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ERALDO ALMEIDA LIMA JÚNIOR

UNIQUENESS OF HYPERSURFACES IMMERSED ON RIEMANNIAN AND LORENTZIAN SPACES:

RESULTS, EXAMPLES AND COUNTER-EXAMPLES

FORTALEZA

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“ Il n’existe pas de sciences appliquées, mais seulement des applications de la science.”(Louis Pasteur)

RESUMO

Neste trabalho, apresentamos resultados de unicidade para hipersuperfícies de curvatura média constante, tanto em um produto Riemanniano como Lorentziano. Tratamos de produtos cuja fibra tenha curvatura seccional limitada por baixo. Para isto, consideramos um certo controle na norma do gradiente da função altura pela norma da segunda forma fundamental com o objetivo de obter que tal hipersuperfície deve ser um slice, i.e., uma “fatia”. Também obtemos a unicidade através de condições de integrabilidade no gradiente da função altura. Apresentamos uma extensão de um lema devido a Nishikawa que utilizamos para provar os resultados no caso das superfícies máximas, ou seja, aquelas com curvatura média nula. Utilizamos como ferramenta essencial, na prova dos resultados, o princípio do máximo generalizado de Omori-Yau em suas versões mais atuais. Finalmente, apresentamos exemplos que justificam a necessidade das hipóteses exigidas nos resultados.
Palavras-chave: Hipersuperfície tipo-espaço. Produtos semi-Riemannianos. Curvatura média constante.

ABSTRACT

In this work we present uniqueness results for constant mean curvature hypersurfaces in Riemannian and Lorentzian products. We dealt with product whose fiber has sectional curvature bounded from below. We considered a certain control in the norm of the gradient of the height function by the norm of the second fundamental form in order to obtain that such a surface is slice. We also obtained uniqueness through integrability conditions in the gradient of the height function. We also presented an extension of a lemma due to Nishikawa which was used to prove the results for the case of maximal surfaces, that is, with zero mean curvature. We have utilized as an essential tool, in the prove of the results, the generalized Omori-Yau maximum principle in one of the latest versions. In the end, we present examples showing and justifying the necessity of required hypothesis in the results.

Keywords : Spacelike hypersurfaces. Semi-Riemannian Products. Constant Mean Curvature.

LISTA DE ABREVIATURAS E SIGLAS

CAPES	Coordenação de aperfeiçoamento de pessoal de ensino superior
CNPq	Conselho Nacional de Desenvolvimento Científico e Tecnológico
PIBIC	Programa Institucional de Bolsas de Iniciação Científica
UFCG	Universidade Federal de Campina Grande

SUMMARY

1 INTRODUCTION

Classically, hypersurfaces, or more simply surfaces, have been studied since the earliest steps of geometry. Formulas for volume and area of symmetric surfaces as the spheres were discovered even before the integral calculus. In particular, Archimedes evaluated the volumes of the sphere on a beautiful and whimsical way (EVES, 1990). It turned out that such a surface maximizes the inner volume among the one sharing the same area (STEINER, 1841).

When the inner volume or between the (hyper)surfaces and a (hyper)plane are maximum considering a fixed area we conceive it to be optimal. While we cannot clearly say the volume of a Lorentzian Manifold we can discuss the area of spacelike (hyper)surfaces when they have a fixed boundary. This concept is natural in the Riemannian case. The mean curvature is related to both phenomenons of critical points of the area function given by

$$\mathcal{A}(M) = \int_M dM$$

whose variations ϕN, for N the unit normal give us

$$d\mathcal{A}(M) = \int_M H\langle N, \nabla\phi\rangle dM$$

where the (hyper)-surface with zero mean curvature minimizes locally the area in the Riemannian ambient and therefore are called Minimals while for the Lorentzian case they maximize the area being called Maximals. The (spacelike) (hyper)surfaces have as a Riemannian manifold its own intrinsic geometric properties as well extrinsic properties as they are immersed into a Riemannian or Lorentzian manifold. In the sequel will define the Lorentzian spaces that are used to describe the universe with a distinguished metric for the time component.

The uniqueness of minimal surfaces in $\mathbb{R}^3$ flourished with Bernstein's theorem in (BERNSTEIN, 1910) stating that the only complete minimal graphs in $\mathbb{R}^3$ are the planes.

In the Lorentzian Geometry we have that only complete maximal spacelike hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes (see (CALABI, 1970), for $n \leq 4$, and (CHENG and YAU, 1976), for arbitrary n).

So the interest in the study of spacelike hypersurfaces in Lorentzian spaces is also motivated by the fact that such hypersurfaces exhibit nice Bernstein-type properties. For example, Xin (1991) and Aiyama (1992) simultaneous and independently characterized the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ having the image of its Gauss map contained in a geodesic ball of the hyperbolic space (see also (PALMER, 1990) for a weaker first version of this result). More recently, de Lima (2011) obtained an extension of the Xin-Aiyama theorem concerning complete spacelike hypersurfaces immersed with bounded mean curvature in $\mathbb{L}^{n+1}$.

In this thesis we present results related to the Riemannian and Lorentzian Product which are characterization of constant mean curvature hypersurfaces satisfying some constrains in order to obtain they are slices, totally geodesic or maximal/minimal.

In our results we utilized recurrently the Omori-Yau maximum principle, which had been generalized in many contexts with that in mind we considered alternative proofs in order that the results would be more easily generalized accordingly to the new generalization and weaker forms available. See (ALÍAS, DAJCZER, and RIGOLI, 2013), (BORBÉLY, 2012) and (BESSA and PESSOA, 2014).

In Chapter 3, we initially show a uniqueness property for hypersurfaces in a Riemannian product such that the height function growth is controlled by the norm of the second fundamental form as we see below:

Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ *be a Riemannian product space whose base* M^n *has sectional curvature* K_M *satisfying* $K_M \geq -\kappa$ *for some* $\kappa > 0$*, and let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a two-sided complete hypersurface with constant mean curvature* H *and* H_2 *bounded from below. Suppose that the angle function* η *of* Σ^n *is bounded away from zero and that its height function* h *satisfies one of the following conditions:*

$$|\nabla h|^2 \leq \frac{\alpha}{(n-1)\kappa}|A|^2, \tag{1}$$

for some constant $0 < \alpha < 1$*; or*

$$|\nabla h|^2 \leq \frac{n}{(n-1)\kappa}H^2. \tag{2}$$

Then, Σ^n *is a slice of* $\overline{M}^{n+1}$*.*

These results extends the main theorem in (De LIMA and PARENTE, 2012) where they considered $\mathbb{R} \times \mathbb{H}^n$ and $|\nabla h|^2 \leq \dfrac{n\alpha}{(n-1)\kappa}H^2$, $\alpha < 1$ instead of inequality (2).

We also described how should behave the mean curvature of a hypersurface in such Riemannian product accordingly with the *a priori* bounds.

Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ *be a Riemannian product space whose base* M^n *has sectional curvature bounded from below, and let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a two-sided complete hypersurface which lies between two slices of* $\overline{M}^{n+1}$*. Suppose that the angle function* η *of* Σ^n *is not adherent to* 1 *or* -1*. If* H_2 *is bounded from below,* H *is bounded and it does not change sign on* Σ^n*, then* $\inf_\Sigma H = 0$*. In particular, if* H *is constant, then* Σ^n *is minimal.*

In Chapter 4 we dealt with the Lorentzian context, where we properly extended the results in (ALBUJER, CAMARGO, and de LIMA, 2010), they dealt with spacelike constant mean curvature hypersurfaces immersed in $-\mathbb{R}\times\mathbb{H}^n$. The final result gained the following configuration:

Let $\psi : \Sigma^n \to -\mathbb{R}\times M^n$ be a complete spacelike hypersurface immersed with constant mean curvature H in a Lorentzian product space $-\mathbb{R}\times M^n$, whose sectional curvature K_M of its fiber M^n is such that $-\kappa \le K_M$ for some positive constant κ. Suppose that one of the following conditions is satisfied:
(a) *The height function h of Σ^n is such that*

$$|\nabla h|^2 \le \frac{n}{\kappa(n-1)}H^2. \tag{3}$$

(b) *H_2 is bounded from below on Σ^n and the height function h of Σ^n is such that, for some constant $0<\alpha<1$,*

$$|\nabla h|^2 \le \frac{\alpha}{\kappa(n-1)}|A|^2. \tag{4}$$

Then, Σ^n is a slice.

As in the Riemannian case we also studied the mean curvature and characterized when a CMC spacelike hypersurface is a slice.

Let $\psi : \Sigma^n \to -\mathbb{R}\times M^n$ be a complete spacelike hypersurface immersed in a Lorentzian product space $-\mathbb{R}\times M^n$, whose sectional curvature K_M of its fiber M^n is such that $-\kappa \le K_M$ for some positive constant κ. Suppose that Σ^n lies between two slices of $-\mathbb{R}\times M^n$ and that $|\nabla h|$ is bounded on Σ^n. If H is bounded and it does not change sign on Σ^n, then H is not globally bounded away from zero. In particular, if H is constant, then Σ^n is maximal.

Here we highlight that the Lorentzian case was previously and in the Riemannian case we also combined with the classical results due to Osserman (1959) in order to obtain uniqueness.

In chapter 5, inspired in the results due to Aledo and Alías (2002) where they studied spacelike hypersurface in de Sitter space, $\mathbb{S}_1^{n+1}$, such that the Normal N is contained in geodesic balls of $\mathbb{H}^{n+1}$ showing their compactness. We decided to present results for spacelike hypersurfaces in the $-\mathbb{R}\times M^n$ where the fiber is compact with positive sectional curvature and compact as we see in the following.

Let Σ^n be a complete spacelike hypersurface immersed with constant mean curvature in a spatially closed Lorentzian product space $-\mathbb{R}\times M^n$, whose fiber M^n has positive sectional curvature. If the normal hyperbolic angle of Σ^n is bounded, then Σ^n is a slice $\{t_0\}\times M^n$ for some $t_0\in\mathbb{R}$.

We also studied whether a CMC hypersurfaces whether they should be maximal and half space properties.

Let Σ^n be a complete spacelike hypersurface immersed with constant mean curvature in Lorentzian product space $-\mathbb{R}\times M^n$, whose fiber M^n has non-negative sectional *curvature. If the normal hyperbolic angle of* Σ^n *is bounded, then* Σ^n *is maximal. Moreover if* $\Sigma(u)$ *is vertically half bounded then it is a slice* $\{t_0\} \times M^n$ *for some* $t_0 \in \mathbb{R}$*.*

We considered integrability conditions in order to obtain the desired uniqueness in this context.

Let Σ^n *be a complete spacelike hypersurface immersed with constant mean curvature in a spatially closed Lorentzian product space* $-\mathbb{R}\times M^n$*, whose fiber* M^n *has positive sectional curvature. If* $\nabla h \in L^1 \cap L^\infty(\Sigma)$ *and* $A \in L^\infty(\Sigma)$ *then* Σ^n *is a slice* $\{t_0\} \times M^n$ *for some* $t_0 \in \mathbb{R}$*.*

Rosenberg, Schulze and Spruck (2013) proved a half-space property for graph in a Riemannian product $\mathbb{R} \times M^n$ whose fiber has non-negative Ricci curvature and sectional curvature bounded from below. Chapter 6 is devoted to graphs and the study of its completeness inspired in the previous work where we considered a half space property:

Let $\overline{M}^{n+1} = -\mathbb{R} \times M^n$ *be a Lorentzian product space, such that the sectional curvature* K_M *of its Riemannian fiber* M^n *satisfies* $K_M \geq -\kappa$*, for some positive constant* κ*. Let* $\Sigma(u)$ *be an entire* H*-graph over* M^n*, with* u *and* H_2 *bounded from below. If*

$$|Du|_M^2 \leq \frac{|A|^2}{\kappa(n-1)+|A|^2}, \tag{5}$$

then $u \equiv t_0$ *for some* $t_0 \in \mathbb{R}$*.*

Inequality (5) is implied either by inequality (3) or (4) therefore its veracity is a weaker assumption than the validity of one of them. However in this result we also assume the hypersurface is a graph and half bounded which will be necessary.

For the case $-\mathbb{R}\times\mathbb{H}^2$, it is known (ALBUJER, 2008b) that there are complete maximal surfaces which are not totally geodesic. Thus, it naturally arises the question to decide what additional assumptions are needed to conclude that a complete maximal surface in $-\mathbb{R} \times M^2$, where $\kappa_M \geq -\kappa$, must be totally geodesic.

In chapter 7 we show Calabi-Bernstein properties of maximal surfaces in a Lorentzian product space where the Gauss curvature of the fiber M^2 satisfies $K_M \geq -\kappa$ for some $\kappa \in \mathbb{R}$, $\kappa \geq 0$.

Let $\overline{M}^3 = -\mathbb{R} \times M^2$ *be a Lorentzian product spacetime, such that the Gauss curvature* K_M *of its Riemannian fiber* M^2 *satisfies* $K_M \geq -\kappa$*, for some positive constant* κ*. Let* Σ *be a complete maximal surface such that it Gauss-Kronecker curvature satisfies* $K_G \leq G(r)$*. If the height function* h *and the shape operator* A *of* Σ *satisfy*

$$|\nabla h|^2 \leq \frac{|A|^2}{\kappa}, \tag{6}$$

then Σ *is a slice.*

Our technique is based on a proper extension of a result by Nishikawa (1984) and relies within the applications of the generalized maximum principle due to Yau (1975) to complete Riemannian manifolds. In fact, we previously proved an extension of (YAU, 1975, Lemma 2) to the case the Ricci curvature is no longer bounded by a constant but by a more general function of the distance on the manifold.

Let M be a complete Riemmanian manifold such that $\mathrm{Ric} \geq -G(r)$ *for G such that $G(0) \geq 1$, $G' \geq 0$ and $G^{-1/2} \notin L^1[0,\infty]$. If u is a non-negative function on M satisfying*

$$\Delta u \geq \beta u^2, \;\; \beta > 0, \tag{7}$$

then $u \equiv 0$.

