, \xi)$;
4. for each OD pair j, draw route flows $\boldsymbol{y}_{tj} \sim \mathrm{MN}(\lceil x_{tj} \rceil, \boldsymbol{p}_{tj}(1 - \pi))$, where π is the probability of a user not choosing a route in the route choice set;
5. assign route flows to links by making $\boldsymbol{z}_t = \boldsymbol{\Delta} \boldsymbol{y}_t + \boldsymbol{\nu}_t$, in which $\boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_{tz})$;
6. update average route costs for next time period by means of the equation $\tilde{\boldsymbol{u}}_{t+1} = \alpha \boldsymbol{\Delta}^{\mathsf{T}} \boldsymbol{\tau}(\boldsymbol{z}_t) + (1 - \alpha) \tilde{\boldsymbol{u}}_t$, in which $\boldsymbol{\tau}(.)$ is a vector-valued function which returns the travel time on links.

We adopt the following values for the parameters of the simulation: $T = 100$ (days); initial mean OD flow vector $\boldsymbol{\theta}_0 = (70.0, 100.0, 80.0)^{\mathsf{T}}$; $\xi = 5.0$; $\alpha = 0.25$; $\pi = 0.01$ (probability of a user not choosing a route in the route choice set). In addition, we assume that mean OD flows are mutually independent and that they vary slightly between times $t-1$ and t, by setting the coefficient of variation equals to 0.01, so that $\mathbf{W}_t = 10^{-4} \times \mathrm{diag}(\boldsymbol{\theta}_{t-1}^2)$. Notice that the mean OD flows will show heteroscedasticity over time, since the variances will be proportional to mean OD levels. We also maintain mean OD flows within the range $[50.0, 150.0]$. With regards to realized OD flows $\boldsymbol{x}_t$, we assume Poisson-like variability, in order that $\boldsymbol{\Sigma}_{tx} = \mathrm{diag}(\boldsymbol{\theta}_t)$. Finally, we assume negligible counting errors in observed volumes, with $\boldsymbol{\Sigma}_{tz} = \mathbf{I}$ (the identity matrix). In applying the updating equations (4.19) and (4.20) at each time t, we start with an *uniformative* prior distribution $\boldsymbol{\theta}_0 \sim \mathrm{N}(\boldsymbol{m}_0, \mathbf{C}_0)$, with $\boldsymbol{m}_0 = (70.0, 100.0, 80.0)^{\mathsf{T}}$ and $\mathbf{C}_0 = 10^4 \times \mathbf{I}$.

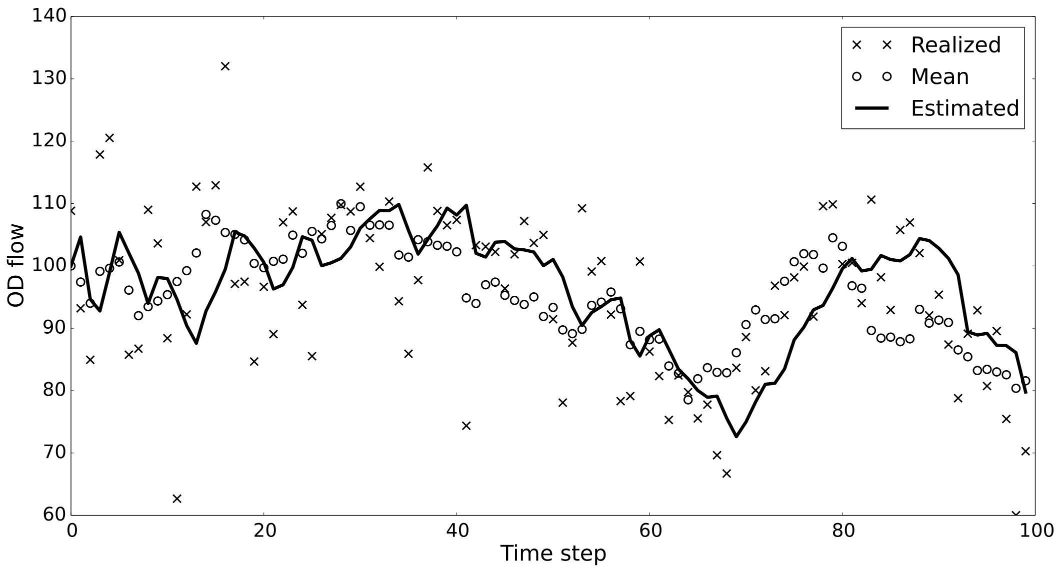
Figures 4, 5 and 6 show the results of the simulation for OD flows, while Figures 7, 8 and 9 show the results for link volumes. In these figures, we can see how the updating equations correctly respond to dynamic changes in OD flows by exploiting information contained in the observed link volumes. The estimated mean OD flows, given by the continuous line in the corresponding figures, is not far from the dots, which correspond to "true" mean OD flows. In Figure 10, we have the probability of choosing route 1 of OD pair 2. As the OD flows vary over the simulated time horizon, the probability also changes in response to congestion. Once again, the estimated probability is not far from the actual probability. As a final comment, we should point out that in this illustrative example the mean OD flows are identifiable, since the number of independent observed link volumes equals the number of OD pairs. In larger and more realistic networks, we should expect facing identifiability problems.

