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Duality, Net Supply and the Directional Distance Function

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Abstract

This paper considers alternative proofs of the Hotelling's lemma and Shephard's lemma using new relations of duality in the theory of the production. The proofs proposals possess two advantages, relatively to the usual demonstrations, which are, not the use of the known enveolpe theorem and the validity of the demonstration for interior points to the production set

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1 Introduction

Duality in the production theory was first investigated in the work of Shephard (1953, 1970, 1973) – the precursor of the axiomatic approach to production theory[1]. His findings, later improved by McFadden (1978), Blackorby & Donaldson (1980) and Färe & Primont (1995), enabled important applications in production theory, which are still the reference guide for many graduate textbooks. With respect to duality, much credit must be devoted to the possibility of obtaining supply and input demand functions through simple derivation of related functions (such as the profit function), without the need to undertake an optimization process.

The vast majority of proofs and demonstrations certifying the existence of duality in production rely on the envelope theorem. This subterfuge, however, requires extensive knowledge about the optimization process of the firm, which contrasts with the simple proposal of duality.

Furthermore, it should be pointed out that the use of the envelope theorem implicitly requires that the choice of output and input vectors be technical and allocatively efficient, that is, that such choices be on border of production.

In this paper, however, new relations of duality shall be explored, between the profit function and the directional distance function more recently-discovered by Chambers, Chung & Färe (1996, 1998). The main objective is to make use of this duality to supply an alternative demonstration of the lemma of Hotelling and Shephard, without using the envelope theorem. In

[1] Diewert (1974) offers and excellent overview of the application of the duality theory in production theory.

fact, as shall be seen throughout this paper, the proposed proof is relatively simple as well as enabling the supply and demand systems generated to be technically inefficient. That is, the vectors of outputs and inputs may be points inside the production set[2]. Further to the above results, it is also shown that Shephard's duality theorem may be seen as a particular case of duality between the profit function and that of directional distance.

In the paper that follows, in addition to the introduction, the layout is as follows: Section 3.2 defines and presents the main properties of the profit and directional distance functions; Section 3.3 shows the main results of the duality between directional distance and profit functions, according to Chambers, Chung & Färe (1996, 1998). Furthermore, alternative proofs of the lemma of Hotelling and Shephard are shown, using the concept of directional distance function, the main result of this paper. Finally, Section 3.4 presents the paper's final conclusions.

2 Profit and Directional Distance Function

In this section the profit function and directional distance function shall be formally defined. In addition, the main structural properties of these functions shall be presented.

2.1 The Profit Function

Let $\mathbf{x} \in \mathbb{R}_+^N$ be an input vector and $\mathbf{y} \in \mathbb{R}_+^M$ an output vector. Production technology is a correspondence $P : \mathbb{R}_+^N \rightrightarrows \mathbb{R}_+^M$ defined as

[2] Allocative efficiency, however, is still maintained.

$$P(\mathbf{x}) = \{\mathbf{y}; \mathbf{x} \text{ } can \text{ } produce \text{ } \mathbf{y}\} \tag{1}$$

which associates to each input vector $\mathbf{x}$ a subset of non-negative $M-$ ortant in which production is feasible for this input vector. Observe that, for the case of a single output and a single input, the graph of this correspondence generates a set of possibilities for production as shown in Figure 1. Denominating as T the correspondence graph $P(\mathbf{x})$, the set T may therefore be defined as:

$$T = \{(\mathbf{x}, \mathbf{y}); \mathbf{y} \in P(\mathbf{x})\} \tag{2}$$

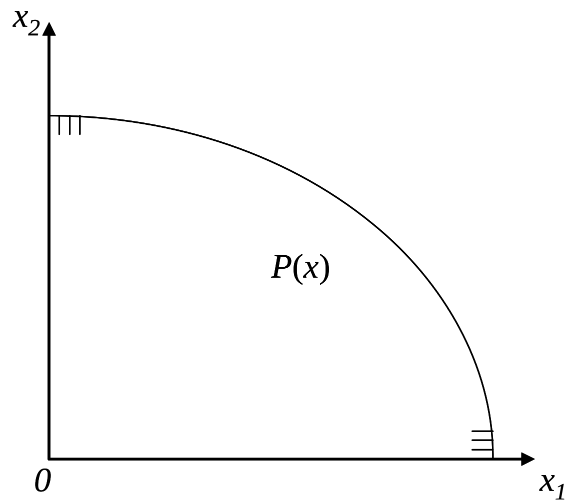

Figure 1: Production Correspondence

The usual axioms decribed below are necessary and sufficient to establish Shephard's (1970) duality theorem to be used later. These axioms are as follows[3]:

[3] See also Färe & Primont (1995).

