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**ρ -EINSTEIN SOLITONS AND CRITICAL METRICS OF THE VOLUME
FUNCTIONAL ON COMPLETE MANIFOLDS**

FORTALEZA

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Tese apresentada ao Programa de Pós-Graduação em Matemática do Centro de Ciências da Universidade Federal do Ceará, como requisito parcial à obtenção do título de doutor em Matemática. Área de Concentração: Geometria Diferencial.

Orientador: Prof. Dr. Ernani de Sousa
Ribeiro Júnior.

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Ao jovem matemático que, anos atrás, abriu mão de muito em sua busca pelo desconhecido. Foi difícil, mas conseguimos.

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"Mais do que provar teoremas, este trabalho foi a arte de aprender a esperar pelo momento em que a confusão se desfaz em clareza inevitável."
(Autor desconhecido)

"A mais bela experiência que podemos ter é o mistério. É a emoção fundamental que está no berço da verdadeira ciência e da verdadeira arte." (Albert Einstein)

RESUMO

O objetivo deste trabalho é investigar propriedades analíticas e geométricas de sólitons ρ -Einstein e métricas V -estáticas em variedades Riemannianas completas. Na primeira parte, estudamos características geométricas e analíticas de sólitons ρ -Einstein completos não compactos, que são soluções autossimilares do fluxo de Ricci–Bourguignon. Estudamos o espectro do operador drift Laplaciano para sólitons ρ -Einstein gradiente shrinking completos. Além disso, similarmente a resultados clássicos devidos a Calabi–Yau e Bishop para variedades Riemannianas completas com curvatura de Ricci não negativa, provamos novas estimativas de crescimento de volume para bolas geodésicas de sólitons ρ -Einstein completos não compactos. Em particular, discutimos o caso de rigidez. Adicionalmente, estabelecemos estimativas de crescimento de volume com peso para bolas geodésicas dessas variedades. Na segunda parte, investigamos métricas críticas do funcional volume (métricas V -estáticas) em variedades completas sem bordo. Provamos que toda métrica crítica do funcional volume em uma variedade conexa e completa com tensor de Ricci paralelo é isométrica a um dos modelos padrão. Além disso, mostramos que uma métrica crítica Bach-flat do funcional volume em uma variedade completa, simplesmente conexa, com função potencial própria é isométrica a uma das seguintes: a esfera padrão \mathbb{S}^n , o espaço Euclidiano \mathbb{R}^n , o espaço hiperbólico \mathbb{H}^n , ou um produto torcido $\mathbb{R} \times_{\varphi} \Sigma_c$, onde Σ_c é uma superfície de nível regular da função potencial. Em particular, estabelecemos resultados de classificação nas dimensões três e quatro sob hipóteses mais fracas sobre o tensor de Bach.

Palavras-chave: sólitons ρ -Einstein; estimativa do primeiro autovalor; estimativa de crescimento de volume; métricas V -estáticas; funcional volume; curvatura de Ricci paralela; variedades Bach-flat.

ABSTRACT

The purpose of this work is to investigate analytic and geometric properties of ρ -Einstein solitons and V -static metrics on complete Riemannian manifolds. In the first part, we study geometric and analytical features of complete non-compact ρ -Einstein solitons, which are self-similar solutions of the Ricci–Bourguignon flow. We study the spectrum of the drifted Laplacian operator for complete gradient shrinking ρ -Einstein solitons. Moreover, similar to classical results due to Calabi–Yau and Bishop for complete Riemannian manifolds with nonnegative Ricci curvature, we prove new volume growth estimates for geodesic balls of complete noncompact ρ -Einstein solitons. In particular, the rigidity case is discussed. In addition, we establish weighted volume growth estimates for geodesic balls of such manifolds. For the second part, we investigate critical metrics of the volume functional (V -static metrics) on complete manifolds without boundary. We prove that every critical metric of the volume functional on a connected complete manifold with parallel Ricci tensor is isometric to one of the standard models. Moreover, we show that a Bach-flat critical metric of the volume functional on a complete, simply connected manifold with proper potential function is isometric to one of the following: the standard sphere \mathbb{S}^n , Euclidean space \mathbb{R}^n , hyperbolic space \mathbb{H}^n , or a warped product $\mathbb{R} \times_{\varphi} \Sigma_c$, where Σ_c is a regular level set of the potential function. In particular, we establish classification results in dimensions three and four under weaker assumptions on the Bach tensor.

Keywords: ρ -Einstein solitons; spectrum gap; volume growth estimate; V -static metrics; volume functional; parallel Ricci curvature; Bach-flat manifolds.

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1 INTRODUCTION

The classical Lichnerowicz theorem [60] states that if (M^n, g) is a compact (without boundary) Riemannian manifold with bounded Ricci curvature $Ric \geq \alpha$, where α is a positive constant, then the first nonzero eigenvalue $\lambda_1(\Delta)$ of the Laplacian operator Δ , also known as spectrum gap, must satisfy $\lambda_1(\Delta) \geq \frac{n}{n-1}\alpha$. Furthermore, Obata's theorem [72] says that the equality holds if and only if (M^n, g) is an n -dimensional sphere with constant sectional curvature $\frac{\alpha}{n-1}$. This raised the question whether a similar result holds true for smooth metric measure spaces. In this context, we recall that a smooth metric measure space $(M^n, g, e^{-f}dV)$ is a complete n -dimensional Riemannian manifold with a potential function $f : M \rightarrow \mathbb{R}$ and the weighted volume $e^{-f}dV$ in M . For such spaces, it is more natural to consider the drifted Laplacian operator

$$\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle.$$

This comes from the fact that Δ_f is a densely defined self-adjoint operator in $L^2(M, e^{-f}dV)$ and hence, for $u, v \in C_0^\infty(M)$, one sees that

$$\int_M u \Delta_f v e^{-f} dV = - \int_M \langle \nabla u, \nabla v \rangle e^{-f} dV.$$

Moreover, instead of the usual Ricci tensor, we may consider the Bakry-Émery Ricci tensor given by

$$Ric_f := Ric + \nabla^2 f,$$

where $\nabla^2 f$ stands for the hessian of the potential function f . However, it is important to highlight that, in this case, certain topological differences arise. For instance, the Bonnet-Myers theorem does not hold under the assumption that $Ric_f \geq \delta > 0$ when f is non-constant, and therefore the manifold may fail to be compact, as exemplified by the Gaussian shrinking soliton $(\mathbb{R}^n, g_{can}, f(x) = \frac{|x|^2}{4})$.

By the works of Bakry-Émery [4], Morgan [66], Hein-Naber [57] and Cheng-Zhou [34], it is known the following Lichnerowicz-Obata type theorem for smooth metric measure spaces.

Theorem 1 ([4],[34],[57],[66]). *Let $(M^n, g, e^{-f}dV)$ be a complete smooth metric measure space with $Ric_f \geq \frac{\alpha}{2}g$ for some positive constant α . Then the spectrum of the drifted Laplacian operator*

Δ_f is discrete and the first nonzero eigenvalue, denoted by $\lambda_1(\Delta_f)$, must satisfy

$$\lambda_1(\Delta_f) \geq \frac{\alpha}{2}. \quad (1.1)$$

Moreover, equality holds in (1.1) with multiplicity $k \geq 1$ if and only if

1. $1 \leq k \leq n$;
2. M is a noncompact manifold which is isometric to $\Sigma^{n-k} \times \mathbb{R}^k$ with the product metric for some complete $(n-k)$ -dimensional manifold (Σ, g_Σ) satisfying $\text{Ric}_f^\Sigma \geq \frac{\alpha}{2} g_\Sigma$ and $\lambda_1(\Delta_f^\Sigma) > \frac{\alpha}{2}$;
3. By passing an isometry, for $(x, t) \in \Sigma^{n-k} \times \mathbb{R}^k$,

$$f(x, t) = f(x, 0) + \frac{\alpha}{2} |t|^2.$$

In this thesis, we begin by investigating geometric properties of gradient ρ -Einstein solitons, where this first part is contained in the paper *Geometric and analytical results for ρ -Einstein solitons* [38]. In particular, we establish a result analogous to Theorem 1 for this class of manifolds. Before stating our main results, let us recall the definition of such manifolds. For a given $\rho \in \mathbb{R}$, we say that (M^n, g, f, λ) is a gradient ρ -Einstein soliton if it satisfies the equation

$$\text{Ric} + \nabla^2 f = (\rho R + \lambda)g. \quad (1.2)$$

Following the terminology used for Ricci solitons, we say that the ρ -Einstein soliton is shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. When $\rho = \frac{1}{2(n-1)}$, it is called *Schouten soliton*; see [30]. Moreover, notice that the gradient Ricci soliton equation is obtained when $\rho = 0$ in (1.2).

Gradient ρ -Einstein solitons were first introduced in [30]. According to the work of Catino, Mazzieri and Mongodi [28], these solitons arise as self-similar solutions to the Ricci-Bourguignon flow, originally introduced by Bourguignon in [20],

$$\frac{\partial}{\partial t} g_t = -2(\text{Ric}_{g_t} - \rho R g_t);$$

see also [31]. In the last years, several topological, geometric and analytical features concerning Schouten solitons have also been proven. For instance, Catino and Mazzieri [30] showed that a complete steady Schouten soliton must be Ricci flat. Moreover, they proved that 3-dimensional Schouten solitons are isometric to $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. While Borges [19] classified gradient Schouten solitons with vanishing Bach tensor. Furthermore, Borges [18] obtained interesting results regarding the asymptotic behavior of the potential function f and the norm of its gradient.

Cunha, Lemos and Roing [44] established conditions under which ρ -Einstein solitons have constant scalar curvature; see also [76]. Despite these results, it remains of interest to show that such manifolds are analytic. In [28], Catino et al. proved that if $\rho \notin \{\frac{1}{n}, \frac{1}{2(n-1)}\}$, then the metric g and the potential function f must be analytic.

Among the examples of ρ -Einstein solitons, we may mention that Einstein manifolds are natural examples of ρ -Einstein solitons with constant potential function f . However, it is interesting to obtain examples with non-constant potential function f , i.e., a *nontrivial* ρ -Einstein soliton. A classical nontrivial example can be obtained in the generalized cylinder (see [18]). To be precise, for $n \geq 3$, $k \leq n$, $\lambda \in \mathbb{R}$ and $\rho \in \mathbb{R}$ with $\rho \neq \frac{1}{k}$, we consider a k -dimensional Einstein manifold (Σ^k, g_Σ) with scalar curvature $R_\Sigma = \frac{k\lambda}{1-\rho k}$. Besides, for $(x, p) \in \mathbb{R}^{n-k} \times \Sigma^k$, it suffices to take the potential function

$$f(x, p) = \frac{1}{2} \left(\frac{\lambda}{1-\rho k} \right) |x|^2$$

to conclude that $(\mathbb{R}^{n-k} \times \Sigma^k, g, f, \lambda)$ is a nontrivial ρ -Einstein soliton, where g is the product metric. At the same time, we highlight that ρ -Einstein solitons with constant scalar curvature are precisely gradient Ricci solitons; see [53]. Hence, it is very important to present examples of ρ -Einstein solitons with non-constant scalar curvature. As it was observed by Agila and Gomes, $\mathbb{H} \times_h \mathbb{F}^2$ and $\mathbb{R} \times_h \mathbb{F}^2$, where \mathbb{F}^2 is a complete 2-dimensional Ricci flat manifold, are nontrivial ρ -Einstein solitons with non-constant scalar curvature; for more details, see Examples 1 and 2 in Section 2.4; see also [1].

