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DISSON SOARES DOS PRAZERES

IMPROVED REGULARITY ESTIMATES IN NONLINEAR ELLIPTIC EQUATIONS

FORTALEZA
2015

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Tese apresentada ao Programa de Pós-graduação em Matemática do Departamento de Matemática da Universidade Federal do Ceará, como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática. Área de concentração: Análise.

Orientador:
Prof. Dr. Eduardo Vasconcelos Oliveira Teixeira.

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RESUMO

Neste trabalho estabelecemos estimativas de regularidade local para soluções ”flat” de equações elípticas totalmente não-lineares não-cônvexas e estudamos equations do tipo cavidade com coeficientes meramente mensuráveis.

Palavras-chave: Propriedades de regularidade de soluções. Estimativas ótimas. EDPs elípticas totalmente não-lineares. Equações do tipo cavidade. Fronteira livre.

ABSTRACT

In this work we establish local regularity estimates for flat solutions to non-convex fully nonlinear elliptic equations and we study cavitation type equations modeled within coefficients bounded and measurable.

Keywords: Smoothness properties of solutions. Optimal estimates. Fully nonlinear elliptic PDEs. Cavitation type equations. Free boundary.

SUMÁRIO

1 INTRODUCTION

The theory of second order partial differential elliptic equations always had a central importance in the study of classical mechanics and differential geometry. Through the last fourty years pioneering works provided us striking results and methods in the theory of PDE in the non-divergent form and in PDE applied in the free boundary problems. This allowed a big advance in the understanding of the theory of fully nonlinear elliptic equations which arise in differential games, optimal cost in a stochastic control problems and in the problems of jet flows and cavity. In this work we provide original results in the theories cited above.

In Chapter 2 we obtain optimal estimates for flat solutions to a class of non-convex fully nonlinear elliptic equations of the form

$$F(X, D^2u) = \mathscr{G}(X, u, \nabla u). \tag{1.1}$$

Under continuous differentiability with respect to the matrix variable and appropriate continuity assumptions on the coefficients and on the source function, we present a Schauder type regularity result for flat solutions, namely for solutions with small enough norm, $|u| \ll 1$.

The nonlinear operator $F\colon B_1 \times \mathrm{Sym}(n) \to \mathbb{R}$ is assumed to be uniformly elliptic, namely, there exist constants $0 < \lambda \leq \Lambda$ such that for any $M, P \in \mathrm{Sym}(n)$, with $P \geq 0$ and all $X \in B_1 \subset \mathbb{R}^n$ there holds

$$\lambda\|P\| \leq F(X, M+P) - F(X, M) \leq \ \Lambda\|P\|. \tag{1.2}$$

Under such condition it follows as a consequence of Krylov-Safonov Harnack inequality that solutions to the homogeneous, constant coefficient equations

$$F(D^2h) = 0 \tag{1.3}$$

are locally of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. Under appropriate hypotheses on $\mathscr{G}\colon B_1 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, the same conclusion is obtained, i.e., viscosity solutions are of class $C^{1,\alpha}$. Thus, insofar as the regularity theory for equation of the form (1.1) is concerned, one can regard the right hand side $\mathscr{G}(X, u, \nabla u)$ as an $\tilde{\alpha}-$Hölder continuous source, $f(X)$. Therefore, within this present work, we choose to look at the RHS $\mathscr{G}(X, u, \nabla u)$ simply as a source term $f(X)$, and equation (1.1) will be written as

$$F(X, D^2u) = f(X). \tag{1.4}$$

Regularity theory for heterogeneous equations (1.4) has been a central target of research for the past three decades. While a celebrated result due to Evans and Krylov assures that solutions to convex equations are classical, i.e., $C^{2,\alpha}$ for some $\alpha > 0$, the problem of establishing continuity of the Hessian of solutions to general equations of the form (1.3) challenged the community for over twenty years. The problem has been settled in the negative by Nadirashvili and Vladut, [23, 24], who exhibit solutions to uniform elliptic equations whose Hessian blows-up.

In view of the impossibility of a general existence theory for classical solutions to all fully nonlinear equations (1.3), it becomes a central topic of research the study of reasonable conditions on F and on u as to assure the Hessian of the solution is continuous. In such perspective the works [16] and [9] on interior $C^{2,\alpha}$ estimates for a particular class of non-convex equations are highlights. A decisive contribution towards Hessian estimates of solutions to fully nonlinear elliptic equations was obtained by Savin in [27]. By means of a robust approach, Savin shows in [27] that small solutions are classical, provided the operator is of class C^2 in all of its arguments.

Inspired by problems of the form (1.1), in the present work, we obtain regularity estimates for flat solutions to heterogeneous equation (1.4), under continuity conditions on the media. We show that if $X \mapsto (F(X, \cdot), f(X))$ is α-Hölder continuous, then flat solutions are locally $C^{2,\alpha}$. In the case $\alpha = 0$, namely when the coefficients and the source are known to be just continuous, we show that flat solutions are locally $C^{1,\text{Log-Lip}}$.

The proofs of both results mentioned above, to be properly stated in Theorem 2.2 and Theorem 2.3 respectively, are based on a combination of geometric tangential analysis and perturbation arguments inspired by compactness methods in the theory of elliptic PDEs.

We conclude this introduction of Chapter 2 explaining the heuristics of the geometric tangential analysis behind our proofs. Given a fully nonlinear elliptic operator F, we look at the family of elliptic scalings

$$F_\mu(M) := \frac{1}{\mu} F(\mu M), \quad \mu > 0.$$

This is a continuous family of operators preserving the ellipticity constants of the original equation. If F is differentiable at the origin (recall, by normalization $F(0) = 0$), then indeed

$$F_\mu(M) \to \partial_{M_{ij}} F(0) M_{ij}, \quad \text{as } \mu \to 0.$$

In other words, the linear operator $M \mapsto \partial_{M_{ij}} F(0) M_{ij}$ is the *tangential equation* of F_μ as $\mu \to 0$. Now, if u solves an equation involving the original operator F, then $u_\mu := \frac{1}{\mu} u$

is a solution to a related equation for F_μ. However, if in addition it is known that the norm of u is at most μ, then it accounts into saying that u_μ is a normalized solution to the μ-related equation, and hence we can access the universal regularity theory available for the (linear) tangential equation by compactness methods. In the sequel we transport such good *limiting* estimates towards u_μ, properly adjusted by the geometric tangential path used to access the *tangential* linear elliptic regularity theory. Similar reasoning has been recently employed in [31, 32, 33, 35].

Already in Chapter 3 given a Lipschitz bounded domain $\Omega \subset \mathbb{R}^n$, a bounded measurable elliptic matrix $a_{ij}(X)$, i.e. a symmetric matrix with varying coefficients satisfying the (λ, Λ)-ellipticity condition

$$\lambda \mathrm{Id} \leq a_{ij}(X) \leq \Lambda \mathrm{Id}, \tag{1.5}$$

and a nonnegative boundary data $\varphi \in L^2(\partial\Omega)$, we are interested in studying local minimizers u to the discontinuous functional

$$\mathscr{F}(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \chi_{\{u>0\}} \right\} dX \to \min, \tag{1.6}$$

among all competing functions $u \in H^1_\varphi(\Omega) := \{u \in H^1(\Omega) \mid \mathrm{Trace}(u) = \varphi\}$.

The variational problem set in (1.6) appears in the mathematical formulation of a great variety of models: jet flows, cavity problems, Bernoulli problems, free transmission problems, optimal designs, just to cite few. Its mathematical treatment has been extensively developed since the epic marking work of Alt and Caffarelli [1]. The program for studying minimization problems for discontinuous functionals of the form (1.6) is nowadays well established in the literature. Existence of minimizer follows by classical considerations. Any minimum is nonnegative provided the boundary data is nonnegative. Minimizers satisfy, in the distributional sense, the Euler-Lagrange equation

$$\mathrm{div}(a_{ij}(X)\nabla u) = \mu, \tag{1.7}$$

where μ is a measure supported along the free boundary. In particular a minimum of the functional $\mathscr{F}$ is a-harmonic within this positive set, i.e.,

$$\mathrm{div}(a_{ij}(X)\nabla u) = 0, \qquad \text{in } \{u > 0\} \cap \Omega.$$

By pure energy considerations, one proves that minimizers grow linearly always from their free boundaries. Finally, if a_{ij} are, say, Hölder continuous, then the free boundary $\partial\{u > 0\}$ is of class $C^{1,\alpha}$ up to a possible negligible singular set. In such a scenario, the free boundary condition

$$\langle a_{ij}(\xi)\nabla u(\xi), \nabla u(\xi)\rangle = \text{Const.}$$

then holds in the classical sense along the regular part of the free boundary, in particular for all $\xi \in \partial_{\text{red}}\{u > 0\} \cap \Omega$.

A decisive, key step though, required within any program for studying variational problems of the form (1.6), concerns Lipschitz estimates of minimizers. However, if no further regularity assumptions upon the coefficients $a_{ij}(X)$ is imposed, even a-harmonic functions, $\text{div}(a_{ij}(X)\nabla h) = 0$, may fail to be Lipschitz continuous. That is, the universal Hölder continuity exponent granted by DeGiorgi-Nash-Moser regularity theory may be strictly less than 1, even for two-dimensional problems. Such a technical constrain makes the study of local minima to (1.6) in discontinuous media rather difficult from a rigorous mathematical viewpoint.

The above discussion brings us to the main goal in Chapter 3. Even though it is hopeless to obtain gradient bounds for minimizers of functional (1.6) in Ω, we shall prove that any minimum is universally Lipschitz continuous along its free boundary, $\partial\{u > 0\} \cap \Omega$. This estimate is strong enough to carry on a geometric-measure analysis near the free boundary, which in particular implies that the non-coincidence set has uniform positive density and that the free boundary has finite $(n - \varsigma)$-Hausdorff measure, for a dimensional number $0 < \varsigma \leq 1$.

In this part of the work we shall carry a slightly more general analysis as to contemplate singular approximations of the minimization problem (1.6). Let $\beta \in L^\infty(\mathbb{R})$ be a bounded function supported in the unit interval $[0, 1]$. For each $\epsilon > 0$, we defined the integral preserving, ϵ-perturbed potential:

$$\beta_\epsilon(t) := \frac{1}{\epsilon}\beta\left(\frac{t}{\epsilon}\right), \tag{1.8}$$

which is now supported in $[0, \epsilon]$. Such a sequence of potentials converges in the distributional sense to $\int \beta$ times the Dirac measure δ_0. Consider further

$$B_\epsilon(\xi) = \int_0^\xi \beta_\epsilon(t)dt \to \left(\int \beta(s)ds\right) \cdot \chi_{\{\xi>0\}}, \tag{1.9}$$

in the distributional sense. We now look at local minimizers u_ε to the variational problem

$$\mathscr{F}_\epsilon(u) = \int_{B_1} \left\{\frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + B_\epsilon(u)\right\} dX \to \min, \tag{1.10}$$

among all competing functions $u \in H^1_\varphi(\Omega) := \{u \in H^1(\Omega) \mid \text{Trace}(u) = \varphi\}$. There is a large literature on such a class of singularly perturbed equations, see for instance [4, 12, 13, 15, 25, 26, 30]. It is well established that the functional $\mathscr{F}$ defined in (1.6) can be recovered by letting ϵ go to zero in (1.10). For each ϵ fixed though, minimizers of the functional $\mathscr{F}_\epsilon$ is related to a number of other physical problems, such as high energy activations and the theory of flame propagation. From the applied point of view, it is more appealing to indeed study the whole family of functionals $(\mathscr{F}_\epsilon)_{0\le\epsilon\le 1}$. We also mention that the study of minimization problem (1.10) with no continuity assumption on the coefficients is also motivated by several applications, for instance if homogenization theory, composite materials, etc.