Figure 4 – Simulation of OD flow in pair 1



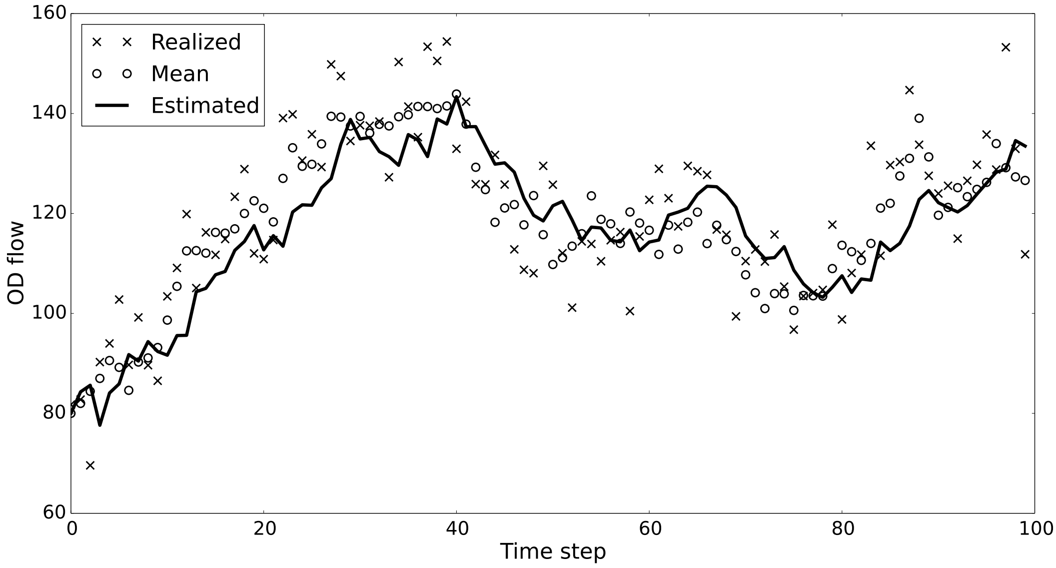
Source: the author

Figure 5 – Simulation of OD flow in pair 2



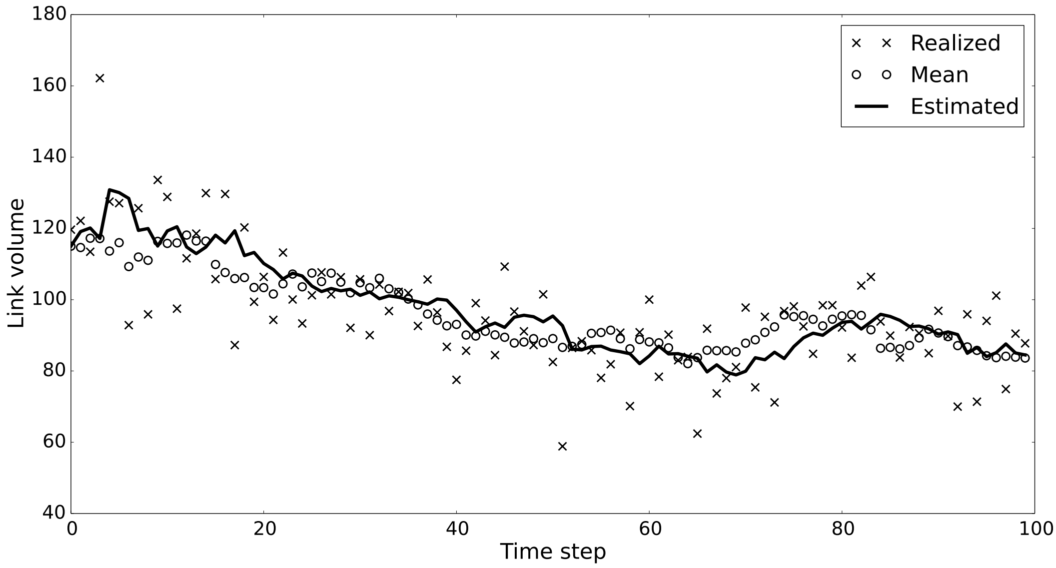
Source: the author

Figure 6 – Simulation of OD flow in pair 3



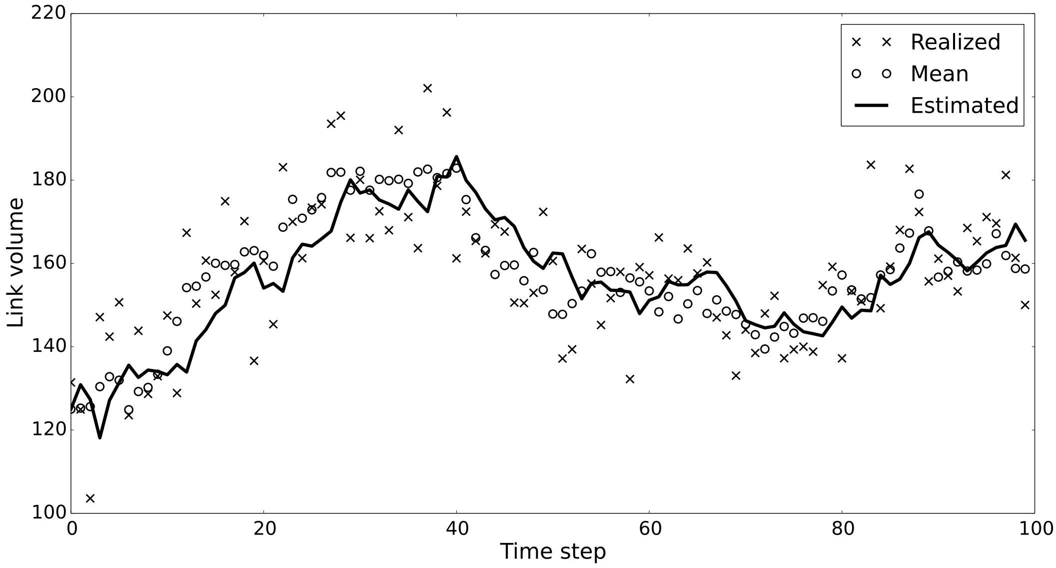
Source: the author

Figure 7 – Simulation of volume on link 1



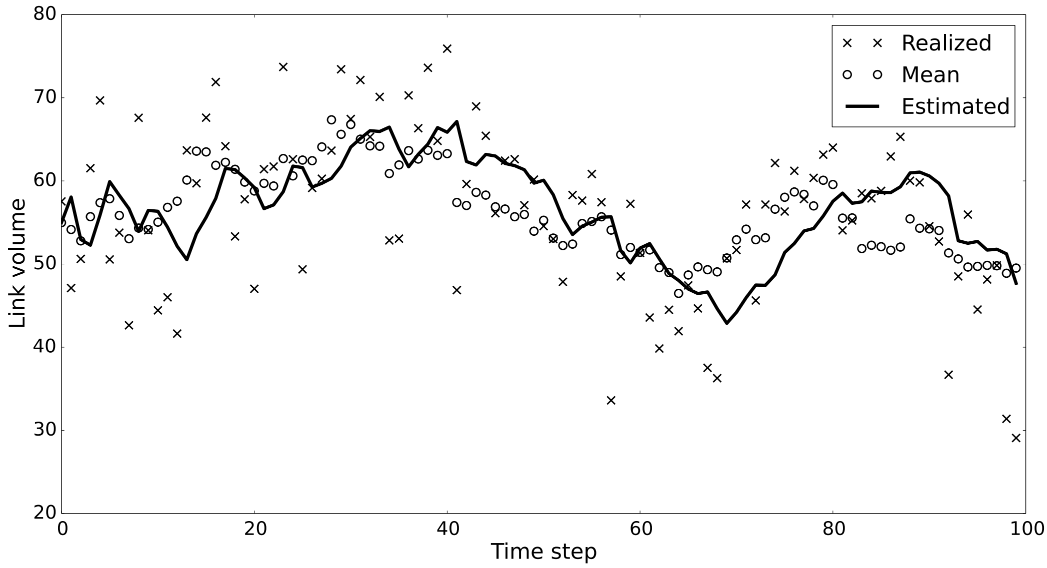
Source: the author

Figure 8 – Simulation of volume on link 2



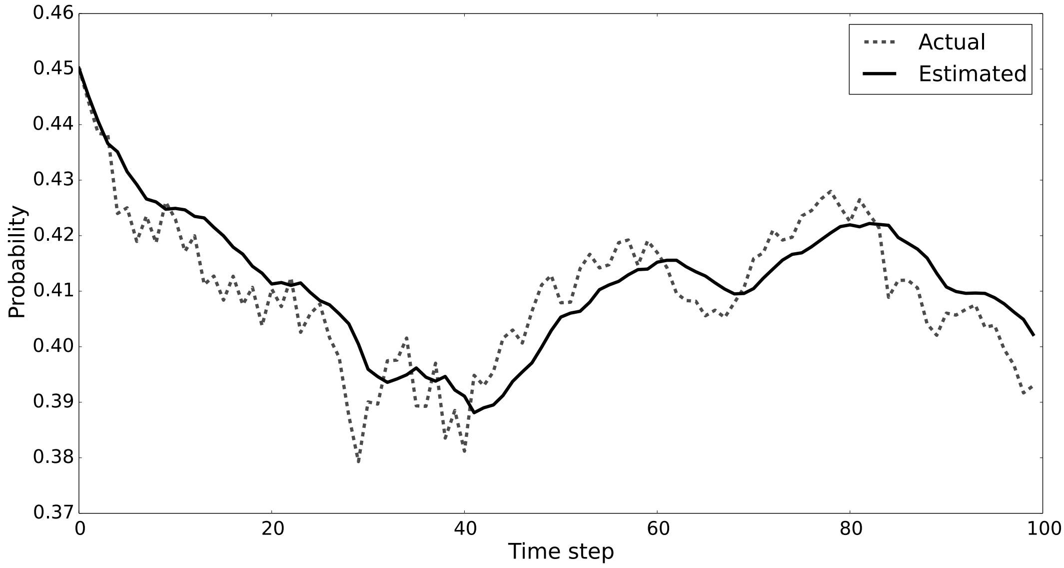
Source: the author

Figure 9 – Simulation of volume on link 3



Source: the author

Figure 10 – Probability of choosing the route 1 in OD pair 2



Source: the author

5 COMPUTATIONAL EXPERIMENTS

In this chapter, we describe three computational experiments we performed. In the first one, our objective is to assess if dynamic models can reduce the non-identifiability (underspecification problem) of the mean OD matrix. This corresponds to our first research question as stated in the introduction. We test our proposed dynamic model and a dynamic version of the GLS model at best conditions, and verify if they are able to estimate the dynamic mean OD matrices over time with low estimation errors. By "best conditions" we mean that the models are tested as if all parameters were known exactly except for the mean OD matrices. In the second experiment, corresponding to our second research question, our objective is to compare static and dynamic models in a realistic scenario to test if dynamic models can produce better estimates of mean OD matrices. The impact of prior information, our third research question, is assessed in both experiments 1 and 2. Finally, in a third experiment, we address our fourth research question, the effect of the assignment matrix on the estimation error, by evaluating the performance of the GLS and dynamic models on three test cases in which we use different assignment matrices. In the following sections, we detail how the models were implemented, the data generated, the experimental design and the performance measures used. We comment on the results in each experiment. We summarize the main findings in the final Section 5.9 of this chapter.

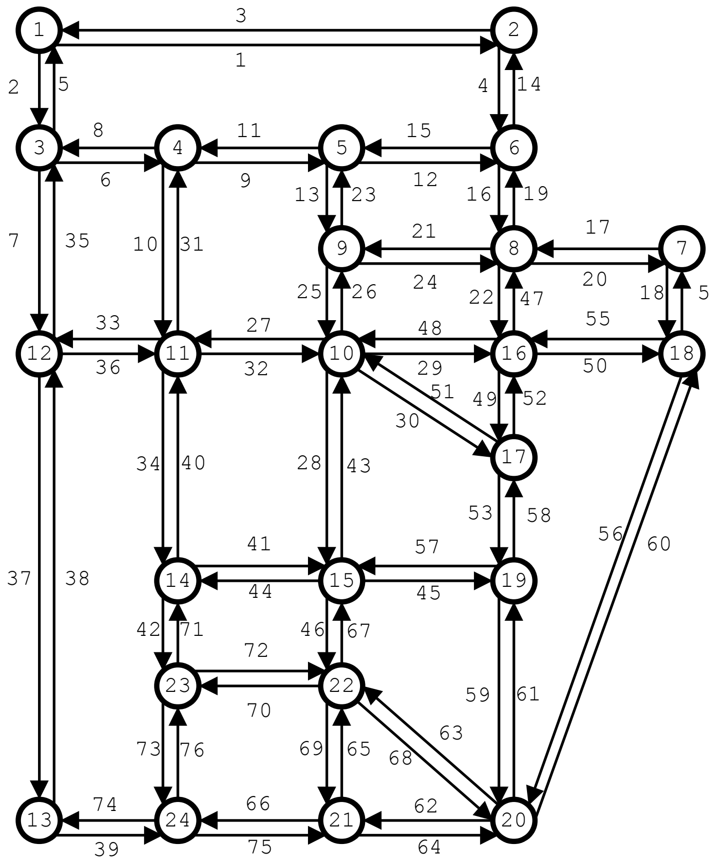
5.1 Implementation details

All algorithms were implemented in the Python programming language version 2.7.7, which is a high level open-source programming language (LUTZ, 2003). We used the Scientific Python library (SciPy), which is a collection of functions and algorithms for scientific computing (JONES et al., 2001). In particular, we used heavily the linear algebra submodule `linalg`, which is a wrapper for the low level functions of the LAPACK (Linear Algebra Package) and BLAS (Basic Linear Algebra Subroutines) written in FORTRAN. This means that, although our high level codes are written in Python, all the "numerical crunching" is effectively outsourced to computationally efficient routines in FORTRAN. We used the version of LAPACK provided by Intel via the Math Kernel Library (MKL). The optimization models were solved by using the limited memory Broyden-Fletcher-Goldfarb-Shanno algorithm for bounded variables, known as L-BFGS-B (NOCEDAL; WRIGHT, 2006). In order to model the network and perform some operations on nodes and links, we used the Python module `networkx` (HAGBERG; SCHULT; SWART, 2008).

5.2 Characterizarization of the test network

We use the *Sioux Falls* network as the test unit in our experiments. It is an abstracted version of a real network in the city of Sioux Falls in the United States. It is routinely used in the literature, which allows for repeatability and comparison. Its first reference is the paper by LeBlanc, Morlok and Pierskalla (1975). Figure 11 shows an schematic representation, while Table 1 shows its main features.

Figure 11 – Schematic representation of Sioux Falls network



Source: Shao et al. (2014)

Table 1 – Main characteristics of Sioux Falls network

Feature	Value
Nodes	24
Links	76
OD pairs	552
Type of link	All bidirectional
Total number of routes	$\approx 1.8 \times 10^6$

Source: the author

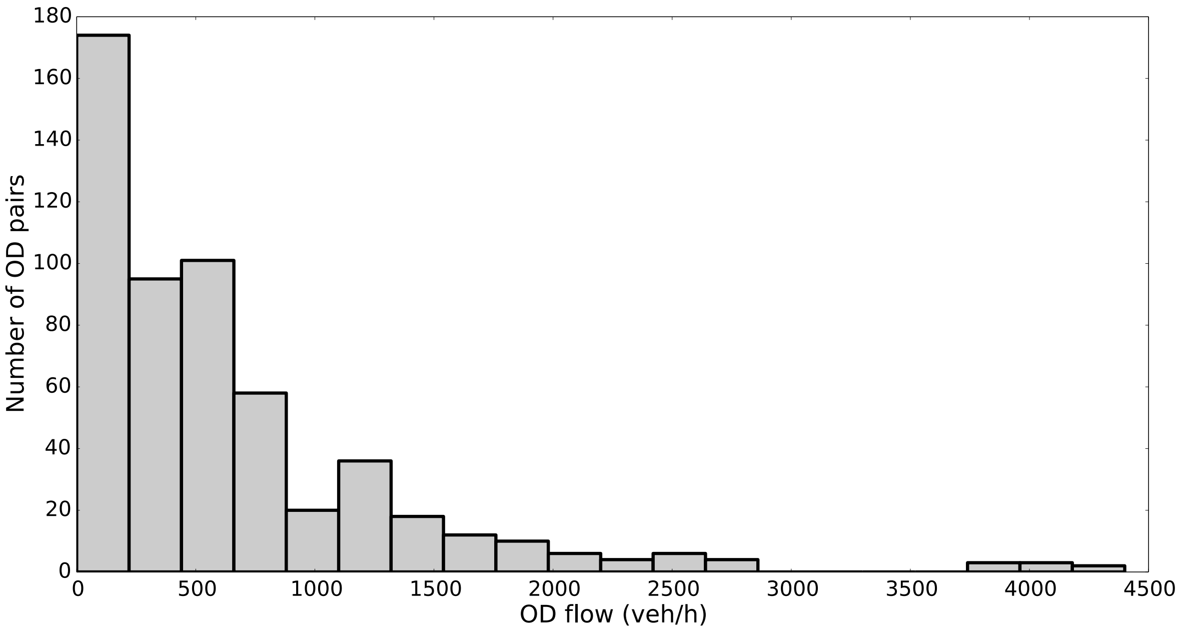
Table 13 in Annex B contains the lengths of all links, their corresponding capacities and other parameters. Table 2 shows some statistics on the OD matrix for Sioux Falls (in vehicles per hour), while Figure 12 shows an histogram of the OD flows. It can be seen from the histogram that the distribution of the OD flows is skewed to the right, with many OD pairs with low flow and just a few with high flow. Table 12 in Annex A contains the values of the OD flows for all OD pairs.

Table 2 – Statistics for the OD matrix of Sioux Falls

Statistic	Value (veh/h)
Total flow	360600
Mean	653
Minimum	0
Maximum	4400
Standard deviation	695
Number of OD pairs with zero OD flow	24

Source: the author

Figure 12 – Histogram of Sioux Falls OD flows



Source: the author

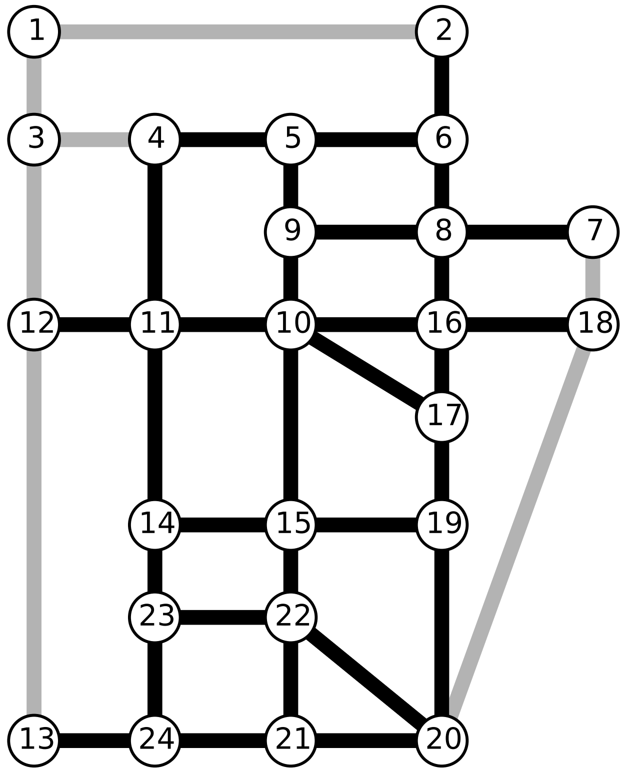
As the total number of feasible routes is intractably large ($\approx 1.8 \times 10^6$, see Table 1), we enumerate only the five shortest routes for each OD pair by mean of Yen's algorithm (YEN, 1971), summing up to 2760 routes (5 routes for each one of the 552 OD pairs). This is a fairly realistic assumption, since in practice the users consider a relatively small set of routes (BEKHOR; BEN-AKIVA; RAMMING, 2006). We also assume that all 76 links are observed, resulting in a link-path incidence matrix $\mathbf{\Delta}$ with 76 rows and 2760 columns.

Figure 13 shows the result of a stochastic user equilibrium assignment of the OD matrix of the Sioux Falls network. In the assignment we used a multinomial logit route choice model with scale factor $\xi = 5.0$ (equation (2.10)) and link-path incidence matrix generated from the 5-shortest routes in each OD pair.

5.3 Tested models

Below we briefly describe the three models we tested in the experiments. A dynamic version of the GLS model described in Section 3.2.2, the static estimation model we developed in

Figure 13 – Congested links in Sioux Falls network (in black)



Source: the author

Section 4.4, and the dynamic model we developed in Section 3.4.

5.3.1 Dynamic *GLS model*

Since the GLS model was designed for reconstruction, we define what we call a *dynamic* GLS: at each time t, we reconstruct an OD matrix based on the observed volumes at t. The prior matrix at time t is the reconstructed matrix from previous time period $t-1$, with an initial prior matrix $\boldsymbol{x}_0$. At the end, we calculate the estimate of the mean OD matrix as the average of all reconstructed matrices:

$$\boldsymbol{\theta}^{\mathrm{GLS}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_t^{\mathrm{GLS}} \tag{5.1}$$

In equation (5.1), $\boldsymbol{x}_t^{\mathrm{GLS}}$ is the reconstructed matrix at time t, obtained as the solution of equation (3.8).

5.3.2 Static estimation model

The estimator of the mean OD matrix given by the static estimation model (SE model) is given by the solution of equation (4.31):

$$\boldsymbol{\theta}^{\mathrm{SE}} = \boldsymbol{m}^* \qquad \text{where} \qquad \boldsymbol{m}^* = \mathrm{E}[\mathrm{p}(\boldsymbol{\theta}|\mathbf{P}(\boldsymbol{m}^*), \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T)] \tag{5.2}$$

It should be emphasized that the estimator $\boldsymbol{\theta}^{\mathrm{SE}}$ is applied in *batch* mode, this is, it needs observations from all T time periods in the time horizon in order to be applied.