We are now ready to state our first main result, which establishes a Lichnerowicz-Obata-type theorem for gradient shrinking ρ -Einstein solitons. More precisely, we have the following result.

Theorem 2. *Let (M^n, g, f, λ) be a complete gradient shrinking ρ -Einstein soliton with $\rho > 0$ and nonnegative scalar curvature. Then the following assertions hold:*

1. *The spectrum of Δ_f is discrete;*
2. *$\lambda_1(\Delta_f) \geq \lambda$;*
3. *Equality holds in assertion (2) if and only if (M^n, g, f) is $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$, i.e., the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{|x|^2}{4})$, up to scaling.*

It is worth noting that the discreteness and lower bound of the spectral gap also follow from Theorem 1. However, a key advantage of our approach lies in the rearrangement of

its proof, which allows us to establish the rigidity case.

As a consequence of Theorem 2 combined with the scalar curvature estimates proved by Catino and Mazzieri [30, Corollary 5.2] and Borges [18, Theorem 1.1], we get the following corollary for Schouten solitons.

Corollary 1. *Let (M^n, g, f, λ) be a complete gradient shrinking Schouten soliton. Then the following assertions hold:*

1. *The spectrum of Δ_f is discrete;*
2. *$\lambda_1(\Delta_f) \geq \lambda$;*
3. *Equality holds in assertion (2) if and only if (M^n, g, f) is $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$, i.e., the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{|x|^2}{4})$, up to scaling.*

A precise understanding of the volume growth rate is a key geometric property from which various other features of the underlying Riemannian manifold can be derived. A classical theorem due to Calabi [22] and Yau [79] asserts that the geodesic balls of complete noncompact manifolds with nonnegative Ricci curvature have at least linear volume growth. While the classical Bishop volume comparison theorem [16] guarantees that the geodesic balls of complete noncompact Riemannian manifolds with nonnegative Ricci curvature have at most polynomial volume growth. Similar results were obtained for gradient Ricci solitons in [23],[32],[67],[68] and quasi-Einstein manifolds in, e.g., [11],[12],[36]. Recently, Borges [18] adapted some ideas by Cao and Zhou [23] in order to prove volume growth estimates for Schouten solitons. To be precise, he showed that given a complete noncompact shrinking Schouten soliton, there are positive constants C_1, C_2 and r_0 , depending only on λ and n , such that

$$C_1 r^{\frac{n}{2} - \frac{(n-2)\theta}{4(n-1)\lambda}} \leq \text{Vol}(B_q(r)) \leq C_2 r^{n - \frac{(n-2)\delta}{2(n-1)\lambda}},$$

for any $r > r_0$, where $q \in M$, $\delta = \inf_{p \in M} R(p)$ and $\theta = \sup_{p \in M} R(p)$. The proof of this result relies on the behavior of the potential function f . We highlight that it is not known whether the same approach is valid for ρ -Einstein solitons in general. In the same context, Munteanu and Wang [69, Theorem 1.4] showed that any n -dimensional smooth metric measure space $(M^n, g, e^{-f} dv)$ with $\text{Ric}_f \geq \frac{1}{2}$ and $|\nabla f|^2 \leq f$ must satisfy $\text{Vol}(B_p(r)) \leq c(n)r^n$, for all $r > 0$, where $c(n)$ is a constant depending only on the dimension of the manifold. While Cheng, Ribeiro and Zhou [36, Theorem 5] obtained the precise value of the constant $c(n)$ for gradient shrinking Ricci solitons. Similar estimates are interesting for ρ -Einstein solitons. Here, by adapting some

techniques outlined in [36], we establish the following volume growth estimate for geodesic balls of complete noncompact ρ -Einstein solitons.

Theorem 3. *Let (M^n, g, f) be a complete noncompact n -dimensional gradient shrinking ρ -Einstein soliton with $\rho > 0$ and $R \geq 0$. Then, for all $r > 0$, the volume of the geodesic ball $B_p(r)$ must satisfy*

$$\text{Vol}(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\Phi} r^{n-1} dr d\theta, \quad (1.3)$$

where

$$\Phi = -\frac{\lambda r^2}{6} + f(\theta, r) + f(p) - \frac{2}{r} \int_0^r f(\theta, s) ds.$$

Moreover, equality holds in (1.3) for all $r > 0$ if and only if (M^n, g, f) is $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda|x-p|^2}{2})$, i.e., the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{|x|^2}{4})$, up to scaling and translation.

Remark 1. *We point out that Theorem 2 and Theorem 3 are also true if we assume $\rho < 0$ and $R \leq 0$, with a minor difference in the rigidity part of Theorem 2. In particular, the proof of this fact is essentially the same.*

In general, as in the case of smooth metric measure spaces, it is also very important to obtain weighted volume growth estimates for geodesic balls for ρ -Einstein solitons. This is due to the fact that the existence of the potential function f provides us useful analytical information associated with the drifted Laplacian operator. In [1, Lemma 1], Agila and Gomes proved an estimate for the weighted volume of geodesic balls under suitable conditions involving the scalar curvature and the potential function. However, for their purpose, they needed to obtain an integral that would diverge when the radius of the geodesic ball tends to infinity. Therefore, it remains interesting to obtain new (refined) volume growth estimates for such manifolds. As a consequence of the proof of Theorem 3, we have the following result.

Theorem 4. *Let (M^n, g, f) be a complete noncompact n -dimensional gradient ρ -Einstein soliton. Then*

$$\text{Vol}_f(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\Psi} r^{n-1} dr d\theta, \quad (1.4)$$

where

$$\Psi = -\frac{\lambda r^2}{6} + f(p) - \frac{2}{r} \int_0^r f(\theta, s) ds - \frac{1}{r} \int_0^r \int_0^s \tau \rho R dt ds.$$

Combining Theorem 4 and the potential function estimates obtained by Munteanu and Wang [70, Theorem 0.4], we get the following corollary.

Corollary 2. *Let (M^n, g, f) be a complete noncompact n -dimensional gradient shrinking ρ -Einstein soliton with $\rho R \geq \delta > -\lambda$. Then there exists $r_0 > 0$ such that, for all $r \geq r_0$,*

$$\text{Vol}_f(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\frac{r^2}{2}} r^{n-1} dr d\theta. \quad (1.5)$$

In other words, Corollary 2 asserts that the weighted volume of a complete noncompact gradient shrinking ρ -Einstein soliton with $\rho R \geq \delta > -\lambda$ is less than or equal to the weighted volume of the Gaussian shrinking soliton.

Remark 2. *We highlight that under the hypothesis of Corollary 2, it follows that $\text{Ric} + \nabla^2 f \geq (\delta + \lambda)g$, which is a positive constant. In this situation, Wei and Wylie [78, Theorem 3.1] proved a similar estimate.*

Another classical approach to studying canonical metrics on a smooth manifold is to analyze the critical points of geometric functionals under appropriate constraints. From the seminal works of Einstein and Hilbert, it is well known that the critical points of the total scalar curvature functional, when restricted to the set of Riemannian metrics g on a compact manifold M^n with unit volume, are precisely the Einstein metrics; that is, those satisfying $\text{Ric} = \lambda g$, where λ is a constant and Ric denotes the Ricci curvature of (M^n, g) ; see [13, Theorem 4.21]. In the same spirit, it follows from Besse [13, p. 127–128] that the Euler–Lagrange equation of the total scalar curvature functional, when restricted to metrics of unit volume and constant scalar curvature, takes the form

$$\mathcal{L}_g^*(f) = -(\Delta f)g + \nabla^2 f - f\text{Ric} = \text{Ric} - \frac{R}{n}g, \quad (1.6)$$

where \mathcal{L}_g^* is the formal L^2 -adjoint of the linearization of the scalar curvature operator \mathcal{L}_g , f is a smooth function on M^n and R stands for the scalar curvature of (M^n, g) . Here, Δ and ∇^2 denote the Laplacian and Hessian, respectively. These foundational results have inspired a rich theory involving other functionals and geometric constraints.

In a related direction, and motivated by a volume comparison theorem due to Fan, Shi, and Tam [47], Miao and Tam [65, 64] initiated the study of critical metrics of the volume functional constrained to the space of metrics with constant scalar curvature on compact manifolds with boundary. Subsequently, Corvino, Eichmair, and Miao [41] employed this framework

to establish a deformation result, which indicates that scalar curvature alone is not sufficient to establish a volume comparison result. In particular, their deformation result implies that Schoen's conjecture cannot be directly extended to manifolds with boundary under only Dirichlet boundary conditions (see also [80]). Recall that *Schoen's conjecture*¹ [75] asserts: *Let (M^n, \bar{g}) be a closed hyperbolic manifold and let g be another metric on M^n with scalar curvature $R(g) \geq R(\bar{g})$, then $Vol(g) \geq Vol(\bar{g})$.* It follows from the work of Corvino, Eichmair and Miao [41] that, for a given constant κ , if a metric g does not admit a nontrivial solution f to the equation

$$\mathcal{L}_g^*(f) = -(\Delta f)g + \nabla^2 f - fRic = \kappa g, \quad (1.7)$$

then it is possible to simultaneously prescribe a compactly supported perturbation of the scalar curvature within a bounded domain Ω , and a prescribed perturbation of the volume, via a small deformation of the metric supported in $\bar{\Omega}$. Despite that, scalar curvature and volume comparison results do hold for certain special metrics. For instance, motivated by the work of Brendle, Marques and Neves [21] on Min-Oo's conjecture, Miao and Tam [63] proved a rigidity theorem for the upper hemisphere with respect to nondecreasing scalar curvature and volume. They also showed that an analogous result holds for Euclidean domains. In particular, these spaces satisfy equation (1.7). More recently, Yuan [80, Theorem A] established a volume comparison result for small geodesic balls under appropriate boundary conditions.

Before proceeding, we fix the following terminology (cf. [41],[64],[65]).

Definition 1. *Let (M^n, g) be a complete Riemannian manifold. We say that g is a V -static metric if there exists a constant κ and a non-constant smooth function f satisfying (1.7). In this case, f is a V -static potential.*

V -static metrics are critical points of the volume functional restricted to the space of metrics with prescribed constant scalar curvature and fixed boundary metric; see [41, Theorem 2.3] and [64, Theorem 2.1]. Moreover, as noted by McCormick [62], they arise in the study of asymptotically hyperbolic manifolds as critical points of the volume-renormalized mass. Observe that when $\kappa = 0$, equation (1.7) reduces to the static vacuum equation, relevant in general relativity (see [2, 42]). Furthermore, when f is constant, the equation forces the metric to be Einstein. Hence, V -static metrics generalize Einstein metrics in a natural way. Interestingly, it follows from [41, Proposition 2.1] and [64, Theorem 7] that any connected Riemannian manifold satisfying (1.7) must have constant scalar curvature.