In chapter 8 we exhibit examples and highlight their properties that are important to the previous presented result. Which are the three examples of spacelike graphs in $-\mathbb{R} \times \mathbb{H}^2$, for $\mathbb{H}^2$ given by the Poincaré model of half plane. Example 8.1, see (De LIMA and LIMA JR, 2013), $u(x,y) = a \ln y$ with $|a| < 1$, this example is also considered in the Riemannian setting, it shows that maximality cannot be removed in the results we required so, as well that we cannot choose $\alpha = 1$ when we required $\alpha < 1$. In fact, we cannot even replace it by constant mean curvature. Considering the previously quoted result we see that Example 8.2, see (ALBUJER, 2008b), $u(x,y) = a \ln(x^2 + y^2)$ for $a < \frac{1}{2}$, lacks, on the general hypothesis, the main inequality we usually required in the control of norm of the gradient of the height function. Finally the Example 8.3, see (ALBUJER, 2008b), $u(x,y) = \ln(y + \sqrt{a + y^2})$ where $a > 0$, fails the control in the growth of the gradient of the height function, in reality its growth is more than exponential. We also emphasized that this example is a graph half bounded which is required in Chapter 6.

2 SOME PRELIMINARIES AND CLASSIC RESULTS

2.1 Semi-Riemannian elements and basic results

We consider hypersurfaces Σ^n immersed into an $(n+1)$-dimensional Semi-riemannian product space $\overline{M}^{n+1}$ of the form $\mathbb{R}\times M^n$, where M^n is an n-dimensional connected Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the Lorentzian metric

$$\langle,\rangle = \epsilon\pi_{\mathbb{R}}^*(dt^2) + \pi_M^*(\langle,\rangle_M),$$

where $\pi_{\mathbb{R}}$ and π_M denote the canonical projections from $\mathbb{R}\times M$ onto each factor, and $\langle,\rangle_M$ is the Riemannian metric on M^n. For simplicity, we will just write $\overline{M}^{n+1} = \epsilon\mathbb{R}\times M^n$ and $\langle,\rangle = \epsilon dt^2 + \langle,\rangle_M$. In this setting, for a fixed $t_0\in\mathbb{R}$, we say that $M^n_{t_0} = \{t_0\}\times M^n$ is a *slice* of $\overline{M}^{n+1}$. It is not difficult to prove that a slice of $\overline{M}^{n+1}$ is a totally geodesic hypersurface (see Proposition 1 in (MONTIEL, 1999)).

In the Lorentzian case a smooth immersion $\psi:\Sigma^n\to -\mathbb{R}\times M^n$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted for $\langle,\rangle$. Since

$$\partial_t = (\partial/\partial_t)_{(t,x)}\,,\quad (t,x)\in -\mathbb{R}\times M^n,$$

is a unitary timelike vector field globally defined on the ambient spacetime, then there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n which is in the same time-orientation as ∂_t. By using Cauchy-Schwarz inequality, we get

$$\langle N,\partial_t\rangle \le -1 < 0 \quad \text{on } \Sigma^n. \tag{8}$$

We will refer to that normal vector field N as the *future-pointing Gauss map* of the spacelike hypersurface Σ^n.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\epsilon\mathbb{R}\times M^n$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for the (spacelike) hypersurface $\psi:\Sigma^n\to\epsilon\mathbb{R}\times M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \epsilon\langle AX,Y\rangle N \tag{9}$$

and

$$AX = -\overline{\nabla}_X N, \tag{10}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator (or Weingarten endomorphism) of Σ^n with respect to the future-pointing Gauss map N.

As in (O'NEILL, 1983), the curvature tensor R of the spacelike hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

The curvature tensor R of the (spacelike) hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of the ambient spacetime $\overline{M}^{n+1} = \epsilon\mathbb{R} \times M^n$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top + \epsilon\langle AX, Z\rangle AY - \epsilon\langle AY, Z\rangle AX, \tag{11}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n.

Now, we consider two particular functions naturally attached to a (spacelike) hypersurface Σ^n immersed into a Semi-Riemannian product space $\epsilon\mathbb{R} \times M^n$, namely, the (vertical) height function $h = (\pi_{\mathbb{R}})|_\Sigma$ and the support function $\langle N, \partial_t\rangle$, where we recall that N denotes the future-pointing Gauss map of Σ^n and ∂_t is the coordinate vector field induced by the universal time on $\epsilon\mathbb{R} \times M^n$.

Let us denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics of $\epsilon\mathbb{R} \times M^n$ and Σ^n, respectively. Then, a simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $-\mathbb{R} \times M^n$ is given by

$$\overline{\nabla}\pi_{\mathbb{R}} = \epsilon\langle\overline{\nabla}\pi_{\mathbb{R}}, \partial_t\rangle\partial_t = \epsilon\partial_t, \tag{12}$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^\top = \epsilon\partial_t^\top = \epsilon\partial_t - \langle N, \partial_t\rangle N. \tag{13}$$

Thus, we get

$$\epsilon|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1, \tag{14}$$

where $|\ |$ denotes the norm of a vector field on Σ^n.

In the Lorentzian case the geometrical interpretation of the norm of the gradient of the height function involves the notion of normal hyperbolic angle. More precisely, if Σ^n is a spacelike hypersurface of $-\mathbb{R} \times M^n$ with future-pointing Gauss map N, we define the *normal hyperbolic angle* θ of Σ^n as being the smooth function $\theta : \psi(\Sigma) \to [0, +\infty)$ given by

$$\cosh\theta = -\langle N, \partial_t\rangle. \tag{15}$$

Consequently, from (14) and (15) we have that

$$|\nabla h|^2 = \cosh^2\theta - 1. \tag{16}$$

In order to evaluate the Laplacian of the height function we consider that ∂_t is parallel on $\epsilon\mathbb{R}\times M^n$, then

$$\overline{\nabla}_X\partial_t = 0, \tag{17}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$. Writing $\epsilon\partial_t = \nabla h + \langle N, \partial_t\rangle N$ along the hypersurface Σ^n and using formulas (9) and (10), we get that

$$\nabla_X\nabla h = \langle N, \partial_t\rangle AX, \tag{18}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$. Therefore, from (18) we obtain that the Laplacian on Σ^n of the height function is given by

$$\Delta h = \epsilon n H\langle N, \partial_t\rangle, \tag{19}$$

where $H = -\epsilon\frac{1}{n}\text{trace}(A)$ is the mean curvature of Σ^n relative to N. The gradient of the support function is given by

$$\begin{aligned} X\langle N, \partial_t\rangle &= \langle\overline{\nabla}_X N, \partial_t\rangle + \langle N, \overline{\nabla}_X\partial_t\rangle \\ &= \langle AX, -\partial_t{}^{\top}\rangle = \langle AX, -\epsilon\nabla h\rangle \\ &= -\epsilon\langle A\nabla h, X\rangle. \end{aligned} \tag{20}$$

Since X is arbitrary we get

$$\nabla\langle N, \partial_t\rangle = -\epsilon A(\nabla h). \tag{21}$$

Moreover, as a particular case of the Proposition 3.1 in (CAMINHA and De LIMA, 2009), we obtain the following suitable formula for the Laplacian on Σ^n of the angle function η.

Lemma 1 *Let $\psi : \Sigma^n \to \epsilon\mathbb{R}\times M^n$ be a hypersurface with orientation N, and let $\eta = \langle N, \partial_t\rangle$ be its angle function. If Σ^n has constant mean curvature H, then*

$$\Delta\eta = -\epsilon\left(\text{Ric}_M(N^*, N^*) + |A|^2\right)\eta, \tag{22}$$

where Ric_M denotes the Ricci curvature of the base M^n, N^* is the projection of the unit normal vector field N onto the base M^n and $|A|$ is the Hilbert-Schmidt norm of the shape operator A.

In order to establish our results, we also need of the following auxiliary lemma for the Lorentzian case.

Lemma 2 *Let $\psi : \Sigma \to -\mathbb{R} \times M^n$ be a spacelike hypersurface immersed in a Lorentzian product space $-\mathbb{R} \times M^n$, whose sectional curvature K_M of its fiber M^n is such that $-\kappa \leq K_M$ for some positive constant κ. Then, for all $X \in \mathfrak{X}(\Sigma)$, the Ricci curvature of Σ^n satisfies the following inequality*

$$\mathrm{Ric(X, X)} \geq -\kappa(n-1)(1+|\nabla h|^2)|X|^2 - \frac{n^2H^2}{4}|X|^2. \tag{23}$$

Proof. Since the curvature tensor R of the spacelike hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of the ambient spacetime $-\mathbb{R} \times M^n$ by the so-called Gauss equation given by[1]

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX, \tag{24}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \cdots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss equation that the *Ricci curvature tensor* Ric is given by

$$\begin{aligned}\mathrm{Ric}(X,X) &= \sum_{i=1}^n \langle \overline{R}(X,E_i)X, E_i\rangle + nH\langle AX, X\rangle + \langle AX, AX\rangle \\ &= \sum_{i=1}^n \langle \overline{R}(X,E_i)X, E_i\rangle - \frac{n^2H^2}{4}|X|^2 + \left|AX + \frac{nH}{2}X\right|^2.\end{aligned}$$

Moreover, we have that

$$\begin{aligned}\langle \overline{R}(X,E_i)X, E_i\rangle &= \langle R(X^*, E_i^*)X^*, E_i^*\rangle_M \\ &= K_M(X^*, E_i^*)\left(\langle X^*, X^*\rangle_M \langle E_i^*, E_i^*\rangle_M - \langle X^*, E_i^*\rangle_M^2\right).\end{aligned}$$

Since $X^* = X + \langle X, \partial_t\rangle\partial_t$, $E_i^* = E_i + \langle E_i, \partial_t\rangle\partial_t$ and $\nabla h = -\partial_t^\top$, with a straightforward computation we see that

$$\langle X^*, X^*\rangle_M \langle E_i^*, E_i^*\rangle_M = (1 + \langle E_i, \nabla h\rangle^2)(|X|^2 + \langle X, \nabla h\rangle^2)$$

and

$$\begin{aligned}\langle X^*, E_i^*\rangle_M \langle E_i^*, E_i^*\rangle_M &= \langle X, E_i\rangle^2 + 2\langle X, \nabla h\rangle\langle E_i, \nabla h\rangle\langle X, E_i\rangle \\ &\quad + \langle X, \nabla h\rangle^2\langle E_i, \nabla h\rangle^2.\end{aligned}$$

[1] As in (O'NEILL, 1983), the curvature tensor R of the spacelike hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Therefore, since we are supposing that $-\kappa \leq K_M$ for some positive constant κ, we obtain inequality (23). ■

Remark 1 *We note that when the ambient spacetime $-\mathbb{R} \times M^n$ is such that its fiber M^n is flat, from Lemma 2 we see that the boundedness of the mean curvature is enough to guarantee that the Ricci curvature of the spacelike hypersurface is bounded from below.*

When we have a maximum point we can have nice properties on any smooth function on manifolds. However when the manifold is not compact we cannot guarantee such a critical point even when the function admits a infimum or a supremum. With some completeness assumptions on the Riemannian manifolds as we see below it is possible to still obtain similar properties.

Lemma 3 *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $u : \Sigma^n \to \mathbb{R}$ be a smooth function which is bounded from above on Σ^n. Then there is a sequence of points $p_k \in \Sigma^n$ such that*

$$\lim_k u(p_k) = \inf u, \quad \lim_k |\nabla u(p_k)| = 0 \quad \text{and} \quad \limsup_k \Delta u(p_k) \geq 0.$$

This is the well known generalized maximum principle of Omori-Yau (OMORI, 1967) and (YAU, 1975). Several authors have developed extension for this principle manly in order obtain the same thesis assuming weaker hypothesis see (ALÍAS, DAJCZER, and RIGOLI, 2013), (BORBÉLY, 2012) and (BESSA and PESSOA, 2014). For simplicity we present one given by

Lemma 4 (Borbély) *Let M^n be a complete Riemannian manifold, $p \in M$ be a fixed point and $r(x)$ be the distance function from p. Let us assume that away from the cut locus of p we have*

$$\mathrm{Ric}(\nabla r, \nabla r) \geq -BG^2(r),$$

where $G(t)$ has the following properties:

$$G \geq 1\ ,\ G' \geq 0 \text{ and } \quad \int_0^\infty \frac{dt}{G(t)} = \infty$$

Then M^n satisfies the Omori-Yau maximum principle.

Since $G(r)$ belongs to the class of function such that $G \geq 0$, $G' \geq \alpha > 0$ and $G^{-\frac{1}{2}} \notin L^1[0, \infty]$, we will identify any function in that class just by $G(r)$.

The next results are due to Yau (1975) and (1976) for the first one we put a simplified version adapted for our purposes

Lemma 5 *The only harmonic semi-bounded functions defined on an n-dimensional complete Riemannian manifold whose Ricci curvature is nonnegative are the constant ones.*

The second one is an extension of H. Hopf theorem to a complete noncompact Riemannian manifold. In what follows, $\mathcal{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 6 *Let Σ^n be an n-dimensional, complete noncompact Riemannian manifold and let $u:\Sigma^n\to\mathbb{R}$ be a smooth function. If u is a subharmonic (or superharmonic) function such that $|\nabla u|\in\mathcal{L}^1(\Sigma)$, then u must actually be harmonic.*

2.2 Vertical Graphs in a Semi-Riemannian Product

Let $\epsilon\mathbb{R}\times M^n$ a Semi-Riemannian product space and let $\Omega\subseteq M^n$ be a connected domain of the fiber M^n. A *vertical graph* over Ω is determined by a smooth function $u\in C^\infty(\Omega)$ and it is given by

$$\Sigma^n(u)=\{(u(x),x);x\in\Omega\}\subset\epsilon\mathbb{R}\times M^n.$$

The metric induced on Ω from the Semi-Riemannian metric on the ambient space via $\Sigma^n(u)$ is

$$\langle,\rangle=\epsilon du^2+\langle,\rangle_M. \tag{25}$$

A graph is said to be entire if $\Omega=M^n$.