5.3.3 Dynamic estimation model

Finally, dynamic estimation (DE) model gives an estimate of the mean OD matrix $\boldsymbol{\theta}_t^{\mathrm{DE}}$ at each time step as the mean of the posterior distribution $p(\boldsymbol{\theta}_t|\boldsymbol{z}_t, \boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}_1)$:

$$\boldsymbol{\theta}_t^{\mathrm{DE}} = \boldsymbol{m}_t \qquad \text{where} \qquad \boldsymbol{m}_t = \mathrm{E}[p(\boldsymbol{\theta}_t|\boldsymbol{z}_t, \boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}_1)] \tag{5.3}$$

And $\boldsymbol{m}_t$ is obtained recursively by means of equation (4.19). It is worth noting that the GLS model is fundamentally different from the other two. First, it assumes that the demand is deterministic, so that it does not seek to estimate a mean OD matrix, but to reconstruct an OD matrix which generated the observed volumes. It takes into account only errors in the observation of volumes. On the other hand, both static and dynamic estimation models are fully stochastic, in the sense that they also consider errors in volumes brought about by the random nature of OD flows and route choices. Table 3 shows a summary of the main characteristics of the tested models:

Table 3 – Main features of the tested models

Model	Paradigm	Demand	Main references
GLS	Frequentist	Deterministic static	Cascetta (1984), Bell (1991), Cascetta and Postorino (2001)
Static Estimation (SE)	Bayesian	Stochastic static	Hazelton (2000), Hazelton (2001), this thesis
Dynamic Estimation (DE)	Bayesian	Stochastic dynamic	This thesis

Source: the author

5.4 Performance measures

In all experiments, we use the relative root mean squared error (RRMSE) and the relative mean absolute error (RMAE) as performance measures of the model. Equations (5.4) and

(5.5) give the root mean squared error (RMSE) and the mean absolute error (MAE), while equations (5.6) and (5.7) give the corresponding RRMSE and RMAE:

$$\mathrm{RMSE_{OD}} = \sqrt{\frac{1}{nT}\sum_{t=1}^{T}\sum_{j=1}^{n}(\hat{\theta}_{tj} - \theta_{tj})^2} \tag{5.4}$$

$$\mathrm{MAE_{OD}} = \frac{1}{nT}\sum_{t=1}^{T}\sum_{j=1}^{n}|\hat{\theta}_{tj} - \theta_{tj}| \tag{5.5}$$

$$\mathrm{RRMSE_{OD}} = \frac{\mathrm{RMSE_{OD}}}{\frac{1}{nT}\sum_{t=1}^{T}\sum_{j=1}^{n}\theta_{tj}} \tag{5.6}$$

$$\mathrm{RMAE_{OD}} = \frac{\mathrm{MAE_{OD}}}{\frac{1}{nT}\sum_{t=1}^{T}\sum_{j=1}^{n}\theta_{tj}} \tag{5.7}$$

In which, θ_{tj} and $\hat{\theta}_{tj}$ are, respectively, the actual mean OD flow and estimated OD flow in OD pair j at time t, and n is the number of OD pairs. The RMSE is the traditional performance measure, but it is sensitive to extreme values. MAE is less sensitive, then in some cases it may provide a more fair comparison between models. We use similar performance measures with relation to volumes:

$$\mathrm{RMSE_{vol}} = \sqrt{\frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{m}(\hat{\mu}_{ti} - \mu_{tj})^2} \tag{5.8}$$

$$\mathrm{MAE_{vol}} = \frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{m}|\hat{\mu}_{ti} - \mu_{ti}| \tag{5.9}$$

$$\mathrm{RRMSE_{vol}} = \frac{\mathrm{RMSE_{vol}}}{\frac{1}{mT}\sum_{t=1}^{T}\sum_{i=1}^{m}\mu_{ti}} \tag{5.10}$$

$$\mathrm{RMAE_{vol}} = \frac{\mathrm{MAE_{vol}}}{\frac{1}{mT}\sum_{t=1}^{T}\sum_{i=1}^{m}\mu_{ti}} \tag{5.11}$$

Where μ_{ti} and $\hat{\mu}_{ti}$ are, respectively, the actual and estimated volumes on link i at time t, and m is the number of observed links.

5.5 Generation of synthetic data

We use simulated data in order to test the models. The procedure of generation is the following: we generate a sequence of OD flow vectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_T$ according to a *gaussian random walk* (See equation (4.1)), with $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t$, and $\boldsymbol{\omega}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{W}_t)$. We assume independent mean OD flows, so that $\mathrm{Cov}(\boldsymbol{\theta}_{ti}, \boldsymbol{\theta}_{tj}) = 0$. Furthermore, we assume that from time $t-1$ to t the standard deviation of the mean flow in an OD pair j is proportional to the realized mean OD flow $\theta_{t-1,j}$ at the previous time $t-1$, with proportionality constant given by the coefficient of variation κ. Thus, the evolution matrix at time t will be a diagonal matrix given by $\mathbf{W}_t = \kappa^2 \text{diag}(\boldsymbol{\theta}_{t-1}^2)$. We adopt $\kappa = 0.01$, i.e., from time $t-1$ to t the standard deviation of the mean OD flow in each OD pair is only 1% of the mean OD flow in the preceding time period. In this way, we simulate a smooth drift of the mean OD matrix over time.

For each time period t we assign the mean OD matrix $\boldsymbol{\theta}_t$ to the network in order to generate observed link volumes. We draw realized OD flows according to a multivariate normal distribution, so that $\boldsymbol{x}_t \sim \text{N}(\boldsymbol{\theta}_t, \boldsymbol{\Sigma}_{tx})$. We assume an independent Poisson-like variance structure, then $\boldsymbol{\Sigma}_{tx} = \text{diag}(\boldsymbol{\theta}_t)$. In the next step, we assign the realized OD flows to routes according to a multinomial distribution, i.e., for each OD pair j, $\boldsymbol{y}_{tj} \sim \text{MN}(\lceil x_{tj} \rceil, \boldsymbol{p}_{tj}(1-\pi))$, and $\boldsymbol{y}_t$ is the vector of realized route flows. We calculate the route choice probabilities by means of a multinomial logit model $\boldsymbol{p}_{tj} = \text{logit}(\tilde{\boldsymbol{u}}_{tj}; \xi)$, based on average route costs $\tilde{\boldsymbol{u}}_{tj}$. We assume a logit scale parameter $\xi = 5.0$, and a probability of not using any of the routes in the route choice set $\pi = 0.01$.

Finally, we assign route flows to links by means of the link-path incidence matrix, $\boldsymbol{z}_t = \boldsymbol{\Delta}\boldsymbol{y}_t + \boldsymbol{\nu}_t$, in which $\boldsymbol{\nu}_t \sim \text{N}(\mathbf{0}, \boldsymbol{\Sigma}_{tz})$. We assume negligible counting errors in observed volumes, so that $\boldsymbol{\Sigma}_{tz} = \mathbf{I}$ (the identity matrix). At each time step, we update the average route costs by means of the expression $\tilde{\boldsymbol{u}}_{t+1} = \alpha \boldsymbol{\Delta}^{\mathsf{T}} \boldsymbol{\tau}(\boldsymbol{z}_t) + (1-\alpha)\tilde{\boldsymbol{u}}_t$, with starting average route costs $\tilde{\boldsymbol{u}}_0$ based on free-flow times, and $\alpha = 0.05$ is the assumed weight users apply in averaging route costs. The starting mean OD matrix $\boldsymbol{\theta}_1$, which is the original OD matrix for the Sioux Falls network, is available in Annex A. We generate *sample paths* for T time periods, and discard the T' first time periods in order to mitigate the influence of starting conditions on future values. Below we summarize in a systematic way the steps just described, while in Table 4 we arrange the values adopted for the parameters used in the generation of the data.

Table 4 – Parameters in the generation of simulated data

Parameter	Value	Description
κ	0.01	Coefficient of variation of the mean OD flows
ξ	5.00	Scale parameter in multinomial logit model
π	0.01	Probability of none of the routes in a route choice set being chosen
α	0.05	Weight in averaging route costs
$\mathbf{W}_t$	$\kappa^2 \text{diag}(\boldsymbol{\theta}_{t-1}^2)$	Evolution matrix
$\boldsymbol{\Sigma}_{tx}$	$\text{diag}(\boldsymbol{\theta}_t)$	Covariance matrix of the realized OD flows
$\boldsymbol{\Sigma}_{tz}$	$\mathbf{I}$	Covariance of counting errors in traffic volumes
$\boldsymbol{\theta}_1$	See Annex A	Initial mean OD matrix

Source: the author

Procedure for the generation of the simulated data:

1. Draw a new mean OD flow vector $\boldsymbol{\theta}_t \sim \mathrm{N}(\boldsymbol{\theta}_{t-1}, \mathbf{W}_t)$ (equation (4.1));
2. draw OD flows $\boldsymbol{x}_t \sim \mathrm{N}(\boldsymbol{\theta}_t, \boldsymbol{\Sigma}_{tx})$;
3. compute route choice probabilities by using a logit route choice model based on current average route costs: $\boldsymbol{p}_{tj} = \mathrm{logit}(\tilde{\boldsymbol{u}}_{tj}, \xi)$;
4. for each OD pair j, draw route flows $\boldsymbol{y}_{tj} \sim \mathrm{MN}(\lceil x_{tj} \rceil, \boldsymbol{p}_{tj}(1-\pi))$ (notice that π is the probability of a user not choosing a route in the route choice set);
5. assign route flows to links by making $\boldsymbol{z}_t = \boldsymbol{\Delta} \boldsymbol{y}_t + \boldsymbol{\nu}_t$, in which $\boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_{tz})$;
6. update average route costs for next time period by means of the equation $\tilde{\boldsymbol{u}}_{t+1} = \alpha \boldsymbol{\Delta}^{\mathsf{T}} \boldsymbol{\tau}(\boldsymbol{z}_t) + (1-\alpha)\tilde{\boldsymbol{u}}_t$, in which $\boldsymbol{\tau}(.)$ is a vector-valued function which returns the travel time on links.

5.6 Experiment 1

In this first experiment, our objective is to assess if dynamic models can reduce the non-identifiability (underspecification problem) of the mean OD matrix. We test our proposed dynamic model described in Chapter 4 and the GLS model, proposed by Cascetta (1984) and described in Section 3.2.2. We used a "dynamic" version of the GLS, which consists simply in applying the GLS at each time period t assuming as prior matrix the matrix reconstructed at time period $t-1$. (See Section 5.3.1). We test both models at best conditions, so as to exclude estimation errors in parameters other than the mean OD flows and link volumes.

We consider one non-informative case and two informative cases. The cases are slightly different according to the model tested. For example, in the GLS model there is no prior probability distribution, only a point estimate of a prior OD matrix, while in the dynamic estimation model the prior information is specified by the mean and covariance matrix of the prior distribution. The non-informative case corresponds to a situation in which there is no prior knowledge on the OD flows or this knowledge is unreliable. This may happen, for example, when no prior OD matrix is available or it is outdated. For the GLS model, we assume a prior OD matrix $\boldsymbol{x}_0 = 100 \times \mathbf{1}$, and for the dynamic model we assume a prior mean OD flow vector $\boldsymbol{m}_0 = 100 \times \mathbf{1}$ and covariance matrix $\mathbf{C}_0 = 10^6 \times \mathbf{I}$.

In contrast, in the informative cases there is a reliable prior OD matrix, and we further consider two situations: a scaled-down case (informative 1), and an exact case (informative 2). In the scaled-down case, the prior OD matrix has the same pattern but its magnitude is 75% of the actual matrix at time $t = 1$; while in exact case, the prior OD matrix is equal to the actual OD matrix at time $t = 1$. In the dynamic estimation model the prior covariance matrix is set to the identity, while in the GLS model in both

Table 5 – Controlled parameters (kept constant) for the dynamic and GLS models in the test cases

Model	Parameter	Values	Description
Common to both models	ξ	5.0	Logit scale parameter
	π	0.01	Probability of a trip occur in a route outside the route choice set
GLS	$\mathbf{\Sigma}_x$	$\mathbf{I}$	Covariance matrix of the OD flows
	$\mathbf{\Sigma}_z$	$\mathbf{I}$	Covariance matrix of the traffic volumes
	ρ	1.0	Weight of the distance to the prior OD matrix
Dynamic estimation	$\mathbf{\Sigma}_x$	$\text{diag}(\bar{\boldsymbol{m}}_t)$	Covariance matrix of OD flows
	$\mathbf{\Sigma}_z$	$\mathbf{I}$	Covariance matrix of the errors in counting traffic volumes
	κ	0.01	Coefficient of variation in the evolution matrix
	$\mathbf{W}$	$\kappa^2\text{diag}(\bar{\boldsymbol{m}}_t^2)$	Estimated evolution matrix

Source: the author

non-informative and informative cases the covariances of OD flows and traffic volumes are set to the identity, since they do not represent subjective uncertainty as in Bayesian models. Table 5 shows the controlled factors for the three tested models, while Table 6 summarizes the test cases. In all cases the assignment matrices are assumed known and equal to the simulated ones. The results for a simulation of $T = 350$ with $T' = 50$ first observations discarded are given in Table 7.