¹ It has been confirmed in dimension three by the works of Hamilton [53] and Perelman [73], and also for metrics C^2 -close to the hyperbolic metric, through results by Besson–Courtois–Gallot [14, 15].

There are explicit examples of V -static metrics on both compact and noncompact manifolds. These include the Schwarzschild and AdS-Schwarzschild metrics restricted to suitable domains, as well as the standard metrics on geodesic balls in \mathbb{R}^n , \mathbb{H}^n , and \mathbb{S}^n (cf. [41],[64],[65]). The classification of V -static metrics plays a central role in understanding the interplay between scalar curvature and volume. As noted in [41],[64],[65], V -static metrics tend to exhibit strong rigidity properties. In recent years, several rigidity results have been established, especially in the compact with boundary case; see, for instance, [5],[7],[8],[9],[10],[41],[46],[48],[56],[58],[64],[65],[77],[80].

In this part of the work, which contains the results obtained in the article *Critical metrics of the volume functional on complete manifolds*, a joint work with R. Diógenes and E. Ribeiro Jr. [39], we focus on rigidity phenomena for V -static metrics (M^n, g, f) on complete manifolds without boundary. A key result in this setting is due to Miao and Tam [65, Theorem 2.2], who showed that if a V -static metric $(\kappa \neq 0)$ on a connected, complete manifold without boundary is Einstein, then (M^n, g) must be isometric to one of the following:

- the standard sphere \mathbb{S}^n ,
- the Euclidean space \mathbb{R}^n ,
- the hyperbolic space \mathbb{H}^n , or
- a warped product space $(\mathbb{R} \times \Sigma^{n-1}, dt^2 + \cosh^2 t g_0)$, where (Σ^{n-1}, g_0) is complete, Einstein with $Ric_{g_0} = -(n-2)g_0$, and the solution is $f(t, x) = A \sinh t + \frac{1}{n-1}$, with $A > 0$.

The result was proved for $\kappa = 1$, but, up to a rescaling of f , it also holds to general V -static metrics $(\kappa \neq 0)$. We note that the case of connected, compact manifolds with boundary was treated in [65, Theorem 1.1].

Since manifolds with parallel Ricci tensor have harmonic curvature (although the converse does not generally hold; see [45]), it is natural to ask whether the Einstein condition assumed in [65, Theorem 2.2] can be relaxed to the weaker assumption of parallel Ricci tensor, that is, $\nabla Ric = 0$. This question was answered affirmatively in the case of compact manifolds with boundary by Baltazar–Ribeiro [5], who proved that if a critical metric of the volume functional on an n -dimensional compact, connected manifold with boundary has parallel Ricci tensor, then the manifold is isometric to a geodesic ball in a simply connected space form \mathbb{R}^n , \mathbb{H}^n , or \mathbb{S}^n . Notably, this result excludes the *Nariai space*—the manifold $I \times \mathbb{S}^n$ with the product metric—as a V -static space with $\kappa \neq 0$, even though it is a static vacuum space (i.e., $\kappa = 0$); see [2, 71].

Our first main result addresses the aforementioned question in the setting of complete manifolds without boundary. More precisely, we prove the following result

Theorem 5. *Let (M^n, g, f) be an n -dimensional connected, complete V -static metric with $\kappa \neq 0$ and parallel Ricci tensor. Then (M^n, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^n , or*
- (2) *the Euclidean space \mathbb{R}^n , or*
- (3) *the hyperbolic space \mathbb{H}^n when $\nabla f(p) = 0$ for some $p \in M$, or*
- (4) *a warped product space $(\mathbb{R} \times \Sigma^{n-1}, dt^2 + \cosh^2 t g_0)$, where (Σ^{n-1}, g_0) is complete, Einstein with $\text{Ric}_{g_0} = -(n-2)g_0$, $f(t, x) = \kappa \left(A \sinh t + \frac{1}{n-1} \right)$, and $A > 0$ is constant.*

Remark 3. *We emphasize that the condition $\kappa \neq 0$ in Theorem 5 is essential and cannot be removed. Following the idea outlined in Costa et al. [43, Example 1], one sees that the product manifold $\mathbb{S}^{p+1} \times \mathbb{S}^q$, with $q > 1$, metric $g = g_{\mathbb{S}^{p+1}} + \frac{q-1}{p+1} g_{\mathbb{S}^q}$ and potential function $f = \cos(r)$, where $r(x)$ is the height function on \mathbb{S}^{p+1} , defines a static vacuum space with parallel Ricci tensor which fails to be Einstein whenever $p+1 \neq q$. In the noncompact setting, a simple calculation shows that the manifold $\mathbb{H}^{p+1} \times \mathbb{H}^q$, $q > 1$, equipped with the metric $g = g_{\mathbb{H}^{p+1}} + \frac{q-1}{p+1} g_{\mathbb{H}^q}$ and potential function $f = \cosh(r)$, where $r(x)$ is the height function on \mathbb{H}^{p+1} , also defines a static vacuum space with parallel Ricci tensor. However, this manifold is not Einstein whenever $p+1 \neq q$.*

For what follows, we recall the definition of the *Bach tensor* on a Riemannian manifold (M^n, g) . Originally introduced in the context of conformal relativity by Bach in the 1920s [3], the Bach tensor plays a key role in conformal geometry. For dimensions $n \geq 4$, it is defined in terms of the Weyl tensor W_{ijkl} by

$$B_{ij} = \frac{1}{n-3} \nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}. \quad (1.8)$$

In dimension $n = 3$, where the Weyl tensor vanishes identically, the Bach tensor is instead defined via the Cotton tensor C_{ijk} as

$$B_{ij} = \nabla_k C_{kij}. \quad (1.9)$$

A manifold (M^n, g) is said to be *Bach-flat* if $B_{ij} = 0$. In the 4-dimensional compact case, Bach-flat

metrics arise as the critical points of the conformally invariant functional

$$\mathcal{W}(g) = \int_M |W_g|^2 dV_g,$$

where W_g denotes the Weyl tensor associated to the metric g . We recall that both locally conformally flat metrics and Einstein metrics are Bach-flat. Moreover, in dimension $n = 4$, it is well known that metrics which are half-conformally flat or locally conformal to Einstein metrics also satisfy the Bach-flat condition.

Inspired by the works of Cao and Chen [24],[25], Barros, Diógenes and Ribeiro [10] proved that a Bach-flat critical metric of the volume functional on a simply connected 4-dimensional manifold with boundary isometric to a standard sphere must be isometric to a geodesic ball in a simply connected space form \mathbb{R}^4 , \mathbb{H}^4 , or \mathbb{S}^4 ; see also [6]. Roughly speaking, they replaced the assumption of locally conformally flat in the Miao–Tam result (cf. [65, Theorem 1.2]) by the weaker condition of Bach-flat (see [10]). Moreover, they established a similar rigidity result in dimension $n = 3$ under a weaker assumption.

A natural question that arises from these comments is what happens when the manifold is complete and has no boundary. In this work, we also address that question. More precisely, we establish the following result.

Theorem 6. *Let (M^n, g, f) , $n \geq 4$, be an n -dimensional complete, simply connected Bach-flat V -static metric with $\kappa \neq 0$ and f be a proper function. Then (M^n, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^n , or*
- (2) *the Euclidean space \mathbb{R}^n , or*
- (3) *the hyperbolic space \mathbb{H}^n or*
- (4) *$\mathbb{R} \times_{\varphi} \Sigma_c$, where Σ_c is a regular level set of the potential function, which is an Einstein manifold with $\text{Ric}_{\Sigma_c} = \lambda g_{\Sigma_c}$, and the warping function φ is a solution of the ODE*

$$\varphi \left(2\varphi'' + \frac{R}{n-1} \varphi \right) + (n-2)(\varphi')^2 = \lambda. \quad (1.10)$$

Remark 4. *We emphasize that Theorem 6 remains valid if the Bach-flat condition is replaced by the weaker assumption that the Bach tensor is radially flat, that is, $B(\nabla f, \nabla f) = 0$; see Lemma 6 and Theorem 14 in Section 4.2.*

In order to prove Theorem 6, which is a special case of Theorem 14, we adapt some techniques outlined in [10],[24], [26],[27],[29],[61]. Our analysis begins by establishing a

connection between the conditions $T \equiv 0$ and Bach-flat, where T denotes an auxiliary 3-tensor (Lemma 6). This step required revisiting and adapting several arguments originally formulated in [10], where the presence of a boundary was essential. In our context, the assumption that the potential function f is proper plays a crucial role in circumventing the absence of boundary conditions. This requirement is also natural because, in our setting, there is no “integrability condition” analogous to Hamilton’s identity for Ricci solitons (see Eq. (4.13) in [37]), which makes it substantially more difficult to control the asymptotic behavior of the potential function. Next, we construct a local warped product structure around the regular points of f (Proposition 10). To extend this local structure to a global one, we adapt techniques from a recent work of Cao–Yu [26], along with methods introduced by Cao–Chen [24] and Fernández-López and García-Río [61] in the study of Ricci solitons, which are well suited to the complete without boundary case.

We now turn our attention to the lower-dimensional cases. In particular, when $n = 4$, equation (1.10) becomes more tractable. As a consequence of Theorem 6, we obtain the following corollary.

Corollary 3. *Let (M^4, g, f) be a complete, simply connected four-dimensional Bach-flat V-static metric with $\kappa \neq 0$ and f be a proper function. Then (M^4, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^4 , or*
- (2) *the Euclidean space \mathbb{R}^4 , or*
- (3) *the hyperbolic space \mathbb{H}^4 , or*
- (4) *the warped product $\mathbb{R} \times_{\varphi} \mathbb{S}^3$, where φ is a solution of the ODE*

$$\varphi \left(\varphi'' + \frac{R}{6} \varphi \right) + (\varphi')^2 = 1.$$

It is natural to ask whether an analogous result holds in the three-dimensional case. In this situation, however, arguments from Cao et al. [27], combined with key steps from the second part of the proof of Theorem 6, allows us to establish our next result under the weaker assumption $\operatorname{div} B(\nabla f) = 0$.