We should also mention the connections this present work has with the theory of free phase transmission problems. This class of problems appear, for instance, in the system of equations modeling an ice that melts submerged in a heated inhomogeneous medium. For problems modeled within an organized medium (say Hölder continuous coefficients), monotonicity formula [6] yields Lipschitz estimates for solutions. However, by physical interpretations of the model, it is natural to consider the problem within discontinuous media. Under such an adversity (monotonicity formula is no longer available), Lipschitz estimate along the free boundary has been an important open problem within that theory, see [2] for discussion. However, if we further assume in the model that the temperature of the ice remains constant, which is reasonable in very low temperatures, then free phase transmission problems fit into the mathematical formulation of this present article; and a Lipschitz estimate becomes available by our main result.

We conclude this introduction by mentioning that the improved, sharp regularity estimate we establish in this work holds true in much more generality. The proof designed is purely nonlinear and uses only the Euler-Lagrange equation associated to the minimization problem (1.10). Hence, it can be directly applied to degenerate discontinuous functionals of the form

$$\int F(X, u, \nabla u)dX \to \min.,$$

where $F(X, u, \xi) \sim |\xi|^{p-2}A(X)\xi\cdot\xi + f(X)\,(u^+)^m + Q(X)\cdot\chi_{\{u>0\}}$, where $A(X)$ is bounded, measurable elliptic matrix, $f \in L^q(\Omega)$, $q > n$, $1 \le m < p$ and Q is bounded away from zero and infinity, see [18, 22]. Nonvariational cavitation problems, as well as parabolic versions of such models can also be tackled by the methods designed here to establish Lipschitz estimate along the free boundary.

2 REGULARITY OF FLAT SOLUTIONS TO FULLY NONLINEAR ELLIPTIC PROBLEMS

The chapter is organized as follows. In Section 2.1 we state all the hypotheses, mathematical set-up and notions to be used throughout the whole chapter. In this section we also state properly the two main theorems proven in this thesis. In Section 2.2 we rigorously develop the heuristics of the geometric tangential analysis explained in the introduction. The proof of $C^{2,\alpha}$ estimates, Theorem 2.2, will be given in Section 2.3. Two applications of such a result will be discussed in Section 2.4. Theorem 2.3 will be proven in Section 2.5.

2.1 Hypotheses and main results

Let us begin by discussing the hypotheses, set-up and main notations used here. For B_1 we denote the open unit ball in the Euclidean space $\mathbb{R}^n$. The space of $n \times n$ symmetric matrices will be denoted by $\mathrm{Sym}(n)$. By modulus of continuity we mean an increasing function $\varpi\colon [0, +\infty) \to [0, +\infty)$, with $\varpi(0^+) = 0$.

Hereafter we shall assume the following conditions on the operator $F\colon B_1 \times \mathrm{Sym}(n) \to \mathbb{R}$ and $f\colon B_1 \to \mathbb{R}$:

(H1) There exist constants $0 < \lambda \leq \Lambda$ such that for any $M, P \in \mathrm{Sym}(n)$, with $P \geq 0$ and all $X \in B_1$, there holds

$$\lambda \|P\| \leq F(X, M + P) - F(X, M) \leq \Lambda \|P\|. \tag{2.1}$$

(H2) $F(X, M)$ is differentiable with respect to M and for a modulus of continuity ω, there holds

$$\|D_M F(X, M_1) - D_M F(X, M_2)\| \leq \omega(\|M_1 - M_2\|), \tag{2.2}$$

for all $(X, M_i) \in B_1 \times \mathrm{Sym}(n)$.

(H3) For another modulus of continuity τ, there holds

$$|F(X, M) - F(Y, M)| \leq \tau(|X - Y|) \cdot \|M\|, \tag{2.3}$$

$$|f(X) - f(Y)| \leq \tau(|X - Y|), \tag{2.4}$$

for all $X, Y \in B_1$ and $M \in \mathrm{Sym}(n)$. It will also be enforced hereafter the following normalization conditions:

$$F(0, 0_{n\times n}) = f(0) = 0; \tag{2.5}$$

though such hypothesis is not restrictive, as one can always reduce the problem as to verify that.

Condition (H1) concerns the notion of uniform ellipticity. Under such a structural condition, the theory of viscosity solutions provides an appropriate notion for weak solutions to such equations.

Definition 2.1. *A continuous function $u \in C^0(B_1)$ is said to be a viscosity subsolution to* (1.4) *in B_1 if whenever one touches the graph of u from above by a smooth function φ at $X_0 \in B_1$ (i.e. $\varphi - u$ has a local minimum at X_0), there holds*

$$F(X_0, D^2\varphi(X_0)) \geq f(X_0).$$

Similarly, u is a viscosity supersolution to (1.4) *if whenever one touches the graph of u from below by a smooth function ϕ at $Y_0 \in B_1$, there holds*

$$F(Y_0, D^2\phi(Y_0)) \leq f(Y_0).$$

We say u is a viscosity solution to (1.4) *if it is a subsolution and a supersolution of* (1.4).

Condition (H2) fixes a modulus of continuity ω to the derivative of F. The regularity estimates proven in this chapter depends upon ω. Condition (H3) sets the continuity of the media. When $\tau(t) \approx t^\alpha$, $0 < \alpha < 1$, the coefficients and the source function are said to be α-Hölder continuous. In such scenario we prove that flat solutions are locally of class $C^{2,\alpha}$ – a sharp Schauder type of estimate for non-convex fully nonlinear equations.

Theorem 2.2 ($C^{2,\alpha}$ regularity)**.** *Let $u \in C^0(B_1)$ be a viscosity solution to*

$$F(X, D^2u) = f(X) \text{ in } B_1,$$

where F and f satisfy (H1)–(H3) with $\tau(t) = Ct^\alpha$ for some $0 < \alpha < 1$. There exists a $\overline{\delta} > 0$, depending only upon $n, \lambda, \Lambda, \omega, \alpha$, and $\tau(1)$, such that if

$$\sup_{B_1} |u| \leq \overline{\delta}$$

then $u \in C^{2,\alpha}(B_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq \overline{C} \cdot \overline{\delta},$$

where $\overline{C}$ depends only upon $n, \lambda, \Lambda, \omega$, and $(1 - \alpha)$.

We should emphasize that the Hölder exponent obtained in Theorem 2.2 is sharp, as it is the same one from the Hölder continuity of the medium and the source function f. If f is merely continuous, then even for the classical Poisson equation

$$\Delta u = f(X),$$

solutions may fail to be of class C^2. In connection to Theorem 5.1 in [32], in this chapter we show that flat solutions in continuous media are locally of class $C^{1,\text{Log-Lip}}$, which corresponds to the optimal regularity estimate under such weaker conditions.

Theorem 2.3 ($C^{1,\text{Log-Lip}}$ estimates)**.** *Let $u \in C^0(B_1)$ be a viscosity solution to*

$$F(X, D^2u) = f(X) \text{ in } B_1.$$

Assume (H1)–(H3). Then there exist a $\overline{\delta} = \overline{\delta}(n, \lambda, \Lambda, \omega, \tau)$ such that if

$$\sup_{B_1} \|u\| \leq \overline{\delta},$$

then $u \in C^{1,\text{Log}-\text{Lip}}(B_{\frac{1}{2}})$ and

$$|u(X) - [u(Y) + \nabla u(Y) \cdot (X - Y)]| \leq -\overline{C}.\overline{\delta} \cdot |X - Y|^2 \log(|X - Y|),$$

for a constant $\overline{C}$ that depends only upon $n, \lambda, \Lambda, \omega,$ and $(1 - \alpha)$.

2.2 Geometric tangential analysis

In this Section we provide a rigorous treatment of the heuristics involved in the geometric tangential analysis explained in the introduction. The next lemmas are key tools for the proof of both Theorem 2.2 and Theorem 2.3.

Lemma 2.4. *Let $F\colon B_1 \times Sym(n) \to \mathbb{R}$ satisfy conditions (H1) and (H2). Given $0 \leq \gamma < 1$, there exists $\eta > 0$, depending only on $n, \lambda, \Lambda, \omega,$ and γ, such that if u satisfies $|u| \leq 1$ in B_1 and solves $\mu^{-1}F(X, \mu D^2u) = f(X)$ in B_1, for*

$$0 < \mu \leq \eta, \qquad \sup_{M \in Sym(n)} \frac{|F(X, M) - F(0, M)|}{\|M\|} \leq \eta \quad \text{and} \quad \|f\|_{L^\infty(B_1)} \leq \eta,$$

then one can find a number $0 < \sigma < 1$, depending only on n, λ and Λ, and a quadratic polynomial P satisfying

$$\mu^{-1}F(0, \mu D^2P) = 0, \quad \text{with} \quad \|P\|_{L^\infty(B_1)} \leq C(n, \lambda, \Lambda),$$

for a universal constant $C(n, \lambda, \Lambda) > 0$, such that

$$\sup_{B_\sigma} |u - P| \leq \sigma^{2+\gamma}.$$

Proof. Let us suppose, for the purpose of contradiction, that the Lemma fails to hold. If so, there would exist a sequence of elliptic operators, $F_k(X, M)$, satisfying hypotheses (H1) and (H2), a sequence $0 < \mu_k = \mathrm{o}(1)$ and sequences of functions

$$u_k \in C(B_1) \text{ and } f_k \in L^\infty(B_1),$$

all linked through the equation

$$\frac{1}{\mu_k} F_k(X, \mu_k D^2 u_k) = f_k(X) \text{ in } B_1, \tag{2.6}$$

in the viscosity sense, such that

$$\|u_k\|_\infty \leq 1, \quad \mu_k \leq \frac{1}{k}, \quad \sup_{M \in \mathrm{Sym}(n)} \frac{|F_k(X,M) - F_k(0,M)|}{\|M\|} \leq \frac{1}{k} \quad \text{and} \quad \|f_k\|_\infty \leq \frac{1}{k}; \tag{2.7}$$

however for some $0 < \sigma_0 < 1$

$$\sup_{B_{\sigma_0}} |u_k - P| > \sigma_0^{2+\gamma}, \tag{2.8}$$

for all quadratic polynomials P that satisfies

$$\frac{1}{\mu_k} F_k(0, \mu_k D^2 P) = 0.$$

Passing to a subsequence if necessary, we can assume $F_k(X, M) \to F_\infty(X, M)$ locally uniform in $\mathrm{Sym}(n)$. From uniform C^1 estimate on F_k and the coefficient oscillation hypothesis in (2.7), we deduce

$$\frac{1}{\mu_k} F_k(X, \mu_k M) \to D_M F_\infty(0,0) \cdot M, \tag{2.9}$$

locally uniform in $\mathrm{Sym}(n)$. In fact, we have that

$$F_k(X_k, \mu_k M) = \|\mu_k M\| \frac{F_k(X_k, \mu_k M) - F_k(0, \mu_k M)}{\|\mu_k M\|} + \frac{d}{dt} \int_0^{\mu_k} F_k(0, tM) dt$$

thus

$$F_k(X_k, \mu_k M) = \|\mu_k M\| \frac{F_k(X_k, \mu_k M) - F_k(0, \mu_k M)}{\|\mu_k M\|} + \int_0^{\mu_k} D_M F_k(0, tM) M dt$$

which, using (2.2), implies that

$$F_k(X_k,\mu_k M) \geq -\|\mu_k M\| \frac{|F_k(X_k,\mu_k M) - F_k(0,\mu_k M)|}{\|\mu_k M\|} + \mu_k D_M F_k(0,0)M - \mu_k \omega(\|\mu_k M\|)$$

Dividing the inequality by μ_k and making $\mu_k \to 0$ we obtain that

$$D_M F_\infty(0,0).M \leq \lim_{\mu_k \to 0} \frac{1}{\mu_k} F_k(X,\mu_k M).$$

Repeating the argument we show that

$$D_M F_\infty(0,0).M \geq \lim_{\mu_k \to 0} \frac{1}{\mu_k} F_k(X,\mu_k M)$$

what concludes the claim.