P.1) $P(\mathbf{x})$ is a closed set, limited and convex for any $\mathbf{x} \in \mathbb{R}_+^N$;

P.2) Given, $\mathbf{y} \in \mathbb{R}_+^M$ and $\mathbf{x}' \geq \mathbf{x}$, we have $P(\mathbf{x}) \subseteq P(\mathbf{x}')$ - free disposal of inputs;

P.3) Given $\mathbf{x} \in \mathbb{R}_+^N$, if, $\mathbf{y} \in P(\mathbf{x})$ and $\mathbf{y} \geq \mathbf{y}'$, then $\mathbf{y}' \in P(\mathbf{x})$ - free disposal of outputs;

P.4) If, $\mathbf{y} \in P(\mathbf{x})$ and $\mathbf{x} = 0$, then $\mathbf{y} = 0$ - *no* free lunch;

P.5) $0 \in P(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}_+^N$ - inanition.

Having defined these technological assumptions, the profit function is now formally defined below.

Definition 1 *Let* $\mathbf{x} \in \mathbb{R}_+^N$ *an input vector,* $\mathbf{y} \in \mathbb{R}_+^M$ *an output vector,* $P(\mathbf{x})$ *the production correspondence,* $\mathbf{w} \in \mathbb{R}_+^N$ *and* $\mathbf{p} \in \mathbb{R}_+^M$ *the vectors for price of inputs and outputs respectively. The function* $\pi : \mathbb{R}_+^N \times \mathbb{R}_+^M \rightarrow \mathbb{R}$ *defined by:*

$$\pi(\mathbf{w}, \mathbf{p}) = \sup_{(\mathbf{x},\mathbf{y})\geq 0} \{\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x};\ \mathbf{y} \in P(\mathbf{x})\} \tag{3}$$

is denominated the profit function in $(\mathbf{w}, \mathbf{p})$.

The profit function has a series of usual properties such as homogeneity of degree 1 in $(\mathbf{w}, \mathbf{p})$, concavity in $(\mathbf{w}, \mathbf{p})$, non-decreasing in $\mathbf{p}$ and non-increasing in $\mathbf{w}$.

2.2 The Directional Distance Function

In a series of studies, Luenberger (1992a, 1992b, 1994, 1995 e 1996), defines and establishes the main structural and algebraic properties of the benefit function as being a measure of availability of an agent to exchange a certain level of utility for a certain amount of comodity, relative to a reference vector (of comodities). The adaptation of this device for production theory is due to the work of Chambers, Chung & Färe (1996, 1998) who renamed the benefit function, calling it the directional distance function.

Since then, a large number of papers have been dedicated to exploring the structure and several applications of the directional distance function.

Using the notation introduced in the previous section, the following definition formally establishes the directional distance function.

Definition 2 *Let* $\mathbf{x} \in \mathbb{R}_+^N$ *an input vector,* $\mathbf{y} \in \mathbb{R}_+^M$ *an output vector,* $P(\mathbf{x})$ *the production correspondence and,* $g = (-g_x, g_y) \in \mathbb{R}_+^N \times \mathbb{R}_+^M$*, with* $g \neq 0$*, a vector denominated direction vector. The function* $\overrightarrow{D} : (\mathbb{R}_+^N \times \mathbb{R}_+^M) \times (\mathbb{R}_+^N \times \mathbb{R}_+^M) \to \mathbb{R}$ *defined by:*

$$\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y) = \sup\{\beta \in \mathbb{R}_+ \ ; (\mathbf{x} - \beta g_x, \mathbf{y} + \beta g_y) \in T\} \tag{4}$$

is denominated the directional distance function where β *is larger or equal to zero.*

In illustrative terms, Figure 2 below shows how the directional distance function is calculated. In this example, the direction vector is represented by the line $\overline{OA}$ nd the directional distance function is given by $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y) = \overline{OB}/\overline{OA}$. Note that in Figure 2, the point projected by the directional distance function, $(\mathbf{x} - \beta g_x, \mathbf{y} + \beta g_y)$, represents the choice which maximizes profit, as it is located at the tangent between iso-profit and production technology[4]. Thus, the point observed $(\mathbf{x}, \mathbf{y})$ although technically inefficient, shows allocative efficiency in the prices that determine the slope of the iso-profit line in Figure 2.