If $\Sigma^n(u)$ is a (spacelike) vertical graph over a domain we verify that the vector field

$$N(x)=\frac{1}{\sqrt{1+\epsilon|Du|^2}}(\partial_t|_{(u(x),x)}-\epsilon Du(x)),\ \ x\in\Omega, \tag{26}$$

defines the Gauss map of $\Sigma^n(u)$, it is future-pointing in the Lorentzian case. For the shape operator A of $\Sigma^n(u)$ with respect its orientation given by (26). For any $X\in\mathfrak{X}(\Sigma(u))$, for X^* the projection onto the tangent space of the fiber M, then $X=X^*+\epsilon\langle Du,X^*\rangle_M\partial_t$, we have that

$$AX=-\overline{\nabla}_XN=\epsilon\langle Du,X^*\rangle_M\overline{\nabla}_{\partial_t}N-\overline{\nabla}_{X^*}N. \tag{27}$$

From (26), (27), and with aid of Proposition 7.35 in (O'NEILL, 1983), we verify that

$$\begin{aligned}
AX &=\epsilon\langle Du,X^*\rangle\overline{\nabla}_{\partial_t}\left(\frac{\partial_t-\epsilon Du}{\sqrt{1+\epsilon|Du|^2}}\right)-\overline{\nabla}_{X^*}\left(\frac{\partial_t-\epsilon Du}{\sqrt{1+\epsilon|Du|^2}}\right)\\
&=-\overline{\nabla}_{X^*}\left(\frac{\partial_t-\epsilon Du}{\sqrt{1+\epsilon|Du|^2}}\right)\\
&=\overline{\nabla}_{X^*}\left(\frac{\epsilon Du}{\sqrt{1+\epsilon|Du|^2}}\right)\\
&=\epsilon\frac{\overline{\nabla}_{X^*}Du}{\sqrt{1+\epsilon|Du|^2}}-\frac{Du}{(1+\epsilon|Du|^2)^{3/2}}\langle\overline{\nabla}_{X^*}Du,Du\rangle.
\end{aligned}$$

Denoting by D the Levi-Civita connection with respect to the metric $\langle\,,\rangle_M$ on $\mathfrak{X}(M)$ we obtain

$$AX=\frac{\epsilon}{\sqrt{1+\epsilon|Du|_M^2}}D_{X^*}Du-\frac{\langle D_{X^*}Du,Du\rangle_M}{(1+\epsilon|Du|_M^2)^{3/2}}Du, \tag{28}$$

From (28) we obtain that the mean curvature of $\Sigma(u)$ is given by

$$nH = -\mathrm{Div}\left(\frac{\epsilon Du}{\sqrt{1+\epsilon|Du|_M^2}}\right), \tag{29}$$

where Div stands for the divergence operator on M^n with respect to the metric $\langle,\rangle_M$. In the Lorentzian context a graph $\Sigma^n(u)$ is a spacelike hypersurface if, and only if, $|Du|^2 < 1$, being Du the gradient of u in Ω and $|Du|$ its norm, both with respect to the metric $\langle,\rangle_M$ in Ω. Note that every complete spacelike hypersurface in $-\mathbb{R}\times M^n$ is an entire spacelike vertical graph in such space. For a proof of this fact see Lemma 3.1 in (ALÍAS, ROMERO, and SANCHEZ, 1995) and Lemma 3.1 in (ALBUJER and ALÍAS, 2009). However, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph in a Lorentzian product space is not necessarily complete, in the sense that the induced Riemannian metric (25) is not necessarily complete on M^n. In fact, Albujer have obtained explicit examples of non-complete entire maximal graphs in $-\mathbb{R}\times\mathbb{H}^2$ (cf. (ALBUJER, 2008b) we also put it in details in Chapter 8).

3 HYPERSURFACES WITH PRESCRIBED ANGLE FUNCTION

3.1 Introduction

One of the most celebrated theorems of the theory of minimal surfaces in $\mathbb{R}^3$ is Bernstein's theorem in (BERNSTEIN, 1910) which establishes that the only complete minimal graphs in $\mathbb{R}^3$ are the planes. This result was extended under the weaker hypothesis that the image of the Gauss map of Σ^2 lies in an open hemisphere of $\mathbb{S}^2$, as we can see in (BARBOSA and Do CARMO, 1974).

Meanwhile, Osserman (1959) answered a conjecture due to Nirenberg, showing that if a complete minimal surface Σ^2 in $\mathbb{R}^3$ is not a plane, then its normals must be everywhere dense on the unit sphere $\mathbb{S}^2$. More generally, Fujimoto (1988) proved that if the Gaussian image misses more than four points, then it is a plane. Meanwhile Hoffman, Osserman and Schoen in (1982) showed that if a complete oriented surface Σ^2 with constant mean curvature in $\mathbb{R}^3$ is such that the image of its Gauss map $N(\Sigma)$ lies in some open hemisphere of $\mathbb{S}^2$, then Σ^2 is a plane. Moreover, if $N(\Sigma)$ lies in a closed hemisphere, then Σ^2 is a plane or a right circular cylinder.

When the ambient space is a Riemannian product $\overline{M}^{n+1} = \mathbb{R} \times M^n$, as it was already observed by Espinar and Rosenberg (2009), the condition that the image of the Gauss map is contained in a closed hemisphere, becomes that the angle function $\eta = \langle N, \partial_t \rangle$ does not change sign. Here, N denotes a unit normal vector field along a hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ and ∂_t stands for the unitary vector field which determines on $\overline{M}^{n+1}$ a codimension one foliation by totally geodesic slices $\{t\} \times M^n$. In this setting, our purpose in this work is to establish analogous results to those ones above described. In other words, we aim to give new satisfactory answers to the following question: *under what reasonable geometric restrictions on the angle function must a complete hypersurface immersed in a certain product space be a slice*?

We can truly say that one of the first remarkable results in this direction was the celebrated theorem of Bombieri, De Giorgi and Miranda (1969), who proved that an entire minimal positive graph over $\mathbb{R}^n$ is a totally geodesic slice.

Many other authors have approached problems in this branch. Hence, in this case, the graph is a horizontal slice or M^2 is a flat $\mathbb{R}^2$ and the graph is a tilted plane. Later on, Bérard and Sá Earp (2008) have described all rotation hypersurfaces with constant mean curvature in $\mathbb{R} \times \mathbb{H}^n$ and used them as barriers to prove existence and characterization of certain vertical graphs with constant mean curvature and to give symmetry and uniqueness results for constant mean curvature compact hypersurfaces whose boundary is one or two parallel submanifolds in slices.

Espinar and Rosenberg (2009) have studied constant mean curvature surfaces Σ^2 in $\mathbb{R} \times M^2$. Under the assumption that the angle function does not change sign, they classified such surfaces according to the infimum of the Gaussian curvature of their horizontal projection.

Recently, Aquino, de Lima and Parente (AQUINO and De LIMA, 2011) and (De LIMA and PARENTE, 2012) have applied the well known generalized maximum principle of Omori-Yau (OMORI, 1967; YAU, 1975) and an extension of it due to Akutagawa (1987) in order to obtain rigidity theorems concerning complete vertical graphs with constant mean curvature in $\mathbb{R} \times \mathbb{Q}^n$. De Lima (2014) also extended the technique developed by Yau (1976) in order to investigate the rigidity of entire vertical graphs in a Riemannian product space $\mathbb{R} \times M^n$, whose base M^n is supposed to have Ricci curvature with strict sign. Under a suitable restriction on the norm of the gradient of the function u which determines such a graph $\Sigma^n(u)$, he proved that $\Sigma^n(u)$ must be a slice $\{t\} \times M^n$.

3.2 Uniqueness for hypersurfaces on a Riemannian product

Inspired in the previous works and motivated by comprehend the theory, we could prove proper extensions and give counter-examples showing the paths where the results cannot be extended. These results are originally in (De LIMA, LIMA JR, and PARENTE, 2014).

In the following $H_2 = \frac{2}{n(n-1)} S_2$ stands for the mean value of the second elementary symmetric function S_2 on the eigenvalues of the Weingarten operator A of the hypersurface Σ^n. Moreover, we recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, there is on it a globally defined unit normal vector field.

Theorem 1 *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a Riemannian product space whose base M^n has sectional curvature K_M such that $K_M \geq -\kappa$ for some $\kappa > 0$, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided complete hypersurface with constant mean curvature H and H_2 bounded from below. Suppose that the angle function η of Σ^n is bounded away from zero and that its height function h satisfies one of the following conditions:*

$$|\nabla h|^2 \leq \frac{\alpha}{(n-1)\kappa} |A|^2, \tag{30}$$

for some constant $0 < \alpha < 1$; or

$$|\nabla h|^2 \leq \frac{n}{(n-1)\kappa} H^2. \tag{31}$$

Then, Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since we are assuming that η is bounded away from zero, we can suppose that $\eta > 0$ and, consequently, $\inf \eta > 0$. From Lemma 1, we have

$$\Delta \eta = -(\mathrm{Ric_M}(N^*, N^*) + |A|^2)\eta. \tag{32}$$

Moreover, since we are also assuming that the sectional curvature K_M of the base M^n is such that $K_M \geq -\kappa$ for some $\kappa > 0$, with a straightforward computation we get

$$\mathrm{Ric_M}(N^*, N^*) \geq -(n-1)\kappa|N^*|^2 = -(n-1)\kappa(1-\eta^2),$$

where the N^* stands for the component of N tangent to M^n. Then, from (14) and (32) we obtain

$$\Delta\eta \leq -(|A|^2 - (n-1)\kappa|\nabla h|^2)\eta. \tag{33}$$

Thus, if we assume that the height function of Σ^n satisfies the hypothesis (30) and from (33), we have that

$$\Delta\eta \leq -(1-\alpha)|A|^2\eta. \tag{34}$$

We claim that the Ricci curvature of Σ^n is bounded from below. Therefore, we are in conditions to apply Lemma 3 to the function η, obtaining a sequence of points $p_k \in \Sigma^n$ such that

$$\liminf_{k\to\infty} \Delta\eta(p_k) \geq 0 \quad \text{and} \quad \lim_{k\to\infty} \eta(p_k) = \inf_{p\in\Sigma} \eta(p).$$

Consequently, since we are assuming that the Weingarten operator A is bounded on Σ^n, from (34), up to a subsequence, we get

$$0 \leq \liminf_{k\to\infty} \Delta\eta(p_k) \leq -(1-\alpha) \lim_{k\to\infty} |A|^2(p_k) \inf_{p\in\Sigma} \eta(p) \leq 0.$$

Thus, we obtain that $\lim_{k\to\infty} |A|(p_k) = 0$ and, from (30), $\lim_{k\to\infty} |\nabla h|(p_k) = 0$. Hence, from (14) we conclude that $\inf_{p\in\Sigma} \eta(p) = 1$ and, consequently, $\eta \equiv 1$. Therefore, Σ is a slice.

It just remains to prove our claim that the Ricci curvature of Σ^n is bounded from below. For this, let us consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \cdots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss Equation (11) that

$$\mathrm{Ric}_\Sigma(X, X) = \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle + nH\langle AX, X\rangle - \langle AX, AX\rangle. \tag{35}$$

Thus, taking into account once more the lower bound of the sectional curvature of the base M^n, we have

$$\langle \overline{R}(X, E_i)X, E_i\rangle \geq \; -\kappa(\langle X^*, X^*\rangle_{M^n}\langle E_i^*, E_i^*\rangle_{M^n} - \langle X^*, E_i^*\rangle^2_{M^n}), \tag{36}$$

where $X^* = X - \langle X, \partial_t\rangle\partial_t$ and $E_i^* = E_i - \langle E_i, \partial_t\rangle\partial_t$ are the projections of the tangent vector fields X and E_i onto M^n, respectively. Then, summing up relation (36) we get

$$\begin{aligned}\sum_i \langle \overline{R}(X, E_i)X, E_i\rangle &\geq -\kappa\big((n-1)|X|^2 - |\nabla h|^2|X|^2 - (n-2)\langle X, \nabla h\rangle^2\big)\\ &\geq -\kappa(n-1)|X|^2.\end{aligned}$$

Therefore, from (35) and using Cauchy-Schwarz inequality we have that the Ricci curvature of Σ^n satisfies the following lower estimate

$$\mathrm{Ric}_\Sigma(X, X) \geq -\big((n-1)\kappa - |A||A - nHI|\big)|X|^2, \tag{37}$$

for all $X \in \mathfrak{X}(\Sigma)$. Therefore, taking into account that

$$|A|^2 = n^2H^2 - n(n-1)H_2, \tag{38}$$

our restrictions on H and H_2 guarantee that the Ricci curvature tensor of Σ^n is bounded from below and, hence, we conclude the first part of the proof of Theorem 1.

Now, let us suppose that the height function of Σ^n satisfies the hypothesis (31). In this case, from (33) and (38) we obtain

$$\Delta\eta \leq -n(n-1)(H^2 - H_2)\eta. \tag{39}$$

Consequently, in a similar way of the previous case, we can apply Lemma 3 in order to obtain a sequence of points $p_k \in \Sigma^n$ such that

$$0 \leq \liminf_{k\to\infty} \Delta\eta(p_k) \leq -n(n-1)\liminf_{k\to\infty}(H^2 - H_2)(p_k)\inf_{p\in\Sigma}\eta(p) \leq 0.$$

Hence, up to a subsequence,

$$\lim_{k\to\infty}(H^2 - H_2)(p_k) = 0.$$

Moreover, since H is supposed to be constant, from (38) we get

$$\lim_{k\to\infty}|A|^2(p_k) = nH^2.$$

At this point, we recall that $|A|^2 = \sum_i \kappa_i^2$, where κ_i are the eigenvalues of A. Thus, up to subsequence, for all $1 \leq i \leq n$ we have that $\lim_k \kappa_i(p_k) = \kappa_i^*$ for some $\kappa_i^* \in \mathbb{R}$. Motivated from such fact, we set

$$\frac{n(n-1)}{2}\overline{H}_2 = \sum_{i<j}\kappa_i^*\kappa_j^*$$

and we note that $H = \frac{1}{n}\sum_i \kappa_i^*$. Then, we have $H^2 = \overline{H}_2$ and, for all $1 \le i \le n$, $\kappa_i^* = H$. So, let $\{e_i\}$ be a local orthonormal frame of eigenvectors associated to the eigenvalues $\{\kappa_i\}$ of A. In this setting, we can write $\nabla h = \sum_i \lambda_i e_i$ for some continuous functions λ_i on Σ^n.