Also, by Krylov-Safonov $C^{0,\gamma}$ bounds for equation (2.6), up to a subsequence, $u_k \to u_\infty$ locally uniform in B_1. Thus, by stability of viscosity solutions, we conclude

$$D_M F_\infty(0,0) \cdot D^2 u_\infty = 0, \quad \text{in } B_1. \tag{2.10}$$

As u_∞ solves a linear, constant coefficient elliptic equation, u_∞ is smooth. Define

$$P := u_\infty(0) + Du_\infty(0) \cdot X + \frac{1}{2} X.D^2 u_\infty(0) X.$$

Since $\|u_\infty\| \leq 1$, it follows from C^3 estimates on u_∞ that

$$\sup_{B_r} |u_\infty - P| \leq Cr^3,$$

for a constant C that depends only upon dimension n and ellipticity constants, λ and Λ. Thus, if we select

$$\sigma := \sqrt[1-\gamma]{\frac{1}{2C}},$$

a choice that depends only on n, λ, Λ and γ, we readily have

$$\sup_{B_\sigma} |u_\infty - P| \leq \frac{1}{2} \sigma^{2+\gamma},$$

Also, from equation (2.10), we obtain

$$D_M F_\infty(0,0) \cdot D^2 P = 0$$

which implies that

$$|\mu_k^{-1} F_k(0, \mu_k D^2 P)| = \mathrm{o}(1).$$

Now, since F_k is uniformly elliptic in $B_1 \times \mathrm{Sym}(n)$ and $F_k(0,0) = 0$, it is possible to find a sequence of real numbers $(a_k) \subset \mathbb{R}$ with $|a_k| = \mathrm{o}(1)$, for which the quadratic polynomial

$$P_k := P + a_k |X|^2$$

do satisfy

$$\mu_k^{-1} F_k(0, \mu_k D^2 P_k) = 0.$$

Finally we have, for any point in B_σ and k large enough,

$$\begin{aligned}\sup_{B_\sigma} |u_k - P_k| &\leq |u_k - u_\infty| + |u_\infty - P| + |P - P_k| \\ &\leq \frac{1}{5}\sigma^{2+\gamma} + \frac{1}{2}\sigma^{2+\gamma} + |a_k|\sigma^2 \\ &< \sigma^{2+\gamma},\end{aligned}$$

which contradicts (2.8). Lemma 2.4 is proved. □

In the sequel, we transfer the geometric tangential access towards a smallness condition of the L^∞ norm of the solution.

Lemma 2.5. *Let F satisfy (H1) and (H2) and $0 \leq \alpha < 1$ be given. There exist a small positive constant $\delta > 0$ depending on n, λ, Λ,and α, and a constant $0 < \sigma < 1$ depending only on n, λ, Λ and $(1-\alpha)$ such that if u is a solution to (1.4) and*

$$\|u\|_{L^\infty(B_1)} \leq \delta, \qquad \sup_{M \in Sym(n)} \frac{|F(X,M) - F(0,M)|}{\|M\|} \leq \delta^{3/2} \quad \textit{and} \quad \|f\|_{L^\infty(B_1)} \leq \delta^{3/2},$$

then one can find a quadratic polynomial P satisfying

$$F(0, D^2 P) = 0, \quad \textit{with} \quad \|P\|_{L^\infty(B_1)} \leq \delta C(n, \lambda, \Lambda) \tag{2.11}$$

for a universal constant $C(n,\lambda,\Lambda) > 0$, and

$$\sup_{B_\sigma} |u - P| \leq \delta \cdot \sigma^{2+\alpha}$$

Proof. Define the normalized function $v = \delta^{-1} u$. We immediate check that

$$\delta^{-1} F(X, \delta D^2 v) = \frac{f(X)}{\delta}.$$

If η is the number from Lemma 2.4, we choose $\delta = \eta^2$ and the Lemma follows. □

2.3 $C^{2,\alpha}$ estimates in $C^{0,\alpha}$ media

In this Section we show that if the coefficients and the source are α-Hölder continuous, then flat solutions are locally of class $C^{2,\alpha}$, i.e. That is, herein we assume

$$\tau(t) \lesssim Ct^{\alpha}, \tag{2.12}$$

for some $0 < \alpha < 1$ and $C > 0$, where τ is the modulus of continuity of the coefficients and the source function appearing in (2.3) and (2.4). Under such condition, we aim to show that flat solutions are locally of class $C^{2,\alpha}$.

The idea of the proof is to employ Lemma 2.5 in an inductive process as to establish the aimed $C^{2,\alpha}$ estimate for flat solutions under an appropriate smallness regime for the oscillation of the coefficients and the source function.

Lemma 2.6. *Let F, f and u be under the hypotheses of Lemma 2.5. Then there exists a $\delta = \delta(n, \lambda, \Lambda, \omega) > 0$, such that if*

$$\sup_{B_1} |u| \leq \delta \quad \text{and} \quad \tau(1) \leq \delta^{3/2},$$

then $u \in C^{2,\alpha}$ at the origin and

$$|u - (u(0) + \nabla u(0) \cdot X + \frac{1}{2} X^t D^2 u(0) X)| \leq C \cdot \delta |X|^{2+\alpha},$$

where $C > 0$ depends only upon $n, \lambda, \Lambda, \omega$ and $(1 - \alpha)$.

Proof. The proof consists in iterating Lemma 2.5 as to produce a sequence of quadratic polynomials

$$P_k = \frac{1}{2} X^t A_k X + b_k \cdot X + c_k \quad \text{with} \quad F(0, D^2 P_k) = 0, \tag{2.13}$$

that approximates u in a $C^{2,\alpha}$ fashion, i.e.,

$$\sup_{B_{\sigma^k}} |u(X) - P_k(X)| \leq \delta \sigma^{(2+\alpha)k}. \tag{2.14}$$

Furthermore, we aim to control the oscillation of the coefficients of P_k as

$$\begin{cases} |A_k - A_{k-1}| & \leq & C\delta\sigma^{\alpha(k-1)} \\ |b_k - b_{k-1}| & \leq & C\delta\sigma^{(1+\alpha)(k-1)} \\ |c_k - c_{k-1}| & \leq & C\delta\sigma^{(2+\alpha)(k-1)} \end{cases} \tag{2.15}$$

where $C > 0$ is universal and σ and δ are the parameters in Lemma 2.5. The proof of the existence of the polynomials P_k verifying (2.13), (2.14) and (2.15) will be delivered by induction. The case $k = 1$ is precisely the statement of Lemma 2.5. Suppose now we have verified the kth step of induction, i.e., that there exists a quadratic polynomial P_k satisfying (2.13), (2.14) and (2.15). We define

$$\tilde{u}(X) := \frac{1}{\sigma^{(2+\alpha)k}}(u(\sigma^k X) - P_k(\sigma^k X)); \tag{2.16}$$

$$\tilde{F}(X, M) := \frac{1}{\sigma^{k\alpha}} F(\sigma^k X, \sigma^{k\alpha} \cdot M + D^2 P_k). \tag{2.17}$$

Notice that

$$\left| D_M \tilde{F}(X, M) - D_M \tilde{F}(X, N) \right| \leq \omega(\sigma^{k\alpha} \|M - N\|) \leq \omega(\|M - N\|),$$

that is, $\tilde{F}$ fulfills (H2). It readily follows from (2.14) that $\tilde{u}$ satisfies

$$|\tilde{u}|_{L^\infty(B_1)} \leq \delta.$$

Moreover, $\tilde{u}$ solves

$$\tilde{F}(X, D^2 \tilde{u}) = \frac{1}{\sigma^{k\alpha}} f(\sigma^k X) =: \tilde{f}(X)$$

in the viscosity sense. From τ-continuity of f and the coefficients of F, together with the smallness condition $\tau(1) \leq \delta^{3/2}$, we verify

$$\|\tilde{f}\|_\infty \leq \delta^{3/2},$$

and likewise,

$$\sup_{M \in \mathrm{Sym}(n)} \frac{|\tilde{F}(X, M) - \tilde{F}(0, M)|}{\|M\|} \leq \delta^{3/2}.$$

Applying Lemma 2.5 to $\tilde{u}$ gives a quadratic polynomial $\tilde{P}$ satisfying $\tilde{F}(0, D^2 \tilde{P}) = 0$ for which

$$|\tilde{u}(X) - \tilde{P}(X)| \leq \delta \sigma^{2+\alpha}, \quad \text{for } |X| \leq \sigma.$$