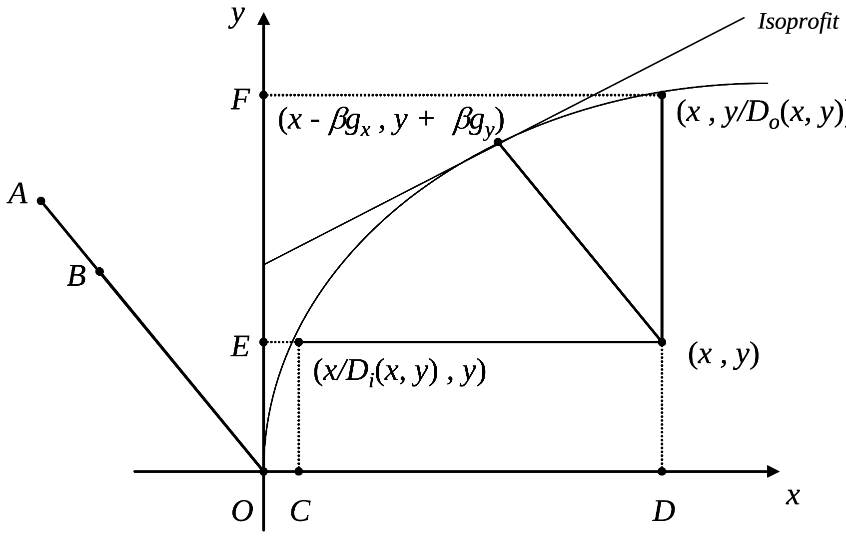

Figure 2: Directional Distance Function and Shephard's Distance Functions

Lemma 1 below establishes the main structural and algebraic properties of the directional distance function. The proof of the results below may be found in Luenberger (1992), Chambers, Chung & Färe (1996, 1998) and Briec (2000).

Lemma 1 *Suppose that technology satisfies the axioms P.1-P.5. The directional distance function satisfies the following properties:*

D.1) (Translation) $\overrightarrow{D}(\mathbf{x}-\alpha g_x, \mathbf{y}+\alpha g_y; -g_x, g_y) = \overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y) - \alpha,\ \forall \alpha \in \mathbb{R}$;

[4]Chambers, Chung & Färe (1998) use the geometric argument in Figure 2 to interpret the duality between the profit function and the directional distance function.

D.2) (Continuity) $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y)$ *is superior semicontinuous in* (x, y);

D.3) (Homogeneity) $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -\lambda g_x, \lambda g_y) = \lambda^{-1}\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y), \ \forall \lambda > 0$;

D.4) (Monotonicity) The directional distance function is non-decreaseing in $\mathbf{x}$ *and non-increasing in* $\mathbf{y}$;

D.5) (Convexity) $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y)$ *is concave in* $(\mathbf{x}, \mathbf{y})$*;*

D.6) (Full characterization) $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y) \geq 0 \Leftrightarrow \mathbf{y} \in P(x)$.

As established by Chambers, Chung & Färe (1996, 1998), the directional distance function generalizes several other distance functions such as Shephard's (1970) input and output distance functions.[5]. According to these authors, the relationship between directional distance function and the input and output distance functions are, respectively, given by:

$$\overrightarrow{D}(\mathbf{x}, \mathbf{y}; 0, \mathbf{y}) = \frac{1}{D_o(\mathbf{x}, \mathbf{y})} - 1 \tag{5}$$

$$\overrightarrow{D}(\mathbf{x}, \mathbf{y}; \mathbf{x}, 0) = 1 - \frac{1}{D_i(\mathbf{x}, \mathbf{y})} \tag{6}$$

where $D_o(\mathbf{x}, \mathbf{y})$ and $D_{\ i}(\mathbf{x}, \mathbf{y})$ are, respectively, the input and output distance functions.[6].

[5] The directional distance function also generalizes Gauge Function of McFadden (1978), the translation function of Blackorby & Donaldson (1980) and the affine distance function of Färe & Lovell (1978).

[6] The output distance function, $D_0(\mathbf{x}, \mathbf{y})$, is defined as, $D_0(\mathbf{x}, \mathbf{y}) = \inf\{\theta \in \mathbb{R}_+ : (\mathbf{x}, \mathbf{y}/\theta) \in T\}$. In other words, $D_0(\mathbf{x}, \mathbf{y})$ measures the maximum proportional expansion of the vector of outputs $\mathbf{y}$ so that the resulting production process is still feasible for a given vector of inputs $\mathbf{x}$. Similarly the input distance function, $D_i(\mathbf{x}, \mathbf{y})$, is defined as $D_i(\mathbf{x}, \mathbf{y}) = \sup\{\theta \in \mathbb{R}_+ : (\mathbf{x}/\theta, y) \in T\}$. This measures the maximum contraction of the vector of inputs $\mathbf{x}$ so that the vector of outputs $\mathbf{y}$ may be produced. With regard to the

3 Dualities and Differential Properties

This section exposes the main results of duality involving the directional distance function and the profit function, originally established in Chambers, Chung & Färe (1996, 1998) and Färe (2000). The main findings of research for this paper shall also be established.