From (8) and (13) we have that

$$X(\eta) = -\langle A(X), \partial_t\rangle = -\langle X, A(\partial_t^\top)\rangle = -\langle X, A(\nabla h)\rangle,$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus,

$$\nabla\eta = -A(\nabla h). \tag{40}$$

Hence, by applying once more Lemma 3 to the function η, from (40) we get

$$\begin{aligned} 0 &= \lim_k |A(\nabla h)|^2(p_k) = \sum_i \lim_k \left(\kappa_i^2\lambda_i^2\right)(p_k) \\ &= \sum_i (\kappa_i^*)^2 \lim_k \lambda_i^2(p_k) = H^2 \sum_i \lim_k \lambda_i^2(p_k), \end{aligned}$$

up to subsequence. If $H = 0$, from hypothesis (30), we have immediately that Σ^n is a slice. If $H^2 > 0$ then, for all $1 \le i \le n$, we have that $\lim_k \lambda_i(p_k) = 0$. Thus, $\lim_k |\nabla h|(p_k) = 0$ and, from equation (14),

$$\inf_{p\in\Sigma} \eta(p) = \lim_{k\to\infty} \eta(p_k) = 1.$$

Therefore, $\eta = 1$ on Σ^n and, hence, Σ^n is a slice. ■

Now we treat the case when the mean curvature H is not assumed to be constant, but it is just supposed not to change sign along the hypersurface.

Theorem 2 *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a Riemannian product space whose base M^n has sectional curvature bounded from below, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided complete hypersurface which lies between two slices of $\overline{M}^{n+1}$. Suppose that the angle function η of Σ^n is not adherent to 1 or -1. If H_2 is bounded from below, H is bounded and it does not change sign on Σ^n, then $\inf_\Sigma H = 0$. In particular, if H is constant, then Σ^n is minimal.*

Proof.

First, we note that, as in the proof of Theorem 1, our restrictions on the sectional curvature of the base M^n jointly with the hypothesis on the mean curvatures H and H_2 guarantee that the Ricci curvature of Σ^n is bounded from below.

Suppose for instance that $H \ge 0$ on Σ^n. Since Σ^n lies between two slices of $\mathbb{R} \times M^n$ the height function is bounded. Therefore on a maximizing sequence of points $p_k \in \Sigma^n$ accordingly with Lemma 3 we obtain the following. Firstly, using (19) we get

$$0 \ge \limsup_{k\to\infty} \Delta h(p_k) = n \limsup_{k\to\infty} (H\eta)(p_k).$$

Moreover, from equation (14) we also have that

$$0=\lim_{k\to\infty}|\nabla h|(p_k)=1-\lim_{k\to\infty}\eta^2(p_k).$$

Thus, if we suppose, for instance, that η is not adhere to -1, we get

$$\lim_{k\to\infty}\eta(p_k)=1.$$

Consequently,

$$0\geq\limsup_{k\to\infty}\Delta h(p_k)=n\limsup_{k\to\infty}H(p_k)\geq 0$$

and, hence, we conclude that

$$\limsup_{k\to\infty}H(p_k)=0.$$

If $H\leq 0$, from (14) and (19), we can apply once more Lemma 3 in order to obtain a sequence $q_k\in\Sigma^n$ such that

$$0\leq\liminf_{k\to\infty}\Delta h(q_k)=n\liminf_{k\to\infty}\left(H\eta\right)(q_k)$$

and, supposing once more that η is not adhere to -1, we get

$$0\leq\liminf_{k\to\infty}\Delta h(p_k)=n\liminf_{k\to\infty}H(p_k)\leq 0.$$

Consequently, we have that

$$\liminf_{k\to\infty}H(p_k)=0.$$

Therefore, in this case, we also conclude that $\inf_\Sigma H=0$. ■

Corollary 1 *The only two-sided complete constant mean curvature surfaces of $\mathbb{R}^3$ with Gaussian curvature bounded from below, lying between two planes and whose Gauss map is not adhere to both poles of $\mathbb{S}^2$ which are orthogonal to such planes, are planes of $\mathbb{R}^3$.*

Through Example 8.1, we see that the assumption of that hypersurface Σ^n lies between two slices of $\mathbb{R}\times M^n$ is a necessary hypothesis in Theorem 2 in order to conclude that the mean curvature of Σ^n cannot be globally bounded away from zero. Moreover, we observe that the horizontal circular cylinder $\mathcal{C}\subset\mathbb{R}^3$ satisfies almost all hypothesis of Corollary 1, except to that one which requires the Gauss map N of $\mathcal{C}$ to be not adhere to both poles of $\mathbb{S}^2$ orthogonal to $\mathcal{C}$. Actually, such cylinder is unbounded in all directions where N is isolated.

Rosenberg, Schulze and Spruck (2013) showed that an entire minimal graph with nonnegative height function in a product space $\mathbb{R}\times M^n$, whose base M^n is a complete Riemannian manifold having non-negative Ricci curvature and with sectional curvature bounded from below, must be a slice. Consequently, from Theorem 2 we obtain the following:

Corollary 2 *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature and whose sectional curvature is bounded from below. Let $\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset \mathbb{R} \times M^n$ be an entire graph of a nonnegative smooth function $u : M^n \to \mathbb{R}$, with H constant and H_2 bounded from below. If u is bounded, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

Furthermore, taking into account once more Theorem 2 jointly with Theorem 1.2 in (ROSENBERG, SCHULZE, and SPRUCK, 2013), we also have:

Corollary 3 *Let M^n be a parabolic complete Riemannian manifold whose sectional curvature is bounded. Let $\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset \mathbb{R} \times M^n$ be an entire graph of a smooth function $u : M^n \to \mathbb{R}$, with H constant and H_2 bounded from below. If u is bounded, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

We point out that, in the context of Theorem 2, the constant mean curvature hypersurface will be indeed minimal provided the asked hypothesis. An interesting question that arises from Theorem 1 is whether the constant mean curvature hypersurface trapped between two planes with Gauss map not adherent to both poles is a graph, then is it trivial? Accordingly to Osserman (1959) that hypersurface is indeed a plane when the ambient space is $\mathbb{R}^3$. When the ambient space is a product whose fiber has nonnegative Ricci curvature and sectional curvature bounded from below, Corollary 2 also give a positive answer for such question, provided that the hypersurface is already a graph of a bounded and nonnegative function, while Corollary 3 deals with the parabolic case using that parabolicity is invariant under conformal changes.

4 CALABI-BERNSTEIN RESULTS FOR HYPERSURFACES ON A LORENTZIAN SPACE

4.1 Uniqueness results in $-\mathbb{R} \times M^n$

Here N denotes the future-pointing Gauss map of a spacelike hypersurface $\psi : \Sigma^n \to -\mathbb{R} \times M^n$. In the sequence we show Calabi-Bernstein results on a Lorentzian ambient dual to the ones in the previous Chapter. The results here in this chapter are published in (De LIMA and LIMA JR, 2013).

Theorem 3 *Let $\psi : \Sigma^n \to -\mathbb{R} \times M^n$ be a complete spacelike hypersurface immersed in a Lorentzian product space $-\mathbb{R} \times M^n$, whose sectional curvature K_M of its fiber M^n is such that $-\kappa \leq K_M$ for some positive constant κ. Suppose that Σ^n lies between two slices of $-\mathbb{R} \times M^n$ and that $|\nabla h|$ is bounded on Σ^n. If H is bounded and it does not change sign on Σ^n, then H is not globally bounded away from zero. In particular, if H is constant, then Σ^n is maximal.*

Proof. First, from Lemma 2 we have that our restriction on the sectional curvature of the fiber M^n jointly with the hypothesis that $|\nabla h|$ and H are bounded on Σ^n guarantee that the Ricci curvature of Σ^n is bounded from below.

Now, suppose for instance that $H \geq 0$ on Σ^n. Thus, since Σ^n lies between two slices of $-\mathbb{R} \times M^n$, from equation (19) and Lemma 3 we get a sequence of points $p_k \in \Sigma^n$ such that

$$0 \leq \lim_k \Delta(-h)(p_k) = n \lim_k \left(H \langle N, \partial_t \rangle\right)(p_k).$$

From equation (14) jointly with Lemma 3, we have that

$$0 = \lim_k |\nabla h|(p_k) = \lim_k \langle N, \partial_t \rangle^2 (p_k) - 1.$$

Thus, since $\langle N, \partial_t \rangle \leq -1$,

$$\lim_k \langle N, \partial_t \rangle (p_k) = -1.$$

Consequently,

$$0 \leq \lim_k \Delta(-h)(p_k) = -n \lim_k H(p_k) \leq 0$$

and, hence, we conclude that

$$\lim_k H(p_k) = 0.$$

If $H \leq 0$, with the aid of equations (14) and (19) then applying Lemma 3 , we have a sequence $q_k \in \Sigma^n$ such that

$$0 \leq \lim_k \Delta h(q_k) = -n \lim_k \left(H \langle N, \partial_t \rangle\right)(q_k)$$

and

$$\lim_k \langle N, \partial_t \rangle (q_k) = -1.$$

Therefore, we conclude again that H is not globally bounded away from zero. ■

Remark 2 *We observe that Example 8.1 shows that the spacelike hypersurface Σ^n to lie between two slices of $-\mathbb{R} \times M^n$ is a necessary hypothesis in Theorem 3 to conclude that the mean curvature of Σ^n can not be globally bounded away from zero.*

In what follows, let us consider the product model of the Minkowski space $\mathbb{L}^{n+1}$, that is, $\mathbb{L}^{n+1} \simeq -\mathbb{R} \times \mathbb{R}^n$.

Taking into account Remark 1 and that the only complete maximal spacelike hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes (see (CALABI, 1970), for $n \leq 4$, and (CHENG and YAU, 1976), for arbitrary n), from Theorem 3 we get the following

Corollary 4 (Theorem 1 in (ALEDO and ALÍAS, 2000)) *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a complete spacelike hypersurface with constant mean curvature H and which lies between two parallel spacelike hyperplanes of $\mathbb{L}^{n+1}$. Then, Σ^n is a hyperplane.*

Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike hypersurface. We note that the future-pointing timelike unit normal vector field $N \in \mathfrak{X}(\Sigma)$ can be regarded as the Gauss map $N : \Sigma^n \to \mathbb{H}^n$ of Σ^n, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space, that is,

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -1, \, x_{n+1} \geq 1\}.$$

In this setting, the image $N(\Sigma)$ is called the *hyperbolic image* of Σ^n. Furthermore, given a hyperbolic geodesic ball $B(a, \varrho) \subset \mathbb{H}^n$ of radius $\varrho > 0$ and centered at a point $a \in \mathbb{H}^n$, we recall that $B(a, \varrho)$ is characterized as the following

$$B(a, \varrho) = \{p \in \mathbb{H}^n; -\cosh \varrho \leq \langle p, a \rangle \leq -1\}.$$

So, if the hyperbolic image of Σ^n is contained into some $B(a, \varrho)$, then

$$1 \leq |\langle N, a \rangle| \leq \cosh \varrho.$$

From Theorem 3 we obtain the following result, which can be regarded as a sort of extension of the result due to Xin (1991) and Aiyama (1992).

Corollary 5 (Theorem 1.1 in (De LIMA, 2011)) *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a complete spacelike hypersurface which lies between two parallel spacelike hyperplanes of $\mathbb{L}^{n+1}$. Suppose that the mean curvature H is bounded and that it does not change sign on Σ^n. If the hyperbolic image of Σ^n is contained in the closure of a hyperbolic geodesic ball of radius ϱ which satisfies $\cosh \varrho \leq 1 + \inf_\Sigma |H|$, then Σ^n is a hyperplane.*

Let S_2 denote the second elementary symmetric function on the eigenvalues of the second fundamental form A of Σ^n, and

$$H_2 = \frac{2}{n(n-1)} S_2$$

denotes the mean value of S_2. Elementary algebra gives

$$|A|^2 = n^2 H^2 - n(n-1) H_2. \tag{41}$$

Our next result is an extension of Theorems 3.1 and 3.2 in (ALBUJER, CAMARGO, and de LIMA, 2010).

Theorem 4 *Let $\psi : \Sigma^n \to -\mathbb{R} \times M^n$ be a complete spacelike hypersurface immersed with constant mean curvature H in a Lorentzian product space $-\mathbb{R} \times M^n$, whose sectional curvature K_M of its fiber M^n is such that $-\kappa \leq K_M$ for some positive constant κ. Suppose that one of the following conditions is satisfied:*

(a) *The height function h of Σ^n is such that*

$$|\nabla h|^2 \leq \frac{n}{\kappa(n-1)} H^2. \tag{42}$$

(b) *H_2 is bounded from below on Σ^n and the height function h of Σ^n is such that, for some constant $0 < \alpha < 1$,*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)} |A|^2. \tag{43}$$

Then, Σ^n is a slice.

Proof. First, let us suppose that the condition of item (a) is satisfied. From hypothesis (42), we get

$$\langle N, \partial_t \rangle^2 = 1 + |\nabla h|^2 \leq 1 + \frac{n}{\kappa(n-1)} H^2.$$

Consequently, we have that the infimum $\inf_{p \in \Sigma} \langle N, \partial_t \rangle(p)$ exists and is a negative number.

We easily see that $\langle N^*, N^* \rangle_M = |\nabla h|^2$, where $N^* = N + \langle N, \partial_t \rangle \partial_t$ is the projection of N onto the fiber M^n. Consequently, taking a local orthonormal frame $\{E_1, \cdots, E_n\}$ on M^n, we have that the Ricci curvature Ric_M of M^n is such that

$$\begin{aligned}
\mathrm{Ric}_M(N^*, N^*) &= \sum_i \langle R_M(N^*, E_i) N^*, E_i \rangle_M \\
&= \sum_i K_M(N^*, E_i) \left(\langle N^*, N^* \rangle_M - \langle N^*, E_i \rangle_M^2 \right) \\
&\geq -\kappa \sum_i \left(\langle N^*, N^* \rangle_M - \langle N^*, E_i \rangle_M^2 \right) \\
&= -\kappa (n-1) |\nabla h|^2,
\end{aligned}$$

where we have used our restriction on the sectional curvature K_M of M^n. Thus, using again hypothesis (42), from Lemma 1 and equation (41) we obtain

$$\begin{aligned}\Delta\langle N,\partial_t\rangle &\leq \left(n^2H^2 - n(n-1)H_2 - \kappa(n-1)|\nabla h|^2\right)\langle N,\partial_t\rangle\\ &= \left(nH^2 + n(n-1)(H^2-H_2) - \kappa(n-1)|\nabla h|^2\right)\langle N,\partial_t\rangle\\ &\leq n(n-1)(H^2-H_2)\langle N,\partial_t\rangle.\end{aligned}$$

Now, taking into account Lemma 2, we can apply Lemma 3 to obtain a sequence of points $p_k \in \Sigma^n$ such that

$$0 \leq \lim_k \Delta\langle N,\partial_t\rangle(p_k) \leq n(n-1)\inf_{p\in\Sigma}\langle N,\partial_t\rangle(p)\lim_k(H^2-H_2)(p_k) \leq 0.$$

Consequently,

$$\lim_k(H^2-H_2)(p_k) = 0.$$

Moreover, since $|A|^2 = nH^2 + n(n-1)(H^2-H_2)$, we get

$$\lim_k |A|^2(p_k) = nH^2.$$

At this point, we recall that $|A|^2 = \sum_i \kappa_i^2$, where κ_i are the eigenvalues of A. Thus, up to subsequence, for all $1 \leq i \leq n$ we have that $\lim_k \kappa_i(p_k) = \kappa_i^*$ for some $\kappa_i^* \in \mathbb{R}$.