Theorem 7. *Let (M^3, g, f) be a complete, simply connected V-static metric with $\kappa \neq 0$ and f be a proper function. If $\operatorname{div} B(\nabla f) = 0$, then (M^3, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^3 , or*
- (2) *the Euclidean space \mathbb{R}^3 , or*

(3) the hyperbolic space \mathbb{H}^3 , or

(4) the warped product $\mathbb{R} \times_{\varphi} \mathbb{S}^2$, where φ is a solution of the ODE

$$\varphi \left(2\varphi'' + \frac{R}{2}\varphi \right) + (\varphi')^2 = 1. \quad (1.11)$$

2 PRELIMINARIES

The purpose of this chapter is to establish some results that will play a central role in the development of this thesis. We are going to present the required Riemannian framework and a few preliminaries results regarding our subjects of investigation

2.1 Riemannian geometry background

We begin with a section that contains a brief survey of the classical Riemannian geometry background that is going to be used. Let us begin with some basic definitions

Definition 2. *Let M^n be a smooth manifold. A Riemannian metric on M^n is a smooth 2-tensor g , which associates at each point $p \in M^n$ an inner product g_p in T_pM . A **Riemannian manifold** is a pair (M^n, g) , where M^n is a smooth manifold and g is a specific choice of Riemannian metric.*

In order to obtain an object that captures the essential properties of the covariant derivative on surfaces in \mathbb{R}^3 , we introduce the notion of affine connection

Definition 3. *Let M^n be a smooth manifold. An affine connection on M^n is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

satisfying the following properties:

$$(i) \quad \nabla_{fX+hY}Z = f\nabla_XZ + h\nabla_YZ;$$

$$(ii) \quad \nabla_X(Y+Z) = \nabla_XY + \nabla_XZ;$$

$$(iii) \quad \nabla_X(fY) = f\nabla_XY + X(f)Y;$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $f, h \in C^\infty(M)$.

While the definition above is suitably general, it is too broad for the usual analytical and geometric purposes. Consequently, we introduce a more restricted class of affine connections in the following definition.

Definition 4. *Let (M^n, g) be a Riemannian manifold and ∇ an affine connection on M^n . Then*

1. We say that connection ∇ is compatible with the metric g if

$$X(g(Y, Z)) = g(\nabla_XY, Z) + g(Y, \nabla_XZ)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

2. We say that the connection ∇ is symmetric if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y \in \mathfrak{X}(M)$.

The following theorem is central to Riemannian geometry theory, as it provides a natural way to select an affine connection ∇ for a given metric g .

Theorem 8 (Levi-Civita). *Let (M^n, g) be a Riemannian manifold. Then, there exists a unique affine connection ∇ on M^n , such that*

(1) ∇ is compatible;

(2) ∇ is symmetric.

This connection is referred as the Levi-Civita connection of (M^n, g) .

Throughout this work (M^n, g) will denote an n -dimensional Riemannian manifold, ∇ the Levi-Civita connection, $\mathfrak{X}(M)$ the space of smooth vector fields, and $C^\infty(M)$ the space of smooth functions over M . Recall that, for (M^n, g) , the Riemann curvature tensor is the (1,3)-tensor defined as

$$Rm(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.1)$$

where X, Y and $Z \in \mathfrak{X}(M)$.

Now, let $\{x_i\}_{i=1}^n$ be a local coordinate system, the components of Rm are given by

$$Rm(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l, \quad (2.2)$$

where $\{\partial_i\}_{i=1}^n$ are the associated vector fields. On the other hand, the Riemannian curvature tensor Rm is often viewed as a (0,4)-tensor, which components are defined as follows

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l) = g(Rm(\partial_i, \partial_j)\partial_l, \partial_k). \quad (2.3)$$

By taking the trace on the Riemann curvature tensor, we arrive at another very important notion of curvature for this thesis, the Ricci curvature, The components of the Ricci tensor are given by

$$R_{ik} = g^{jl} R_{ijkl}. \quad (2.4)$$

Taking the trace one more time, we find the expression for the scalar curvature R , which is a smooth function on M defined as

$$R = g^{ij} R_{ij}. \quad (2.5)$$

These are the basic curvature notions on a Riemannian manifold that we are going to be using in this work. Moving forward, we present other tensors that are closely related to those three above and play a crucial role in the development of this thesis. We begin with the Weyl tensor W , which is the traceless component of the Riemann curvature tensor R_{ijkl} . It is defined by the following decomposition

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2} \left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il} \right) - \frac{R}{(n-1)(n-2)} \left(g_{jl}g_{ik} - g_{il}g_{jk} \right). \quad (2.6)$$

Next, we consider the Cotton tensor C_{ijk} , which is defined as

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} \left(\nabla_i R g_{jk} - \nabla_j R g_{ik} \right). \quad (2.7)$$

For dimensions $n \geq 4$, the Cotton tensor is related to the Weyl tensor by

$$C_{ijk} = -\frac{n-2}{n-3} \nabla_l W_{ijkl}. \quad (2.8)$$

Another important tensor is the Schouten tensor, defined by

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}. \quad (2.9)$$

In terms of A , we can express the Riemann curvature tensor as

$$R_{ijkl} = \frac{1}{n-2} (A \odot g)_{ijkl} + W_{ijkl}. \quad (2.10)$$

For what follows, we recall the definition of the Bach tensor on a Riemannian manifold (M^n, g) . Originally introduced in the context of conformal relativity by Bach in the 1920s [3], the Bach tensor plays a key role in conformal geometry. For dimensions $n \geq 4$, it is defined in terms of the Weyl tensor W_{ijkl} by

$$B_{ij} = \frac{1}{n-3} \nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}. \quad (2.11)$$

In dimension $n = 3$, where the Weyl tensor vanishes identically, the Bach tensor is instead defined via the Cotton tensor C_{ijk} as

$$B_{ij} = \nabla_k C_{kij}. \quad (2.12)$$

A manifold (M^n, g) is said to be *Bach-flat* if $B_{ij} = 0$. In the 4-dimensional compact case, Bach-flat metrics arise as the critical points of the conformally invariant functional

$$\mathcal{W}(g) = \int_M |W_g|^2 dV_g,$$

where W_g denotes the Weyl tensor associated to the metric g . We recall that both locally conformally flat metrics and Einstein metrics are Bach-flat. Moreover, in dimension $n = 4$, it is well known that metrics which are half-conformally flat or locally conformal to Einstein metrics also satisfy the Bach-flat condition.

Now, given a function $f \in C^\infty(M)$, the Hessian of f is the 2-tensor defined by

$$\nabla^2 f(X, Y) = Y(Xf) - (\nabla_Y X)f, \quad (2.13)$$

for any $X, Y \in \mathfrak{X}(M)$. Meanwhile, taking the trace of $\nabla^2 f$ we get the Laplacian of f , in other words

$$\Delta f = g^{ij} \nabla^2 f(\partial_i, \partial_j). \quad (2.14)$$

2.1.1 Warped product metrics

In what follows, we briefly discuss Riemannian manifolds that admit warped structures and their consequences. As we shall see, the existence of such a structure has important consequences regarding the Ricci and scalar curvature. This will be crucial for this work, particularly in the last chapter.

Let (I, dt^2) be an interval $I \subset \mathbb{R}$ with the standard metric, (N, g_N) a Riemannian manifold, and $\varphi : I \rightarrow (0, \infty)$ a smooth function. The warped product $I \times_\varphi N$ is the product manifold with the warped metric

$$g = dt^2 + \varphi^2 g_N.$$

In this context, I is called the basis manifolds, N the fiber, and φ the warping function. Notice that if $\varphi = 1$, then the warped product becomes the usual Riemannian product. We point out that the warped product can be defined in a more general setting — where the basis is any Riemannian manifold — but for the purpose of this work it is enough to deal with interval basis.

We now present two important formulas regarding warped metrics that will be useful in what follows. We compile these formulas into a proposition that we state without proof, since this is a well established result in the theory (see [17]).

Proposition 1. *Consider the warped manifold $(I \times_\varphi N, g)$, where (I, dt^2) is an interval, (N, g_N) is an $(n-1)$ -dimensional Riemannian manifold, and g is the warped metric. Then we have*

(i) *The Ricci curvature of g is given by*

- $Ric_g(\partial_t, \partial_t) = -(n-1) \frac{\varphi''}{\varphi},$

- $Ric_g(X, Y) = Ric_{g_N}(X, Y) - \left[(n-2) \left(\frac{\varphi'}{\varphi} \right)^2 + \frac{\varphi''}{\varphi} \right] g(X, Y), \quad \forall X, Y \in \mathfrak{X}(N)$
- $Ric_g(\partial_t, X) = 0, \quad \forall X \in \mathfrak{X}(N).$

(ii) The scalar curvature of g is given by the formula

$$R_g = -2(n-1) \frac{\varphi''}{\varphi} + \frac{R_{g_N}}{\varphi^2} - (n-1)(n-2) \left(\frac{\varphi'}{\varphi} \right)^2. \quad (2.15)$$

2.1.2 Variation formulas

We now derive the first and second variation formulas for the area of a submanifold of a Riemannian manifold. After presenting the general framework, we specialize to the geometric setting considered in this thesis. These variational formulas are fundamental to the proof of Theorem 10. Our exposition follows the approach of Peter Li's book [59], which we refer the reader for additional details.

Let (M^m, g) be a Riemannian manifold and (N^n, g) a submanifold of M . Consider a family of one-parameter deformations of N given by $N_t = \phi(N, t)$, with $t \in (-\varepsilon, \varepsilon)$ and $N_0 = N$. Now, let $\{x_i\}_{i=1}^n$ be a normal coordinate system near a point $p \in N$, thus we can consider $\{x_1, \dots, x_n, t\}$ to be a local coordinate system of $N \times (-\varepsilon, \varepsilon)$ near the point $(p, 0)$. Let us denote $e^i = d\phi(\partial/\partial x^i)$ and let $T = d\phi(\partial_t)$. Since $\{x^i\}$ is normal at $p \in N$, then $g_{ij}(p, 0) = \delta_{ij}$, where $g_{ij} = \langle e_i, e_j \rangle$ is the induced metric on N_t and $\nabla_{e_i} e_j(p, 0) = 0$. If we define dA_t as the area element of N_t , then for t close enough to 0, we can write $dA_t = J(x, t) dA_0$, where the function $J(x, t)$, with respect to the coordinates $\{x_i\}$, is given by

$$J(x, t) = \frac{\sqrt{g(x, t)}}{\sqrt{g(x, 0)}},$$

where $g(x, t) = \det(g_{ij}(x, t))$. Now, since $g_{ij}(p, 0) = \delta_{ij}$, using the cofactor expansion in the first line for the determinant, we arrive at the following identity

$$\begin{aligned} g'(p, 0) &= \sum_{j=1}^n \left(g'_{1j}(p, 0) c_{1j}(p, 0) + g_{1j}(p, 0) c'_{1j}(p, 0) \right) \\ &= g'_{11}(p, 0) + c'_{11}(p, 0), \end{aligned}$$

where c_{ij} are the cofactors of g_{ij} . Repeating the argument for $c'_{11}(p, 0)$, we see that $g'(p, 0) = \sum_{i=1}^n g'_{ii}$. Now, notice that

$$\begin{aligned} g'_{ii} &= T \langle e_i, e_i \rangle \\ &= 2 \langle \nabla_{e_i} T, e_i \rangle. \end{aligned}$$

On the other hand, if we decompose $T = T^t + T^n$ into tangential and normal components, we find

$$\begin{aligned} \sum_{i=1}^n \langle \nabla_{e_i} T, e_i \rangle &= \sum_{i=1}^n \langle \nabla_{e_i} T^t, e_i \rangle + \langle \nabla_{e_i} T^n, e_i \rangle \\ &= \operatorname{div}(T^t) + \sum_{i=1}^n e_i \langle T^n, e_i \rangle - \sum_{i=1}^n \langle T^n, \nabla_{e_i} e_i \rangle \\ &= \operatorname{div}(T^t) + \langle T^n, \vec{H} \rangle. \end{aligned}$$