3.1 Dualities

The finding presented below show that the directional distance function generalizes the dualities between the input and output distance functions and the revenue and cost functions, respectively. Färe & Primont (1995) show that the axioms P.1 - P.5 are necessary and sufficient conditions for establishment of Shephard's duality theorem (1970)[7] (1970) as a pair of non-conditioned optimizations.

Theorem 1 (Färe and Primont (1995)) *Supposing that technology satisfies axioms P.1 - P.5. Then, provided that $D_o(\mathbf{x},\mathbf{y})$ and $D_i(\mathbf{x},\mathbf{y})$ are the input and output distance functions, respectively, we have:*

$$R(\mathbf{x},\mathbf{p}) = \sup_{\mathbf{y}\geq 0}\left\{\frac{\mathbf{p}^T\mathbf{y}}{D_o(\mathbf{x},\mathbf{y})}\right\} \tag{7.a}$$

$$D_o(\mathbf{x},\mathbf{y}) = \sup_{\mathbf{p}\geq 0}\left\{\frac{\mathbf{p}^T\mathbf{y}}{R(\mathbf{x},\mathbf{p})}\right\} \tag{7.b}$$

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example given in Figure 2, it is given that $D_0(\mathbf{x},\mathbf{y}) = \overline{OE}/\overline{OF}$ and $D_i(\mathbf{x},\mathbf{y}) = \overline{OD}/\overline{OC}$. The properties of these distance functions may be found in Shephard (1970) and Färe & Primont (1995).

[7] Shephard show that if the tecnology satisfy the axioms P1-P5, then: $C(y,w) = \inf_{x\geq 0}\{w^Tx; D_i(x,y)\geq 1\}$ and $R(x,p) = \sup_{y\geq 0}\{p^Ty; D_o(x,y)\leq 1\}$.

and

$$C(\mathbf{y},\mathbf{w}) = \inf_{\mathbf{x}\geq 0}\left\{\frac{\mathbf{w}^T\mathbf{x}}{D_i(\mathbf{x},\mathbf{y})}\right\} \tag{8.a}$$

$$D_i(\mathbf{x},\mathbf{y}) = \inf_{\mathbf{w}\geq 0}\left\{\frac{\mathbf{w}^T\mathbf{x}}{C(\mathbf{y},\mathbf{w})}\right\} \tag{8.b}$$

where $R(\mathbf{x},\mathbf{p})$ *and* $C(\mathbf{y},\mathbf{w})$ *are, respectively, the functions of revenue and production cost.*

It should be noted that the results of duality of Theorem 1 are special cases of duality between directional distance function and profit function, as shall be seen later. Theorem 2, below, according to Chambers, Chung & Färe (1998) is the main result of duality to be explored in this paper.

Theorem 2 (Chambers, Chung, and Färe (1998)) *Suppose that technology satisfies the axioms P.1-P.5, then*

$$\pi(\mathbf{w},\mathbf{p}) = \sup_{(\mathbf{x},\mathbf{y})\geq 0}\left\{\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x} + \overrightarrow{D}(\mathbf{x},\mathbf{y};-g_x,g_y)(\mathbf{p}^T g_y + \mathbf{w}^T g_x)\right\} \text{and}$$

$$\overrightarrow{D}(\mathbf{x},\mathbf{y};g_x,g_y) = \inf_{(\mathbf{p},\mathbf{w})}\left\{\frac{\pi(\mathbf{w},\mathbf{p}) - (\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x})}{\mathbf{p}^T g_y + \mathbf{w}^T g_x}\right\} \tag{10}$$

Based on the above results and using equations (5) and (6), it is possible to demonstrate that the duality relationships (7) and (8) are specific cases of duality between the profit function and directional distance function. This is the content of the following corollary:

Corollary 1 *Let* $\mathbf{x}\in\mathbb{R}_+^N$ *and* $\mathbf{y}\in\mathbb{R}_+^N$ *the vectors of inputs and outputs, respectively,* $\overrightarrow{D}(\mathbf{x},\mathbf{y};-g_x,g_y)$ *the directional distance function on the direction vector* $(-g_x,g_y)$*,* $D_o(\mathbf{x},\mathbf{y})$ *and* $D_i(\mathbf{x},\mathbf{y})$ *the input and output distance functions, respectively and finally,* $\pi(\mathbf{w},\mathbf{p})$*,* $R(\mathbf{x},\mathbf{p})$ *and* $C(\mathbf{y},\mathbf{w})$ *the functions* *of profit, revenue and cost, respectively. Then, the following conditions are verified:*

a) If $(-g_x, g_y) = (0, \mathbf{y})$ then equations (9) and (10) imply the duality relationship (7);

b) If $(-g_x, g_y) = (\mathbf{x}, 0)$ then equations (9) e (10) imply the duality relationship (8).