The $(k+1)$th step of induction is verified if we define

$$P_{k+1}(X) := P_k(X) + \sigma^{(2+\alpha)k} \tilde{P}(\sigma^{-k} X).$$

To conclude the proof of this lemma, notice that (2.15) implies that

$$\{A_k\} \subset \mathrm{Sym}(n), \quad \{b_k\} \subset \mathbb{R}^n, \quad \text{and} \quad \{c_k\} \subset \mathbb{R}$$

are Cauchy sequences. Let us label the limiting quadratic polynomial

$$P_\infty(X) := \frac{1}{2} X^t A_\infty X + b_\infty X + c_\infty,$$

where $A_k \to A_\infty, b_k \to b_\infty$ and $c_k \to c_\infty$. It further follows from (2.15)

$$|P_k(X) - P_\infty(X)| \le C\delta(\sigma^{\alpha k}|X|^2 + \sigma^{(1+\alpha)k}|X| + \sigma^{(2+\alpha)k}), \tag{2.18}$$

whenever $|X| \le \sigma^k$. Finally, fixed $X \in B_\sigma$, take $k \in \mathbb{N}$ such that $\sigma^{k+1} < |X| \le \sigma^k$ and conclude, by means of (2.14) and (2.18), that

$$|u(X) - P_\infty(X)| \le C_1\delta\sigma^{(2+\alpha)k} \le \frac{C_1\delta}{\sigma^{2+\alpha}}|X|^{2+\alpha},$$

as desired. □

We conclude the proof of Theorem 2.2 by verifying that if $\tau(t) = \tau(1)t^\alpha$, the smallness condition of Lemma 2.6, namely

$$\tau(1) \le \delta^{3/2},$$

is not restrictive. In fact, if $u \in C^0(B_1)$ is a viscosity solution to

$$F(X, D^2u) = f(X) \text{ in } B_1, \tag{2.19}$$

the auxiliary function

$$v(X) := \frac{u(\mu X)}{\mu^2} \tag{2.20}$$

solves

$$F_\mu(X, D^2v) = f_\mu(X),$$

where

$$F_\mu(X, M) := F(\mu X, M) \quad \text{and} \quad f_\mu(X) := f(\mu X).$$

Clearly the new operator F_μ satisfies the same assumptions (H1)–(H3) as F, with the same universal parameters λ, Λ and ω. Note however that

$$\max\left\{|f_\mu(X) - f_\mu(Y)|, \frac{|F_\mu(X, M) - F_\mu(Y, M)|}{\|M\|}\right\} \le \tau(1)\mu^\alpha|X - Y|^\alpha,$$

for $M \in \mathrm{Sym}(n)$. Thus if τ_μ is the modulus of continuity for f_μ and F_μ,

$$\tau_\mu(1) = \tau(1)\mu^\alpha.$$

Finally, we take

$$\mu := \min\left\{1, \frac{\sqrt[2\alpha]{\delta^3}}{\sqrt[\alpha]{\tau(1)}}\right\},$$

where δ is the universal number from Lemma 2.5. In conclusion, if u solves (2.19) and satisfies the flatness condition

$$\|u\|_{L^\infty(B_1)} \leq \overline{\delta} := \delta\mu^2,$$

then Lemma 2.6 applied to v gives $C^{2,\alpha}$ estimates for v, which is then transported to u according to (2.20). □

2.4 Applications

Probably an erudite way to comprehend Theorem 2.2 is by saying that if u solves a fully nonlinear elliptic equation with C^α coefficients and source, then if it is close enough to a $C^{2,\alpha}$ function, then indeed u is $C^{2,\alpha}$. This is particularly meaningful in problems involving some *a priori* set data.

In this intermediary section, we comment on two applications of Theorem 2.2. The first one concerns an improvement of regularity for classical solutions in Hölder continuous media.

Corollary 2.7 (C^2 implies $C^{2,\alpha}$). *Let $u \in C^2(B_1)$ be a classical, pointwise solution to*

$$F(X, D^2u) = f(X)$$

where $F(X,\cdot) \in C^1(Sym(n))$ satisfy (H1)–(H2). Assume further that condition (H3) holds with $\tau(t) = Ct^\alpha$ for some $0 < \alpha < 1$. Then, $u \in C^{2,\alpha}(B_{1/2})$, and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(n, \lambda, \Lambda, \alpha, \omega, \tau(1), \|u\|_{C^2(B_1)}).$$

Proof. We shall proof that u is $C^{2,\alpha}$ at the origin. To this end, define, for an $r > 0$ to be chosen soon, $v\colon B_1 \to \mathbb{R}$, by

$$v(X) := \frac{1}{r^2}u(rX) - \left[\frac{1}{r^2}u(0) + \frac{1}{r}\nabla u(0)\cdot X + \frac{1}{2}X^tD^2u(0)X\right].$$

We clearly have

$$v(0) = |\nabla v(0)| = 0 \quad \text{and} \quad |D^2v(0)| \leq \varsigma(r), \tag{2.21}$$

where ς is the modulus of continuity for D^2u. Now, we choose $0 < r \ll 1$ so small that

$$\varsigma(r) \leq c_n\overline{\delta},$$

where c_n is a dimensional constant and $\overline{\delta}$ is the number appearing in Theorem 2.2. With such choice, v is under the condition of Theorem 2.2, for $\tilde{F}(X,M) := F(rX, M + D^2u(0))$

and $\tilde{f}(X) = f(rX)$. □

Remark 2.8. *We remark that in the proof of Corollary 2.7, we can estimate the absolute value of v using integral remainders of the Taylor expansion. Thus, the very same conclusion of the above corollary holds true if we just consider VMO condition on D^2u. It is also interesting to highlight that Corollary 2.7 implies that if u is a viscosity solution in B_1 of a non-convex, fully nonlinear equation under hypotheses (H1)–(H3). Then if u is C^2 at a point $p \in B_1$, then indeed u is $C^{2,\alpha}$ in a neighborhood of p.*

The second application we explore here regards a mild extension of a recent result due to Armstrong, Silvestre, and Smart [3], on partial regularity for solutions to uniform elliptic PDEs.

Corollary 2.9 (Partial regularity)**.** *Let $u \in C^0(B_1)$ be a viscosity solution to $F(D^2u) = f(X)$ where $F \in C^1(Sym(n))$ satisfy $c \le D_{u_iu_j}F(M) \le c^{-1}$ for some constant $c > 0$ and the source function f is Lipschitz continuous. Then, $u \in C^{2,1^-}(B_1 \setminus \Sigma)$ for a closed set $\Sigma \subset B_1$, with Hausdorff dimension at most $(n - \epsilon)$ for an $\epsilon > 0$ universal.*

Remark 2.10. *Since we consider f Lipschitz continuous we have*

$$\mathcal{P}^{+}_{\lambda,\Lambda}\left(D^2(u_e)\right) \ge -\overline{C} \text{ and } \mathcal{P}^{-}_{\lambda,\Lambda}\left(D^2(u_e)\right) \le \overline{C},$$

thus, by Lemma 7.8 in [8] and Lemma 5.1 in [3], we get

$$|\{x \in B_{\frac{1}{2}} : \Psi(u, B_1)(X) > t\}| \le C.t^{-\epsilon}.$$

for $C > 0$ universal and $t > 0$, where

$$\begin{aligned} \Psi(u,\Omega)(X) = \inf\{A \ge 0;\ &\text{there exist } p \in \mathbb{R}^n \text{ e } M \in \mathcal{M}_n \text{ such that for all } Y \in \Omega, \\ &\|u(Y) - u(X) + p(X-Y) + (X-Y).M(X-Y)\| \le \frac{1}{6}A\|X-Y\|^3\}. \end{aligned}$$

To prove the partial regularity, first we use the Theorem 2.2 for establish a relation between Ψ and the local $C^{2,\alpha}$ regularity of u, after we use a covering argument. We divide this process in two step as follows to simplify the exposure.

Lemma 2.11. *Let $u \in C^0(B_1)$ be a viscosity solution of*

$$F(X, D^2u) = f \quad in \quad B_1,$$

satisfies $\sup_{B_1}|u| \le 1$ *and* $0 < \alpha < 1$. *There is a universal constant* $\delta(\alpha) > 0$, *such that for every* $Y \in B_{\frac{1}{2}}$ *and* $0 < r < \frac{1}{16}$, *if*

$$\{\Psi(u, B_1) \le r^{-1}\delta\} \cap B(Y, r) \neq \emptyset$$

implies that

$$u \in C^{2,\alpha}(B(Y, r)).$$

Proof. Let $0 < r < \frac{1}{16}$, $Y \in B_{\frac{1}{2}}$ and $B(Y, r)$ such that

$$\Psi(u, B_1) \le r^{-1}\delta$$

then there exist $z \in B(Y, r), p \in \mathbb{R}^n$ and $M \in \mathcal{M}_n$ such that for any $X \in B_1$

$$|u(X) - u(Z) + p.(Z - X) + (Z - X).M(Z - X)| \le \frac{1}{6}r^{-1}\delta|Z - X|^3.$$

We may assume, without loss of generality, that $M \in Sym(n)$. Define the function

$$v(X) = \frac{1}{16r^2}(u(Z + 4rX) - u(Z) + 4rp.X + 16r^2X.MX).$$

Thus $|v| \le \frac{1}{3}\delta$. Define the operator

$$\widetilde{F}(X, N) = F(Z + 4rX, N - M) - F(Z + 4rX, -M),$$

observe that $\widetilde{F}$ satisfies the conditions of the Theorem 2.2 and

$$\widetilde{F}(X, D^2v) = f(Z + 4rX) - F(Z + 4rX, -M) = \widetilde{f}(X) \in C^{0,\alpha}$$

Thus, pelo Theorem 2.2, $v \in C^{2,\alpha}(B_{1/2})$ what implies that $u \in C^{2,\alpha}(B(Y, r))$. □

Proof of Corollary 2.9. Suppose, without loss of generality, that $\sup_{B_1}|u| \le 1$. Let

$$\Sigma = \{X \in B_{\frac{1}{2}} : u \notin C^{2,\alpha}(B(X, r)) \text{for any } r > 0\}.$$

For $0 < r < \frac{1}{16}$, by the Vitalli covering Lemma, there exist $\{B(X_i, r)\}_{i=1}^m$ of disjoint balls, with $X_i \in \Sigma$, such that

$$\Sigma \subset \bigcup_{i=1}^{m} B(X_i, 3r).$$

By the Lemma 2.11 there is a universal constant $\delta(\alpha) > 0$ that

$$\Psi(u, B_1)(Y) > r^{-1}\delta(\alpha) \text{ for every } Y \in \bigcup_{i=1}^{m} B(X_i, r).$$

For the Remark 2.10 we have that

$$m|B_r| \leq |\{X \in B_{\frac{1}{2}} : \Psi(u, B_1)(Y) > r^{-1}\delta(\alpha)\}| \leq Cr^{\epsilon},$$

thus

$$\sum_{i=1}^{m} |B(X_i, 3r)|^{n-\epsilon} \leq C$$

where C and ϵ are universal. □

2.5 Log-Lipschitz estimates in continuous media

In this section we prove Theorem 2.3. Initially we show that under continuity assumption on the coefficients of F and on the source f, after a proper scaling, solutions are under the smallness regime requested by Lemma 2.5, with $\alpha = 0$. For that define

$$v(X) = \frac{u(\mu X)}{\mu^2}, \quad F_\mu(X, M) := F(\mu X, M) \quad \text{and} \quad f_\mu(X) := f(\mu X),$$

for a parameter μ to be determined. Equation

$$F_\mu(X, D^2 v) = f_\mu(X),$$

is satisfied in the viscosity sense. Now we choose μ so small that

$$\tau(\mu) \leq \delta^{3/2},$$

where τ is the modulus of continuity of the media and $\delta > 0$ is the number appearing in Lemma 2.5 with $\alpha = 0$. In the sequel, define

$$\tau_\mu(t) := \tau(\mu t)$$

and note that

$$\max\left\{|f_\mu(X) - f_\mu(Y)|, \frac{|F_\mu(X, M) - F_\mu(Y, M)|}{\|M\|}\right\} \leq \tau_\mu(|X - Y|).$$

Thus,

$$\sup_{M\in \mathrm{Sym}(n)} \frac{|F_\mu(X,M)-F_\mu(0,M)|}{\|M\|} \leq \delta^{3/2} \quad \text{ and } \quad \|f_\mu\|_{L^\infty(B_1)} \leq \delta^{3/2}.$$

Now if we take

$$\|u\|_{L^\infty(B_1)} \leq \overline{\delta} := \delta\mu^2$$

then

$$\|v\|_{L^\infty(B_1)} \leq \delta.$$

Estimates proven for v gives the desired ones for u.