Table 6 – Cases studied in the computational experiment 1

Case	Model	Prior OD matrix	Prior cov. matrix
Non-inform.	GLS	$\boldsymbol{x}_0 = 100 \times \mathbf{1}$	Not applicable
	Dynamic estimation	$\boldsymbol{m}_0 = 100 \times \mathbf{1}$	$\mathbf{C}_0 = 10^6 \times \mathbf{I}$
Informative 1	GLS	$\boldsymbol{x}_0 = 0.75\boldsymbol{\theta}_1$	Not applicable
	Dynamic estimation	$\boldsymbol{m}_0 = 0.75\boldsymbol{\theta}_1$	$\mathbf{C}_0 = \mathbf{I}$
Informative 2	GLS	$\boldsymbol{x}_0 = \boldsymbol{\theta}_1$	Not applicable
	Dynamic estimation	$\boldsymbol{m}_0 = \boldsymbol{\theta}_1$	$\mathbf{C}_0 = \mathbf{I}$

Source: the author

As we can see from the RMAE of the OD flows in Table 7, the dynamic estimation model performed better in both informative cases, with 25% and 15% smaller error in informative cases 1 and 2, respectively. In the non-informative case, the performances of both models were equivalent. This suggests that, in non-informative situations, we may alternatively use the dynamic GLS or the dynamic estimation model, while in informative scenarios we may achieve smaller errors by using the dynamic estimation model. We show graphs of the evolution of OD flows over time for the informative case 2 in three selected OD pairs (See Figure 11): OD pair 10-16 (high flows, Figures 14 and 15); OD pair 1-10

Table 7 – Results for the application of the models to the Sioux Falls Network in the computational experiment 1

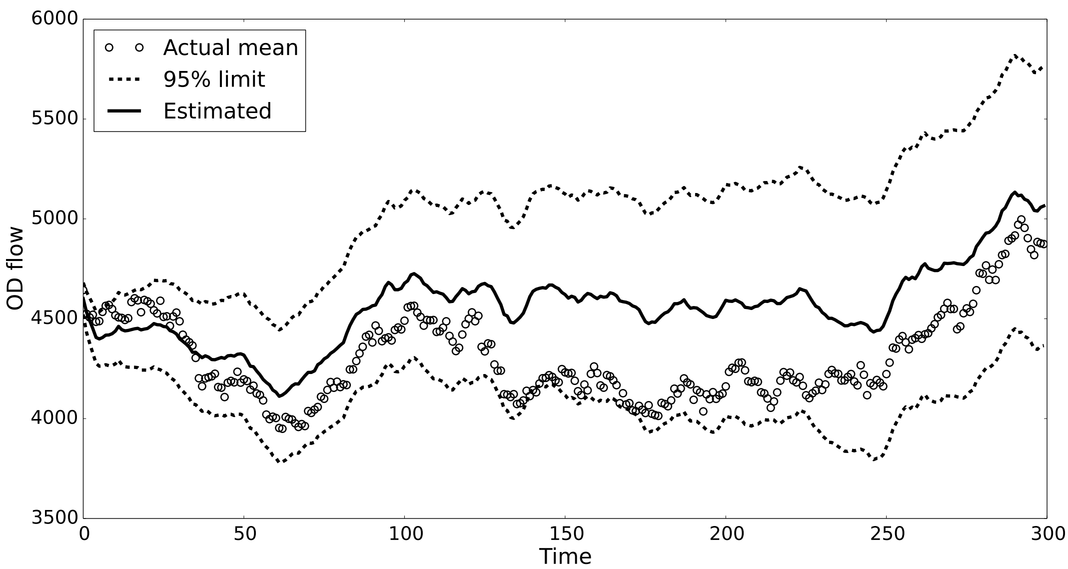
		Performance measures			
Case	Model	RRMSE_{OD}	RMAE_{OD}	$\text{RRMSE}_{\text{vols}}$	$\text{RMAE}_{\text{vols}}$
Non-informative	GLS	1.1016	0.7088	0.0113	0.0086
	Dynamic estimation	1.1075	0.7150	0.0116	0.0087
Informative 1	GLS	0.2945	0.1909	0.0113	0.0091
	Dynamic estimation	0.2134	0.1441	0.0121	0.0085
Informative 2	GLS	0.1754	0.1018	0.0115	0.0088
	Dynamic estimation	0.1377	0.0866	0.0113	0.0086

Source: the author

(medium flows, Figures 16 and 17), and OD pair 24-4 (low flows, Figures 18 and 19). All graphs for the dynamic model show the smoothed curve. We can see from these figures that the dynamic model was able to adapt to the variation of the mean OD flows, while the GLS did not follow the variation, remaining at an approximately constant level. However, the 95% limits for the dynamic model are wide and increasing, indicating high uncertainty in the estimated mean OD flows. In the non-informative case, both models are not able to track the mean OD flows, as shown in Figures 20 and 21.

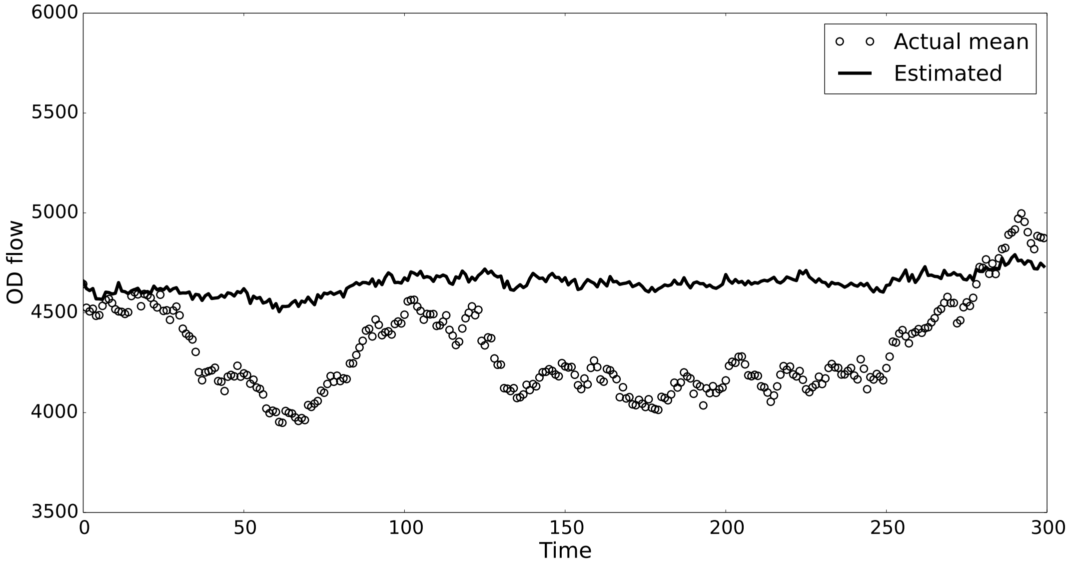
Regarding link volumes, Figures 22 and 23 show the estimation of link volumes in link 32 by the dynamic and GLS models, respectively. We can see that the dynamic model discarded part of the variability in volumes, estimating their means, while the GLS model tracked the observed volumes more closely, being sensitive to the variability.

Figure 14 – Estimation of flows in OD pair 10-16 (high flow) by the dynamic estimation model



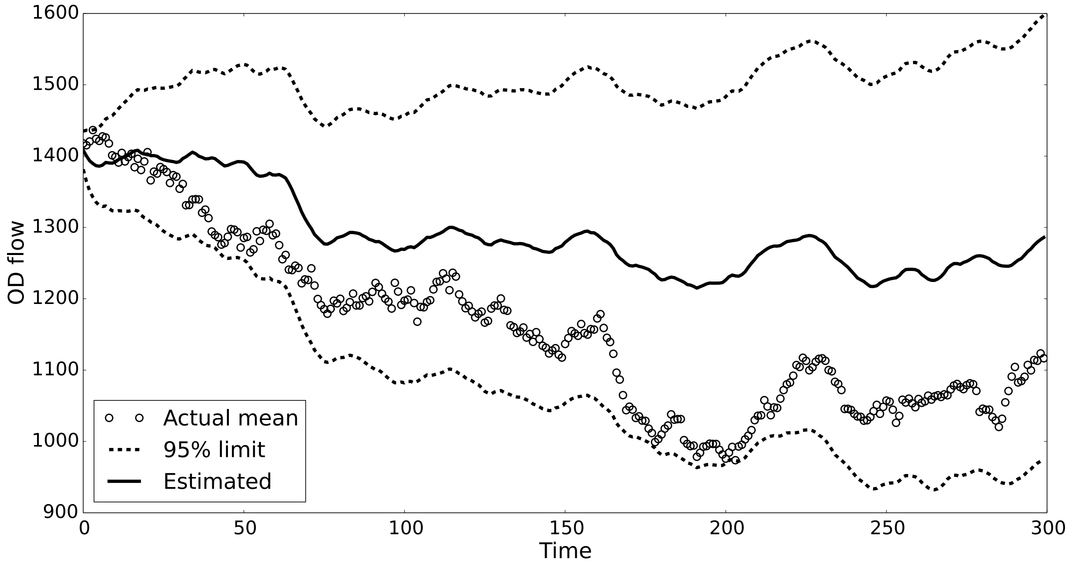
Source: the author

Figure 15 – Estimation of flows in OD pair 10-16 (high flow) by the GLS model



Source: the author

Figure 16 – Estimation of flows in OD pair 1-10 (medium flow) by the dynamic estimation model

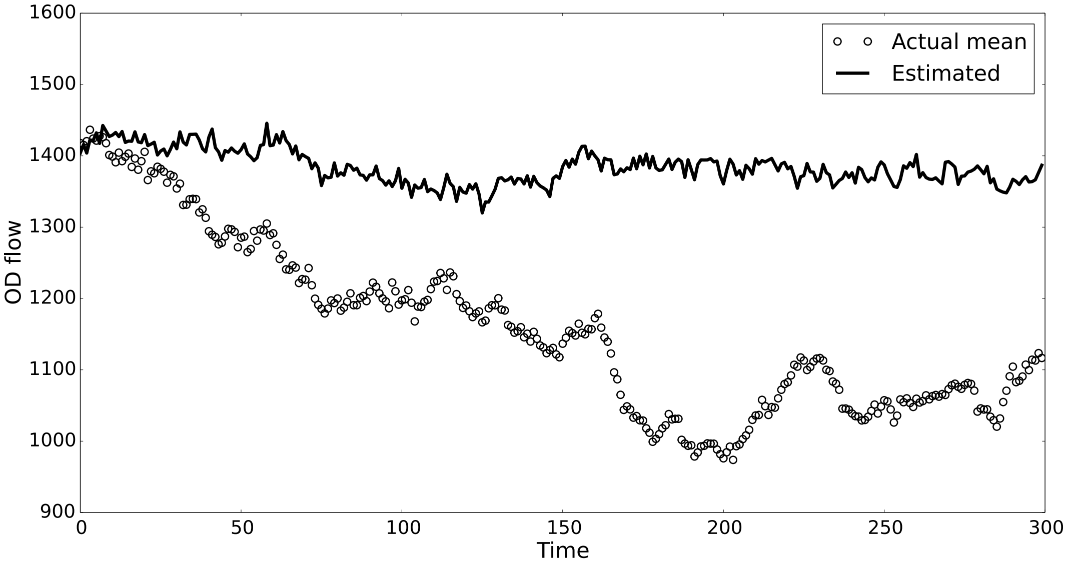


Source: the author

5.7 Experiment 2

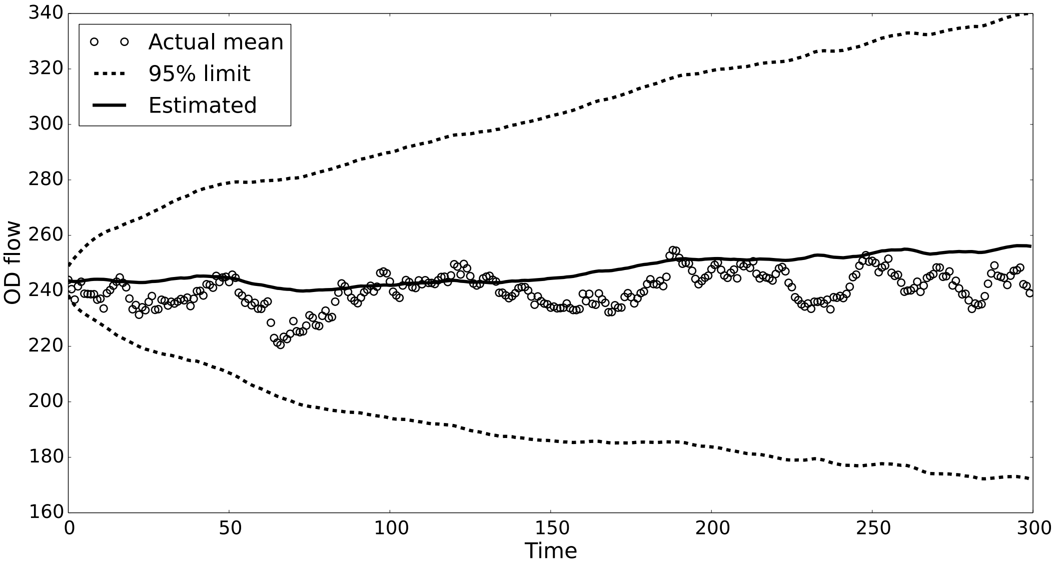
In this experiment, our objective is to compare the models in a realistic scenario. The practical situation we have in mind is one in which a practitioner want to estimate the mean OD matrix for a base-year from a sample of $T = 100$ days of observed volumes on links. In addition to the GLS and dynamic estimation models, we also test the static estimation model we developed in Section 4.4. Once again, we evaluate the models in three test cases: one non-informative and two informatives. The non-informative and informative 1 cases are identical to the ones treated in experiment 1, but the informative case 2 is different: the prior OD matrix is equal to the actual initial mean OD matrix

Figure 17 – Estimation of flows in OD pair 1-10 (medium flow) by the GLS model



Source: the author

Figure 18 – Estimation of flows in OD pair 24-4 (low flow) by the dynamic estimation model

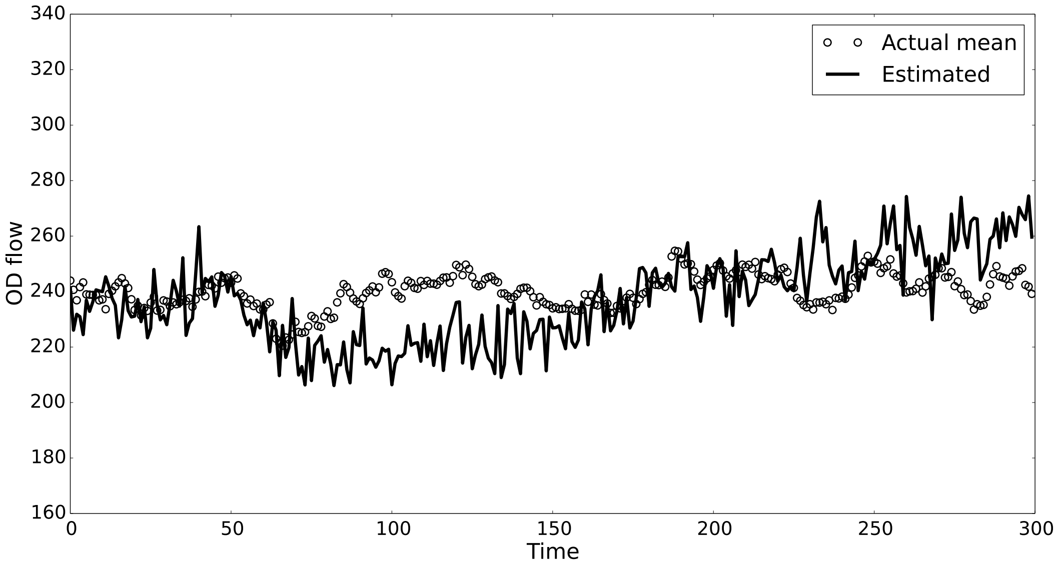


Source: the author

$\boldsymbol{\theta}_1$ at time $t = 1$ plus some random perturbation in the mean OD flows. The random perturbation is given by a gaussian zero-mean error with coefficient of variation equals to 10% of the actual mean flow in each OD pair. In both static and dynamic estimation models the prior covariances are set to the identity. Table 8 shows the controlled factors for the three tested models, while Table 9 summarizes the test cases.