Proof. Provided $(-g_x, g_y) = (0, \mathbf{y})$, by (5) we have $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; 0, \mathbf{y}) = 1/D_o(\mathbf{x}, \mathbf{y})$ -1. Substituting the latter expression in (10) we have:

$$\frac{1}{D_o(\mathbf{x}, \mathbf{y})} - 1 = \inf_{(\mathbf{p}, \mathbf{w})} \left\{ \frac{\pi(\mathbf{w}, \mathbf{p}) - (\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x})}{py} \right\}$$

$$= \inf_{(\mathbf{p}, \mathbf{w})} \left\{ \frac{\pi(\mathbf{w}, \mathbf{p}) + \mathbf{w}^T\mathbf{x})}{\mathbf{p}^T\mathbf{y}} \right\} - 1$$

Therefore,

$$D_o(x, y) = \sup_{(\mathbf{p}, \mathbf{w})} \left\{ \frac{\mathbf{p}^T\mathbf{y}}{\pi(\mathbf{w}, \mathbf{p}) + \mathbf{w}^T\mathbf{x}} \right\} \tag{11}$$

In the event that the input vector $\mathbf{x}$ minimizes cost of production $\mathbf{w}^T\mathbf{x}$, it is shown in the Appendix that $R(\mathbf{x}, \mathbf{p}) = \pi(\mathbf{w}, \mathbf{p}) + \mathbf{w}^T\mathbf{x}$. In these terms, substitution of this latter expression in (11) implies that:

$$D_o(\mathbf{x}, \mathbf{y}) = \sup_{(\mathbf{p}, \mathbf{w})} \left\{ \frac{\mathbf{p}^T\mathbf{y}}{R(\mathbf{x}, \mathbf{p})} \right\}$$

Which is precisely the second relationship of (7). To demonstrate the first relationship of (7), it is considered that $(-g_x, g_y) = (0, \mathbf{y})$.In this case, substituting (5) in (9), we have:

$$\pi(\mathbf{w},\mathbf{p}) = \sup_{(\mathbf{x},\mathbf{y})\geq 0}\left\{\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x} + \left(\frac{1}{D_o(\mathbf{x},\mathbf{y})} - 1\right)\mathbf{p}^T\mathbf{y}\right\}$$

$$= \sup_{(\mathbf{x},\mathbf{y})\geq 0}\left\{\frac{\mathbf{p}^T\mathbf{y}}{D_o(\mathbf{x},\mathbf{y})} - \mathbf{w}^T\mathbf{x}\right\}$$

Thus, for every $(\mathbf{x},\mathbf{y}) \in T$ we have $\pi(\mathbf{w},\mathbf{p}) \geq \mathbf{p}^T\mathbf{y}/D_o(\mathbf{x},\mathbf{y}) - \mathbf{w}^T\mathbf{x}$ implying $\pi(\mathbf{w},\mathbf{p}) + \mathbf{w}^T\mathbf{x} \geq \mathbf{p}^T\mathbf{y}/D_o(\mathbf{x},\mathbf{y})$. Thus, if $\mathbf{x}$ minimizes the cost $\mathbf{w}^T\mathbf{x}$, then::

$$R(\mathbf{x},\mathbf{p}) \geq \frac{\mathbf{p}^T\mathbf{y}}{D_o(\mathbf{x},\mathbf{y})}$$

This implies that

$$R(\mathbf{x},\mathbf{p}) = \sup_{\mathbf{y}\geq 0}\left\{\frac{\mathbf{p}^T\mathbf{y}}{D_o(\mathbf{x},\mathbf{y})}\right\}$$

which is precisely the first expression of (7). Thus (7) is completely proven. To demonstrate the second relationship of (8), consider now $(-g_x, g_y) = (\mathbf{x}, 0)$. Then, substituting from (6) in (10) and using the same logic to demonstrate the second relationship of (7), we find:

$$D_i(\mathbf{x},\mathbf{y}) = \inf_{(\mathbf{p},\mathbf{w})}\left\{\frac{\mathbf{w}^T\mathbf{x}}{C(\mathbf{y},\mathbf{w})}\right\} \tag{12}$$