We set

$$\frac{n(n-1)}{2}\overline{H}_2 = \sum_{i<j}\kappa_i^*\kappa_j^*.$$

We note that $H = -\frac{1}{n}\sum_i \kappa_i^*$. Then, we have $H^2 = \overline{H}_2$ and, for all $1 \leq i \leq n$, $\kappa_i^* = -H$. So, let $\{e_i\}$ be a local orthonormal frame of eigenvectors associated to the eigenvalues $\{\kappa_i\}$ of A. In this setting, we can write $\nabla h = \sum_i \lambda_i e_i$ for some smooth functions λ_i on Σ^n. Thus, up to subsequence, from Lemma 3 we get

$$\begin{aligned}0 &= \lim_k |A(\nabla h)|^2(p_k) = \sum_i \lim_k\left(\kappa_i^2\lambda_i^2\right)(p_k)\\ &= \sum_i (\kappa_i^*)^2\lim_k \lambda_i^2(p_k) = H^2\sum_i \lim_k \lambda_i^2(p_k).\end{aligned}$$

If $H=0$, from hypothesis (42), we have immediately that Σ^n is a slice. If $H^2>0$ then, for all $1 \leq i \leq n$, we have that $\lim_k \lambda_i(p_k) = 0$. Thus, $\lim_k |\nabla h|(p_k) = 0$ and, from equation (14),

$$\inf_{p\in\Sigma}\langle N,\partial_t\rangle(p) = \lim_k\langle N,\partial_t\rangle(p_k) = -1.$$

Therefore, $\langle N,\partial_t\rangle = -1$ on Σ^n, that is, Σ^n is a slice.

Let us consider the case that the item (*b*) is satisfied. Reasoning as in the previous case, we show that $\inf_{p\in\Sigma}\langle N,\partial_t\rangle(p)$ exists and it is negative. Moreover, using again that $\mathrm{Ric}_M(N^*,N^*)\geq -\kappa(n-1)|\nabla h|^2$, from Lemma 1 and hypothesis (43) we get

$$\begin{aligned}\Delta\langle N,\partial_t\rangle &\leq \left(|A|^2-\kappa(n-1)|\nabla h|^2\right)\langle N,\partial_t\rangle\\ &\leq (1-\alpha)|A|^2\langle N,\partial_t\rangle\leq 0.\end{aligned}$$

Taking into account once more Lemma 2, we can use Lemma 3 to guarantee the existence of a sequence of points $p_k\in\Sigma^n$ such that $\lim_k\Delta\langle N,\partial_t\rangle(p_k)\geq 0$ and $\lim_k\langle N,\partial_t\rangle(p_k)=\inf_{p\in\Sigma}\langle N,\partial_t\rangle$. Consequently, $\lim_k\langle N,\partial_t\rangle^2(p_k)=\sup_{p\in\Sigma}\langle N,\partial_t\rangle^2$.

Thus,

$$0\leq\lim_k\Delta\langle N,\partial_t\rangle(p_k)\leq(1-\alpha)\lim_k|A|^2(p_k)\inf_{p\in\Sigma}\langle N,\partial_t\rangle\leq 0.$$

It follows that $\lim_k|A|^2(p_k)=0$. Now, by using the hypothesis (43), we obtain that $\lim_k|\nabla h|^2(p_k)=0$, what it implies by equation (14) that $\sup_{p\in\Sigma}\langle N,\partial_t\rangle^2=\lim_k\langle N,\partial_t\rangle^2(p_k)=1$. But $\langle N,\partial_t\rangle^2\geq 1$, hence, $\langle N,\partial_t\rangle^2=1$ on Σ^n and, therefore, Σ^n is a slice. ■

Remark 3 *As observed in (ALBUJER, CAMARGO, and de LIMA, 2010), in Theorems 3 and 4, a geometrical interpretation of our restriction on the norm of the gradient of the height function h involves the notion of normal hyperbolic angle. More precisely, if $\psi:\Sigma^n\to-\mathbb{R}\times M^n$ is a spacelike hypersurface oriented by the timelike unit vector field N such that $\langle N,\partial_t\rangle<0$, the* normal hyperbolic angle *θ of ψ is the smooth function $\theta:\psi(\Sigma)\to[0,+\infty)$ such that $\cosh\theta=-\langle N,\partial_t\rangle\geq 1$. Thus, from equation (14), we have that $|\nabla h|^2=\cosh^2\theta-1$. Consequently, the conditions on the growth of the height function h can be interpreted geometrically as a boundedness of the normal hyperbolic angle θ of the spacelike hypersurface Σ^n.*

Theorem 5 *Let $\psi:\Sigma^n\to-\mathbb{R}\times M^n$ be a complete spacelike hypersurface such that its mean curvature H does not change sign. If $|\nabla h|\in\mathcal{L}^1(\Sigma)$, then Σ^n is maximal. In addition, if H_2 is bounded from below on Σ^n and the Ricci curvature Ric_M of the fiber M^n is non-negative, then Σ^n is totally geodesic. Moreover, if Ric_M is strictly positive, then Σ^n is a slice.*

Proof. Since we are supposing that H does not change sign on Σ^n, from equation (19) we conclude that Δh also does not change sign on Σ^n. Thus, from Lemma 6, our hypothesis $|\nabla h|\in\mathcal{L}^1(\Sigma)$ guarantees that h is harmonic and, using again equation (19), we conclude that Σ^n is maximal.

From (13) and (17) we have that

$$X\left(\langle N,\partial_t\rangle\right)=-\langle A(X),\partial_t\rangle=-\langle X,A(\partial_t^\top)\rangle=\langle X,A(\nabla h)\rangle,$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus,

$$\nabla\langle N, \partial_t\rangle = A(\nabla h).$$

Consequently, since $H = 0$ and H_2 bounded from below imply that $|A|$ is bounded on Σ^n, we get

$$|\nabla\langle N, \partial_t\rangle| \leq |A||\nabla h| \in \mathcal{L}^1(\Sigma).$$

So, if the Ricci curvature Ric_M of the fiber M^n is non-negative, from Lemmas 1 and 6, we conclude that $\langle N, \partial_t\rangle$ is harmonic. Hence, using Lemma 1, we have $|A| = 0$ on Σ^n, that is, Σ^n is totally geodesic. Moreover, if Ric_M is strictly positive, $\mathrm{Ric}_M(N^*, N^*) = 0$ implies that $N^* = 0$ on Σ^n. Therefore, N is parallel to ∂_t, that is, Σ^n is a slice. ■

From Theorem 5 jointly with Theorem 3.3 in (ALBUJER and ALÍAS, 2009) (see also (ALBUJER, 2008a) for another approach of such result), we get the following

Corollary 6 *Let M^2 be a complete Riemannian surface with nonnegative Gaussian curvature K_M, and let $\psi : \Sigma^2 \to -\mathbb{R} \times M^2$ be a complete spacelike hypersurface such that its mean curvature H does not change sign. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then Σ^2 is totally geodesic. In addition, if $K_M > 0$ at some point on M^2, then Σ^2 is a slice.*

As another consequence of Theorem 5, we also obtain a sort of extension of the classical theorem of Cheng-Yau (1976).

Corollary 7 *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a complete spacelike hypersurface such that the mean curvature H does not change sign. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then Σ^n is a hyperplane.*

4.2 Entire spacelike vertical graphs in $-\mathbb{R} \times M^n$

In this context, we obtain a non-parametric version of Theorem 3.

Corollary 8 *Let $\Sigma^n(u)$ be an entire spacelike vertical graph in a Lorentzian product space $-\mathbb{R} \times M^n$, whose fiber M^n is complete and such that its sectional curvature K_M satisfies $-\kappa \leq K_M \leq 0$ for some positive constant κ. Suppose that $\Sigma^n(u)$ lies between two slices of $-\mathbb{R} \times M^n$. If $|Du| \leq \alpha$, for some constant $0 < \alpha < 1$, and H does not change sign on $\Sigma^n(u)$, then $\Sigma^n(u)$ is complete and H is not globally bounded away from zero. In particular, if H is constant, then $\Sigma^n(u)$ is maximal.*

Proof.

Observe first that, under the assumptions of the theorem, $\Sigma^n(u)$ is a complete hypersurface. In fact, from (25) and the Cauchy-Schwarz inequality we get

$$\langle X, X\rangle = \langle X, X\rangle_M - \langle Du, X\rangle_M^2 \geq (1 - |Du|^2)\langle X, X\rangle_M$$

for every tangent vector field X on $\Sigma^n(u)$. Therefore,

$$\langle X, X\rangle \geq (1 - \alpha^2)\langle X, X\rangle_M.$$

This implies that $L \geq \sqrt{c}L_M$, where L and L_M denote the length of a curve on $\Sigma^n(u)$ with respect to the Riemannian metrics $\langle , \rangle$ and $\langle , \rangle_M$, respectively, and $c = 1 - \alpha^2$. As a consequence, since we are supposing that M^n is complete, then the induced metric on $\Sigma^n(u)$ from the metric of $-\mathbb{R} \times M^n$ is also complete.

Since $N = -\langle N, \partial_t \rangle \partial_t + N^*$ where N^* denotes the projection of N onto the fiber M^n, from equation (13) we get

$$N^{*\top} = -\langle N, \partial_t \rangle \nabla h$$

and

$$|\nabla h|^2 = \langle N^*, N^* \rangle_M.$$

Thus, since

$$N = \frac{1}{\sqrt{1 - |Du|^2}}(\partial_t + Du),$$

we obtain that

$$|\nabla h|^2 = \frac{|Du|^2}{1 - |Du|^2}.$$

Therefore, if $|Du| \leq \alpha$ for some constant $0 < \alpha < 1$, we conclude that

$$|\nabla h|^2 \leq \frac{\alpha^2}{1 - \alpha^2}$$

and, hence, the result follows from Theorem 3. ■

In an analogous way, we can also obtain the following non-parametric version of Theorems 4 and 5,

Corollary 9 *Let $\Sigma^n(u)$ be an entire spacelike vertical graph immersed with constant mean curvature H in a Lorentzian product space $-\mathbb{R} \times M^n$, whose fiber M^n is complete and such that its sectional curvature K_M satisfies $-\kappa \leq K_M$ for some positive constant κ. Suppose that one of the following conditions is satisfied:*

(a) $|Du|^2 \leq \dfrac{nH^2}{\kappa(n-1) + nH^2}$.

(b) *H_2 is bounded from below on $\Sigma^n(u)$ and $|Du|^2 \leq \dfrac{\alpha |A|^2}{\kappa(n-1) + \alpha|A|^2}$, for some constant $0 < \alpha < 1$.*

Then, $\Sigma^n(u)$ is a slice.

Corollary 10 *Let $\Sigma^n(u)$ be an entire spacelike vertical graph in a Lorentzian product space $-\mathbb{R} \times M^n$, whose fiber M^n is complete. Suppose that the mean curvature H does not change sign on $\Sigma^n(u)$. If $|Du| \leq \alpha$, for some constant $0 < \alpha < 1$, and $|Du| \in \mathcal{L}^1(M)$, then $\Sigma^n(u)$ is complete and maximal. In addition, if H is constant, H_2 is bounded from below on $\Sigma^n(u)$ and the Ricci curvature Ric_M of the fiber M^n is non-negative, then $\Sigma^n(u)$ is totally geodesic. Moreover, if Ric_M is strictly positive, then $\Sigma^n(u)$ is a slice.*

Remark 4 *Salavessa (2008) have described an explicit foliation of* $-\mathbb{R} \times \mathbb{H}^n$ *by complete spacelike graphs with constant mean curvature c, for any constant c.*

Proceeding as before from Corollary 10 and the results in (ALBUJER and ALÍAS, 2009) and (ALBUJER and ALÍAS, 2011)), we obtain

Corollary 11 *Let* $\Sigma^2(u)$ *be an entire spacelike vertical graph in a Lorentzian product space* $-\mathbb{R} \times M^2$*, whose fiber* M^2 *is a complete Riemannian surface with nonnegative Gaussian curvature* K_M*. Suppose that the mean curvature* H *does not change sign on* $\Sigma^2(u)$*. If* $|Du| \leq \alpha$*, for some constant* $0 < \alpha < 1$*, and* $|Du| \in \mathcal{L}^1(M)$*, then* $\Sigma^2(u)$ *is complete and totally geodesic. In addition, if* $K_M > 0$ *at some point of* M^2*, then* $\Sigma^2(u)$ *is a slice.*

5 UNIQUENESS FOR SPATIALLY CLOSED SPACES

5.1 Introduction

It was conjectured by Goddard (1977) that in the de Sitter space $\mathbb{S}_1^{n+1}$, the only complete spacelike hypersurfaces with constant mean curvature should be the totally umbilical ones. This conjecture, regardless the veracity, motivated the work of an impressive number of authors who considered the problem of characterizing the totally umbilical spacelike hypersurfaces of de Sitter space in terms of some appropriate geometric assumptions. In particular, Akutagawa (1987) showed that Goddard's conjecture is true if the constant mean curvature H of the hypersurface satisfies $H^2 \leq 4(n-1)/n^2$. As an application of it, Akutagawa also proved that when $n = 2$ Goddard's conjecture is also true under the additional hypothesis of the compactness of the surface (see also (RAMANATHAN, 1987) for a simultaneous and independent alternative proof for $n = 2$). Afterwards, Montiel (1988) extended this last result to the general case by showing that the only compact spacelike hypersurfaces in de Sitter space are the totally umbilical round spheres.