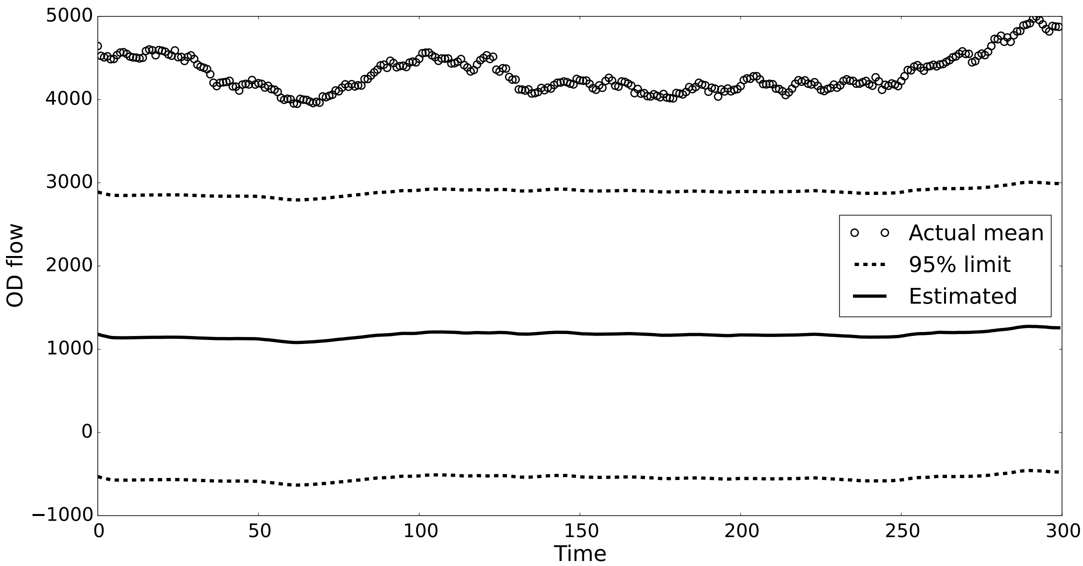
Table 10 exhibits the results of the computational experiments on the three models tested. We can make the following considerations on the results. First, the most noteworthy result is the good performance of the GLS model in estimating the OD flows relative to the static and dynamic estimation models. In the three tested cases, it outperformed the other

Figure 19 – Estimation of flows in OD pair 24-4 (low flow) by the GLS model



Source: the author

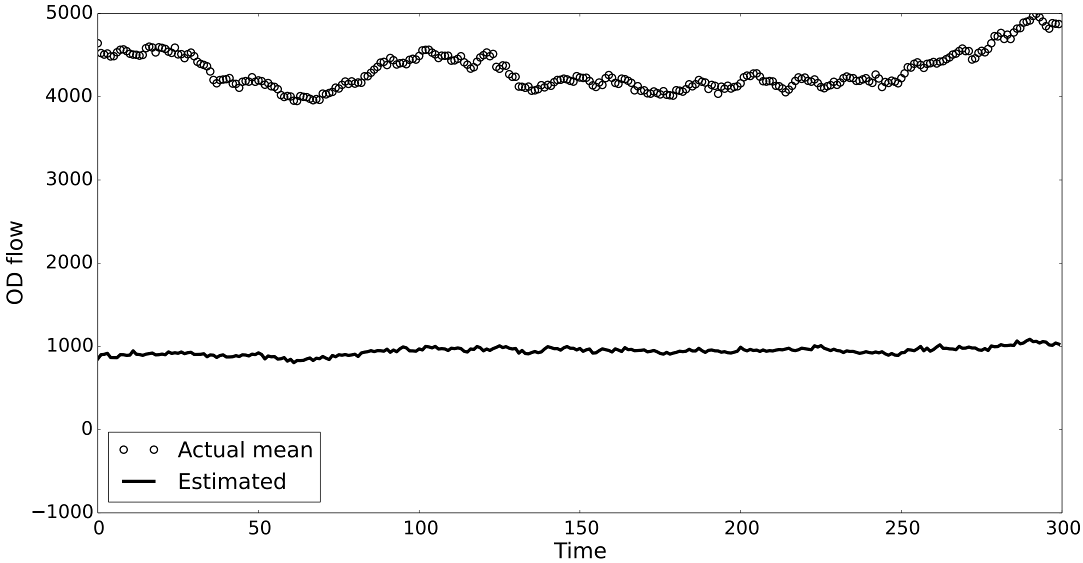
Figure 20 – Estimation of flows in OD pair 10-16 (high flow) by the dynamic model in the non-informative case



Source: the author

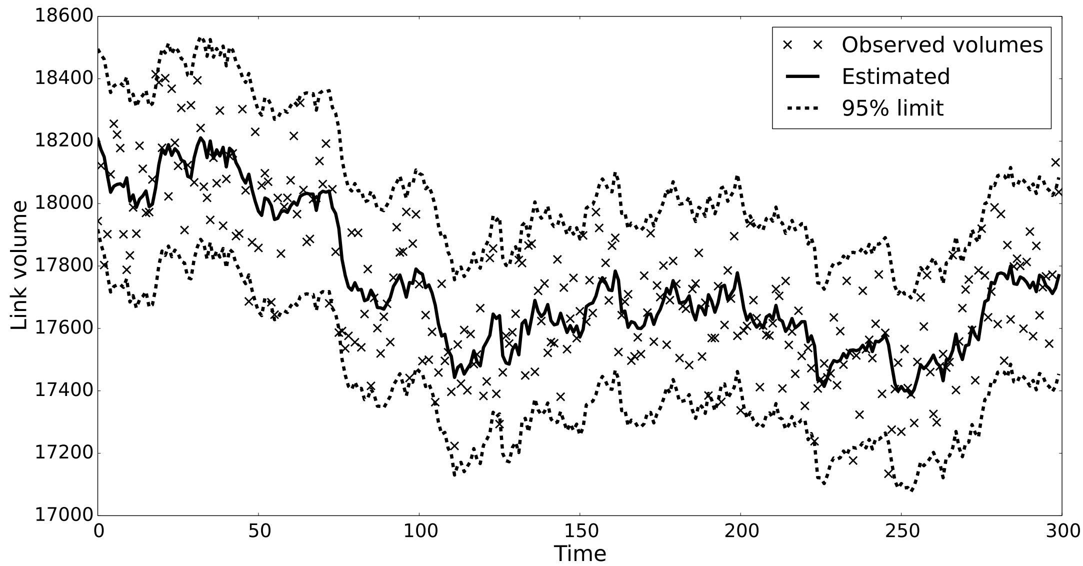
two models in both RRMSE and RMAE measures. This is surprising since we applied the GLS model assuming identity covariance matrices, i.e., we did not inform to the method the degree of uncertainty on the prior OD matrix or on the observed volumes. Although worse in all measures, the static estimation model was not much far from the GLS. The dynamic model showed a low performance, with both RRMSE and RMAE worse than the the GLS and the static estimation models in all cases. The difference in performance is more pronounced in the informative cases. On the other hand, in the non-informative case the relative performance of the dynamic model was seemingly only slightly worse. Our main hypothesis for this low performance of the dynamic model is that it has more

Figure 21 – Estimation of flows in OD pair 10-16 (high flow) by the GLS model in the non-informative case



Source: the author

Figure 22 – Estimation of link volumes in link 32 by the dynamic model

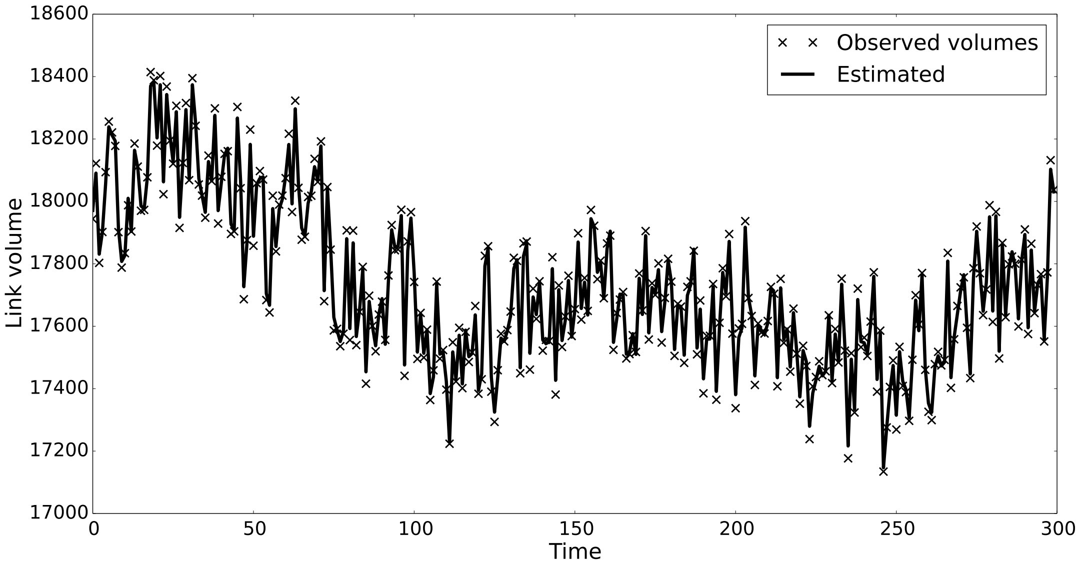


Source: the author

parameters to tune, and the setting of good values for the parameters may demand a fair amount of trial-and-error.

With relation to the traffic volumes, in most cases the performances of the models were very good, with relative errors around 1%. One noteworthy exception, though, is the performance of the static model in informative case 1, with relative errors between 20% and 25%. Our main hypothesis for the cause of this adverse result has to do with prior information. In the static model, we specify prior information by setting the mean and covariance matrix of the prior distribution. According to Table 9, in informative case 1 we set the covariance matrix to the identity matrix, putting much confidence on the prior

Figure 23 – Estimation of link volumes in link 32 by the GLS model



Source: the author

mean. The static estimation model then tried to be near the prior mean, which does not correspond to the observed volumes. We repeated the experiment assuming larger prior variances, thus less confidence on the prior mean, and the relative errors with relation to the traffic volumes were small, though there were no improvement with relation to the estimation of OD flows. It is should point out that, notwithstanding the fact the we also used small prior variances, the dynamic model did not exhibit this problem. The reason is that the prior distribution in the dynamic model refers only to the initial mean OD flows, and not the mean OD flows over the whole time horizon as it is the case for the static estimation model.

Finally, we should mention that, in consonance with experiments reported in the literature, estimation errors reduced significantly when more prior information was provided. In particular, the RMAE of the OD flows was only 10% for the GLS and static estimation models in informative case 2. The results for the other two cases show a substantial decrease in performance down to RMAE of 70% in the non-informative case. This is in agreement with the literature that the lack of identifiability in the OD matrix estimation problem from traffic volumes, which is the reason why only in highly informative scenarios we are able to estimate the OD matrix with low errors, is one of the main hurdles to effective application of the models.

5.8 Experiment 3

In this experiment, we assess the effect of the assignment matrix on the estimation error. We tested the GLS and dynamic models on three test cases, according to the nature of the assignment matrix: estimated, free flow (uncongested), and exact. The static model is

Table 8 – Controlled parameters (kept constant) for the models in the test cases in experiment 2

Model	Parameter	Values	Description
Common to all models	ξ	5.0	Logit scale parameter
	π	0.01	Probability of a trip occur in a route outside the route choice set
GLS	$\mathbf{\Sigma}_x$	$\mathbf{I}$	Covariance matrix of OD flows
	$\mathbf{\Sigma}_z$	$\mathbf{I}$	Covariance matrix of traffic volumes
	ρ	1.0	Weight of the distance to the prior OD matrix
Static estimation	$\mathbf{\Sigma}_x$	$\text{diag}(\bar{\boldsymbol{m}}_t)$	Covariance matrix of OD flows
	$\mathbf{\Sigma}_z$	$\mathbf{I}$	Covariance matrix of the errors in counting traffic volumes
Dynamic estimation	$\mathbf{\Sigma}_x$	$\text{diag}(\bar{\boldsymbol{m}}_t)$	Covariance matrix of OD flows
	$\mathbf{\Sigma}_z$	$\mathbf{I}$	Covariance matrix of the errors in counting traffic volumes
	κ	0.1	Coefficient of variation in the evolution matrix (10 times greater than the simulated one)
	$\mathbf{W}$	$\kappa^2\text{diag}(\bar{\boldsymbol{m}}_t^2)$	Estimated evolution matrix
	α	0.01	Weight in the estimation of the route costs (smaller than the actual one)

Source: the author

Table 9 – Cases studied in the computational experiment 2

Case	Model	Prior OD matrix	Prior cov. matrix
Non-inform.	GLS	$\boldsymbol{x}_0 = 100 \times \mathbf{1}$	Not applicable
	Static estimation	$\boldsymbol{m}_0 = 100 \times \mathbf{1}$	$\mathbf{C}_0 = 10^6 \times \mathbf{I}$
	Dynamic estimation	$\boldsymbol{m}_0 = 100 \times \mathbf{1}$	$\mathbf{C}_0 = 10^6 \times \mathbf{I}$
Informative 1	GLS	$\boldsymbol{x}_0 = 0.75\boldsymbol{\theta}_1$	Not applicable
	Static estimation	$\boldsymbol{m}_0 = 0.75\boldsymbol{\theta}_1$	$\mathbf{C}_0 = \mathbf{I}$
	Dynamic estimation	$\boldsymbol{m}_0 = 0.75\boldsymbol{\theta}_1$	$\mathbf{C}_0 = \mathbf{I}$
Informative 2	GLS	$\boldsymbol{x}_0 = \boldsymbol{\theta}_1 + \boldsymbol{\epsilon}$	Not applicable
	Static estimation	$\boldsymbol{m}_0 = \boldsymbol{\theta}_1 + \boldsymbol{\epsilon}$	$\mathbf{C}_0 = \mathbf{I}$
	Dynamic estimation	$\boldsymbol{m}_0 = \boldsymbol{\theta}_1 + \boldsymbol{\epsilon}$	$\mathbf{C}_0 = \mathbf{I}$

Source: the author

not tested since it does not consider different assignment matrices at each time step.