Combining the above computation with the fact that $J'(p, 0) = \frac{1}{2}g'(p, 0)$, we conclude the first variational formula for the area, that is,

$$\left. \frac{d}{dt} A_t \right|_{(p,0)} = \left(\operatorname{div}(T^t) + \langle T^n, \vec{H} \rangle \right) dA_0 \Big|_{(p,0)}. \quad (2.16)$$

Now, we compute the second variational formula for the area. We begin by considering $\phi : N \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \rightarrow M$ a two-parameter family of variations of N . Adopting the same notation as before, we set the variational vector fields $T = d\phi(\partial/\partial t)$ and $S = d\phi(\partial/\partial s)$. Using the same computation as before and the formula for the derivative of a determinant, we conclude that

$$\frac{\partial J}{\partial t}(x, t, s) = \sum_{i,j=1}^n g^{ij} \langle \nabla_{e_i} T, e^j \rangle J(x, t, s).$$

Therefore, by differentiating with respect to s and evaluating at $(p, 0, 0)$, we find

$$\begin{aligned} \frac{\partial^2 J}{\partial s \partial t} &= \sum_{i,j=1}^n S \left(g^{ij} \langle \nabla_{e_i} T, e^j \rangle J \right) \\ &= \sum_{i,j=1}^n (Sg^{ij}) \langle \nabla_{e_i} T, e_j \rangle + \sum_{i=1}^n S \langle \nabla_{e_i} T, e_i \rangle \\ &\quad + \left(\sum_{i=1}^n \langle \nabla_{e_i} T, e_i \rangle \right) \left(\sum_{j=1}^n \langle \nabla_{e_j} S, e_j \rangle \right). \end{aligned} \quad (2.17)$$

However, a straightforward computation shows that

$$\sum_{k=1}^n (Sg^{ik}) g_{kj} = - \sum_{k=1}^n g^{ik} (Sg_{kj}),$$

so that

$$\begin{aligned} S(g^{ij}) &= - \sum_{k,l=1}^n g^{ik} g^{jl} (Sg_{kl}) \\ &= -S(g_{ij}) \\ &= -\langle \nabla_S e_i, e_j \rangle - \langle e_i, \nabla_S e_j \rangle \\ &= -\langle \nabla_{e_i} S, e_j \rangle - \langle \nabla_{e_j} S, e_i \rangle. \end{aligned}$$

Now, we analyze the terms of (2.17) separately. First, notice that

$$\begin{aligned} \sum_{i,j=1}^n (Sg^{ij}) \langle \nabla_{e_i} T, e_j \rangle &= - \sum_{i,j=1}^n \langle \nabla_{e_i} S, e_j \rangle \langle \nabla_{e_i} T, e_j \rangle \\ &\quad - \sum_{i,j=1}^n \langle \nabla_{e_j} S, e_i \rangle \langle \nabla_{e_i} T, e_j \rangle. \end{aligned} \quad (2.18)$$

At the same time, we have

$$\begin{aligned} \sum_{i=1}^n S \langle \nabla_{e_i} T, e_i \rangle &= \sum_{i=1}^n \langle \nabla_S \nabla_{e_i} T, e_i \rangle + \sum_{i=1}^n \langle \nabla_{e_i} T, \nabla_S e_i \rangle \\ &= \sum_{i=1}^n \langle Rm(S, e_i) T, e_i \rangle + \sum_{i=1}^n \langle \nabla_{e_i} \nabla_S T, e_i \rangle + \sum_{i=1}^n \langle \nabla_{e_i} T, \nabla_{e_i} S \rangle. \end{aligned} \quad (2.19)$$

Finally, combining (2.18) and (2.19) with (2.17), we arrive at the general second variational formula for the area

$$\begin{aligned} \frac{\partial^2 J}{\partial S \partial t} &= - \sum_{i,j=1}^n \langle \nabla_{e_i} S, e_j \rangle \langle \nabla_{e_i} T, e_j \rangle - \sum_{i,j=1}^n \langle \nabla_{e_j} S, e_i \rangle \langle \nabla_{e_i} T, e_j \rangle \\ &\quad + \sum_{i=1}^n \langle Rm(S, e_i) T, e_i \rangle + \sum_{i=1}^n \langle \nabla_{e_i} \nabla_S T, e_i \rangle + \sum_{i=1}^n \langle \nabla_{e_i} T, \nabla_{e_i} S \rangle \\ &\quad + \left(\sum_{i=1}^n \langle \nabla_{e_i} T, e_i \rangle \right) \left(\sum_{j=1}^n \langle \nabla_{e_j} S, e_j \rangle \right). \end{aligned} \quad (2.20)$$

For the purpose of this thesis, we only need a specific case of the above formulas; therefore, we present the required setting as a proposition.

Proposition 2. *Let (M^n, g) be an oriented Riemannian manifold and N be an oriented hypersurface. Consider N_t variations given by hypersurfaces that are constant distant from N . Then, the first and second variational formulas for the area are given by*

$$i) \quad \frac{\partial J}{\partial t}(x, 0) = H(x)J(x, 0),$$

$$ii) \quad \frac{\partial^2 J}{\partial t^2}(x, 0) = - \sum_{i,j=1}^n h_{ij}^2(x)J(x, 0) - Ric(\partial_t, \partial_t)J(x, 0) + H^2(x)J(x, 0),$$

here h_{ij} are the components of the second fundamental form, H is the mean curvature, and $Ric(\partial_t, \partial_t)$ is the Ricci curvature in the radial direction.

Proof. To prove this proposition, we just need to realize that in this setting the variational vector fields are normal to N , we denote by ∂_t (the radial direction), with $\nabla_{\partial_t} \partial_t \equiv 0$ (constant acceleration), and the mean curvature vector is given by $\vec{H} = H \partial_t$.

To prove (i) we just need to show that ∂_t has no tangential component and

$$\langle T^n, \vec{H} \rangle = H.$$

Plugging this into (2.16), one concludes the proof of the first equation. For item (ii), we apply our variational conditions to (2.20), which implies

$$-\sum_{i,j=1}^n \langle \nabla_{e_i} S, e_j \rangle \langle \nabla_{e_i} T, e_j \rangle - \sum_{i,j=1}^n \langle \nabla_{e_j} S, e_i \rangle \langle \nabla_{e_i} T, e_j \rangle + \sum_{i=1}^n \langle \nabla_{e_i} T, \nabla_{e_i} S \rangle = -\sum_{i,j=1}^n h_{ij}^2(x).$$

On the other hand, we have

$$\sum_{i=1}^n \langle Rm(\partial_t, e_i) \partial_t, e_i \rangle = -Ric(\partial_t, \partial_t) \quad \text{and} \quad \sum_{i=1}^n \langle \nabla_{e_i} \nabla_S T, e_i \rangle = 0.$$

Finally, since

$$\left(\sum_{i=1}^n \langle \nabla_{e_i} T, e_i \rangle \right) \left(\sum_{j=1}^n \langle \nabla_{e_j} S, e_j \rangle \right) = H^2(x),$$

we conclude the proof. \square

Following Proposition 2, we establish a key inequality for the area element in polar coordinates. For a fixed $p \in M$, in terms of the polar normal coordinates at p , we may write the volume element as $J(\theta, r) dr \wedge d\theta$, where $d\theta$ is the volume element of the $(n-1)$ -dimensional sphere \mathbb{S}^{n-1} . From the Gauss lemma, it is known that the area element of the geodesic sphere is given by $J(\theta, r) d\theta$. Now, consider $x = (\theta, r)$, a point outside the cut-locus $\mathcal{C}(p)$ of p , and define

$$w(\theta, r) = \frac{\partial}{\partial r} \log J(\theta, r) = \frac{\partial J}{\partial r}(\theta, r) / J(\theta, r).$$

The second area variational formula in polar coordinates is

$$\frac{\partial^2 J}{\partial r^2}(\theta, r) = -\sum_{i,j=1}^{n-1} h_{ij}^2(\theta, r) J(\theta, r) - Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) J(\theta, r) + \frac{\left(\frac{\partial J}{\partial r}\right)^2}{J}(\theta, r), \quad (2.21)$$

where $h_{ij}(\theta, r)$ stands for the second fundamental form of $\partial B_p(r)$. Moreover, notice that

$$\frac{\partial w}{\partial r}(\theta, r) = \frac{\partial^2 J}{\partial r^2}(\theta, r) / J(\theta, r) - w^2(\theta, r). \quad (2.22)$$

Plugging (2.22) into (2.21), one sees that

$$\frac{\partial w}{\partial r}(\theta, r) + w^2(\theta, r) = -\sum_{i,j=1}^{n-1} h_{ij}^2(\theta, r) - Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + \frac{\left(\frac{\partial J}{\partial r}\right)^2}{J^2}(\theta, r).$$

Since h_{ij} is a 2-tensor, we have $|h|^2 \geq \frac{H}{n-1}$, where $H(\theta, r)$ is the mean curvature of $\partial B_p(r)$.

Therefore, it follows that

$$\frac{\partial w}{\partial r}(\theta, r) + \frac{1}{n-1} H^2(\theta, r) + Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq 0. \quad (2.23)$$

Finally, observe that the first area variational formula gives $H(\theta, r) = w(\theta, r)$. From this, one concludes that $w(\theta, r)$ must satisfy

$$w'(\theta, r) + \frac{1}{n-1}w^2(\theta, r) + Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq 0, \quad (2.24)$$

where $w' := \frac{\partial w}{\partial r}$ and $Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$ stands for the Ricci curvature in the radial direction; for more details see, e.g., [59].

Another important variational approach involves considering variations of the metric by a symmetric (0,2)-tensor. In the following lemma, we address this problem and derive variational formulas that will be useful later.