which coincides with the second expression of (8). On the other hand, provided $(-g_x, g_y) = (0, y)$, we have $\overrightarrow{D}(\mathbf{x},\mathbf{y};\mathbf{x},0) = 1 - (1/D_i(\mathbf{x},\mathbf{y}))$.The substitution of this latter relationship in (9) implies that:

$$\pi(\mathbf{p},\mathbf{w}) = \sup_{(\mathbf{x},\mathbf{y})\geq 0}\left\{\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x} + \left(1 - \frac{1}{D_i(\mathbf{x},\mathbf{y})}\right)\mathbf{w}^T\mathbf{x}\right\}$$

$$= \sup_{(\mathbf{x},\mathbf{y})\geq 0} \left\{ \mathbf{p}^T\mathbf{y} - \frac{\mathbf{w}^T\mathbf{x}}{D_i(\mathbf{x},\mathbf{y})} \right\}$$

Therefore, for every $(\mathbf{x},\mathbf{y}) \in T$ it is given that $\pi(\mathbf{p},\mathbf{w}) \geq \mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x}/D_i(\mathbf{x},\mathbf{y})$ implying $\mathbf{p}^T\mathbf{y} - \pi(\mathbf{p},\mathbf{w}) \leq \mathbf{w}^T\mathbf{x}/D_i(\mathbf{x},\mathbf{y})$. Thus, if $\mathbf{y}$ maximizes revenue $\mathbf{p}^T\mathbf{y}$ then:

$$C(\mathbf{y},\mathbf{w}) \leq \frac{\mathbf{w}^T\mathbf{x}}{D_i(\mathbf{x},\mathbf{y})}$$

This implies that

$$C(\mathbf{y},\mathbf{w}) = \inf_{\mathbf{x}\geq 0} \left\{ \frac{\mathbf{w}^T\mathbf{x}}{D_i(\mathbf{x},\mathbf{y})} \right\}.$$

This latter expression is exactly the relation (8.a). Thus, relationships in (8) are proven. ■

In summary, Corollary 1 above shows that the results of Theorem 1 are cases specific to Theorem 2. Below some of the differential properties of the directional distance function are discussed, as is the proof of the Hotteling and Shephard lemma.

3.2 Differential Properties

This section establishes the main findings of the research, the lemma of Hotelling and Shephard. It is always assumed that the directional distance function is twice continually differentiable in all its arguments[8].

Proposition 1 (Hotelling's lemma) *Suppose that* $\mathbf{x} \in \mathbb{R}_+^N$ *and* $\mathbf{y} \in \mathbb{R}_+^M$ *are, respectively, vectors of inputs and outputs,* $\overrightarrow{D}(\mathbf{x},\mathbf{y};-g_x,g_y)$ *the direc-*

[8]Regarding differentiability conditions of the directional distance function, consult Courtalt, Crettez and Hayek (2004).

tional distance function with direction vector $(-g_x, g_y)$ *and,* $\pi(\mathbf{p}, \mathbf{w})$*, the profit function in prices* $(\mathbf{p}, \mathbf{w})$*. Furthermore, suppose that technology satisfies axioms 1-5 with strict convexity, then:*

$$\begin{pmatrix} \nabla_p \pi(\mathbf{p}, \mathbf{w}) \\ \nabla_w \pi(\mathbf{p}, \mathbf{w}) \end{pmatrix} = \begin{pmatrix} \mathbf{y}(\mathbf{p}, \mathbf{w}) \\ -\mathbf{x}(\mathbf{p}, \mathbf{w}) \end{pmatrix} \tag{13}$$

Proof. From first order conditions of the minimization problem in (9), it is found that:

$$\begin{aligned} &[\nabla_w \pi(\mathbf{p}, \mathbf{w}) + \mathbf{x}(\mathbf{p}, \mathbf{w})] - \\ &\tfrac{1}{(\mathbf{p}^T g_y + \mathbf{w}^T g_x)} \left[\pi(\mathbf{p}, \mathbf{w}) - \left(\mathbf{p}^T \mathbf{y}(\mathbf{p}, \mathbf{w}) - \mathbf{w}^T \mathbf{x}(\mathbf{p}, \mathbf{w})\right)\right] g_x = \mathbf{0}_{N \times 1} \end{aligned} \tag{14}$$

$$\begin{aligned} &[\nabla_p \pi(\mathbf{p}, \mathbf{w}) - \mathbf{y}(\mathbf{p}, \mathbf{w})] - \\ &\tfrac{1}{(\mathbf{p}^T g_y + \mathbf{w}^T g_x)} \left[\pi(\mathbf{p}, \mathbf{w}) - \left(\mathbf{p}^T \mathbf{y}(\mathbf{p}, \mathbf{w}) - \mathbf{w}^T \mathbf{x}(\mathbf{p}, \mathbf{w})\right)\right] g_y = \mathbf{0}_{M \times 1} \end{aligned} \tag{15}$$