In the first case, the assignment matrix is estimated at each time step, and the estimation procedure is dependent on the model: in the dynamic GLS, the assignment matrix corresponds to an equilibrium state at each time step obtained by means of equation (3.8), while in the dynamic estimation model it is estimated by means of equations (4.23) and (4.24). In the free flow case, we used a constant assignment matrix for the whole time

Table 10 – Results for the application of the models to the Sioux Falls Network in the computational experiment 2

		Performance measures			
Case	Model	$RRMSE_{OD}$	$RMAE_{OD}$	$RRMSE_{vols}$	$RMAE_{vols}$
Non-informative	GLS	1.0936	0.6996	0.0114	0.0088
	Static Estimation	1.1202	0.7309	0.0163	0.0118
	Dynamic Estimation	1.1856	0.8152	0.0116	0.0089
Informative 1	GLS	0.2905	0.1881	0.0114	0.0088
	Static Estimation	0.3596	0.2211	0.2454	0.2171
	Dynamic Estimation	0.8253	0.5503	0.0116	0.0090
Informative 2	GLS	0.1807	0.1089	0.0114	0.0088
	Static Estimation	0.1853	0.0940	0.0516	0.0363
	Dynamic Estimation	0.6633	0.4578	0.0116	0.0089

Source: the author

horizon based on a uncongested stochastic traffic assignment with a logit route choice model (equation (2.10)). Finally, in the exact case both GLS and dynamic estimates are obtained with the exact assignment matrix at each time step. All other factors are kept constant at the levels given in informative case 2 from Table 9.

Table 11 – Effect of the assignment matrix on the estimation errors in experiment 3

		Performance measures			
Case	Model	$RRMSE_{OD}$	$RMAE_{OD}$	$RRMSE_{vols}$	$RMAE_{vols}$
Estimated	GLS	0.1807	0.1089	0.0114	0.0088
	Dynamic Estimation	0.6633	0.4578	0.0116	0.0089
Free flow	GLS	2.3409	1.3785	0.0615	0.0345
	Dynamic Estimation	3.4790	1.5969	0.2698	0.2277
Exact	GLS	0.1754	0.1018	0.0114	0.0088
	Dynamic Estimation	0.1377	0.0866	0.0114	0.0088

Source: the author

The corresponding results are given in Table 11, in which we can see that the quality of the estimated assignment matrix has a high detrimental impact on the estimation of OD flows. In particular, the performance measures show a high deterioration in estimation error when we used a constant assignment matrix corresponding to an uncongested condition for both the GLS and dynamic models. This gives us evidence that the premise of uncongested network adopted by some models revised in Chapter 3 may result in really bad estimates in practice. In the case we used an estimated assignment matrix, the GLS had a good performance, unlike the dynamic model. Our main hypothesis to this result lies in the fact that the GLS model is based on the equilibrium hypothesis, while the dynamic model does not. Perhaps assuming equilibrium may give good estimates in practice even though the network is not fully equilibrated. Finally, in the case we used an exact assignment matrix, both models gave low errors, with a marginal advantage to the dynamic model.

5.9 Comments on the main results

The first question we addressed in the experiments concerned whether dynamic models could reduce the non-identifiability of mean OD matrices by incorporating the variability of the link volumes over time. As can be seen in Figures 14, 16 and 18, the dynamic model was able to reasonably track the variation of the mean OD matrix if we provide it with highly informative prior information. Nonetheless, the 95% credible limits are increasing, which is a sign that the information in link volumes are not contributing to decreasing the uncertainty on OD flows. Moreover, as shown for a selected OD pair in Figure 20, in the non-informative case the dynamic model was not able of estimating the true level of the mean OD flows.

With regards to the evaluation if dynamic models can provide estimates of the OD matrix with lower errors than static models, according to the results of experiment 2, given in Table 10, the dynamic GLS model produced better estimates than the static model in both the non-informative and informative 1 cases, with a comparable error in informative case 2, while the estimates obtained by the dynamic estimation model had high errors. We should emphasize that this experiment tried to simulate practical conditions, and part of the error in the case of the dynamic estimation model may be due to the fact that it has more parameters to tune. The setting of good values for the parameters may demand a fair amount of trial-and-error.

In addition, the static estimation model seems to be more sensitive to the prior distribution, as can be seen in the high errors in estimated volumes in informative case 1. In this case we used an informative prior mean whose magnitude was 75% of the actual initial mean and a small covariance matrix, which tilted the model towards fitting the estimates to the prior mean and neglecting part of the information on link volumes. As the static model assumes that the mean OD matrix is constant, it assigns all the variability to the uncertainty in the prior information or in the link volumes, not allowing for part of the variation to be attributed to the temporal changes in mean OD flows. We think that the results will be more favorable to dynamic models in environments in which the trip pattern changes faster, while if the pattern is almost constant there should be no marginal gain in using a dynamic model.

Considering prior information, the results in all experiments and models indicated that the prior information is a key factor in obtaining good estimates. For example, in the non-informative case in experiment 1, in which the models are tested at best conditions, the relative mean absolute error (RMAE) of the OD flows were around 70%. This magnitude of error may be unacceptable in practice, so that the availability of useful prior information is central for the applicability of any reconstruction or estimation model. This also means that the naive application of these models in “black box” computational packages, without the proper understanding of their limitations, may bring about severe errors in decision making by planners.

Finally, we should also point out the importance of obtaining good estimates of assignment matrices. As indicated by the results of experiment 3, in which we tested the dynamic estimation and GLS models with estimated, free flow and exact assignment matrices, the use of a poorly estimated assignment matrix may greatly increase estimation errors. For example, in Table 11 we can see that the estimation errors measured by the RMAE in the case of the free flow assignment matrix are very high. This suggests that using a model which is based on an assumption of uncongested network may yield low quality estimates in a congested environment. This highlights the importance of putting effort in more precise methods of estimation of the assignment matrix, such as more realistic route choice models or the use of traffic flow simulation software. Surprisingly, the GLS model with assignment matrices estimated by an equilibrium assignment had a good performance, which may suggest that the equilibrium assumption may render good results even if the real network under analysis is not in equilibrium.

6 CONCLUSION

This doctoral thesis started from the hypothesis that day-to-day dynamic models could provide better estimates of OD matrices and be more responsive to changes than static models, since they should be able to account for the evolution of transportation systems over time and make use of the information provided by temporal changes. With this in mind, its general objective was to assess the potential gains and limitations of day-to-day dynamic Bayesian statistical models for the estimation of the OD matrix based on link volumes. We developed a dynamic linear model which considers the variation of OD flows and link volumes over time. In particular, our model is more general than previous static estimation models proposed by Vardi (1996) and Hazelton (2000), and than the dynamic model of Hazelton (2008). Its main characteristics are the following:

- it assumes Gaussian OD flows, allowing more flexible covariance structures;
- it allows the updating of the estimate of the OD matrix *online* as soon as a new vector of link volumes is available;
- it may be applied to congested networks, since we proposed a method to estimate online the route choice probabilities and the assignment matrix, and
- it may be used to forecast future OD flows and link volumes.

In the following paragraphs, we give tentative answers to the research questions which drove our research and make suggestions for further developments.

The first question was whether dynamic models were capable of reducing the non-identifiability of mean OD matrices by incorporating the variation of the link volumes over time. Our experiments did not provide evidence that this is the case. In contrast to our expectation, the availability of samples of volumes on links over the days did not reduce the uncertainty in OD flows. This is flagrant in the results of the application of the dynamic estimation model, in which the variances of the OD flows kept increasing over time. Moreover, in non-informative cases there was no sign of convergence of estimates to the actual mean OD flows with the observation of link volumes over time.

The second question was if dynamic estimation models could produce better estimates of mean OD matrices than static estimation models. In our experiments, dynamic models gave better estimates than static models in informative cases. Our main hypothesis for this result is that dynamic models take into account the variability in link volumes originated in the variability of the mean OD flows, while static models do not consider this possibility. However, the performances were similar in non-informative cases.

The third question was related to the impact of prior information, and as expected, the experiments showed a high deterioration in estimation errors when no prior information was used. This draws attention to the fact that an analyst must be aware of this limitation of the models, in particular when using software packages with "black box" models, which may lead he or she to trust in estimates which may be very far from the reality.

One more critical aspect of the problem is the importance of obtaining good estimates of the assignment matrices, corresponding to our fourth research question. In particular, our experiments showed that using a constant assignment matrix based on free flows, which is suitable in uncongested networks, produced high estimation errors in OD flows in congested networks. As assignment matrices have been estimated through traffic assignment models, this suggests that special care should be taken in choosing and calibrating these models.

Finally, we think that the key for developing more accurate models is finding strategies to overcome the underspecification/identifiability problem. We envisage two promising directions: the development of more parsimonious models and the incorporation of other sources of information in addition to the observed link volumes.

The principle of parsimony states that "among theories fitting the data equally well, researchers should choose the simplest theory". (GAUCH, 2003, p.269). In statistics, this means that among models which fit the data, we should choose the one with least parameters. In the models for the estimation of OD matrix, the data are the observed link volumes and the parameters are the OD mean flows, covariances, logit scale and others. In most of the cases tested in our experiments, the models were able to fit the observed volumes, i.e., they were able to explain the data. In this context, the underspecification/identifiability problem in fact means that our models may be over-parameterized and that we should seek to develop more parsimonious models.

We see as a promising direction to extend our dynamic linear model in a similar way as proposed by Hazelton (2008), who parameterized the OD flows in terms of a set of static parameters. We may define the dynamic OD flows as functions of seasonal effects and linear trends, regarded as time-invariant in a defined time horizon. Another direction is the one followed by Marzano, Papola and Simonelli (2009), who proposed a parsimonious model in which the total trips generated in each origin are assumed to vary in the short term, while the trip distribution among destinations are assumed constant within the time horizon.

Regarding other sources of information, a research direction which is worth investigating is the incorporation of data on land use and on the activity system. As land use supposedly changes at a lower rate over time than OD flows, this may allow the estimation of long term correlations among OD flows which may be considered constant over shorter time horizons. These correlations can potentially reduce the non-identifiability of mean OD flows, since the covariances are taken into account when updating the posterior estimates. Moreover, with the development of new technologies which allow the tracking of vehicles over space and time, one can acquire data on origins and destinations, measure travel times and other variables which may be used to calibrate route choice models and other relevant parameters.

Appendix

APPENDIX A – THE MULTIVARIATE NORMAL PROBABILITY DENSITY

Given a vector of random variables $\boldsymbol{x} \in \mathbb{R}^n$, we say that it follows a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ if its density function has the following form:

$$f(\boldsymbol{x}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \tag{A.1}$$

The covariance matrix $\boldsymbol{\Sigma}$ must be positive-definite in order that equation (A.1) is a proper multivariate density, otherwise it will be degenerate.

Let $\boldsymbol{x} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the conditional distribution of a random vector $\boldsymbol{y}$ given $\boldsymbol{x}$ be $\mathrm{N}(\mathbf{A}\boldsymbol{x} + \boldsymbol{b}, \mathbf{C})$. Then the joint distribution of $(\boldsymbol{x}, \boldsymbol{y})$ is given by (SÄRKKÄ, 2013):

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{A}\boldsymbol{\mu} + \boldsymbol{b} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{A}^{\mathsf{T}} \\ \mathbf{A}\boldsymbol{\Sigma} & \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathsf{T}} + \mathbf{C} \end{bmatrix}\right) \tag{A.2}$$

The marginal distribution of $\boldsymbol{y}$, from the joint distribution in (A.2) is:

$$\boldsymbol{y} \sim \mathrm{N}(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathsf{T}} + \mathbf{C}) \tag{A.3}$$

Let $\boldsymbol{\nu} = \mathbf{A}\boldsymbol{\mu} + \boldsymbol{b}$ and $\boldsymbol{\Phi} = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathsf{T}} + \mathbf{C}$. The conditional distribution of $\boldsymbol{x}$ given $\boldsymbol{y}$ is also multivariate normal:

$$\boldsymbol{x}|\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{\mu} + \mathbf{R}(\boldsymbol{y} - \boldsymbol{\nu}), \boldsymbol{\Sigma} - \mathbf{R}\boldsymbol{\Phi}\mathbf{R}^{\mathsf{T}}) \tag{A.4}$$

Where $\mathbf{R} = \boldsymbol{\Sigma}\mathbf{A}^{\mathsf{T}}\boldsymbol{\Phi}^{-1}$ is the *regression matrix* of $\boldsymbol{x}$ on $\boldsymbol{y}$.