The conclusion of the above reasoning is that we can start off the proof of Theorem 2.3 out from Lemma 2.5. That is, the proof of the current Theorem begins with the existence of a quadratic polynomial P_1 satisfying $F(0,D^2P_1)=0$ and a number $\sigma>0$ for which the following estimate

$$\sup_{B_\sigma} |u-P_1| \leq \sigma^2\delta, \tag{2.22}$$

holds, provided δ is small enough, depending only on universal parameters. As in Lemma 2.6, we shall prove by induction process the existence of a sequence of polynomials

$$P_k(X) = \frac{1}{2}X^tA_kX + b_kX + c_k$$

satisfying $F(0,D^2P_k)=0$ such that

$$|u(X)-P_k(X)| \leq \delta\sigma^{2k} \quad \text{ for } |X|\leq \sigma^k. \tag{2.23}$$

Moreover, we have the following estimates on the coefficients

$$\left\{ \begin{array}{rcl} |A_k - A_{k-1}| & \leq & C\delta \\ |b_k - b_{k-1}| & \leq & C\delta\sigma^{(k-1)} \\ |c_k - c_{k-1}| & \leq & C\delta\sigma^{2(k-1)}. \end{array} \right. \tag{2.24}$$

The case $k=1$ is precisely the conclusion enclosed in (2.22). Assume we have verified the kth step of induction. Define the scaled function and the scaled operator

$$\tilde{u}(X) := \frac{1}{\sigma^{2k}}(u(\sigma^kX)-P_k(\sigma^kX)) \quad \text{ and } \quad \tilde{F}(X,M) := F(\sigma^kX, M+D^2P_k).$$

Easily one verifies that $\tilde{u}$ is a viscosity solution to

$$\tilde{F}(X,D^2\tilde{u}) = f(\sigma^kX) := \tilde{f}(X).$$

From the induction hypothesis, (2.23), $\tilde{u}$ is flat, i.e., $|\tilde{u}|_{L^\infty(B_1)} \leq \delta$. Also, clearly

$$\sup_{M \in \mathrm{Sym}(n)} \frac{|\tilde{F}(X,M) - \tilde{F}(0,M)|}{\|M\|} \leq \delta^{3/2} \quad \text{and} \quad \|\tilde{f}\|_{L^\infty(B_1)} \leq \delta^{3/2}.$$

And thus $\tilde{u}$ is satisfies (2.22), namely there exists a quadratic polynomial $\tilde{P}$ with $\tilde{F}(0, D^2\tilde{P}) = 0$ and

$$|\tilde{u}(X) - \tilde{P}(X)| \leq \delta\sigma^{2k} \quad \text{for } |X| \leq \sigma.$$

The $(k+1)$th step of induction follows by defining

$$P_{k+1}(X) := P_k(X) + \sigma^{2k}\tilde{P}(\sigma^{-k}X).$$

In view of the coefficient oscillation control (2.24), we conclude b_k converges in $\mathbb{R}^n$ to a vector b_∞ and c_k converges in $\mathbb{R}$ to a real number c_∞. Also

$$|c_k - c_\infty| \leq C\delta\sigma^{2k}, \tag{2.25}$$

$$|b_k - b_\infty| \leq C\delta\sigma^{k}. \tag{2.26}$$

The sequence of matrices A_k may diverge, however, we can at least estimate

$$\|A_k\|_{\mathrm{Sym(n)}} \leq kC\delta. \tag{2.27}$$

In the sequel, we define the tangential affine function

$$\ell_\infty(X) := c_\infty + b_\infty \cdot X$$

and estimate, in view of (2.25), (2.26) and (2.27), for $|X| \leq \sigma^k$,

$$\begin{aligned} |u(X) - \ell_\infty(X)| &\leq |u(X) - P_k(X)| + |c_k - c_\infty| + |(b_k - b_\infty)||X| + |A_k||X|^2 \\ &\leq \delta\sigma^{2k} + 2C\delta\sigma^{2k} + kC\delta\sigma^{2k} \\ &\leq C\delta(k\sigma^{2k}). \end{aligned} \tag{2.28}$$

Finally, fixed $X \in B_\sigma$, take $k \in \mathbb{N}$ such that $\sigma^{k+1} < |X| \leq \sigma^k$. From (2.28), we find

$$|u(X) - \ell_\infty(X)| \leq -(C_1\delta) \cdot |X|^2 \log |X|,$$

as desired. The proof of Theorem 2.3 is concluded. □

3 CAVITY PROBLEMS IN DISCONTINUOUS MEDIA

The chapter is organized as follows. In Section 3.1 we guarantee the existence of uniform Hölder estimates and the linear growth close to the level set $\partial\{u_\epsilon \geq \epsilon\}$ for minimizers to 1.10. In Section 3.2 we establish the main goal of this chapter that is the Lipschitz regularity along the free boundary. In sequel we comment how the same results in the Section 3.2 can be obtained for all minima for functional (1.6). To finish, in Section 3.4 we show consequences of our Lipschitz estimates.

3.1 Existence, Uniform Hölder regularity and Linear growth

In this Section we gather some results and tools available for the analysis of minimizers to the functional (1.10). The results stated herein follow by methods and approaches available in the literature.

Theorem 3.1 (Existence of minimizers)**.** *For each $\epsilon > 0$ fixed, there exists at least one minimizer $u_\epsilon \in H^1_\varphi(\Omega)$ to the function* (1.10)*. Furthermore u_ε satisfies*

$$\mathrm{div}(a_{ij}(X)\nabla u_\epsilon) = \beta_\epsilon(u_\epsilon), \quad \textit{in } \Omega, \tag{3.1}$$

in the distributional sense. Furthermore, each u_ϵ in a nonnegative function, provided the boundary data φ is nonnegative.

Proof. Existence of minimizer as well as the Euler-Lagrange equation associated to the functional follow by classical methods in the Calculus of Variations. Non-negativity of a minimum is obtained as follows. Suppose, for the sake of contradiction, the set $\{u_\epsilon < 0\}$ were not empty. Since $\varphi \geq 0$ on $\partial\Omega$, one sees that $\partial\{u_\epsilon < 0\} \subset \{u_\epsilon = 0\} \cap \Omega$. Since β_ε is supported in $[0, \epsilon]$, from the equation we conclude that u_ϵ satisfies the homogeneous equation $\mathrm{div}(a_{ij}(X)\nabla u_\epsilon) = 0$ in $\{u_\epsilon < 0\}$. By the maximum principle we conclude $u_\epsilon \equiv 0$ in such a set, which gives a contradiction. □

Regarding higher regularity for minimizers, it is possible to show uniform-in-ϵ L^∞ bounds and also a uniform-in-ϵ $C^{0,\alpha}$ estimate, for a universal exponent $0 < \alpha < 1$. To this we use the follow energy estimate.

Lemma 3.2. *Let $u \in H^1(B_R(X_0))$ and $h \in H^1(B_R(X_0))$ weak solution to*

$$div(a_{ij}(X)\nabla h) = 0 \ in \ B_R(X_0),$$

then there exist a constant $C(n,\lambda,\Lambda)>0$ such that for $0<r<R$ holds

$$\int_{B_r(X_0)} |\nabla u-(\nabla u)_r|^2dx \le C(\lambda,\Lambda,n)\left(\frac{r}{R}\right)^{n-2+2\alpha_{\lambda,\Lambda}}\int_{B_R(X_0)}|\nabla u-(\nabla u)_R|^2dx$$
$$+\int_{B_R(X_0)}|\nabla u-\nabla h|^2dx,$$

where $\alpha_{\lambda,\Lambda}$ is the best exponent for the homogeneous equation.

Proof. See Lemma 4.2 in [2] □

Theorem 3.3 (Uniform Hölder regularity of minimizers)**.** *Fixed a subdomain $\Omega'\Subset\Omega$, there exists a constant $C>0$, depending on dimension, ellipticity constants, $\|\varphi\|_{L^2}$ and Ω', but independent of ϵ, such that*

$$\|u_\epsilon\|_{L^\infty(\Omega')}+[u_\epsilon]_{C^\alpha(\Omega')}<C,$$

where $0<\alpha<1$ is a universal number.

Proof. Fix $X_0\in\Omega$, $R>0$ such that $R>dist(X_0,\partial\Omega)$ and take h a function such that

$$div(A(X)\nabla h)=0 \text{ in } B_R(X_0) \text{ and } h-u\in H^1_0(B_R(X_0)).$$

Thus

$$\int_{B_R(X_0)}\frac{1}{2}(\langle A(X)\nabla u,\nabla u\rangle-\langle A(X)\nabla h,\nabla h\rangle)dx=\int_{B_R(X_0)}\frac{1}{2}\langle A(X)\nabla(u-h),\nabla(u-h)\rangle dx$$

and by ellipticity

$$\int_{B_R(X_0)}\frac{1}{2}\langle A(X)\nabla(u-h),\nabla(u-h)\rangle dx\ge\frac{\lambda}{2}\int_{B_R(X_0)}|\nabla u-\nabla h|^2dx.$$

Now, using the minimality of u we obtain that

$$\int_{B_R(X_0)}\frac{1}{2}(\langle A(X)\nabla u,\nabla u\rangle-\langle A(X)\nabla h,\nabla h\rangle)dx\le\int_{B_R(X_0)}(\mathbf{B}_\epsilon(h)-\mathbf{B}_\epsilon(u)).$$

Therefore, using the inequalities above together with the Lemma 3.2 we have

$$\int_{B_r(X_0)} |\nabla u - (\nabla u)_r|^2 dx \le C(\lambda, \Lambda, n) \left(\frac{r}{R}\right)^{n-2+2\alpha_{\lambda,\Lambda}} \int_{B_R(X_0)} |\nabla u - (\nabla u)_R|^2 dx$$
$$+ C(\lambda, \Lambda, n, \|\beta\|_{L^1})|B_R|$$
$$\le C\left(\frac{r}{R}\right)^{n-2+2\alpha_{\lambda,\Lambda}} \int_{B_R(X_0)} |\nabla u - (\nabla u)_R|^2 dx + CR^n.$$

So the theorem follows by Lemma 8.23 in [20] and the classical Morrey embedding Theorem. □

As a consequence of Theorem 3.3, up to a subsequence, u_ϵ converges locally uniformly in Ω to a nonnegative function u_0. By linear interpolation techniques, see for instance [30, Theorem 5.4], one verifies that u_0 is a minimizer for the functional (1.6).

The final result we state in this section gives the sharp lower bound for the grow of u_ϵ away from ϵ-level surfaces.

Theorem 3.4 (Linear Growth). *Let $\Omega' \Subset \Omega$ be a given subdomain and $X_0 \in \Omega' \cap \{u_\epsilon \ge \epsilon\}$ then*

$$u_\epsilon(X_0) \ge c \cdot \mathrm{dist}(X_0, \partial\{u_\epsilon \ge \epsilon\}), \tag{3.2}$$

where c is a constant that depends on dimension and ellipticity constants, but it is independent of ϵ.