Later on, Aledo and Alías (2002) studied complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ whose image of the Gauss mapping is contained in a geodesic ball of the hyperbolic space $\mathbb{H}^{n+1}$, showing that such a hypersurface Σ^n is necessarily compact and obtaining sharp estimates concerning the n-dimensional volume of Σ^n. As an application of their result, they also concluded that Goddard's conjecture is true under the assumption that the image of the Gauss mapping of the spacelike hypersurface is bounded. Next, S. Montiel (2003) have proved that if a complete spacelike hypersurface Σ^n in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H \geq 1$ is such that the image of its Gauss mapping is contained in the closure of the interior domain enclosed by a horosphere of $\mathbb{H}^{n+1}$, then its mean curvature is, in fact, equal to 1. When $n = 2$, this implies that Σ^2 is also an umbilical surface.

In this chapter, we study the geometry of complete hypersurfaces immersed with constant mean curvature in a *spatially closed* Lorentzian product space $-\mathbb{R} \times M^n$, that is, the Riemannian fiber M^n is compact; for a thorough discussion about this class of spacetimes, see for example (ALÍAS, ROMERO, and SANCHEZ, 1995) and (ALÍAS and COLARES, 2007).

5.2 Characterization of CMC hypersurfaces on spatially closed spaces

Recall that the normal hyperbolic angle θ of the spacelike hypersurface Σ^n is defined by $\cosh\theta = -\langle N, \partial_t\rangle$, where N stands for the future-pointing Gauss map of Σ^n and ∂_t denotes the coordinate vector field induced by the universal time on the Lorentzian product space $-\mathbb{R} \times M^n$. The following results are in (AQUINO, De LIMA, and LIMA, 2014).

Theorem 6 *Let Σ^n be a complete spacelike hypersurface immersed with constant mean curvature in a spatially closed Lorentzian product space $-\mathbb{R} \times M^n$, whose fiber M^n has positive sectional curvature. If the normal hyperbolic angle of Σ^n is bounded, then Σ^n is a slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$.*

Proof.

From the Gauss equation (11), taking a (local) orthonormal frame $\{E_1, \cdots, E_n\}$ in $\mathfrak{X}(\Sigma)$, we have that the Ricci curvature Ric of Σ^n is given by

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n} \langle \overline{R}(X,E_i)X, E_i\rangle + nH\langle AX, X\rangle + |AX|^2, \tag{44}$$

for $X \in \mathfrak{X}(\Sigma)$. Moreover, we have that

$$\begin{aligned} \langle \overline{R}(X,E_i)X,E_i\rangle &= \langle R(X^*,E_i^*)X^*,E_i^*\rangle_M \\ &= K_M(X^*,E_i^*)(\langle X^*,X^*\rangle_M\langle E_i^*,E_i^*\rangle_M - \langle X^*,E_i^*\rangle_M^2). \end{aligned} \tag{45}$$

where $X^* = X + \langle X, \partial_t\rangle\partial_t$ and $E_i^* = E_i + \langle E_i, \partial_t\rangle\partial_t$ are the projections of the tangent vector fields X and E_i onto the fiber M^n, respectively.

Now, taking into account (13), with a straightforward computation we see that

$$\langle X^*,X^*\rangle_M\langle E_i^*,E_i^*\rangle_M = (1+\langle E_i,\nabla h\rangle^2)(|X|^2+\langle X,\nabla h\rangle^2)$$

and

$$\begin{aligned} \langle X^*,E_i^*\rangle_M^2 &= \langle X,E_i\rangle^2 + 2\langle X,\nabla h\rangle\langle E_i,\nabla h\rangle\langle X,E_i\rangle \\ &\quad + \langle X,\nabla h\rangle^2\langle E_i,\nabla h\rangle^2. \end{aligned}$$

Thus, since M^n is compact with $K_M > 0$, there exists a positive constant κ such that

$$\sum_{i=1}^{n} \langle \overline{R}(X,E_i)X,E_i\rangle \geq \kappa\left((n-1)|X|^2 + (n-2)\langle X,\nabla h\rangle^2 + |X|^2|\nabla h|^2\right). \tag{46}$$

Hence, from (44) and (46) we obtain

$$\begin{aligned} \mathrm{Ric}(X,X) &\geq \kappa\left((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X,\nabla h\rangle^2\right) \\ &\quad + nH\langle AX,X\rangle + |AX|^2. \end{aligned} \tag{47}$$

Here we can observe that we can write

$$nH\langle AX,X\rangle + |AX|^2 = \left|AX + \frac{nH}{2}X\right|^2 - \frac{n^2H^2}{4}|X|^2. \tag{48}$$

Thus, from equation (47) and (48) we obtain that

$$\mathrm{Ric}_\Sigma(X,X) \geq -\frac{n^2H^2}{4}|X|^2, \tag{49}$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus, since H is constant, we conclude from (49) that the Ricci curvature of Σ^n is bounded from below. Now consider Bochner's formula, (BOCHNER, 1946),

$$\frac{1}{2}\Delta\left(|\nabla h|^2\right) = |\nabla^2 h|^2 + \mathrm{Ric}(\nabla h, \nabla h) + \langle \nabla(\Delta h), \nabla h\rangle. \tag{50}$$

Since H is constant, from (19) and (21) we also have

$$\nabla(\Delta h) = -nHA(\nabla h). \tag{51}$$

Thus, from (47) and (51) we get

$$\mathrm{Ric}(\nabla h, \nabla h) \geq (n-1)\kappa|\nabla h|^2(1+|\nabla h|^2) + nH\langle A(\nabla h), \nabla h\rangle + |A(\nabla h)|^2. \tag{52}$$

Then from (50), (51) and (52) we get

$$\frac{1}{2}\Delta\left(|\nabla h|^2\right) \geq |\nabla^2 h|^2 + (n-1)\kappa|\nabla h|^2(1+|\nabla h|^2) + |A(\nabla h)|^2. \tag{53}$$

At this point, we observe that from (16) our hypothesis that the normal hyperbolic angle of Σ^n is bounded implies that the function $|\nabla h|^2$ is also bounded on Σ^n. Hence, from Lemma 3 we have that there exists a sequence of points $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k |\nabla h|^2(p_k) = \sup_\Sigma |\nabla h|^2 \quad \text{and} \quad \lim_k \sup \Delta\left(|\nabla h|^2\right)(p_k) \leq 0.$$

Thus, from (53) we have that

$$0 \geq \lim_k \sup \Delta\left(|\nabla h|^2\right)(p_k) \geq \kappa \sup_\Sigma |\nabla h|^2 \geq 0.$$

Consequently, we obtain that $\sup_\Sigma |\nabla h|^2 = 0$ and, hence, h is constant on Σ^n. Therefore, Σ^n is a slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. ■

The next result deals with the half space property, that is, when hypersurface is vertically half bounded it must be a slice.

Theorem 7 *Let Σ^n be a complete spacelike hypersurface immersed with constant mean curvature in Lorentzian product space $-\mathbb{R}\times M^n$, whose fiber M^n has non-negative sectional curvature. If the normal hyperbolic angle of Σ^n is bounded, then Σ^n is maximal. Moreover if $\Sigma(u)$ is vertically half bounded then it is a slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$.*

Proof. Analogously to the proof of Theorem 6 we see that the Ricci curvature is bounded from below and by the same bounds. Thus, taking into account that

$n|\nabla^2 h|^2 \geq (\Delta h)^2$ from (53) we get

$$\Delta|\nabla h|^2 \geq \frac{2}{n}(\Delta h)^2 \tag{54}$$

From (19) and (54) we get

$$\Delta|\nabla h|^2 \geq 2nH^2\langle N, \partial_t\rangle^2 \geq 2nH^2. \tag{55}$$

As before for a sequence $(p_k)_{k\geq 1} \in \Sigma^n$ given by Lemma 3 we have from (55) that

$$0 \geq \limsup_k \Delta\left(|\nabla h|^2\right)(p_k) \geq 2nH^2 \geq 0.$$

Then Σ is Maximal. Consequently, from (49) we obtain that Ric is non-negative on Σ and from (51) we get h is harmonic. Using Lemma 5 h must be constant. ■

As observed by Espinar and Rosenberg (2009), we see that our restriction on θ can be interpreted as the image $N(\Sigma)$ of the Gauss map N of Σ^n being bounded. Consequently, Theorem 6 can be regarded as a natural version of theorems of Xin (1991) and Aiyama (1992), and Aledo-Alías (2002) to the context of Lorentzian product spaces. Through Example 4.4 in (De LIMA, 2014) we see that Theorem 6 does not hold when the fiber M^n of the ambient spacetime $-\mathbb{R} \times M^n$ has negative sectional curvature.

Using the integrability condition we can also obtain the following:

Theorem 8 *Let Σ^n be a complete spacelike hypersurface immersed with constant mean curvature in a spatially closed Lorentzian product space $-\mathbb{R} \times M^n$, whose fiber M^n has positive sectional curvature. If $\nabla h \in L^1 \cap L^\infty(\Sigma)$ and $A \in L^\infty(\Sigma)$ then Σ^n is a slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$.*

Proof. From inequality (53) we get

$$\frac{1}{2}\Delta\left(|\nabla h|^2\right) \geq (n-1)\kappa|\nabla h|^2 + |A(\nabla h)|^2. \tag{56}$$

Note that

$$\nabla|\nabla h|^2 = \nabla\langle N, \partial_t\rangle^2 = 2\langle N, \partial_t\rangle\nabla\langle N, \partial_t\rangle = 2\langle N, \partial_t\rangle A(\nabla h)$$

So it is integrable. Integrating (56) obtain

$$0 \geq \int_\Sigma \kappa|\nabla h|^2 d\Sigma.$$

Since $\kappa > 0$ we see that h must be constant. ■

Remark 5 *Albujer and Alías (2009) established Calabi-Bernstein results for maximal surfaces immersed into a Lorentzian product space $-\mathbb{R} \times M^2$. In particular, when M^2 is a Riemannian surface with nonnegative Gaussian curvature, they proved that any complete maximal surface in $-\mathbb{R} \times M^2$ must be totally geodesic. Besides, if M^2 is non-flat, the* *authors concluded that it must be a slice* $\{t\} \times M^2$*. Li and Salavessa (2009) generalized such results of Albujer and Alías to higher dimension and codimension.*

From inequality (53), we can apply Hopf's theorem in order to obtain the following:

Corollary 12 *The only compact spacelike hypersurfaces immersed with constant mean curvature in a spatially closed Lorentzian product space* $-\mathbb{R} \times M^n$*, whose fiber* M^n *has positive sectional curvature, are the slices* $\{t\} \times M^n$*.*

6 BOUNDED HYPERSURFACES IN VERTICAL REGION OF $-\mathbb{R} \times M^n$

6.1 CMC Graphs in $-\mathbb{R} \times M^n$

The next results shows that we can improve the Calabi-Bernstein results in Chapter 4 for the case the graphs are vertically bounded these results are accepted to be published in (De LIMA and LIMA JR, 2015).

Theorem 9 *Let $\overline{M}^{n+1} = -\mathbb{R} \times M^n$ be a Lorentzian product space, such that the sectional curvature K_M of its Riemannian fiber M^n satisfies $K_M \geq -\kappa$, for some positive constant κ. Let $\Sigma(u)$ be an entire H-graph over M^n, with u and H_2 bounded from below. If*

$$|Du|_M^2 \leq \frac{|A|^2}{\kappa(n-1) + |A|^2}, \tag{57}$$

then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.

Proof.

Observe first that, under the assumptions of the theorem, $\Sigma(u)$ is indeed a complete spacelike hypersurface. In fact, from (25) and the Cauchy-Schwarz inequality we get

$$\langle X, X \rangle = \langle X^*, X^* \rangle_M - \langle Du, X^* \rangle_M^2 \geq (1 - |Du|_M^2)\langle X^*, X^* \rangle_M, \tag{58}$$

for every tangent vector field X on $\Sigma(u)$.

Recall that the Hilbert-Schmidt norm of the shape operator A of $\Sigma(u)$ satisfies the following algebraic identity

$$|A|^2 = n^2 H^2 - n(n-1)H_2. \tag{59}$$

Since H is constant and H_2 is supposed to be bounded from below, from (59) it holds that $\sup_{p \in \Sigma(u)} |A_p|^2 < +\infty$. From (57) we see that there exists a constant $0 < \alpha < 1$ such that $|Du|_M \leq \alpha$. Hence, from (58) we get

$$\langle X, X \rangle \geq (1 - \alpha^2)\langle X^*, X^* \rangle_M.$$

This implies that $L \geq \sqrt{c} L_M$, where L and L_M denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics $\langle , \rangle$ and $\langle , \rangle_M$, respectively, and $c = 1 - \alpha^2$. As a consequence, since M^n is complete, the induced metric on $\Sigma(u)$ from the metric of $-\mathbb{R} \times M^n$ is also complete.