Due to the duality between the profit function and directional distance function, $\mathbf{y}(\mathbf{p}, \mathbf{w})$ and $\mathbf{x}(\mathbf{p}, \mathbf{w})$, are the vectors of outputs and inputs that maximize the profit according to the expression (9). Thus,

$$\pi(\mathbf{p}, \mathbf{w}) = \mathbf{p}^T \mathbf{y}(\mathbf{p}, \mathbf{w}) - \mathbf{w}^T \mathbf{x}(\mathbf{p}, \mathbf{w})$$

and, therefore, equations (14) and (15) imply that:

$$\begin{pmatrix} \nabla_w \pi(\mathbf{p}, \mathbf{w}) \\ \nabla_p \pi(\mathbf{p}, \mathbf{w}) \end{pmatrix} = \begin{pmatrix} -\mathbf{x}(\mathbf{p}, \mathbf{w}) \\ \mathbf{y}(\mathbf{p}, \mathbf{w}) \end{pmatrix}$$

■

Note that in the proof above, the only stronger assumption used was the strict convexity of the technology, so as to guarantee that the profit function be well defined.

Proposition 1 has three important implications. The first is that the result known as Hotelling's lemma is proven without making use of the envelope theorem and, thus, without direct use of the structure of the optimization process inherent in the problem of maximizing profit restricted to technological conditions. In fact, only the first order conditions are used, in consideration of the duality between the profit function and directional distance function. Second, it shows how the directional distance function, through its duality with the profit function, is used as a way of obtaining a net supply of products through partial derivatives of the profit function. Finally, it is observed that the method of proof used enables finding supplies of products and demands for inputs that are technically inefficient, although the maximization of profit requires allocative efficiency at the point of choice, as illustrated in Figure 2 above.

Below, using Corollary 1, the directional distance function may also be used to establish Shephard's lemma (1953, 1970), according to which the conditional supply and demand functions are found through partial derivatives, with respect to input and output prices, of the cost and revenue functions respectively.

Proposition 2 (Shephard's lemma) *Let* $\mathbf{x} \in \mathbb{R}_+^N$ *and* $\mathbf{y} \in \mathbb{R}_+^M$ *be the input and output vectors combined under technology* T*. Also, let* $C(\mathbf{w}, \mathbf{y})$ *be the cost function,* $R(\mathbf{x}, \mathbf{p})$ *the revenue function and* $\overrightarrow{D}(\mathbf{x}, \mathbf{y}; -g_x, g_y)$ *the* *directional distance function. Thus, it is given that:*

a) If $(-g_x, g_y) = (\mathbf{x}, 0)$*, then* $\nabla_w C(\mathbf{w}, \mathbf{y}) = x(\mathbf{w}, \mathbf{y})$;

b) If $(-g_x, g_y) = (0, \mathbf{y})$*, then* $\nabla_p R(\mathbf{p}, \mathbf{x}) = y(\mathbf{p}, \mathbf{x})$;

Proof. According to Corollary 1, duality relationships (8) may be found using the directional distance function by

$$\frac{1}{1 - \overrightarrow{D}(\mathbf{x}, \mathbf{y}; \mathbf{x}, 0)} = \inf_{\mathbf{w} \geq 0} \left\{ \frac{\mathbf{w}^T \mathbf{x}}{C(\mathbf{y}, \mathbf{w})} \right\} \tag{16}$$

Using the first order conditions of the minimization problem in (16) we have :

$$\left\{ \mathbf{x}(\mathbf{w}, \mathbf{y}) - \nabla_w C(\mathbf{w}, \mathbf{y}) \left[\frac{\mathbf{w}^T \mathbf{x}(\mathbf{w}, \mathbf{y})}{C(\mathbf{w}, \mathbf{y})} \right] \right\} = 0 \tag{17}$$

By duality, $\mathbf{x}(\mathbf{w}, \mathbf{y})$ resolves the problem in (8.a) and therefore

$$\mathbf{w}^T \mathbf{x}(\mathbf{w}, \mathbf{y}) = C(\mathbf{w}, \mathbf{y})$$

Thus (17) implies that:

$$\nabla_w C(\mathbf{w}, \mathbf{y}) = \mathbf{x}(\mathbf{w}, \mathbf{y})$$

which demonstrates item a). To demonstrate b), the same logic is applied, using the first order conditions of the maximization problem in (7.b) and the duality relationship in (7.a). ■

4 Conclusions

This paper establishes alternative proofs of traditional results in theory of production. The use of duality between the profit function and the directional distance function enables the latter to be used to establish Hotelling's lemma, a finding which obtains net supply of outputs through partial derivatives of the profit function with regard to prices of inputs and outputs.