Lemma 1. *Let $g(t)$ be a one-parameter family of metrics on a Riemannian manifold (M^n, g) .*

Suppose that $g(0) = g$ and $\frac{\partial}{\partial t}g_{ij} = h_{ij}$, where h is a symmetric 2-tensor. Then

$$\begin{aligned} \text{i)} \quad \frac{\partial}{\partial t}\Big|_{t=0} dV_g &= \frac{1}{2}(tr_g h)dV_g, \\ \text{ii)} \quad \frac{\partial}{\partial t}\Big|_{t=0} R_{ij} &= \frac{1}{2}\left(-\Delta h_{ij} + \nabla_i(\text{div}h)_j + \nabla_j(\text{div}h)_i - \nabla_i\nabla_j(tr_g h)\right) \\ &\quad - 2R_{ikjp}h_{kp} + R_{ip}h_{jp} + R_{jp}h_{ip}, \\ \text{iii)} \quad \frac{\partial}{\partial t}\Big|_{t=0} R_{g(t)} &= -\Delta(tr_g h) + \text{div}^2 h - \langle Ric, h \rangle. \end{aligned}$$

Proof. To begin with, recall that for a local coordinate system $\{x_i\}$ the volume element is given by

$$dV_{g(t)} = \sqrt{\det(g(t))}dx_1 \wedge \dots \wedge dx_n,$$

Thus

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} dV_g &= \frac{1}{2} \frac{1}{\sqrt{\det g(t)}} \frac{\partial}{\partial t}\Big|_{t=0} (\det g(t)) \\ &= \frac{1}{2} \sqrt{\det g(t)} tr(g^{-1}h) \\ &= \frac{1}{2} \sqrt{\det g(t)} g^{ij}h_{ij} \\ &= \frac{1}{2} (tr_g h) dV_g. \end{aligned}$$

Next, we shall prove item (ii). To do so, we need to establish a variational formula for the Christoffel symbols of the metric. We recall that

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \quad (2.25)$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \Gamma_{ij}^k &= \frac{1}{2} \left(\frac{\partial}{\partial t} \Big|_{t=0} g^{kl} \right) \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij} \right) \\ &\quad + \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial t} g_{jl} + \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} g_{il} - \frac{\partial}{\partial x_l} \frac{\partial}{\partial t} g_{ij} \right). \end{aligned} \quad (2.26)$$

Assuming further that $\{x_i\}$ is a normal coordinate system around $p \in M$, we have $\frac{\partial}{\partial x_i} g_{jk}(p) = 0$, $\Gamma_{ij}^k(p) = 0$ and $\frac{\partial}{\partial x_i} h_{jk}(p) = \nabla_i h_{jk}(p)$. Then, (2.26) becomes

$$\frac{\partial}{\partial t} \Big|_{t=0} \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\nabla_i g_{jl} + \nabla_j g_{il} - \nabla_l g_{ij} \right), \quad (2.27)$$

where we used $\frac{\partial}{\partial t} g^{ij} = -g^{ik} h_{kl} g^{jl}$.

After establishing the variational formula for the Christoffel symbols, notice that, in coordinates, the Ricci tensor is given by

$$R_{ij} = \frac{\partial}{\partial x_l} \Gamma_{ij}^l - \frac{\partial}{\partial x_i} \Gamma_{lj}^l + \Gamma_{lm}^l \Gamma_{ij}^m - \Gamma_{im}^l \Gamma_{lj}^m. \quad (2.28)$$

Differentiating (2.28), combining with (2.27) and manipulating the equations, we arrive at

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} R_{ij} &= \frac{1}{2} g^{lm} \left(\nabla_l \nabla_i - \nabla_i \nabla_l \right) h_{jm} + \frac{1}{2} g^{lm} \left(\nabla_l \nabla_j h_{im} + \nabla_i \nabla_m h_{lj} \right) \\ &\quad - \frac{1}{2} \left(\Delta h_{ij} - \nabla_i \nabla_j (tr_g h) \right). \end{aligned}$$

Accordingly, applying the Ricci identity in the above equality, we conclude that

$$\frac{\partial}{\partial t} \Big|_{t=0} R_{ij} = \frac{1}{2} g^{lm} \left(-R_{lij}^p h_{pm} - R_{lim}^p h_{jp} \right) - \frac{1}{2} \left(\Delta h_{ij} - \nabla_i \nabla_j (tr_g h) \right) \quad (2.29)$$

$$+ \frac{1}{2} g^{lm} \left(\nabla_l \nabla_j h_{im} - R_{lji}^p h_{pm} - R_{ljm}^p h_{ip} + \nabla_i \nabla_m h_{lj} \right). \quad (2.30)$$

Rearranging the terms in (2.29), we have

$$\frac{\partial}{\partial t} \Big|_{t=0} R_{ij} = \frac{1}{2} \left(-\Delta h_{ij} + \nabla_i (div h)_j + \nabla_j (div h)_i - \nabla_i \nabla_j (tr_g h) \right) \quad (2.31)$$

$$- 2R_{ikjp} h_{kp} + R_{ip} h_{jp} + R_{jp} h_{ip}. \quad (2.32)$$

This concludes (ii). Finally, to prove (iii), we just need to recall that $R = g^{ij} R_{ij}$, consequently,

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} R_{g(t)} &= \left(\frac{\partial}{\partial t} \Big|_{t=0} g^{ij} \right) R_{ij} + g^{ij} \left(\frac{\partial}{\partial t} \Big|_{t=0} R_{ij} \right) \\ &= \left(-g^{ik} h_{kl} g^{jl} \right) R_{ij} + \frac{1}{2} g^{ij} \left(-\Delta h_{ij} + \nabla_i (div h)_j + \nabla_j (div h)_i - \nabla_i \nabla_j (tr_g h) \right) \\ &\quad - 2R_{ikjp} h_{kp} + R_{ip} h_{jp} + R_{jp} h_{ip} \\ &= -\Delta (tr_g h) + div^2 h - \langle Ric, h \rangle. \end{aligned}$$

Hence, the proof is completed. \square

2.2 Einstein manifolds

In this section, we briefly discuss Einstein manifolds and their properties. Let us begin with the following definition

Definition 5. *Let (M^n, g) be a Riemannian manifold. We say that the metric g is Einstein if it satisfies*

$$\text{Ric}_g(X, Y) = \lambda g(X, Y), \quad (2.33)$$

for every $X, Y \in \mathfrak{X}(M)$, where λ is a smooth function on M .

Einstein manifolds are a very relevant subject in both physics and mathematics. From the physical point of view, they appear in general relativity theory related to the Einstein field equation given by

$$\text{Ric} - \frac{1}{2}Rg + \Lambda g = 8\pi T,$$

where T is a symmetric (0,2)-tensor called stress-energy tensor and Λ is the cosmological constant. When considering the vacuum condition, that is, $T \equiv 0$ the Einstein field equation becomes

$$\text{Ric} = \left(\frac{R}{2} - \Lambda\right)g. \quad (2.34)$$

Hence, the vacuum metric is an Einstein metric in the sense of Definition 5.

From a mathematical perspective, the seminal work of Hilbert called very much interest in these manifolds. Investigating critical points of the scalar curvature functional, which is given by

$$\mathcal{R}(g) = \int_M R_g dV_g,$$

he proved that Einstein metric are precisely the critical points when restricted to the space of compact manifolds with unit volume.

Einstein metrics are often regarded as canonical metrics, and it is natural to ask whether a given manifold admits one. It is well established, from the solution of the Yamabe problem, that a compact manifold always admits a metric of constant scalar curvature. However, it remains an open and very interesting problem to determine whether a given manifold admits an Einstein metric. This search has motivated a variety of research directions, among which we

highlight the study of Ricci solitons, quasi-Einstein manifolds, and ρ -Einstein solitons. In fact, there exists a more general framework, often called Einstein-type manifolds, which encompasses a broad class of metrics that generalize, in some way, the concept of Einstein metrics.

Now, we present two important properties of Einstein metrics, which we compile in the following proposition.

Proposition 3. *Let (M^n, g) be an Einstein Riemannian manifold with $n \geq 3$. Then, the following assertions are true:*

- (1) *The function λ given in the definition is constant.*
- (2) *The scalar curvature of the metric g is constant and satisfies $R = \lambda n$.*

Proof. To prove (1), we recall that the twice contracted Bianchi identity asserts that $2\text{divRic} = \nabla R$. Thus, by taking the divergence in (2.33), we infer

$$\nabla R = 2\nabla\lambda. \quad (2.35)$$

On the other hand, by taking the trace in (2.33), we conclude that

$$R = n\lambda. \quad (2.36)$$

Therefore, combining (2.35), together with (2.36), we discover $(n-2)\nabla\lambda = 0$. Since we assumed that $n \geq 3$ we get the desired assertion. Now, notice that (2) is straightforward from the proof of (1) and our proposition is complete. \square

This concludes our brief overview of Einstein manifolds. We strongly suggest Besse's book [13] for those who are interested in more details regarding Einstein metrics.

The following two sections present preliminary results concerning the class of manifolds investigated in this thesis. Key properties will be established, and examples will be provided to clarify their geometric and analytic structure.

2.3 Gradient ρ -Einstein solitons

In this section, we present basic facts that are useful for establishing the main results on ρ -Einstein solitons. Moreover, we will describe some examples of gradient ρ -Einstein solitons. To begin with, let us briefly present a background.

The Ricci-Bourguignon flow was originally introduced by Bourguignon in [20] and it is given by the the following differential equation

$$\frac{\partial}{\partial t} g_t = -2(\text{Ric}_{g_t} - \rho R g_t),$$

where $g(t)$ is a one-parameter family of metrics on a given manifold. In the study of geometric flows, it is often useful to examine special types of solution, in particular self-similar solutions. It is well known that for the Ricci flow ($\rho = 0$), the gradient shrinking Ricci solitons are precisely the self-similar solutions. According to the work of Catino, Mazzieri and Mongodi [28], the ρ -Einstein solitons arises as self-similar solutions do the Ricci-Bourguignon flow. In that direction, we have the following definition.

Definition 6. Let (M^n, g) be a Riemannian manifold, f a nonconstant smooth function on M^n and λ a real constant. For a given $\rho \in \mathbb{R}$, we say that (M^n, g, f, λ) is a gradient ρ -Einstein soliton if it satisfies the equation

$$\text{Ric} + \nabla^2 f = (\rho R + \lambda)g, \quad (2.37)$$

where R is the scalar curvature of (M^n, g) .

Following the terminology used for Ricci solitons, we say that the ρ -Einstein soliton is *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. When $\rho = \frac{1}{2(n-1)}$, it is called *Schouten soliton*; see [30]. Moreover, notice that the gradient Ricci soliton equation is obtained when $\rho = 0$ in (2.37).

For the reader's convenience, we present the proof that ρ -Einstein solitons are self similar solution to the Ricci-Bourguignon flow; see also[28].

Theorem 9 ([28]). Let (M^n, g_0, f_0) be a complete gradient ρ -Einstein soliton with constant λ . Then there exist a solution $g(t)$ to the Ricci-Bourguignon flow, a family of diffeomorphisms $\phi(t, \cdot)$, with $\phi(0, \cdot) = \text{Id}_M$ and a family of functions $f(t, \cdot)$, with $f(0, \cdot) = f_0(\cdot)$, defined for every t such that $\tau(t) := 1 - 2\lambda t > 0$ satisfying:

i) the family $\phi(t, \cdot)$ is generated by the vector fields $\frac{\nabla f_0}{\tau(t)}$, that is,

$$\frac{\partial}{\partial t} \phi(t, \cdot) = \frac{1}{\tau(t)} \nabla f_0(\phi(t, \cdot)); \quad (2.38)$$

ii) the metric $g(t)$ is given by the pull-back through $\phi(t, \cdot)$ and rescaling through $\tau(t)$, in other word

$$g(t) = \tau(t) \phi(t, \cdot)^* g_0; \quad (2.39)$$

iii) the family $f(t, \cdot)$, given by the pull-back through $\phi(t, \cdot)$, that is,

$$f(t, \cdot) = (f \circ \phi)(t, \cdot), \quad (2.40)$$

defines a gradient ρ -Einstein soliton for every t where the solution is well defined.