Proof. Denote $d = d(X_0, \partial\{u_\epsilon \ge \epsilon\})$, $u_\epsilon(X_0) = \alpha d$ and define

$$v(X) = \frac{u_\epsilon(X_0 + dX)}{d}.$$

So

$$\mathrm{div}(a_{ij}(X_0 + dX)\nabla v) = 0$$

and $v(0) = \alpha$, therefore by Harnack inequality

$$\underline{c}\alpha \le v \le \bar{c}\alpha \text{ in } B_{1/2}$$

Now, consider a smooth function ψ, non-negative and radially symmetric such that

$$0 < \psi < 1, \psi = 0 \text{ in } B_{1/8} \text{ and } \psi = 1 \text{ in } B_1 \setminus B_{1/2}$$

and define

$$g(X) = \begin{cases} \min\{v, \bar{c}\alpha\psi\} & \text{in} \quad B_{1/2} \\ v & \text{in} \quad B_1 \setminus B_{1/2} \end{cases}$$

Since u_ϵ is a minimum to (1.10) we have that

$$F(v) \le F(g)$$

where

$$F(u) = \int_{B_1} \{\frac{1}{2}\langle a_{ij}(X_0 + dX)\nabla u, \nabla u\rangle + \mathbf{B}_\epsilon(du)\}.$$

In the other terms

$$\int_{B_1} \frac{1}{2}\left(\langle a_{ij}(X_0 + dX)\nabla g, \nabla g\rangle - \langle a_{ij}(X_0 + dX)\nabla v, \nabla v\rangle\right) \ge \int_{B_1} \mathbf{B}_\epsilon(dv) - \int_{B_1} \mathbf{B}_\epsilon(dg).$$

If on the one hand we have

$$\int\limits_{B_{1/2}\cap\{\bar{c}\alpha\psi\le v\}} \frac{1}{2}\left(\bar{c}^2\alpha^2\langle a_{ij}(X_0 + dX)\nabla\psi, \nabla\psi\rangle - \langle a_{ij}(X_0 + dX)\nabla v, \nabla v\rangle\right) \le \alpha^2 C\Lambda\|\nabla\psi\|^2_{L^2},$$

on the other hand, since that $v \ge g$ and B_ϵ is non-decreasing we get that

$$\begin{aligned} \int_{B_1} (\mathbf{B}_\epsilon(dv) - \mathbf{B}_\epsilon(dg)) &\ge \int_{B_{1/8}} (\mathbf{B}_\epsilon(dv) - \mathbf{B}_\epsilon(dg)) \\ &\ge \int_{B_{1/8}} \mathbf{B}_\epsilon(dv) \ge \mathbf{B}_\epsilon(\underline{c}\alpha)|B_{1/8}| \ge |B_{1/8}|\mathbf{B}_\epsilon(\underline{c}\epsilon) \\ &\ge \mathbf{B}_1(\underline{c}) = \mathbf{c}. \end{aligned}$$

Thus

$$\alpha \ge \mathbf{C}$$

which finishes the proof.

□

3.2 Lipschitz regularity along the free boundary

This section is the heart of this part of the thesis, where we deliver a proof of the main, key result of the chapter, namely that uniform limits of solutions to (3.1) are locally Lipschitz continuous along their free boundaries.

Theorem 3.5 (Lipschitz regularity). *Let u_0 be a uniform limit point of solutions to*

$$\mathrm{div}(a_{ij}(X)\nabla u_\epsilon) = \beta_\epsilon(u_\epsilon) \quad \textit{in } \Omega$$

and assume that $u_0(\xi) = 0$. Then there exists a universal constant $C > 0$, depending only on dimension, ellipticity constants, $\mathrm{dist}(\xi, \partial\Omega)$ and L^∞ bounds of the family such that

$$|u_0(X)| \leq C|X - \xi|,$$

for all point $X \in \Omega$.

Our strategy is based on a flatness improvement argument, within whom the next lemma plays a decisive role.

Lemma 3.6. *Fixed a ball $B_r(Y) \Subset \Omega$ and given $\theta > 0$, there exists a $\delta > 0$, depending only on $B_r(Y)$, dimension, ellipticity constants and L^∞ bounds for u_ϵ, such that if*

$$div(a_{ij}(X)\nabla u_\epsilon) = \delta \cdot \beta_\epsilon(u_\epsilon)$$

and

$$\max\{\epsilon, \inf_{B_r(Y)} u_\epsilon\} \leq \delta.$$

Then

$$\sup_{B_{\frac{r}{2}}(Y)} u_\epsilon \leq \theta.$$

Proof. Let us suppose, for the sake of contradiction, that the Lemma fails to hold. There would then exist a sequence of functions u_{ϵ_k} satisfying

$$div(a^k_{ij}(X)\nabla u_{\epsilon_k}) = \delta_k \beta_{\epsilon_k}(u_{\epsilon_k})$$

with a^k_{ij} (λ, Λ)-elliptic, $\delta_k = \mathrm{o}(1)$, and

$$\max\{\epsilon_k, \inf_{B_r(Y)} u_{\epsilon_k}\} =: \eta_k = \mathrm{o}(1),$$

but

$$\sup_{B_r/2(Y)} u_{\epsilon_k} \geq \theta_0 > 0, \tag{3.3}$$

for some $\theta_0 > 0$ fixed. Let X_k be the point where u_{ϵ_k} attains its minimum in $\bar{B}_r(Y)$ and denote $\sigma := \operatorname{dist}(B_r(Y), \partial\Omega) > 0$. Define the scaled function $v_k \colon B_{\sigma\epsilon_k^{-1}} \to \mathbb{R}$, by

$$v_k(X) := \frac{u_{\epsilon_k}(X_k + \epsilon_k X)}{\eta_k}$$

One simply verifies that $v_k \geq 0$ and it solves, in the distributional sense,

$$\begin{aligned} \operatorname{div}(a_{ij}^k(X)\nabla v_k) &= \delta_k \cdot \left(\tfrac{\epsilon_k}{\eta_k}\beta_1(\tfrac{\eta_k}{\epsilon_k}v_k)\right) \\ &= \mathrm{o}(1), \end{aligned}$$

as $k \to \infty$, in the L^∞-topology. Also, one easily checks that $v_k(0) \leq 1$. Hence, by Harnack inequality, the sequence v_k is uniform-in-k locally bounded in $B_{\sigma\epsilon_k^{-1}}(0)$. From De Giorgi, Nash, Moser regularity theory, up to a subsequence, v_k converges locally uniformly to an entire v_∞. In addition, by standard Caccioppoli energy estimates, the sequence v_k is locally bounded in H^1, uniform in k. Also by classical truncation arguments, up to a subsequence, $\nabla v_k(X) \to \nabla v_\infty(X)$ a.e. (see [29] and [31] for similar arguments). By ellipticity, passing to another subsequence, if necessary, a_{ij} converges weakly in L^2_{loc} to a (λ, Λ)-elliptic matrix b_{ij}. Summarizing we have the following convergences:

$$v_k \to v_\infty \text{ locally uniformly in } B_{\sigma\epsilon_k^{-1}} \tag{3.4}$$

$$v_k \rightharpoonup v_\infty \text{ weakly in } H^1(B_{\sigma\epsilon_k^{-1}}) \tag{3.5}$$

$$\nabla v_k(X) \to \nabla v_\infty(X) \text{ almost everywhere in } B_{\sigma\epsilon_k^{-1}} \tag{3.6}$$

$$a_{ij}^k(X) \rightharpoonup b_{ij} \text{ weakly in } L^2(B_{\sigma\epsilon_k^{-1}}) \tag{3.7}$$

Passing to the limits, we conclude,

$$\operatorname{div}(b_{ij}(X)\nabla v_\infty) = 0, \quad \text{in } \mathbb{R}^n.$$

Applying Liouville Theorem to v_∞, we conclude that

$$v_\infty \equiv \text{Const.} < +\infty,$$

for a bounded constant, in the whole space. The corresponding limiting function u_∞ obtained from u_{ϵ_k} must therefore be identically zero. We now reach a contradiction with (3.3) for $k \gg 1$. The Lemma is proven. □

Before continuing, we remark that if u_ϵ is a solution to the original equation (3.1) and a positive number $\bar{\delta} > 0$ is given, then the zoomed-in function

$$\tilde{u}_\epsilon(X) = u_\epsilon(\sqrt{\bar{\delta}}X)$$

satisfies in the distributional sense the equation

$$\operatorname{div}(a_{ij}(X)\nabla \tilde{u}_\epsilon) = \bar{\delta}\beta_\epsilon(\tilde{u}_\epsilon)$$

We are in position to start delivering the proof of Theorem 3.5. Let u_ϵ be a bounded sequence of distributional solutions to (3.1) and u_0 a limit point in the uniform convergence topology. We assume, with no loss, that $\xi = 0$, that is $u_0(0) = 0$. Within the statement of Lemma 3.6, select

$$\theta = \frac{1}{2}.$$

Since $u_\epsilon(0) \to 0$ as $\varepsilon \to 0$, Lemma 3.6 together with the above remark, gives the existence of a positive, universal number $\delta_\star > 0$, such that if $0 < \varepsilon \le \epsilon_0 \ll 1$, for $\tilde{u}_\epsilon(X) := u_\varepsilon(\sqrt{\delta_\star}X)$ we have

$$\sup_{B_{1/2}} \tilde{u}_\epsilon(X) \le \frac{1}{2}.$$

Passing to the limit as $\epsilon \to 0$, we obtain

$$\sup_{B_{\frac{\sqrt{\delta_\star}}{2}}} u_0(X) \le \frac{1}{2}.$$

Define the rescaled function

$$v^1(X) := 2u_\varepsilon(\frac{\sqrt{\delta_\star}}{2}X).$$

It is simple to verify that v^1 satisfies

$$\operatorname{div}(a^1_{ij}(X)\nabla v^1(X)) = \delta_\star\beta_{2\epsilon}(v^1),$$

in the distributional sense, where $a^1_{ij}(X) = a_{ij}(\sqrt{\delta_\star}/2X)$ is another (λ, Λ)-elliptic matrix. Once more, $v^1(0) \to 0$ as $\epsilon \to 0$, hence, for $\epsilon \le \epsilon_1 < \epsilon_0 \ll 1$, we can apply Lemma 3.6 to v^1 and deduce, after scaling the inequality back,

$$\sup_{B_{\frac{\sqrt{\delta_\star}}{4}}} u_0(X) \le \frac{1}{4}.$$

Continuing this process inductively, we conclude that for any $k \ge 1$, that holds

$$\sup_{B_{\frac{\sqrt{\delta_\star}}{2^k}}} u_0(X) \le \frac{1}{2^k}. \tag{3.8}$$

Finally, given $X \in B_{1/2}$ let $k \in \mathbb{N}$ be such that

$$\frac{\sqrt{\delta_\star}}{2^{k+1}} < |X| \le \frac{\sqrt{\delta_\star}}{2^k}.$$

We estimate from (3.8)

$$\begin{aligned} u_0(X) &\leq \sup_{B_{\frac{\sqrt{\delta_\star}}{2^k}}} u_0(X) \\ &\leq \frac{1}{2^k} \\ &\leq \frac{2}{\sqrt{\delta_\star}}|X|, \end{aligned}$$

and the proof of Theorem 3.5 is concluded.