Now, let us consider on $\Sigma(u)$ the functions $\eta = 1 - e^{-ku}$ and $W = \sqrt{1 - |Du|_M^2}$. Since we are supposing that u is bounded from below, we have that the function $\vartheta = \eta W$ is bounded from below. We claim that the Ricci curvature of $\Sigma(u)$ is also bounded from below. Hence, we are in conditions to apply Lemma 3 to the function ϑ, obtaining a sequence of points $\{p_{k,\varepsilon}\}$ in $\Sigma(u)$ such that, for each fixed $k > 0$,

$$|\nabla\vartheta|(p_{k,\varepsilon}) \leq \varepsilon \quad , \quad \vartheta(p_{k,\varepsilon}) \leq \inf_{\Sigma(u)} \vartheta + \varepsilon \quad \text{and} \quad \Delta\vartheta(p_{k,\varepsilon}) \geq -\varepsilon.$$

Hence, along this minimizing sequence $\{p_{k,\varepsilon}\}$, we have

$$\nabla\vartheta = \nabla\eta W = -\eta\nabla W + \varepsilon v_{k,\varepsilon}, \tag{60}$$

for $v_{k,\varepsilon}$ vectors satisfying $|v_{k,\varepsilon}| \leq 1$. Computing $\Delta\vartheta$ we obtain

$$\Delta\vartheta = \Delta(\eta W) = W\Delta\eta + \eta\Delta W + 2\langle\nabla W, \nabla\eta\rangle \tag{61}$$

Therefore, from (60) and (61) we get

$$\Delta\vartheta = W\Delta\eta + \eta\left(\Delta W - 2\frac{|\nabla W|^2}{W}\right) + \frac{2\varepsilon}{W}\langle\nabla W, v_{k,\varepsilon}\rangle. \tag{62}$$

Since $N = -\langle N, \partial_t\rangle\partial_t + N^*$ where N^* denotes the projection of N onto the fiber M^n, from equation (13) it is not difficult to see that $N^{*\top} = -\langle N, \partial_t\rangle\nabla h$ where $\top$ denotes the tangent part with respect to the graph. From (26) we obtain

$$|\nabla h|^2 = \langle N^*, N^*\rangle_M \text{ and } \langle N, \partial_t\rangle = -\frac{1}{W}. \tag{63}$$

Here we used that

$$|\nabla h|^2 = \frac{|Du|_M^2}{1 - |Du|_M^2}. \tag{64}$$

Hence, taking into account that

$$\Delta\left(\frac{1}{W}\right) = -\frac{1}{W^2}\left(\Delta W - \frac{2|\nabla W|^2}{W}\right),$$

we can use formula (22) to rewrite (62) as

$$\Delta\vartheta = W\Delta\eta - \eta(\mathrm{Ric}_M(N^*, N^*) + |A|^2)W + \frac{2\varepsilon}{W}\langle\nabla W, v_{k,\varepsilon}\rangle. \tag{65}$$

Hence, along the minimizing sequence $\{p_{k,\varepsilon}\}$, we get

$$-\varepsilon \leq W\Delta\eta - \eta(\mathrm{Ric}_M(N^*, N^*) + |A|^2)W + \frac{2\varepsilon}{W}\langle\nabla W, v_{k,\varepsilon}\rangle. \tag{66}$$

Since we are assuming that $K_M \geq -\kappa$ for some positive constant κ, we have

$$\mathrm{Ric}_M(N^*, N^*) \geq -\kappa(n-1)|N^*|^2 = -\kappa(n-1)|\nabla h|^2.$$

But, from (57) and (64) it holds that

$$|\nabla h|^2 \leq \frac{|A|^2}{\kappa(n-1)}.$$

Consequently,

$$\mathrm{Ric}_M(N^*, N^*) + |A|^2 \geq -\kappa(n-1)|\nabla h|^2 + |A|^2 \geq 0. \tag{67}$$

Using inequalities (66) and (67), on the minimizing Omori-Yau sequence $\{p_{k,\varepsilon}\}$ we obtain

$$-\varepsilon\left(\frac{W + 2|\nabla W|}{W^2}\right) \leq \Delta\eta$$

or, equivalently,

$$-\,\varepsilon\left(\frac{W + 2|\nabla W|}{W^2}\right) \leq e^{-ku}(k\Delta h - k^2|\nabla h|^2). \tag{68}$$

Hence, taking into account (19) and (29), from (68) we must have

$$-\,\varepsilon e^{ku}\left(W + 2|\nabla W|\right) \leq (-nHkW - k^2|Du|_M^2). \tag{69}$$

We claim that ∇W is also bounded. In fact, from (63) we have that

$$\nabla W = -\frac{1}{\langle N, \partial_t\rangle^2}\nabla\langle N, \partial_t\rangle$$

and, hence,

$$|\nabla W| \leq W^2|A||\nabla h| \leq W^2\frac{|A|^2}{\sqrt{\kappa(n-1)}}.$$

Thus, letting $\varepsilon \to 0$ in (69) and taking the lim inf on ε, we obtain the following estimate

$$0 \leq n|H|\liminf_{\varepsilon\to 0} W - k\limsup_{\varepsilon\to 0}|Du|_M^2. \tag{70}$$

Now, multiplying (70) by $\dfrac{1}{k}$ and making $k \to \infty$ as we take the lim inf over k we get the next

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}|Du|_M^2 = 0. \tag{71}$$

Consequently, since $W^2 = 1 - |Du|_M^2$, we have

$$\liminf_{k\to\infty}\liminf_{\varepsilon\to 0} W^2 = 1. \tag{72}$$

Since these sequences are minimizing, by Lemma 3 on an arbitrary point we have the ensuing

$$\eta^2(p_{k,\varepsilon})W^2(p_{k,\varepsilon}) \leq \eta^2 W^2 + \varepsilon,$$

which implies that

$$\begin{aligned}|Du|_M^2 &\leq 1 - \frac{\eta_*^2}{\eta^2} W^2(p_{k,\varepsilon}) + \frac{\varepsilon}{\eta^2}\\ &\leq 1 - (1 - e^{-ku_*})^2 W^2(p_{k,\varepsilon}) + \frac{\varepsilon}{(1 - e^{-ku_*})^2},\end{aligned}$$

where $\eta_* = \inf_{\Sigma(u)} \eta$ and $u_* = \inf_{\Sigma(u)} u$. Without loss of generality we can suppose that $u \geq u_* > 0$. Thus,

$$|Du|_M^2 \leq 1 - (1 - e^{-ku_*})^2 W^2(p_{k,\varepsilon}) + \frac{\varepsilon}{(1 - e^{-ku_*})^2}. \tag{73}$$

Since ε does not appear in the left hand side of (73), we can take $\limsup_{\varepsilon \to 0}$ on both sides of that obtaining

$$|Du|_M^2 \leq 1 - (1 - e^{-ku_*})^2 \liminf_{\varepsilon \to 0} W^2(p_{k,\varepsilon}). \tag{74}$$

In an analogous way, taking $\limsup_{k \to \infty}$ on (74), we finally conclude that $|Du|_M^2 = 0$ on $\Sigma(u)$, that is, $u \equiv t_0$ for some $t_0 > 0$.

It just remains to prove our claim that the Ricci curvature of $\Sigma(u)$ is bounded from below. Since H is constant and taking into account the hypothesis (57) jointly with (81) that follows directly from Lemma 2 we conclude that Ric is bounded from below. ■

Remark 6 *We recall that the Cheeger constant $\mathfrak{h}(M)$ of a complete Riemannian manifold M^n is given by*

$$\mathfrak{h}(M) = \inf_D \frac{A(\partial D)}{V(D)},$$

where D ranges over all open submanifolds of M^n with compact closure in M^n and smooth boundary, and where $V(D)$, $A(\partial D)$ are the volume of D and the area of ∂D, respectively, relative to the metric of M^n.

Returning to the context of Theorem 9, assuming that there exists an entire H-graphs with $H > 0$ and such that (57) holds, from (29) we can apply an argument due to Salavessa (1989) to get

$$\begin{aligned}nHV(D) &\leq \int_D nH dV = \int_D \operatorname{Div}\left(\frac{Du}{\sqrt{1 - |Du|_M^2}}\right) dV\\ &= \oint_{\partial D} \left\langle \frac{Du}{\sqrt{1 - |Du|_M^2}}, \nu \right\rangle dA \leq \sqrt{\frac{n}{(n-1)\kappa}} H A(\partial D),\end{aligned}$$

where ν is the outward unit normal of ∂D. Yielding the following lower estimate for the Cheeger constant of the fiber M^n

$$\sqrt{n(n-1)\kappa} \leq \mathfrak{h}(M).$$

Furthermore, recalling the stability operator $\mathcal{J} = \Delta + \mathrm{Ric}(N, N) - |A|^2$, *a* H*-hypersurface* Σ^n *is said to be stable if*

$$\int_\Sigma \mathcal{J} f.f \geq 0, \quad \forall f \in C_0^2(\Sigma). \tag{75}$$

We also note that, under the stated hypothesis of Theorem 9, entire H*-graph is, in fact, a slice and therefore* $\mathrm{Ric}(\partial_t, \partial_t) = 0$ *and* $|A|^2 \equiv 0$. *Hence, in this case, from (75) we see that such graph is stable.*

Remark 7 *According to Example 8.1 originally in (De LIMA and LIMA JR, 2013), that family of complete vertical* H*-graphs given by*

$$\Sigma(u) = \{(a \ln y, x, y); y > 0\} \subset -\mathbb{R} \times \mathbb{H}^2$$

with $H = -\dfrac{a}{2\sqrt{1-a^2}}$, $H_2 = 0$ *and satisfying*

$$|Du|^2_{\mathbb{H}^2} = \frac{|A|^2}{1 + |A|^2}. \tag{76}$$

Shows that the semi-bound on u *is actually necessary on Theorem 9. Furthermore* $\langle N, \partial_t \rangle$ *is constant on* $\Sigma(u)$, *from (22), (64) and (76), we get*

$$\Delta \langle N, \partial_t \rangle = (|A|^2 - |\nabla h|^2) \langle N, \partial_t \rangle = 0. \tag{77}$$

Consequently, according to the stability criteria given in (75), from equation (77) we also conclude that $\Sigma(u)$ *constitutes a nontrivial example of stable surface in* $-\mathbb{R} \times \mathbb{H}^2$. *Therefore, concerning the context of Theorem 9, we see that the stability of the entire* H*-graph cannot alone guarantee the uniqueness result.*

7 MAXIMAL SURFACES IN A LORENTZIAN PRODUCT

7.1 Introduction

Albujer and Alías (2011), established Calabi-Bernstein results for maximal surfaces in a Lorentzian product space $-\mathbb{R} \times M^2$. In particular, when the Riemannian surface M^2 has non-negative Gauss curvature, they proved that any complete maximal surface must be totally geodesic. Besides, if M^2 is non-flat, the authors have concluded that it must be a slice $\{t\} \times M^2$. The necessity of the assumption on the Gauss curvature can be observed from the examples of maximal surfaces in $-\mathbb{R} \times \mathbb{H}^2$, constructed in (ALBUJER, 2008b).

In (De LIMA and LIMA JR, 2013) see Example 8.1 the author and de Lima exhibit a (non totally geodesic) complete spacelike surface of constant mean curvature (CMC) in $-\mathbb{R} \times \mathbb{H}^2$ such that the hyperbolic angle function is constant. Caballero, Romero and Rubio (2013) worked in the generalized Robertson-Walker spaces considering maximal surfaces with uniqueness results for the case the fiber is of non-negative Gauss curvature generalizing results of Albujer and Alías (2011)].

The main aim of this section is to present Calabi-Bernstein properties of maximal surfaces in a Lorentzian product space where the Gauss curvature of the fiber M^2 satisfies $K_M \geq -\kappa$ for some $\kappa \in \mathbb{R}$, $\kappa \geq 0$.

Since it is known that there are complete maximal surfaces which are not totally geodesic in $-\mathbb{R} \times \mathbb{H}^2$. That naturally arises the question to decide what additional assumptions are needed to conclude that a complete maximal surface in $-\mathbb{R} \times M^2$, where $\kappa_M \geq -\kappa$, must be totally geodesic.

Our technique is based on a proper extension of a result by Nishikawa (1984) and relies within the applications of the generalized maximum principle due to Yau (1975) to complete Riemannian manifolds. In fact, we previously proved an extension of (NISHIKAWA, 1984, Lemma 2) to the case the Ricci curvature is no longer bounded by a constant but by a more general function of the distance on the manifold (Lemma 7), that will also be in (LIMA JR and ROMERO, 2015).

7.2 Gauss equation

The Gauss curvature K_Σ of the surface Σ is described in terms of A and the curvature of the ambient spacetime by the Gauss equation, which is given by

$$K_\Sigma = \overline{K} + K_G, \tag{78}$$

where $\overline{K}$ denotes the sectional curvature in $-\mathbb{R} \times M^2$ of the tangent plane to Σ and $K_G = -\det A$. We can also write $\overline{K}$ in terms of the Gauss curvature of M as

$$\overline{K} = \kappa_M(1 + |\nabla h|^2) \tag{79}$$

where, for simplicity, κ_M stands for the Gauss curvature of M along the surface Σ for the projection of the tangent plane to Σ onto the tangent plane to M.

Combining equations (78) and (79) we obtain

$$K_\Sigma = \kappa_M \cosh^2 \theta + K_G, \tag{80}$$

where θ is the hyperbolic angle between N and ∂_t. We also have the relation

$$|A|^2 = 2H^2 + 2(H^2 + K_G). \tag{81}$$

7.3 An Omori-Yau-Borbély-Nishikawa generalized maximum principle

The following Lemma is a generalization of a result due to Nishikawa (1984). Using a generalized maximum principle given by Borbély (2012).

Lemma 7 *Let M be a complete Riemannian manifold such that* Ric $\geq -G^2(r)$ *for G such that $G(0) \geq 1$, $G' \geq 0$ and $G^{-1} \notin L^1[0, \infty]$. If u is a non-negative function on M satisfying*

$$\Delta u \geq \beta u^2, \;\; \beta > 0, \tag{82}$$

then $u \equiv 0$.

Proof. Under this conditions (M, g) satisfies the Omori-Yau-Borbély generalized maximum principle. Since $u \in C^\infty(M)$ and non-negative, consider the following function

$$F = \frac{1}{(1+u)^{\frac{1}{2}}}, \;\; F \in C^\infty(M), \;\; F > 0, \;\; \inf(F) \geq 0.$$

Therefore

$$\nabla u = -\frac{2}{F^3} \nabla F$$

and

$$\Delta F = -\frac{\Delta u}{2(1+u)^{\frac{3}{2}}} + \frac{|\nabla F|^2}{(1+u)^{\frac{5}{2}} F^6},$$

then

$$F \Delta F = -\frac{1}{2} F^4 \Delta u + 3|\nabla F|^2. \tag{83}$$

Then for an Omori-Yau sequence we have

$$|\nabla F|(p_m) < \frac{1}{m} \tag{84}$$

$$\Delta F(p_m) > -\frac{1}{m} \tag{85}$$

and

$$0 \leq \inf F \leq F(p_m) < \inf F + \frac{1}{m}. \tag{86}$$

By definition of F

$$\lim F(p_m) = \inf F \Leftrightarrow \lim u(p_m) = \sup u. \tag{87}$$

Combining 82, 83 and 84 we obtain

$$\begin{aligned} -\frac{1}{m}F(p_m) + c\frac{1}{2}u^2(p_m)F^4(p_m) &< \frac{1}{m} \\ -\frac{1}{m}F(p_m) + c\frac{1}{2}\frac{u^2(p_m)}{(1+u(p_m))^2} &< \frac{1}{m} \end{aligned}$$

Letting $m \to \infty$ we get $\lim u(p_m) = 0$ therefore $u \equiv 0$. ■

7.4 Maximal surfaces in products with Gauss curvature bounded from below on the fiber

In order to use Lemma 7 note that if we assume $K_M \geq -\kappa$ for some positive constant κ, then (80) gives the following inequality

$$K_\Sigma \geq -\kappa(1 + |\nabla h|^2) + K_G. \tag{88}$$

Theorem 10 *Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let Σ be a complete maximal surface such that $K_G \leq G(r)$. If*

$$|\nabla h|^2 \leq \frac{|A|^2}{\kappa}, \tag{89}$$

then Σ is a slice.