This paper also shows that the directional distance function generalizes the dualities between the input distance function and cost function, and between the output distance function and the revenue function. Based on this fact, it is proven that the directional distance function may be used to establish Shephard's lemma, through which conditional supplies and demands are found.

All the results cited here are established without making any reference to the envelope theorem, which is traditionally the most commonly-used resource to prove such results. Furthermore, the systems for input demand and supply of outputs generated through the methodology used, do not need to be technically efficient. That is, the proof is valid for points which are within the production set, although the allocative efficiency hypothesis is maintained.

Appendix

In this appendix, it shall be proven that $\pi(\mathbf{w},\mathbf{p}) = R(\mathbf{x}^*,\mathbf{p}) - C(\mathbf{y}^*,w)$. *Thus*, let

$$(\mathbf{x}^*,\mathbf{y}^*) = \arg\sup_{(\mathbf{x},\mathbf{y})\in T}\left\{\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x}\right\}$$

$$\mathbf{y}' = \arg\sup_{(\mathbf{x},\mathbf{y})\in T}\left\{\mathbf{p}^T\mathbf{y}\right\}$$

$$\mathbf{x}' = \arg\inf_{(\mathbf{x},\mathbf{y})\in T}\left\{\mathbf{w}^T\mathbf{x}\right\}$$

and $\pi(p,w)$, $R(p,x)$ *and* $C(w,y)$ be the profit, revenue and cost functions, respectively. *Necessarily* $\pi(p,w) = R(p,x^*) - C(w,y^*) = R(p,x') - C(w,y')$.

Proof. Provided $\mathbf{y}' = \arg\sup_{(\mathbf{x},\mathbf{y})\in T}\left\{\mathbf{p}^T\mathbf{y}\right\}$, $\mathbf{x}' = \arg\inf_{(\mathbf{x},\mathbf{y})\in T}\left\{\mathbf{w}^T\mathbf{x}\right\}$ and $(\mathbf{x}^*,\mathbf{y}^*) = \arg\sup_{(\mathbf{x},\mathbf{y})\in T}\left\{\mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x}\right\}$, then it is given that, respectively:

$$R(\mathbf{p},\mathbf{x}) = \mathbf{p}^T\mathbf{y}' \tag{A.1}$$

$$C(\mathbf{w},\mathbf{y}) = \mathbf{w}^T\mathbf{x}' \tag{A.2}$$

$$\pi(p,w) = \mathbf{p}^T\mathbf{y}^* - \mathbf{w}^T\mathbf{x}^* \tag{A.3}$$

By construction of any feasible combinations of inputs and outputs and in particular for $(\mathbf{x}^*,\mathbf{y}^*)$ it is found that $\mathbf{p}^T\mathbf{y}' \geq \mathbf{p}^T\mathbf{y}^*$ and $\mathbf{w}^T\mathbf{x}' \leq \mathbf{w}^T\mathbf{x}^*$ implying that:

$$\mathbf{p}^T\mathbf{y}' - \mathbf{w}^T\mathbf{x}' \geq \mathbf{p}^T\mathbf{y}^* - \mathbf{w}^T\mathbf{x}^* \tag{A.4}$$

Using the expressions (A1), (A.2) and (A.3) the inequality in (A.4) may be re-written as:

$$\pi(\mathbf{p},\mathbf{w}) \leq R(\mathbf{p},\mathbf{x}') - C(\mathbf{w},\mathbf{y}') \tag{A.5}$$

On the other hand, for any feasible pair $(\mathbf{x},\mathbf{y})$ fit is given that $\pi(p,w) \geq \mathbf{p}^T\mathbf{y} - \mathbf{w}^T\mathbf{x}$ thus, in particular, it is given that $\pi(\mathbf{p},\mathbf{w}) \geq \mathbf{p}^T\mathbf{y}' - \mathbf{w}^T\mathbf{x}'$ which, according to (A.1) e (A.2) implies that::

$$\pi(\mathbf{p},\mathbf{w}) \geq R(\mathbf{p},\mathbf{x}') - C(\mathbf{w},\mathbf{y}') \tag{A.6}$$

Finally, combining (A.5) and (A.6) it is necessarily true that:

$$\pi(\mathbf{p},\mathbf{w}) = R(\mathbf{p},\mathbf{x}') - C(\mathbf{w},\mathbf{y}')$$

■

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