Definition 3.7. *Given a large constant $K > 0$, we say that a uniform elliptic matrix $a_{ij}(X)$ satisfies (K-Lip) property if for any $0 < d < 1$, $h \in H^1(B_d)$ solves*

$$\mathrm{div}\,(a_{ij}(X)\nabla h) = 0 \text{ in } B_d$$

in the distributional sense, then

$$\|\nabla h\|_{L^\infty(B_{d/2})} \leq \frac{K}{d} \cdot \|h\|_{L^\infty(B_d)}.$$

It is classical that Dini continuity of the medium is enough to assure that a_{ij} satisfies (K-Lip) property, for some $K > 0$ that depends only upon dimension, ellipticity constants and the Dini-modulus of continuity of a_{ij}.

Our next Corollary says that uniform limits of singularly perturbed equation (3.1) is Lipschitz continuous up to the free boundary provided a_{ij} satisfies (K-Lip) property for some $K > 0$. The, not obvious, message being that when it comes to Lipschitz estimates, the homogeneous equation and the free boundary problem $\mathrm{div}(a_{ij}(X)\nabla u) \sim \delta_0(u)$ require the same amount of organization of the medium.

Corollary 3.8. *Under the assumptions of Theorem 3.5, assume further that $a_{ij}(X)$ satisfies (K-Lip) property for some K. Then, given a subdomain $\Omega' \Subset \Omega$,*

$$|\nabla u_0(X)| \leq C,$$

for a constant that depends only on dimension, ellipticity constants, $\mathrm{dist}(\partial\Omega', \partial\Omega)$*, L^∞ bounds of the family and K.*

Proof. It follows from Theorem 3.5 and property K that u_0 is pointwise Lipschitz continuous, i.e.,

$$|\nabla u_0(\xi)| \leq C(\xi).$$

We have to show that $C(\xi)$ remains bounded as ξ goes to the free boundary. For that, let ξ be a point near the free boundary $\partial\{u_0 > 0\}$ and denote by $Y \in \partial\{u_0 > 0\}$ a point such that

$$|Y - \xi| =: d = \text{dist}(\xi, \partial\{u_0 > 0\}).$$

From Theorem 3.5, we can estimate

$$\sup_{B_{d/2}(\xi)} u_0(\xi) \leq \sup_{B_{2d}(Y)} u_0(\xi) \leq C \cdot 2d.$$

Applying (K-Lip) property to the ball $B_{d/2}(\xi)$, we obtain

$$|\nabla u_0(\xi)| \leq \frac{2K}{d} \cdot 2Cd = 4C \cdot K,$$

and the proof is concluded. □

3.3 Lipschitz estimates for the minimization problem

Limiting functions u_0 obtained as ϵ goes to zero from a sequence u_ϵ of minimizers of functional (1.10) are minima of the discontinuous functional (1.6). Hence, limiting minima are Lipschitz continuous along their free boundaries. Nonetheless, as previously advertised in Theorem 3.5, the sharp Lipschitz regularity estimate holds indeed for *any* minima of the functional (1.6), not necessarily for limiting functions.

In this intermediate section we shall comment on how one can deliver this estimate directly from the analysis employed in the proof of Theorem 3.5. In fact, the proof of Lipschitz estimate for minima of the functional (1.6) is simpler than the proof delivered in previous section, which has been based solely on the singular equation satisfies.

Theorem 3.9. *Let $u_0 \geq 0$ be a minimum to*

$$\mathscr{F}(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \chi_{\{u>0\}} \right\} dX$$

and assume that $u_0(\xi) = 0$. Then there exists a universal constant $C > 0$, depending only on dimension, ellipticity constants, $\text{dist}(\xi, \partial\Omega)$ *and its L^∞ norm such that*

$$u_0(X) \leq C|X - \xi|,$$

for all point $X \in \Omega$.

The proof follows the lines designed in Section 3.2. We obtain the corresponding flatness Lemma as follows:

Lemma 3.10. *Fixed a ball $B_r(Y) \Subset \Omega$ and given $\theta > 0$, there exists a $\delta > 0$, depending only on $B_r(Y)$, dimension, ellipticity constants and L^∞ norm of u_0, such that if u_0 is* *nonnegative a minimum of*

$$\mathscr{F}^{\delta}(u) = \int_{\Omega} \left\{ \frac{1}{2} \langle a_{ij}(X) \nabla u, \nabla u \rangle + \delta \cdot \chi_{\{u>0\}} \right\} dX,$$

and $u_0(Y) = 0$*, then*

$$\sup_{B_{\frac{r}{2}}(Y)} u_0 \leq \theta.$$

Proof. The proof follows by a similar tangential analysis of the proof of Lemma 3.6, but in fact in a simpler fashion. The tangential functional, obtained as $\delta \to 0$, satisfies minimum principle, hence the limiting function, from the contradiction argument, must be identically zero.

Here are some more details: suppose, for the sake of contradiction, that the Lemma fails to hold. It means, for a sequence (λ, Λ)-elliptic matrices, a_{ij}^k, and a sequence of minimizers u_k of

$$\mathscr{F}^{k}(u) = \int_{\Omega} \left\{ \frac{1}{2} \langle a^k_{ij}(X) \nabla u, \nabla u \rangle + \delta_k \cdot \chi_{\{u>0\}} \right\} dX,$$

with $\delta_k = \mathrm{o}(1)$, and, say $\|u_k\|_\infty \leq 1$,

$$\sup_{B_r/2(Y)} u_k \geq \theta_0 > 0, \tag{3.9}$$

for some $\theta_0 > 0$ fixed. As in Lemma 3.6, by compactness, up to a subsequence, $u_k \to u_0$. Passing the limits we conclude u_0 is a local minimum of

$$\mathscr{F}^{\infty}(u) = \int \frac{1}{2} \langle b_{ij}(X) \nabla u_0, \nabla u_0 \rangle dX.$$

Since, $u_0 \geq 0$ and $u_0(Y) = 0$, by the maximum principle, $u_0 \equiv 0$. We now reach a contradiction with (3.9) for $k \gg 1$. The Lemma is proven. □

Once we have obtained Lemma 3.10, the proof of Theorem 3.9 follows exactly as the final steps in the proof of Theorem 3.5.

3.4 Further consequences of Theorem 3.5

We start off this part by commenting that Theorem 3.5 as well as Theorem 3.9 hold for two-phase problems, under the assumption that the negative values of u is universally controlled by below:

$$\inf_{\Omega} u \geq -\delta_\star, \tag{3.10}$$

for a universal value $\delta_\star > 0$. Such a condition is realistic for models involving very low temperatures, i.e., for physical problem near the absolute zero for thermodynamic temperature scale (zero Kelvin).

Let us briefly comment on such generalization. Within the proof of Lemma 3.10, one includes condition (3.10) in the compactness argument. Here is the two-phase version of Lemma 3.10:

Lemma 3.11. *Fixed a ball $B_r(Y) \Subset \Omega$ and given $\theta > 0$, there exists a $\delta > 0$, depending only on $B_r(Y)$, dimension, ellipticity constants and L^∞ norm of u, such that if u is a changing sign minimum of*

$$\mathscr{F}^\delta(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \delta \cdot \chi_{\{u>0\}} \right\} dX,$$

with

$$u_0(Y) = 0 \quad \text{and} \quad \inf_\Omega u \geq -\delta,$$

then

$$\sup_{B_{\frac{r}{2}}(Y)} |u| \leq \theta.$$

The proof of Lemma 3.11 follows the lines of Lemma 3.10, noticing that, by letting $\delta = \mathrm{o}(1)$ in the compactness approach, the tangential configuration is too a non-negative minima of a functional which satisfies minimum principle.

In the sequel we show how the improved estimate given by Theorem 3.5 implies some geometric estimates on the free boundary. Hereafter in this section, u_0 will always denote a limit point obtained from a sequence of minimizers of the functional (1.10). We will denote by Ω_0 the non coincidence set, $\Omega_0 := \{u_0 > 0\} \cap \Omega$. Unless otherwise stated, no continuity assumption is imposed upon the medium a_{ij},

Theorem 3.12 (Nondegeneracy). *Let $\Omega' \Subset \Omega$ be a given subdomain and $Y \in \Omega' \cap \{u_\epsilon > 2\epsilon\}$ then*

$$\sup_{B_r(Y)} u_\epsilon \geq c \cdot r.$$

for $r <$ dist$(\Omega', \partial\Omega)$. Moreover, for $Y \in \Omega' \cap \overline{\{u_0 > 0\}}$ holds

$$\sup_{B_r(Y)} u_0 \geq c \cdot r.$$

Proof. First we going to prove that there exist a universal constant δ_0 such that

$$\sup_{B_{d_\epsilon(X)}(X)} u_\epsilon \geq (1 + \delta_0) u_\epsilon(X)$$

for $X \in B_{1/2} \cap \{u_\epsilon \geq 2\epsilon\}$, where $d_\epsilon(X) = dist(X, \partial u_\epsilon \geq \epsilon)$. Suppose that the claim is not true. Then there exist sequences u_{ϵ_k}, X_k and $\delta_k \to 0$ for which

$$\sup_{B_{d_\epsilon(X_k)}(X_k)} u_{\epsilon_k} < (1+\delta_k) u_{\epsilon_k}(X_k).$$

Thus, define

$$v_k(X) = \frac{1}{u_{\epsilon_k}(X_k)} u_{\epsilon_k}(X_k + d_\epsilon(X_k)X)$$

and observe that $\sup_{B_1} v_k \leq (1+\delta_k)$, $v_k(0) = 1$ and $v_k > 0$, moreover

$$\text{div}(a_{ij}(X_0 + d_\epsilon X)\nabla v_k) = 0.$$

So, v_k converges locally uniformly to a function v_∞, and applying the Harnack inequality to $|X| < r < 1$ we obtain

$$0 \leq (1+\delta_k) - v_k \leq C_r(1 + \delta_k - v_k(0)) = C_r \delta_k,$$

and making $k \to \infty$ we conclude that $v_\infty = 1$. We take then $Z_k \in \{u_{\epsilon_k} = \epsilon_k\}$ such that $|Z_k - X_k| = d_\epsilon(X_k)$ and define $w_k = \frac{Z_k - X_k}{d_\epsilon(X_k)}$ note that $v_k(w_k) \to 1$, in the other hand we have

$$v_k(w_k) = \frac{\epsilon_k}{u_{\epsilon_k}(X_k)} \leq \frac{1}{2} < 1$$

that lead us to a contradiction. Now, joining the claim above with Theorem 3.4 we can construct a sequence of points $\{X_k\}$ beginning with X_0 such that

1. $u_\epsilon(X_k) \geq (1+\delta_0)^k u_\epsilon(X_0)$
2. $|X_k - X_{k-1}| = d_\epsilon(X_{k-1})$
3. $u_\epsilon(X_k) - u_\epsilon(X_{k-1}) \geq c|X_k - X_{k-1}|$.