Proof. Consider the function $\tau(t) := 1 - 2\lambda t$ and the one-parameter family of vector fields $X_t := \frac{1}{\tau(t)}\nabla f_0$. Since ∇f_0 is a complete vector field, X_t generates a one-parameter family of diffeomorphisms $\phi(t, \cdot)$ defined for every t such that $\tau(t) > 0$. Now, define the metric $g(t) := \tau(t)\phi(t, \cdot)^*g_0$ and notice that

$$\frac{\partial}{\partial t}g(t) = \sigma'(t)\phi_t^*(g_0) + \sigma(t)\frac{\partial}{\partial t}\phi_t^*(g_0). \quad (2.41)$$

By the definition of the Lie derivative, we know that

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t, \cdot)^*(g_0) &= \mathcal{L}_{(\phi(t, \cdot)^{-1})^*(\frac{\partial}{\partial t}\phi(t, \cdot))}\phi(t, \cdot)^*(g_0) \\ &= \mathcal{L}_{\phi(t, \cdot)^*(\frac{\partial}{\partial t}\phi(t, \cdot))}\phi(t, \cdot)^*(g_0) \\ &= \mathcal{L}_{(\phi(t, \cdot)^*(\frac{1}{\tau(t)}\nabla f_0))}\phi(t, \cdot)^*(g_0). \end{aligned}$$

Hence,

$$\frac{\partial g(t)}{\partial t} = \tau'(t)\phi(t, \cdot)^*(g_0) + \tau(t)\mathcal{L}_{(\phi(t, \cdot)^*(\frac{1}{\tau(t)}\nabla f_0))}\phi(t, \cdot)^*(g_0). \quad (2.42)$$

On the other hand, since (M^n, g_0, f_0) is a ρ -Einstein soliton, compute

$$\begin{aligned} -2\text{Ric}(g(t)) &= -2\text{Ric}(\tau(t)\phi(t, \cdot)^*(g_0)) = -2\text{Ric}(\phi(t, \cdot)^*(g_0)) = -2\phi(t, \cdot)^*(\text{Ric}(g_0)) \\ &= -2\phi(t, \cdot)^*((\rho R + \lambda)g_0 - \frac{1}{2}\mathcal{L}_{\nabla f_0}g_0) \\ &= \tau'(t)\phi(t, \cdot)^*(g_0) + \mathcal{L}_{(\phi(t, \cdot)^*\nabla f_0)}\phi(t, \cdot)^*(g_0) - \frac{\rho}{\tau(t)}R(\tau(t)^{-1}g(t))g(t) \\ &= \tau'(t)\phi(t, \cdot)^*(g_0) + \tau(t)\mathcal{L}_{(\phi(t, \cdot)^*(\frac{1}{\tau(t)}\nabla f_0))}\phi(t, \cdot)^*(g_0) - \rho R(g(t))g(t), \end{aligned}$$

where the last equality follows from the fact that $R(\tau(t)^{-1}g(t)) = \tau(t)R(g(t))$. Combining this with (2.42) we conclude that $g(t)$ is in fact a self-similar solution to the Ricci-Bourguignon flow.

Now, notice that

$$\nabla^{g(t)}f(t, \cdot) = \phi(t, \cdot)^*\left(\frac{1}{\tau(t)}\nabla f_0\right), \quad (2.43)$$

which finishes the proof. \square

Remark 5. We point out that the existence of short time solutions to the Ricci-Bourguignon flow is only known for $-\infty < \rho < \frac{1}{2(n-1)}$. However, as far as the subject of our investigation are self-similar solutions, no constraint on ρ is required.

Proceeding, we recall special features established by Catino, Mazzieri and Mongodi [28] for ρ -Einstein solitons. The case of gradient Ricci solitons was proved by Hamilton [55].

Lemma 2 ([28]). *Let (M^n, g, f, λ) be a gradient ρ -Einstein soliton. Then the following equations hold:*

- 1) $\Delta f = (n\rho - 1)R + n\lambda$;
- 2) $(1 - 2(n - 1)\rho)\nabla R = 2Ric(\nabla f)$;
- 3) $(1 - 2(n - 1)\rho)\Delta R = \langle \nabla R, \nabla f \rangle + 2(\rho R^2 - |Ric|^2 + \lambda R)$.

Proof. By taking the trace of the fundamental equation (2.37), we discover that

$$R + \Delta f = \rho Rn + \lambda n.$$

Rearranging the terms we get the first assertion. For the second statement we need to take divergence in the soliton equation. To do so, recall that

$$div(\nabla^2 f) = \nabla \Delta f + Ric(\nabla f). \quad (2.44)$$

Hence, by taking the divergence in (2.37) and using the twice contracted Bianchi identity, we arrive at

$$\begin{aligned} \rho \nabla R &= \frac{1}{2} \nabla R + \nabla \Delta f + Ric(\nabla f) \\ &= \frac{1}{2} \nabla R + (n\rho - 1) \nabla R + Ric(\nabla f). \end{aligned}$$

Therefore

$$(1 - 2(n - 1)\rho)\nabla R = 2Ric(\nabla f),$$

which is precisely the second assertion. Finally, for the last assertion, notice that

$$\Delta R = div(\nabla R),$$

thus

$$\begin{aligned} (1 - 2(n - 1)\rho)\Delta R &= (1 - 2(n - 1)\rho)div(\nabla R) \\ &= 2div(Ric(\nabla f)) \\ &= 2\langle div Ric, \nabla f \rangle + 2\langle Ric, \nabla^2 f \rangle \\ &= \langle \nabla R, \nabla f \rangle + 2\langle Ric, (\rho R + \lambda)g - Ric \rangle \\ &= \langle \nabla R, \nabla f \rangle + 2(\rho R^2 + \lambda R - |Ric|^2). \end{aligned}$$

Hence, the proof is completed. □

In [18], Borges obtained some useful properties regarding the scalar curvature and the norm of the gradient of the potential function.

Proposition 4 ([18]). *Let (M^n, g, f, λ) be a complete noncompact Schouten soliton. Suppose that $\lambda > 0$ ($\lambda < 0$, respectively). Then the potential function f attains a global minimum (maximum, respectively) and it is unbounded from above (below, respectively). Furthermore, we have:*

$$0 \leq R\lambda \leq 2(n-1)\lambda^2$$

and

$$2\lambda(f - f_0) \leq |\nabla f|^2 \leq 4\lambda(f - f_0),$$

where $f_0 = \min_{p \in M} f(p)$ ($f_0 = \max_{p \in M} f(p)$, respectively).

For the case $\lambda > 0$, Catino and Mazzieri [30] established lower bounds for the scalar curvature using the Ricci-Bourguignon flow. To the best of our knowledge, there is no version of Proposition 4 for ρ -Einstein solitons in general.

In [18], Borges also obtained the following proposition concerning the asymptotic behavior of the potential function of a shrinking Schouten soliton.

Proposition 5 ([18]). *Let (M^n, g, f, λ) be a complete noncompact shrinking Schouten soliton. Then we have:*

$$\frac{\lambda}{4}(d(p, q) - A_1)^2 \leq f(p) - f_0 \leq \lambda(d(p, q) + A_2)^2,$$

where $f_0 = \min_{p \in M} f(p)$ and q is a point in M , A_1 and A_2 are positive constants depending only on λ and the unit ball $B_q(1)$ with $d(p, q) > 2$.

In the rest of this section, we are going to present some nontrivial examples of ρ -Einstein solitons; for more details, see, e.g., [1].

Example 1. *Consider the standard 1-dimensional hyperbolic space \mathbb{H} and let \mathbb{F}^2 be a complete 2-dimensional Ricci flat manifold. So, we choose $h(x) = \coth(x)$ and $f(x) = \frac{2}{3} \log(\cosh(x))$. Besides, $\mathbb{H} \times_h \mathbb{F}^2$ with the warping metric and potential function f satisfies*

$$\text{Ric} + \nabla^2 f = -\frac{2 + 4 \cosh(2x)}{3 \cosh^4(x)}.$$

Therefore, it defines a steady ρ -Einstein soliton with $\rho = \frac{1}{3}$ and nonconstant scalar curvature given by

$$R = -\frac{2 + 4 \cosh(2x)}{\cosh^4(x)}.$$

Reasoning as in the previous case, it is not hard to check the following example.

Example 2. Consider the metric $\cosh^2(x)\delta$ in \mathbb{R} and let \mathbb{F}^2 be a complete 2-dimensional Ricci flat manifold. By taking $h = \cosh(x)$ and $f = \frac{1}{12}(8 \log(\cosh(x) + \cos(2x)))$, one sees that $\mathbb{R} \times_h \mathbb{F}^2$ satisfies

$$\text{Ric} + \nabla^2 f = -\frac{3 - \sinh^4(x)}{3 \cosh^4(x)},$$

with h as warping function and f as potential function. So, $\mathbb{R} \times_h \mathbb{F}^2$ is a gradient shrinking ρ -Einstein soliton with $\rho = \lambda = \frac{1}{3}$ and scalar curvature given by

$$R = -\frac{3 + \cosh(2x)}{3 \cosh^4(x)}.$$

As it was pointed out by Agila and Gomes [1], Examples 1 and 2 have complete metrics. In the sequel, we present an example endowed with non-complete metric.

Example 3. Consider \mathbb{R}^n , $n \geq 3$ with coordinates $x = (x_1, \dots, x_n)$ and metric $g = e^{2\xi}\delta$, where $\xi = \sum_{i=1}^n \alpha_i x_i$ and $\sum_{i=1}^n \alpha_i^2 = 1$. Again, let \mathbb{F}^m be a Ricci flat manifold. Besides, choosing $h = e^\xi$ and $f = \frac{c}{2}e^\xi - \frac{(2-m-n)}{2}\xi$ as warping function and potential function, respectively. So, it is not difficult to check that $\mathbb{R}^n \times_h \mathbb{F}^m$ must satisfy

$$\text{Ric} + \nabla^2 f = c + \frac{2-m-n}{2}e^{-2\xi}.$$

Then $\mathbb{R}^n \times_h \mathbb{F}^m$ is a gradient Schouten soliton with $\lambda = c$. Furthermore, the scalar curvature is given by

$$R = -(m+n-2)(m+n-1)e^{-2\xi}.$$

Example 3 guarantees that the completeness hypothesis cannot be removed in the results obtained by Catino and Mazzieri [30, Theorem 1.5] and Borges [18, Theorem 1.1].

2.4 Critical metrics of the volume functional

In this section, we recall the background material required for our analysis. We begin by presenting fundamental properties and examples critical metrics of the volume functional,

for simplicity, V -static metrics. We then recall an auxiliary tensor that is related to those in the classical Riemannian geometry theory, but plays a central role in this work. Finally, we state several important results that will be used throughout the thesis.