APPENDIX B – BAYESIAN INFERENCE

Bayesian Statistics has a different approach to parameter estimation. Unlike *frequentist* statistics, unknown parameters are not treated as constants, but as random variables for which two probability distributions are defined: a *prior* and a *posterior* distribution. The prior distribution synthesizes the *degree of belief* in different values for the parameter *before* any empirical data is collected, and the posterior distribution is an update of the prior after the empirical data is considered (GELMAN et al., 2003). The update rule is given by Bayes Theorem:

$$f(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{f(\boldsymbol{y}|\boldsymbol{\theta})f(\boldsymbol{\theta})}{f(\boldsymbol{y})} \tag{B.1}$$

In which $\boldsymbol{\theta}$ is the parameter to be estimated and $\boldsymbol{y}$ is the empirical data. $f(\boldsymbol{\theta})$ is the prior distribution of the parameter, $f(y)$ is the marginal distribution of the data, $f(\boldsymbol{y}|\boldsymbol{\theta})$ is the likelihood function and $f(\theta|y)$ is the posterior distribution. An estimator for the parameter of interest may be defined as the mean or mode (also called a *posterior maximum*) of the posterior distribution.

For *conjugate* distributions, it is possible to obtain analytically the posterior $f(\boldsymbol{\theta}|\boldsymbol{y})$. For example, if we specify a gamma distribution as a prior and the likelihood as Poisson, then the posterior will also be from the gamma family and we can calculate the posterior parameters analytically. However, in complex models it is in general difficult to specify conjugate distributions, so that we must resort to Monte Carlo simulation (in particular, Markov chain Monte Carlo) in order to estimate moments of the posterior distribution.

APPENDIX C – DYNAMIC LINEAR MODELS

A general DLM is defined by a *system evolution* model and a *measurement* model (WEST; HARRISON, 1997). Let $\boldsymbol{\theta}_t$ be a vector of parameters in time t. What we call by *parameter* is any unobserved quantity, i.e., any variable that is important to our analysis but we cannot observe directly. The parameters are related in some way to quantities we can observe, which are represented by the vector $\boldsymbol{z}_t$. The system evolution model describes how the vector of parameters evolves over time, according to equation (C.1):

$$\boldsymbol{\theta}_t = \mathbf{G}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t \tag{C.1}$$

In (C.1), $\mathbf{G}_t$ is a matrix which describes the *deterministic* evolution of the parameters, while $\boldsymbol{\omega}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{W}_t)$ is a gaussian random error term which describes their *stochastic* evolution. $\mathbf{W}_t$ is a covariance matrix.

The measurement model, which relates the vector of parameters $\boldsymbol{\theta}_t$ to the vector of observed quantities $\boldsymbol{z}_t$, is given by:

$$\boldsymbol{z}_t = \mathbf{F}_t\boldsymbol{\theta}_t + \boldsymbol{\nu}_t \tag{C.2}$$

In (C.2), $\mathbf{F}_t$ is the *regression* matrix at time t and $\boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{V}_t)$ is a gaussian random error term which represents the observation error of $\boldsymbol{z}_t$ relative to its expected value $\mathbf{F}_t\boldsymbol{\theta}_t$.

In order to complete the model, we must also specify a prior probability density function for the initial state $\boldsymbol{\theta}_0$ at time $t = 0$, given by $\mathrm{p}(\boldsymbol{\theta}_0|D_0) = \mathrm{N}(\boldsymbol{m}_0, \mathbf{C}_0)$, in which $\boldsymbol{m}_0$ and $\mathbf{C}_0$ are, respectively, the prior mean and covariance matrix, and D_0 is the set of prior data.

DLMs are Markovian models, in the sense that the current and future states of the parameter vector $\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_{t+2}\dots$ depend only on the previous state $\boldsymbol{\theta}_{t-1}$, and not on the past states $\boldsymbol{\theta}_{t-2}, \boldsymbol{\theta}_{t-3}, \dots$. This is often represented by the following conditional independence condition:

$$\mathrm{p}(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}, \boldsymbol{\theta}_{t-1}, \dots, \boldsymbol{\theta}_0) = \mathrm{p}(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) \tag{C.3}$$

In addition, $\boldsymbol{z}_t$ is also conditionally independent of past states $\boldsymbol{\theta}_{t-1}, \boldsymbol{\theta}_{t-2}, \dots$ and past observations $\boldsymbol{z}_{t-1}, \boldsymbol{z}_{t-2}, \dots$, given the current state $\boldsymbol{\theta}_t$:

$$\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1}, \dots, \boldsymbol{\theta}_0, \boldsymbol{z}_{t-1}, \boldsymbol{z}_{t-2}, \dots, \boldsymbol{z}_1) = \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t) \tag{C.4}$$

In our case, we are interested in *gaussian* DLMs, so that $\mathrm{p}(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) = \mathrm{N}(\mathbf{G}_t\boldsymbol{\theta}_{t-1}, \mathbf{W}_t)$ and $\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t) = \mathrm{N}(\mathbf{F}_t\boldsymbol{\theta}_t, \mathbf{V}_t)$.

The application of DLMs to time series occurs by means of recurrence equations, given in the following section. Given a times series of observed vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n$, the estimation of the sequence of parameter vectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_n$ occurs by repeated application of Bayes updating. At time $t-1$, let $\mathrm{p}(\boldsymbol{\theta}_{t-1}|D_{t-1})$ be the posterior distribution of the parameter vector $\boldsymbol{\theta}_{t-1}$ given the data set $D_{t-1} = \{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{t-1}\}$. The prior distribution of the parameter vector next time period t is given by:

$$
\begin{aligned}
\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1}) &= \int \mathrm{p}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1}|D_{t-1})\mathrm{d}\boldsymbol{\theta}_{t-1} \\
&= \int \mathrm{p}(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})\mathrm{p}(\boldsymbol{\theta}_{t-1}|D_{t-1})\mathrm{d}\boldsymbol{\theta}_{t-1} \qquad \text{(C.5)}
\end{aligned}
$$

In equation (C.5), $\mathrm{p}(\boldsymbol{\theta}_{t-1}|D_{t-1}) = \mathrm{N}(\boldsymbol{m}_{t-1}, \mathbf{C}_{t-1})$ and $\mathrm{p}(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) = \mathrm{N}(\mathbf{G}_t\boldsymbol{\theta}_{t-1}, \mathbf{W}_t)$, so that, from equation (A.3) (Appendix A), $\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1}) = \mathrm{N}(\bar{\boldsymbol{m}}_t, \bar{\mathbf{C}}_t)$ with $\bar{\boldsymbol{m}}_t = \mathbf{G}_t\boldsymbol{m}_{t-1}$ and $\bar{\mathbf{C}}_t = \mathbf{G}_t\mathbf{C}_{t-1}\mathbf{G}_t^{\mathsf{T}} + \mathbf{W}_t$.

Let $\mathrm{p}(\boldsymbol{z}_t|D_{t-1})$ be the *one-step forecasting* distribution of the observation vector, given by:

$$
\begin{aligned}
\mathrm{p}(\boldsymbol{z}_t|D_{t-1}) &= \int \mathrm{p}(\boldsymbol{z}_t, \boldsymbol{\theta}_t|D_{t-1})\mathrm{d}\boldsymbol{\theta}_t \\
&= \int \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t)\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1})\mathrm{d}\boldsymbol{\theta}_t \qquad \text{(C.6)}
\end{aligned}
$$

In equation (C.6), $\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t) = \mathrm{N}(\mathbf{F}_t\boldsymbol{\theta}_t, \mathbf{V}_t)$ and $\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1}) = \mathrm{N}(\bar{\boldsymbol{m}}_t, \bar{\mathbf{C}}_t)$ given by equation (C.5). Once more, from equation (A.3) $\mathrm{p}(\boldsymbol{z}_t|D_{t-1}) = \mathrm{N}(\boldsymbol{f}_t, \mathbf{Q}_t)$ in which $\boldsymbol{f}_t = \mathbf{F}_t\bar{\boldsymbol{m}}_t$ and $\mathbf{Q}_t = \mathbf{F}_t\bar{\mathbf{C}}_t\mathbf{F}_t^{\mathsf{T}} + \mathbf{V}_t$.

Finally, by Bayes' theorem, the posterior distribution of $\boldsymbol{\theta}_t$, given the observed vector $\boldsymbol{z}_t$ and the prior data set D_{t-1} is the following:

$$
\mathrm{p}(\boldsymbol{\theta}_t|D_t) = \frac{\mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t, D_{t-1})\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1})}{\int \mathrm{p}(\boldsymbol{z}_t|\boldsymbol{\theta}_t, D_{t-1})\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1})\mathrm{d}\boldsymbol{\theta}_t} \qquad \text{(C.7)}
$$

In (C.7), $D_t = D_{t-1} \cup \{\boldsymbol{z}_t\}$. From equation (A.4), then $\mathrm{p}(\boldsymbol{\theta}_t|D_t) = \mathrm{N}(\boldsymbol{m}_t, \mathbf{C}_t)$, so that:

$$
\boldsymbol{m}_t = \bar{\boldsymbol{m}}_t + \mathbf{A}_t\boldsymbol{e}_t \qquad \text{(C.8)}
$$

$$
\mathbf{C}_t = \bar{\mathbf{C}}_t - \mathbf{A}_t\mathbf{Q}_t\mathbf{A}_t^{\mathsf{T}} \qquad \text{(C.9)}
$$

In which $\boldsymbol{e}_t = \boldsymbol{z}_t - \boldsymbol{f}_t$ is the forecasting error and $\mathbf{A}_t = \bar{\mathbf{C}}_t\mathbf{F}_t^{\mathsf{T}}\mathbf{Q}_t^{-1}$ is an *adjustment matrix* (or alternatively, the regression matrix of $\boldsymbol{\theta}_t$ on $\boldsymbol{z}_t$).

In summary, given a prior data set D_{t-1}, the posterior distribution $\boldsymbol{\theta}_{\mathrm{t}-1}|\mathrm{D}_{\mathrm{t}-1}$ and the recently observed vector $\boldsymbol{z}_t$, the parameters $\boldsymbol{m}_t$ and $\boldsymbol{C}_t$ of the posterior distribution $\mathrm{p}(\boldsymbol{\theta}_t|D_t)$ are calculated by the following succession of computations:

Compute the parameters of the prior $\mathrm{p}(\boldsymbol{\theta}_t|D_{t-1})$:

$$\begin{aligned}\bar{\boldsymbol{m}}_t &= \mathbf{G}_t\boldsymbol{m}_{t-1}\\ \bar{\mathbf{C}}_t &= \mathbf{G}_t\mathbf{C}_{t-1}\mathbf{G}_t^{\mathsf{T}} + \mathbf{W}_t\end{aligned}$$

Compute the parameters of the one-step forecasting distribution $\mathrm{p}(\boldsymbol{z}_t|D_{t-1})$:

$$\begin{aligned}\boldsymbol{f}_t &= \mathbf{F}_t\bar{\boldsymbol{m}}_t\\ \mathbf{Q}_t &= \mathbf{F}_t\bar{\mathbf{C}}_t\mathbf{F}_t^{\mathsf{T}} + \mathbf{V}_t\end{aligned}$$

Finally, compute the parameters of the posterior distribution $\mathrm{p}(\boldsymbol{\theta}_t|D_t)$:

$$\begin{aligned}\boldsymbol{e}_t &= \boldsymbol{z}_t - \boldsymbol{f}_t\\ \mathbf{A}_t &= \bar{\mathbf{C}}_t\mathbf{F}_t^{\mathsf{T}}\mathbf{Q}_t^{-1}\\ \boldsymbol{m}_t &= \bar{\boldsymbol{m}}_t + \mathbf{A}_t\boldsymbol{e}_t\\ \mathbf{C}_t &= \bar{\mathbf{C}}_t - \mathbf{A}_t\mathbf{Q}_t\mathbf{A}_t^{\mathsf{T}}\end{aligned}$$

Annex

ANNEX A – OD MATRIX OF SIOUX FALLS NETWORK

Table 12 – OD matrix of the Sioux Falls network

Origin 1				
1 : 0.0	2 : 100.0	3 : 100.0	4 : 500.0	5 : 200.0
6 : 300.0	7 : 500.0	8 : 800.0	9 : 500.0	10 : 1300.0
11 : 500.0	12 : 200.0	13 : 500.0	14 : 300.0	15 : 500.0
16 : 500.0	17 : 400.0	18 : 100.0	19 : 300.0	20 : 300.0
21 : 100.0	22 : 400.0	23 : 300.0	24 : 100.0	

Origin 2				
1 : 100.0	2 : 0.0	3 : 100.0	4 : 200.0	5 : 100.0
6 : 400.0	7 : 200.0	8 : 400.0	9 : 200.0	10 : 600.0
11 : 200.0	12 : 100.0	13 : 300.0	14 : 100.0	15 : 100.0
16 : 400.0	17 : 200.0	18 : 0.0	19 : 100.0	20 : 100.0
21 : 0.0	22 : 100.0	23 : 0.0	24 : 0.0	

Origin 3				
1 : 100.0	2 : 100.0	3 : 0.0	4 : 200.0	5 : 100.0
6 : 300.0	7 : 100.0	8 : 200.0	9 : 100.0	10 : 300.0
11 : 300.0	12 : 200.0	13 : 100.0	14 : 100.0	15 : 100.0
16 : 200.0	17 : 100.0	18 : 0.0	19 : 0.0	20 : 0.0
21 : 0.0	22 : 100.0	23 : 100.0	24 : 0.0	

Origin 4				
1 : 500.0	2 : 200.0	3 : 200.0	4 : 0.0	5 : 500.0
6 : 400.0	7 : 400.0	8 : 700.0	9 : 700.0	10 : 1200.0
11 : 1400.0	12 : 600.0	13 : 600.0	14 : 500.0	15 : 500.0
16 : 800.0	17 : 500.0	18 : 100.0	19 : 200.0	20 : 300.0
21 : 200.0	22 : 400.0	23 : 500.0	24 : 200.0	

Origin 5				
1 : 200.0	2 : 100.0	3 : 100.0	4 : 500.0	5 : 0.0
6 : 200.0	7 : 200.0	8 : 500.0	9 : 800.0	10 : 1000.0
11 : 500.0	12 : 200.0	13 : 200.0	14 : 100.0	15 : 200.0
16 : 500.0	17 : 200.0	18 : 0.0	19 : 100.0	20 : 100.0