Therefore, since $u_\epsilon(X_k) \to \infty$, we can find a last point X_{k_0} in B_r and it satisfies

$$|X_{k_0} - X_0| \geq \mathbf{c} r$$

where $\mathbf{c}$ is universal. Thus

$$\sup_{B_r(X_0)} u_\epsilon \geq u_\epsilon(X_{k_0}) \geq u_\epsilon(X_0) + |X_{k_0} - X_0| \geq \mathbf{c} r$$

Since $u_\epsilon \to u_0$ uniformly and $\mathbf{c}$ is independent of ϵ we can conclude the last claim of the theorem to $Y \in \Omega' \cap \{u_0 > 0\}$. For $Y_0 \in \Omega' \cap \partial\{u_0 > 0\}$ consider $X \in \partial B_{r/4}(Y_0) \cap \{u_0 > 0\}$, thus by the previous estimate

$$\sup_{B_r(Y)} u_0 \geq \sup_{B_{r/4}(X)} u_0 \geq \mathbf{c}\frac{r}{4}.$$

□

Theorem 3.13. *Given a subdomain $\Omega' \Subset \Omega$, there exists a constant $\theta > 0$, such that if $X_0 \in \partial\Omega_0$ is a free boundary point then*

$$\mathscr{L}^n(\Omega_0 \cap B_r(X_0)) \geq \theta r^n,$$

for all $0 < r < \operatorname{dist}(\partial\Omega', \partial\Omega)$. Furthermore there is a universal constant $0 < \varsigma \leq 1$ such that

$$dim_{\mathcal{H}}(\partial\Omega_0) \leq n - \varsigma,$$

where $dim_{\mathcal{H}}(E)$ means the Hausdorff dimension to the set E.

Proof. It follows readily from non-degeneracy property, Theorem 3.12, there exists a point $\xi_r \in \partial B_r(X_0)$ such that

$$u_0(\xi_r) \geq cr,$$

for a constant $c > 0$ depending only on the data of the problem. Now, for $0 < \mu \ll 1$, small enough, there holds

$$B_{\mu r}(\xi_r) \subset \Omega_0. \tag{3.11}$$

Indeed, one simply verifies that if

$$B_{\mu r}(\xi_r) \cap \partial\{u_0 > 0\} \neq \emptyset,$$

then from Theorem 3.5 we can estimate

$$cr \leq u_0(\xi_r) \leq \sup_{B_{\mu r}(\xi_r)} u_0 \leq C\mu r$$

which is a lower bound for μ. Hence, if $\mu < c \cdot C^{-1}$, (3.11) must hold. Now, with such $\mu > 0$ fixed, we estimate

$$\mathscr{L}^n\left(B_r(X_0) \cap \Omega_0\right) \geq \mathscr{L}^n\left(B_r(X_0) \cap B_{\mu r}(\xi_r)\right) \geq \theta r^n$$

and the uniform positive density is proven.

Let us turn our attention to the Hausdorff dimension estimate. Given $\sigma = X_0$

in $\partial\{u > 0\}$, we choose

$$\sigma\prime = t\xi_r + (1-t)X_0,$$

with t close enough to 1 as to

$$B_{\frac{1}{2}\mu\cdot r}(\sigma\prime) \subset B_\mu(\xi_r) \cap B_r(\sigma) \subset B_r(\sigma) \setminus \partial\{u > 0\}.$$

We have verified $\partial\{u > 0\} \cap B_{1/2}$ is $(\mu/2)$-porous, hence by a classical result, see for instance [21, Theorem 2.1], its Hausdorff dimension is at most $n - C\mu^n$, for a dimensional constant $C > 0$. □

For problems modeled in a merely measurable medium, one should not expect an improved Hausdorff estimate for the free boundary. When diffusion is governed by the Laplace operator, then Alt-Caffarelli theory gives that $\varsigma = 1$. A natural question is what is the minimum organization of the medium as to obtain perimeter estimates of the free boundary. Next Theorem gives an answer to that issue.

Theorem 3.14. *Assume a_{ij} satisfy (K-Lip) property for some $K > 0$. Then the free boundary has local finite perimeter. In particular $dim_{\mathcal{H}}(\partial\Omega_0) = n - 1$.*

Proof. The proof is divided in three steps. First, fix a free boundary point $X_0 \in \partial\Omega_0$ and given a small, positive number μ one checks that

$$\int_{\{0<u_0<\mu\}\cap B_r(X_0)} |\nabla u_0|^2 \leq C\mu r^{n-1}. \tag{3.12}$$

To see this, consider the following test function, for $C_1 > 0$,

$$\phi = \begin{cases} u_0; \text{ if } 0 \leq u_\epsilon < \mu \\ \mu; \text{ if } u_0 \geq \mu \end{cases}$$

then

$$0 \leq \int_{B_r(X_0)} \mathrm{div}(a_{ij}\nabla u_0))\phi dX.$$

Now using integration by parts we have that

$$\int_{B_r(X_0)} \mathrm{div}(a_{ij}\nabla u_0))\phi dX = \frac{1}{r}\int_{\partial B_r(X_0)} a_{ij}\phi\partial_i u_0(X^i - X_0^i)dH^{n-1} - \int_{B_r(X_0)} a_{ij}\partial_i u_0 \partial_j \phi$$

and, for the Lipschitz estimates of u_0 and the uniform ellipticity, we obtain that

$$\int_{\{0<u_0<\mu\}\cap B_r(X_0)} a_{ij}\partial_i u_0 \partial_j u_0 dX \leq C\mu r^{n-1}.$$

For conclude it is enough observe that, by ellipticity,

$$\int_{\{0<u_0<\mu\}\cap B_r(X_0)} |\nabla u_0|^2 dX \leq C \int_{\{0<u_0<\mu\}\cap B_r(X_0)} a_{ij}\partial_i u_0 \partial_j u_0 dX.$$

In the sequel, we compare the left hand side of (3.12) with $|\{0 < u_0 < \mu\} \cap B_r(X_0)|$. To achieve this goal consider a finite converging, $\{B_j\}$, of $\partial\Omega_0$ by balls of radius proportional to μ and centered on $\partial\Omega_0 \cap B_r(X_0)$. It's enough to show that for a universal constant $c > 0$ holds

$$⨍_{B_j} |\bar{u}_0 - [\bar{u}_0]_j|^2 dX > c\mu^2, \tag{3.13}$$

where $\bar{u}_0 = \min\{u_0, \mu\}$ and $[v]_j := ⨍_{B_j} v$. Indeed, by the Poincaré inequality,

$$c\mu^2 \leq ⨍_{B_j} |\bar{u}_0 - [\bar{u}_0]_j|^2 dX \leq C\mu^2 ⨍_{B_j} |\nabla \bar{u}_0|^2 dX,$$

so

$$\int_{B_j\cap\{0<u_0<\mu\}} |\nabla u_0|^2 dX = \int_{B_j} |\nabla \bar{u}_0|^2 dX \geq c|B_j|.$$

Now, for $\mu \ll r$, we have

$$B_r(X_0) \cap \{0 < u_0 < \mu\} \subset \bigcup 2B_j \subset B_{4r}(X_0).$$

Finally, if we call $A := \{0 < u_0 < \mu\}$, the above gives

$$\begin{aligned}\int_{B_{4r}(X_0)\cap A} |\nabla u_0|^2 dX &\geq \int_{\left(\cup 2B_j\right)\cap A} |\nabla u_0|^2 dX \\ &\geq \frac{1}{m}\sum \int_{2B_j\cap A} |\nabla u_0|^2 dX \\ &\geq c\sum \mathscr{L}^n(B_j) \\ &\geq c\mathscr{L}^n(B_r(X_0)\cap A),\end{aligned} \tag{3.14}$$

where m is the total number of balls, which can be taken universal, by Heine-Borel's Theorem. Remains to prove the existence of the constant c in (3.13), for this we going to show that for each B_j exist sub-balls B_j^1 and B_j^2 to which hold the following statements:

1. The radius of B_j^1 and B_j^2 are $r_1^j = K_1\mu$ and $r_2^j = K_2\mu$, where K_1 and K_2 are universal.

2. $\bar{u}_0 \geq \frac{3}{4}\mu$ in B_j^1 and $\bar{u}_0 \leq \frac{2}{3}\mu$ in B_j^2.

Observe that, by non-degeneracy, there is a point $X_1^j \in \frac{1}{4}B_j$ such that

$$u_0(X_1^j) \geq c\frac{C^\star \mu}{4}.$$

Considering $C^\star$ large, since μ can be taken small, so that

$$cC^\star > 4 \text{ and } K := \sup_{\mathcal{N}_{\frac{d'}{8}}(\Omega')} |\nabla u_0| > \frac{1}{C^\star}$$

and taking $r_j^1 = \frac{1}{8K}\mu$ and $r_j^2 = \frac{1}{3K}\mu$ then the sub-balls that we seek are

$$B_j^1 = B_{r_j^1}(X_1^j) \text{ and } B_j^2 = B_{r_j^2}(X_0).$$

Now suppose that there is sequences $\{X_n\} \in B_j^1$ and $\{Y_n\} \in B_j^2$ such that

$$\frac{|\bar{u}_0(X_n) - [\bar{u}_0]_j\,|}{\mu} \leq \frac{1}{n} \text{ and } \frac{|\bar{u}_0(Y_n) - [\bar{u}_0]_j\,|}{\mu} \leq \frac{1}{n}$$

thus we have

$$\frac{|\bar{u}_0(X_n) - \bar{u}_0(Y_n)|}{\mu} \to 0$$

which is a contradiction with the second statement that the sub-balls satisfy. Then we can conclude that for one of the two sub-balls, for example B_j^1, there is a constant c such that

$$|\bar{u}_0(X) - [\bar{u}_0]_j\,| \geq c\mu \text{ for } X \in B_j^1$$

which implies (3.13) and completes the proof. Therefore (3.14) with (3.12), gives

$$|\mathscr{L}^n(\{0 < u_0 < \mu\} \cap B_r(X_0))| \leq C\mu r^{n-1},$$

To the third step denote

$$\mathcal{N}_\delta(E) := \{X \in \mathbb{R}^n; dist(X, E) < \delta\}.$$

As consequence of Theorem 3.13 we have, for a constant M

$$|\mathcal{N}_\mu(\partial\Omega_0 \cap B_r(X_0))| \leq \frac{1}{2^n\theta}|\mathcal{N}_\mu(\partial\Omega_0 \cap B_r(X_0)) \cap \Omega_0| + M\mu r^{n-1}.$$

By Lipschitz continuity to u_0 there exist, Corollary 3.8, a constant C such that

$$\mathcal{N}_\mu(\partial\Omega_0) \cap B_r(X_0) \cap \Omega_0 \subset \{\{0 < u_0 < \mu\} \cap B_r(X_0)\}$$

Thus

$$|\mathcal{N}_\mu(\partial\Omega_0) \cap B_r(X_0) \cap \Omega_0| \leq C\mu r^{n-1}$$

So the desired Hausdorff estimate follows. $\square$

CONCLUSION

The use of compactness associated with the method of tangent equations have been a important toll in the study of regularity of solutions of partial differential equations. This technique was applied in this thesis providing contributions for the PDE's theory and consequently for the problems that have as mathematical model the equations studied here.

In the first part we obtain estimates $C^{2,\alpha}$ for a class of fully non-linear equations no necessarily convex. To the case of the flat solutions we prove $C^{2,\alpha}$ regularity estimates with α optimal. When we leave of the scenario flat, we show C^{α} estimates to D^2u for u classical solution, ensuring the maxima regularity for this case. Already when u is only a viscosity solution we get estimates for the Hausdorff dimension of the set where the hessiana of the solution blows-up.

In the study relative to the free boundary we treat equations that modelling cavity problems in discontinuous media. The result obtained here guarantee that, although the regularity of the solutions for this case is at most C^{α}, the regularity along of the free boundary is $C^{0,1}$. This allow us show geometric informations about the free boundary.

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