Proof.

Initially we prove that the Gauss curvature of Σ is bounded from below by $G(r)$. From (88) it is enough to prove that $|\nabla h|^2$ is bounded by $G(r)$. Using (89) we need to show that $|A|^2$ has such a bound. This follows directly from (81). Therefore we can use Lemma 3 in the function $|\nabla u|^2$.

Now we recall the classical Bochner-Lichnerowicz's formula which holds for any smooth function on a Riemannian manifold Σ

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u \rangle,$$

where Δ stands for the Laplacian operator, Hess(u) the Hessian of u and Ric the Ricci tensor on Σ.

This formula specializes in our case applied to the height function as follows

$$\frac{1}{2}\Delta|\nabla h|^2 = |\operatorname{Hess}(h)|^2 + K_\Sigma|\nabla h|^2 + \langle \nabla h, \nabla \Delta h \rangle \tag{90}$$

From (LATORRE and ROMERO, 2002) we have

$$|\text{Hess}(h)|^2 = \cosh^2\theta |A|^2 \tag{91}$$

Moreover, from (90) and (91) we have

$$\frac{1}{2}\Delta \sinh^2\theta \quad \geq |A|^2 \cosh^2\theta - \kappa \cosh^2\theta \sinh^2\theta + \tfrac{1}{2}|A|^2 \sinh^2\theta$$

Making use of (88) and (89)

$$\Delta \sinh^2\theta \geq \kappa \sinh^4\theta \tag{92}$$

therefore using Lemma 7 we obtain case $\theta \equiv 0$. ■

Corollary 13 *Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let Σ be a complete H-surface such that $K_G \leq G(r)$. If*

$$|\nabla h|^2 \leq 2\frac{K_G}{\kappa}, \tag{93}$$

then Σ is a slice.

With an analogous argument of the proof of Theorem 10 we obtain that such a surface must be maximal and then we use directly the previous result.

Corollary 14 *Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let Σ be a complete maximal surface such that K_G is bounded from above. If* (89) *holds then Σ is a slice.*

The principal theorem uses three main hypothesis: maximality, the inequality (89) and the controlled growth of K_G. From the proof of Theorem 10 we observe that the condition on the growth of K_G can be replaced by the same condition on the growth of the norm of gradient of the height function (see Corollary 15).

In order to see that the assumptions in Theorem 10 cannot be dropped, it suffices to see in the Chapter 8 the three examples of spacelike graphs in $-\mathbb{R} \times \mathbb{H}^2$, for $\mathbb{H}^2$ given by the Poincaré model of half plane. Example 8.1, originally in (De LIMA and LIMA JR, 2013): $u(x, y) = a \ln y$ with $|a| < 1$. It shows that maximality cannot be removed in Theorem 10. In fact, we cannot even replace it by constant mean curvature. We also consider two examples from (ALBUJER, 2008b) the first one is Example 8.2: $u(x, y) = a \ln(x^2 + y^2)$ for $a < \frac{1}{2}$. It lacks only the hypothesis of the inequality (89). There is also Example 8.3: $u(x, y) = \ln(y + \sqrt{a + y^2})$ where $a > 0$. It satisfies maximality and the inequality (89) however fails the control in the growth, in reality its growth is more than exponential. For more details see Chapter 8 and the original paper.

Remark 8 *In the proof of Theorem 10 we notice that the bound on K_G bounded can be replaced by a bound on the hyperbolic angle. Indeed, the assumption $K_G \leq G(r)$ is used only to guarantee that $\cosh^2\theta$ has such a bound, when we have this for granted we obtain the following consequence.*

Corollary 15 *Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let Σ be a complete maximal surface such that the hyperbolic angle between N and ∂_t is bounded by $G(r)$ and additionally inequality* (89) *holds then Σ is a slice.*

For the special case when $\langle N, \partial_t \rangle$ is constant we have the following result.

Corollary 16 *A complete maximal surface with constant hyperbolic angle in $-\mathbb{R} \times \mathbb{H}^2$ must be a slice.*

Remark 9 *In Example* 8.2, *we have all hypothesis of Corollary 15 except the inequality (89) showing that we cannot withdraw this hypothesis even when the fiber is of constant Gauss curvature -1. That example also shows Corollary 16 cannot be extended to the case bounded hyperbolic angle.*

7.5 Calabi-Bernstein's type results

As a direct consequence of the previous results we have non-parametric uniqueness results.

Corollary 17 *Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let $\Sigma(u)$ be an entire graph over $\times M^2$ such that K_Σ is bounded from below. If*

$$|\nabla u|^2 \leq \frac{|A|^2}{\kappa + |A|^2}, \tag{94}$$

then Σ is a slice.

Analogously to the previous section we have the following

Corollary 18 *Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let $\Sigma(u)$ be an entire maximal graph over M^2 such that the hyperbolic angle between N and ∂_t is bounded and additionally inequality* (94) *holds then u is constant.*

For the special case when $\langle N, \partial_t \rangle$ is constant we have the following result.

Corollary 19 *A complete maximal graph with constant hyperbolic angle in $-\mathbb{R} \times \mathbb{H}^2$ must be a slice.*

8 EXAMPLES OF SURFACES IN $-\mathbb{R}\times\mathbb{H}^2$

In order to give the examples we consider the hyperbolic space with the model of half space in $\mathbb{R}^2$ given by

$$\mathbb{H}^2 = \{(x,y)\in\mathbb{R}^2; y>0\}$$

with the metric

$$ds^2_{\mathbb{H}^2} = \frac{dx^2+dy^2}{y^2}.$$

Therefore for a graph in $-\mathbb{R}\times\mathbb{H}^2$, $\Sigma(u) = \{(u(x,y),x,y);(x,y)\in\mathbb{H}^2\}$ we evaluate in coordinates the second fundamental form as the following

$$\begin{cases} AX_x &= aX_x + bX_y \\ AX_y &= cX_x + dX_y \end{cases}$$

here $X_x = \partial_x + \langle N,\partial_x\rangle N$ and $X_y = \partial_y + \langle N,\partial_y\rangle N$. Therefore we obtain the following

$$\begin{cases} \langle AX_x, X_x\rangle &= a\langle X_x,X_x\rangle + b\langle X_y,X_x\rangle \\ \langle AX_x, X_y\rangle &= a\langle X_x,X_y\rangle + b\langle X_y,X_y\rangle \\ \langle AX_y, X_x\rangle &= c\langle X_x,X_x\rangle + d\langle X_y,X_x\rangle \\ \langle AX_y, X_y\rangle &= c\langle X_x,X_y\rangle + d\langle X_y,X_y\rangle \end{cases}$$

In this way we have

$$\det(A) = ad - bc = -a^2 - bc,$$

since A is traceless and then $d=-a$. Observing that X_x and X_y are linearly independent we can solve this system using only the first three equations and obtain:

$$\begin{cases} a &= \frac{1}{Q(X_y,X_y)}\left(\langle AX_x,X_x\rangle|X_y|^2 - \langle AX_x,X_y\rangle\langle X_x,X_y\rangle\right) \\ b &= \frac{1}{Q(X_y,X_y)}\left(\langle AX_x,X_y\rangle|X_x|^2 - \langle AX_x,X_x\rangle\langle X_x,X_y\rangle\right) \\ c &= \frac{1}{Q(X_y,X_y)}\left(\langle AX_x,X_y\rangle|X_y|^2 - \langle AX_x,X_x\rangle\langle X_x,X_y\rangle\frac{\langle X_y,X_y\rangle}{\langle X_x,X_x\rangle}\right) \end{cases}$$

For $Q(X_y,X_y) = \langle X_y,X_y\rangle\langle X_x,X_x\rangle - \langle X_x,X_y\rangle^2$.

Combining the previous equations we obtain

$$-\det(A) = \frac{1}{Q(X_y,X_y)}\big(\langle AX_x,X_y\rangle^2(|X_y|^2|X_x|^2 + \langle X_x,X_y\rangle^2) \tag{95}$$
$$+\frac{|X_y|^2}{|X_x|^2}\langle AX_x,X_x\rangle^2 Q(X_y,X_y) - 2\langle AX_x,X_y\rangle\langle AX_x,X_x\rangle|X_y|^2\langle X_x,X_y\rangle\Big)$$

The following example shows that the assumption (43) on the norm of gradient of the height function in Theorem 1 and in Theorem 4 can not be extended for $\alpha = 1$.

Example 8.1 *Let us consider the smooth function $u : \mathbb{H}^2 \to \mathbb{R}$ given by $u(x, y) = a \ln y$, and its respective entire vertical graph*

$$\Sigma^2(u) = \{(a \ln y, x, y); y > 0\} \subset \epsilon\mathbb{R} \times \mathbb{H}^2.$$

We have that $Du(x, y) = (0, ay)$ and, hence, $|Du(x, y)|^2 = |a|^2$. If we take $0 < |a| < 1$, we have that $\Sigma^2(u)$ will be a complete spacelike surface in $-\mathbb{R} \times \mathbb{H}^2$. Moreover, its height function h satisfies

$$|\nabla h|^2 = \frac{|Du|^2}{1 + \epsilon|Du|^2} = \frac{|a|^2}{1 + \epsilon|a|^2}.$$

Consequently,

$$\langle N, \partial_t\rangle = \epsilon\frac{1}{\sqrt{1 + \epsilon|a|^2}}.$$

The mean curvature H of $\Sigma^2(u)$ is given by

$$nH = -\mathrm{Div}\left(\frac{\epsilon Du}{\sqrt{1 + \epsilon|Du|^2}}\right),$$

where Div *is the divergent on $\mathbb{H}^2$. So, using that $\mathrm{Div} = \mathrm{Div}_0 - \frac{2}{y}dy$, where Div_0 denotes the divergent on $\mathbb{R}^2$, we get*

$$2Hr^3 = r^2y^2\Delta_0 u - \epsilon y^3(yQ(u) + u_y|D_0u|_0^2), \tag{96}$$

where $r = \sqrt{1 + \epsilon|Du|^2} = \sqrt{1 + \epsilon a^2}$, Δ_0, D_0 and $|.|_0$ are the Laplacian, the gradient and the norm in the Euclidian metric, and $Q(u) = u_x^2u_{xx} + 2u_xu_yu_{xy} + u_y^2u_{yy}$. Replacing $u(x, y) = a \ln y$ in equation (96), we obtain

$$H = \epsilon\frac{a}{2\sqrt{1 + \epsilon a^2}}$$

and, since $\langle N, \partial_t\rangle$ is constant, from Lemma 1 we get

$$0 = \Delta\langle N, \partial_t\rangle = -\epsilon(|A|^2 - |\nabla h|^2)\langle N, \partial_t\rangle.$$

Consequently,

$$|\nabla h|^2 = |A|^2.$$

Furthermore, from equation (81) we easily see that $H_2 = 0$ on $\Sigma^2(u)$. But, $H_2 = \kappa_1\kappa_2$, where κ_1, κ_2 denote the eigenvalues of A. Therefore, considering $\kappa_2 = 0$ and using that $H = \epsilon\frac{\kappa_1+\kappa_2}{2} = \epsilon\frac{\kappa_1}{2}$, we obtain that $\kappa_1 = \frac{a}{\sqrt{1+\epsilon a^2}}$.

Now follow two examples of maximal surfaces in $\mathbb{R} \times \mathbb{H}^2$ given by Albujer (2008b),

Example 8.2 *Consider the function* $u : \mathbb{H}^2 \to \mathbb{R}$ *given by* $u(x, y) = a\ln(x^2 + y^2)$*, and its respective entire graph*

$$\Sigma(u) = \{(a\ln(x^2 + y^2), x, y); y > 0\} \subset -\mathbb{R} \times \mathbb{H}^2.$$

We have that $Du(x, y) = 2a\frac{y^2}{x^2+y^2}(x, y)$ *and, hence,* $|Du(x, y)|^2 = 4a^2\frac{y^2}{x^2+y^2}$*. If we take* $0 < |a| < \frac{1}{2}$*, we have that* $\Sigma(u)$ *will be a complete spacelike surface in* $-\mathbb{R} \times \mathbb{H}^2$*.*

Notice that $|\nabla h|^2 = \frac{|Du(x,y)|^2}{1-|Du(x,y)|^2}$ *is bounded. Making use of (95) gives* $A_{(0,y)} \equiv 0$ *and* $|\nabla h|^2_{(0,y)} = \frac{4a^2}{1-4a^2} > 0$*. Therefore, inequality* (89) *does not hold for this example. Meanwhile,* $|A_{(y,y)}|^2 = O(y^4)$ *for* y *big enough, which implies* $\det(A)$ *is not bounded from below.*

The following example presented here is not complete even though it is maximal.

Example 8.3 *Here the function* u *is given by* $u(x, y) = \ln(y + \sqrt{a + y^2})$*, for* a *a positive constant. Observing that* u *depends only of* y*, we obtain from (95) that* $\langle AX_x, X_y\rangle = 0$ *we also have* $\langle X_x, X_y\rangle = 0$*. Since this graph is maximal, the norm of* A *is given by*

$$|A|^2 = 2|X_x|^{-4}\langle AX_x, X_x\rangle^2 = \frac{2}{W^2}y^2u_y^2 \tag{97}$$

Since

$$|\nabla h|^2 = \frac{|Du|^2}{W^2} = \frac{1}{W^2}y^2u_y^2,$$

we obtain

$$|\nabla h|^2 = \frac{1}{2}|A|^2.$$

Therefore inequality (89) holds for this example, in fact for any maximal graph in $-\mathbb{R}\times\mathbb{H}^2$ *such that* u *depends only on* y *. It shows that we cannot drop out the condition* $K_G \leq G(r)$*.*

9 CONCLUSION

The examples show the necessity of the main hypothesis in all the study. However there are many more options to be explored since we can see example and counter examples arises naturally in this context. In the context of surfaces in Chapter 7 we see that the hypothesis: maximality, $|\nabla h|^2 \leq \dfrac{|A|^2}{\kappa}$ and $K_G \leq G(r)$, cannot be dropped out, meanwhile it leaves room to examine the same results for bigger dimensions since the technique is specific for surfaces.

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