21 : 100.0	22 : 200.0	23 : 100.0	24 : 0.0	

Origin 6				
1 : 300.0	2 : 400.0	3 : 300.0	4 : 400.0	5 : 200.0
6 : 0.0	7 : 400.0	8 : 800.0	9 : 400.0	10 : 800.0
11 : 400.0	12 : 200.0	13 : 200.0	14 : 100.0	15 : 200.0
16 : 900.0	17 : 500.0	18 : 100.0	19 : 200.0	20 : 300.0
21 : 100.0	22 : 200.0	23 : 100.0	24 : 100.0	

Origin 7				
1 : 500.0	2 : 200.0	3 : 100.0	4 : 400.0	5 : 200.0
6 : 400.0	7 : 0.0	8 : 1000.0	9 : 600.0	10 : 1900.0
11 : 500.0	12 : 700.0	13 : 400.0	14 : 200.0	15 : 500.0
16 : 1400.0	17 : 1000.0	18 : 200.0	19 : 400.0	20 : 500.0
21 : 200.0	22 : 500.0	23 : 200.0	24 : 100.0	

Origin 8				
1 : 800.0	2 : 400.0	3 : 200.0	4 : 700.0	5 : 500.0
6 : 800.0	7 : 1000.0	8 : 0.0	9 : 800.0	10 : 1600.0
11 : 800.0	12 : 600.0	13 : 600.0	14 : 400.0	15 : 600.0
16 : 2200.0	17 : 1400.0	18 : 300.0	19 : 700.0	20 : 900.0
21 : 400.0	22 : 500.0	23 : 300.0	24 : 200.0	

Origin 9				
1 : 500.0	2 : 200.0	3 : 100.0	4 : 700.0	5 : 800.0
6 : 400.0	7 : 600.0	8 : 800.0	9 : 0.0	10 : 2800.0
11 : 1400.0	12 : 600.0	13 : 600.0	14 : 600.0	15 : 900.0
16 : 1400.0	17 : 900.0	18 : 200.0	19 : 400.0	20 : 600.0
21 : 300.0	22 : 700.0	23 : 500.0	24 : 200.0	

Origin 10				
1 : 1300.0	2 : 600.0	3 : 300.0	4 : 1200.0	5 : 1000.0
6 : 800.0	7 : 1900.0	8 : 1600.0	9 : 2800.0	10 : 0.0
11 : 4000.0	12 : 2000.0	13 : 1900.0	14 : 2100.0	15 : 4000.0
16 : 4400.0	17 : 3900.0	18 : 700.0	19 : 1800.0	20 : 2500.0
21 : 1200.0	22 : 2600.0	23 : 1800.0	24 : 800.0	

Origin 11				
1 : 500.0	2 : 200.0	3 : 300.0	4 : 1500.0	5 : 500.0
6 : 400.0	7 : 500.0	8 : 800.0	9 : 1400.0	10 : 3900.0
11 : 0.0	12 : 1400.0	13 : 1000.0	14 : 1600.0	15 : 1400.0
16 : 1400.0	17 : 1000.0	18 : 100.0	19 : 400.0	20 : 600.0
21 : 400.0	22 : 1100.0	23 : 1300.0	24 : 600.0	

Origin 12				
1 : 200.0	2 : 100.0	3 : 200.0	4 : 600.0	5 : 200.0
6 : 200.0	7 : 700.0	8 : 600.0	9 : 600.0	10 : 2000.0
11 : 1400.0	12 : 0.0	13 : 1300.0	14 : 700.0	15 : 700.0
16 : 700.0	17 : 600.0	18 : 200.0	19 : 300.0	20 : 400.0
21 : 300.0	22 : 700.0	23 : 700.0	24 : 500.0	

Origin 13				
1 : 500.0	2 : 300.0	3 : 100.0	4 : 600.0	5 : 200.0
6 : 200.0	7 : 400.0	8 : 600.0	9 : 600.0	10 : 1900.0
11 : 1000.0	12 : 1300.0	13 : 0.0	14 : 600.0	15 : 700.0
16 : 600.0	17 : 500.0	18 : 100.0	19 : 300.0	20 : 600.0
21 : 600.0	22 : 1300.0	23 : 800.0	24 : 800.0	

Origin 14				
1 : 300.0	2 : 100.0	3 : 100.0	4 : 500.0	5 : 100.0
6 : 100.0	7 : 200.0	8 : 400.0	9 : 600.0	10 : 2100.0
11 : 1600.0	12 : 700.0	13 : 600.0	14 : 0.0	15 : 1300.0
16 : 700.0	17 : 700.0	18 : 100.0	19 : 300.0	20 : 500.0
21 : 400.0	22 : 1200.0	23 : 1100.0	24 : 400.0	

Origin 15				
1 : 500.0	2 : 100.0	3 : 100.0	4 : 500.0	5 : 200.0
6 : 200.0	7 : 500.0	8 : 600.0	9 : 1000.0	10 : 4000.0
11 : 1400.0	12 : 700.0	13 : 700.0	14 : 1300.0	15 : 0.0
16 : 1200.0	17 : 1500.0	18 : 200.0	19 : 800.0	20 : 1100.0
21 : 800.0	22 : 2600.0	23 : 1000.0	24 : 400.0	

Origin 16				
1 : 500.0	2 : 400.0	3 : 200.0	4 : 800.0	5 : 500.0
6 : 900.0	7 : 1400.0	8 : 2200.0	9 : 1400.0	10 : 4400.0
11 : 1400.0	12 : 700.0	13 : 600.0	14 : 700.0	15 : 1200.0
16 : 0.0	17 : 2800.0	18 : 500.0	19 : 1300.0	20 : 1600.0
21 : 600.0	22 : 1200.0	23 : 500.0	24 : 300.0	

Origin 17				
1 : 400.0	2 : 200.0	3 : 100.0	4 : 500.0	5 : 200.0
6 : 500.0	7 : 1000.0	8 : 1400.0	9 : 900.0	10 : 3900.0
11 : 1000.0	12 : 600.0	13 : 500.0	14 : 700.0	15 : 1500.0
16 : 2800.0	17 : 0.0	18 : 600.0	19 : 1700.0	20 : 1700.0
21 : 600.0	22 : 1700.0	23 : 600.0	24 : 300.0	

Origin 18				
1 : 100.0	2 : 0.0	3 : 0.0	4 : 100.0	5 : 0.0
6 : 100.0	7 : 200.0	8 : 300.0	9 : 200.0	10 : 700.0
11 : 200.0	12 : 200.0	13 : 100.0	14 : 100.0	15 : 200.0
16 : 500.0	17 : 600.0	18 : 0.0	19 : 300.0	20 : 400.0
21 : 100.0	22 : 300.0	23 : 100.0	24 : 0.0	

Origin 19				
1 : 300.0	2 : 100.0	3 : 0.0	4 : 200.0	5 : 100.0
6 : 200.0	7 : 400.0	8 : 700.0	9 : 400.0	10 : 1800.0
11 : 400.0	12 : 300.0	13 : 300.0	14 : 300.0	15 : 800.0
16 : 1300.0	17 : 1700.0	18 : 300.0	19 : 0.0	20 : 1200.0
21 : 400.0	22 : 1200.0	23 : 300.0	24 : 100.0	

Origin 20				
1 : 300.0	2 : 100.0	3 : 0.0	4 : 300.0	5 : 100.0
6 : 300.0	7 : 500.0	8 : 900.0	9 : 600.0	10 : 2500.0
11 : 600.0	12 : 500.0	13 : 600.0	14 : 500.0	15 : 1100.0
16 : 1600.0	17 : 1700.0	18 : 400.0	19 : 1200.0	20 : 0.0
21 : 1200.0	22 : 2400.0	23 : 700.0	24 : 400.0	

Origin 21				
1 : 100.0	2 : 0.0	3 : 0.0	4 : 200.0	5 : 100.0
6 : 100.0	7 : 200.0	8 : 400.0	9 : 300.0	10 : 1200.0
11 : 400.0	12 : 300.0	13 : 600.0	14 : 400.0	15 : 800.0
16 : 600.0	17 : 600.0	18 : 100.0	19 : 400.0	20 : 1200.0
21 : 0.0	22 : 1800.0	23 : 700.0	24 : 500.0	

Origin 22				
1 : 400.0	2 : 100.0	3 : 100.0	4 : 400.0	5 : 200.0
6 : 200.0	7 : 500.0	8 : 500.0	9 : 700.0	10 : 2600.0
11 : 1100.0	12 : 700.0	13 : 1300.0	14 : 1200.0	15 : 2600.0
16 : 1200.0	17 : 1700.0	18 : 300.0	19 : 1200.0	20 : 2400.0
21 : 1800.0	22 : 0.0	23 : 2100.0	24 : 1100.0	

Origin 23				
1 : 300.0	2 : 0.0	3 : 100.0	4 : 500.0	5 : 100.0
6 : 100.0	7 : 200.0	8 : 300.0	9 : 500.0	10 : 1800.0
11 : 1300.0	12 : 700.0	13 : 800.0	14 : 1100.0	15 : 1000.0
16 : 500.0	17 : 600.0	18 : 100.0	19 : 300.0	20 : 700.0
21 : 700.0	22 : 2100.0	23 : 0.0	24 : 700.0	

Origin 24				
1 : 100.0	2 : 0.0	3 : 0.0	4 : 200.0	5 : 0.0
6 : 100.0	7 : 100.0	8 : 200.0	9 : 200.0	10 : 800.0
11 : 600.0	12 : 500.0	13 : 700.0	14 : 400.0	15 : 400.0
16 : 300.0	17 : 300.0	18 : 0.0	19 : 100.0	20 : 400.0
21 : 500.0	22 : 1100.0	23 : 700.0	24 : 0.0	

ANNEX B – SIOUX FALLS NETWORK PARAMETERS

Table 13 – Sioux Falls network parameters

Init node	Term node	Capacity	Length	F.F. time	α	β	Speed limit	Toll	Type
1	2	25900	6	6	0.15	4	0	0	1
1	3	23404	4	4	0.15	4	0	0	1
2	1	25900	6	6	0.15	4	0	0	1
2	6	4958	5	5	0.15	4	0	0	1
3	1	23403	4	4	0.15	4	0	0	1
3	4	17111	4	4	0.15	4	0	0	1
3	12	23404	4	4	0.15	4	0	0	1
4	3	17111	4	4	0.15	4	0	0	1
4	5	17783	2	2	0.15	4	0	0	1
4	11	4909	6	6	0.15	4	0	0	1
5	4	17783	2	2	0.15	4	0	0	1
5	6	4948	4	4	0.15	4	0	0	1
5	9	10000	5	5	0.15	4	0	0	1
6	2	4958	5	5	0.15	4	0	0	1
6	5	4948	4	4	0.15	4	0	0	1
6	8	4899	2	2	0.15	4	0	0	1
7	8	7841	3	3	0.15	4	0	0	1
7	18	23404	2	2	0.15	4	0	0	1
8	6	4899	2	2	0.15	4	0	0	1
8	7	7842	3	3	0.15	4	0	0	1
8	9	5050	10	10	0.15	4	0	0	1
8	16	5046	5	5	0.15	4	0	0	1
9	5	10000	5	5	0.15	4	0	0	1
9	8	5050	10	10	0.15	4	0	0	1
9	10	13916	3	3	0.15	4	0	0	1
10	9	13916	3	3	0.15	4	0	0	1
10	11	10000	5	5	0.15	4	0	0	1
10	15	13512	6	6	0.15	4	0	0	1
10	16	4855	4	4	0.15	4	0	0	1
10	17	4994	8	8	0.15	4	0	0	1
11	4	4909	6	6	0.15	4	0	0	1
11	10	10000	5	5	0.15	4	0	0	1
11	12	4909	6	6	0.15	4	0	0	1
11	14	4877	4	4	0.15	4	0	0	1
12	3	23404	4	4	0.15	4	0	0	1

12	11	4909	6	6	0.15	4	0	0	1
12	13	25900	3	3	0.15	4	0	0	1
13	12	25900	3	3	0.15	4	0	0	1
13	24	5091	4	4	0.15	4	0	0	1
14	11	4877	4	4	0.15	4	0	0	1
14	15	5128	5	5	0.15	4	0	0	1
14	23	4925	4	4	0.15	4	0	0	1
15	10	13512	6	6	0.15	4	0	0	1
15	14	5128	5	5	0.15	4	0	0	1
15	19	14565	3	3	0.15	4	0	0	1
15	22	9599	3	3	0.15	4	0	0	1
16	8	5046	5	5	0.15	4	0	0	1
16	10	4855	4	4	0.15	4	0	0	1
16	17	5230	2	2	0.15	4	0	0	1
16	18	19680	3	3	0.15	4	0	0	1
17	10	4994	8	8	0.15	4	0	0	1
17	16	5230	2	2	0.15	4	0	0	1
17	19	4824	2	2	0.15	4	0	0	1
18	7	23404	2	2	0.15	4	0	0	1
18	16	19680	3	3	0.15	4	0	0	1
18	20	23404	4	4	0.15	4	0	0	1
19	15	14565	3	3	0.15	4	0	0	1
19	17	4824	2	2	0.15	4	0	0	1
19	20	5003	4	4	0.15	4	0	0	1
20	18	23404	4	4	0.15	4	0	0	1
20	19	5003	4	4	0.15	4	0	0	1
20	21	5060	6	6	0.15	4	0	0	1
20	22	5076	5	5	0.15	4	0	0	1
21	20	5060	6	6	0.15	4	0	0	1
21	22	5230	2	2	0.15	4	0	0	1
21	24	4885	3	3	0.15	4	0	0	1
22	15	9599	3	3	0.15	4	0	0	1
22	20	5076	5	5	0.15	4	0	0	1
22	21	5230	2	2	0.15	4	0	0	1
22	23	5000	4	4	0.15	4	0	0	1
23	14	4925	4	4	0.15	4	0	0	1
23	22	5000	4	4	0.15	4	0	0	1
23	24	5079	2	2	0.15	4	0	0	1
24	13	5091	4	4	0.15	4	0	0	1

24	21	4885	3	3	0.15	4	0	0	1
24	23	5079	2	2	0.15	4	0	0	1

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