Before presenting the preliminary results, we shall show how to derive the V -static equation from a variational viewpoint. Although this thesis focuses on manifolds without boundary, V -static metrics naturally arise in the context of compact manifolds with boundary; for that reason, we include this discussion to clarify their geometric origin. Therefore, we consider critical points of the volume functional restricted to metrics with constant scalar curvature and prescribed boundary metric, that is, the volume functional restricted to the space

$$\mathcal{M}_\gamma^R = \{g \text{ Riemannian metric}; R_g \text{ is constant and } g|_{\partial M} = \gamma\}. \quad (2.45)$$

To find critical points with pointwise constraints, we need to introduce a Lagrange multiplier function $\lambda \in C^\infty(M)$. We define the Lagrangian for the volume functional

$$L(g, \lambda) = \int_M [1 - \lambda(R_g - R_0)] dV_g.$$

We seek (g, λ) such that $\delta L = 0$ for all admissible variations. Consider $g(t) = g + th$, where h is a symmetric 2-tensor with $h|_{\partial M} = 0$ (this is important to preserve the condition $g|_{\partial M} = \gamma$). Let us compute the variation with respect to λ . We fix g and vary $\lambda \mapsto \lambda + t\eta$, then

$$\delta_\lambda L = - \int_M \eta (R_g - R_0) dV_g.$$

Thus, for a critical point, we must have $R_g \equiv R_0$ (this was expected since we are restricted to the space of constant scalar curvature). Now, we compute the variation in the direction of the metric, that is, $\delta_g L$.

$$\delta_g L = \int_M \delta(dV_g) - \lambda \delta(R_g) dV_g - \lambda (R_g - R_0) dV_g. \quad (2.46)$$

Applying Lemma 1 in (2.46) and using the already known fact that at a critical point we must have $R_g = R_0$, we arrive at

$$\delta_g L = \int_M \left[\frac{1}{2} (tr_g h) - \lambda (-\Delta(tr_g h) + div^2 h - \langle Ric, h \rangle) \right] dV_g. \quad (2.47)$$

Integrating by parts and using $h|_{\partial M} = 0$ to eliminate boundary terms, we find

$$\int_M \lambda (-\Delta(tr_g h) + div^2 h) dV_g = \int_M \langle \nabla^2 \lambda - (\Delta \lambda)g, h \rangle dV_g.$$

Thus, by writing the above equation in coordinates, we find the following.

$$\delta_g L = \int_M \left[\frac{1}{2} g_{ij} + (\Delta \lambda) g_{ij} - \nabla_i \nabla_j \lambda + \lambda R_{ij} \right] h_{ij} dV_g. \quad (2.48)$$

Since h_{ij} is arbitrary (up to our restriction), for a critical point the integrand must vanish pointwise; therefore,

$$\frac{1}{2} g_{ij} + (\Delta \lambda) g_{ij} - \nabla_i \nabla_j \lambda + \lambda R_{ij} = 0. \quad (2.49)$$

Setting $f = 2\kappa\lambda$ and multiplying (2.49) by -2κ , we conclude that

$$-(\Delta f)g + \nabla^2 f - fRic = \kappa g. \quad (2.50)$$

This leads to the following definition

Definition 7. Let (M^n, g) be a complete Riemannian manifold and f be a smooth function on M . We say that (M^n, g, f) is a *V-static metric* if it satisfies the following equation

$$-(\Delta f)g + \nabla^2 f - fRic = \kappa g. \quad (2.51)$$

Remark 6. We point out that this is the classical construction of V-static metrics, of which details can be found in the works of Corvino, Eichmair and Miao [41] and Miao and Tam [64]. More recently, McCormick [62] showed that the V-static metric also appears as critical points of the volume-renormalised mass when considering asymptotically hyperbolic manifolds.

V-static metrics are also closely related to Schoen's conjecture. Recall that Schoen's conjecture [75] asserts: Let (M^n, \bar{g}) be a closed hyperbolic manifold and let g be another metric on M^n with scalar curvature $R(g) \geq R(\bar{g})$, then $Vol(g) \geq Vol(\bar{g})$. Corvino, Eichmair, and Miao [41] employed this variational framework to establish a deformation result, which indicates that scalar curvature alone is not sufficient to establish a volume comparison result. In particular, their deformation result implies that Schoen's conjecture cannot be directly extended to manifolds with boundary under only Dirichlet boundary conditions (see also [80]) In fact, it follows from their work that, for a given constant κ , if a metric g does not admit a nontrivial solution f to the equation

$$\mathcal{L}_g^*(f) = -(\Delta f)g + \nabla^2 f - fRic = \kappa g, \quad (2.52)$$

then it is possible to simultaneously prescribe a compactly supported perturbation of the scalar curvature within a bounded domain Ω , and a prescribed perturbation of the volume, via a small deformation of the metric supported in $\bar{\Omega}$.

Now, from Definition 7, we may write (2.51) in coordinate notation as

$$-(\Delta f)g_{ij} + \nabla_i \nabla_j f - fR_{ij} = \kappa g_{ij}. \quad (2.53)$$

Taking the trace in (2.53), one sees that

$$\Delta f = -\frac{fR + \kappa n}{n-1}. \quad (2.54)$$

Moreover, it is easy to check that

$$f\mathring{Ric} = \mathring{\nabla}^2 f, \quad (2.55)$$

where $\mathring{Z} = Z - \frac{\text{tr}Z}{n}g$ stands for the traceless of tensor Z . As Barros, Diógenes and Ribeiro [10], we define the a function σ in M^n given by

$$\sigma = |\nabla f|^2 + \frac{2\kappa}{n-1}f + \frac{R}{n-1}f^2. \quad (2.56)$$

We claim that

$$\nabla \sigma = 2f\text{Ric}(\nabla f) \quad (2.57)$$

In fact, by taking the covariant derivative in (2.56) and using that R is constant (see Proposition 6 below) we conclude that

$$\frac{1}{2}\nabla \sigma = \nabla^2(\nabla f) + \frac{\kappa}{n-1}\nabla f + \frac{R}{n-1}f\nabla f. \quad (2.58)$$

Our claim then follows from (2.53) together with (2.54).

Remark 7. *It should be mentioned that, choosing appropriate coordinates, f and g are real analytic; see Proposition 2.1 in [41]. Consequently, the set of regular points of f is dense in M^n .*

A very important feature of V -static metrics is the fact that the scalar curvature is constant. In fact, this plays a central role in our classification result. Before proceeding to example, we present the proof of this property.

Proposition 6. *Let (M^n, g, f) be a V -static metric. Then the scalar curvature of (M^n, g) is constant.*

Proof. By taking the divergence on (2.51), we find

$$-\nabla \Delta f + \text{div}(\nabla^2 f) - \text{Ric}(\nabla f) - f\text{Ric} = 0.$$

Notice that, it yields from the Ricci identity that

$$-\nabla\Delta f + \operatorname{div}(\nabla^2 f) - \operatorname{Ric}(\nabla f).$$

Hence, the twice contracted Bianchi identity guarantees that

$$f\nabla R = 0. \quad (2.59)$$

Therefore, since the metric and the potential function are real analytic, we must have $f \equiv 0$ or $\nabla R \equiv 0$. Taking into account that f is not constant, one sees that $\nabla R \equiv 0$. Thus, the scalar curvature R is constant, which concludes the proof. \square

We now recall some examples of V -static metrics obtained by Miao–Tam [65] (see also Corino–Eichmair–Miao [41]) for complete manifolds without boundary. Let us start with the Euclidean space \mathbb{R}^n with its canonical metric.

Example 4 ([65]). Consider (\mathbb{R}^n, δ) , where δ is its canonical metric. Define the function

$$f(x) = \frac{1}{(n-1)} \left(A - \frac{\kappa}{2}|x|^2 \right), \quad (2.60)$$

where A is a constant. It is straightforward to verify that the triple $(\mathbb{R}^n, \delta, f)$ satisfies Definition 1 with $\kappa \neq 0$.

We now present a similar example as before on the standard sphere \mathbb{S}^n .

Example 5 ([65]). Consider (\mathbb{S}^n, g) the sphere equipped with its canonical metric g . Define the function f along a geodesic $\gamma(s)$ emanating from a point $p \in \mathbb{S}^n$ by

$$f(\gamma(s)) = \frac{1}{n-1} \left(A \cos r - \kappa \right), \quad (2.61)$$

where r denotes the geodesic distance from the point p , such that $\nabla f(p) = 0$, and A is a constant. Thus, (\mathbb{S}^n, g, f) is a V -static metric.

Reasoning as in the spherical case, we have the following example on the hyperbolic space \mathbb{H}^n .

Example 6 ([65]). Consider $(\mathbb{H}^n, g_{\mathbb{H}^n})$ the hyperbolic space with its canonical metric and define the function

$$f(\gamma(s)) = \frac{1}{n-1} \left(\kappa - A \cosh r \right), \quad (2.62)$$

where $\gamma(s)$ is a geodesic emanating from a fixed point $p \in \mathbb{H}^n$ such that $\nabla f(p) = 0$, and A is a constant. Hence, $(\mathbb{H}^n, g_{\mathbb{H}^n}, f)$ is a V -static metric.

The next example is constructed using a warped product metric.

Example 7 ([65]). Consider $\mathbb{R} \times \Sigma^{n-1}$ equipped with the warped product metric $g = dt^2 + \cosh^2 t g_0$, where (Σ^{n-1}, g_0) is complete, Einstein with $\text{Ric}_{g_0} = -(n-2)g_0$, and define the potential function $f(t, x) = \frac{\kappa}{n-1} (A \sinh t + 1)$, and $A > 0$ is constant. Thus, $(\mathbb{R} \times \Sigma^{n-1}, g, f)$ is a V -static metric.

Aligned with the main motivation for V -static metrics, we also present examples for V -static metrics on compact manifolds with nonempty boundary. Let us start with (\mathbb{R}^n, g) , where g is the canonical metric.

Example 8 ([65]). Let Ω be a Euclidean ball in \mathbb{R}^n of radius r_0 . Define

$$f(x) := \frac{\kappa}{2(n-1)} (r_0^2 - |x|^2),$$

where $x \in \mathbb{R}^n$. It is not hard to check that (Ω, g, f) satisfies the V -static equation.

Reasoning as before, we present a similar example on the standard sphere (\mathbb{S}^n, g_0) , where g_0 is the round metric.

Example 9 ([65]). Let Ω be a geodesic ball in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with radius $r_0 \neq \frac{\pi}{2}$, such that Ω is contained in a hemisphere. Suppose that

$$f(x) := \frac{\kappa}{n-1} \left(\frac{\cos r(x)}{\cos r_0} - 1 \right),$$

where $r(x)$ is the geodesic distance from the pole $(0, \dots, 0, 1)$. In this setting, one can see that (Ω, g_0, f) is a V -static metric.

As in the spherical case, it is possible to build an example on the hyperbolic space \mathbb{H}^n .

Example 10 ([65]). Consider the hyperbolic space \mathbb{H}^n embedded in \mathbb{R}^{n+1} , the Minkowski space, with the standard metric $g = dx_1^2 + \dots + dx_n^2 - dt^2$, such that

$$\mathbb{H}^n = \{(x_1, \dots, x_n, t); x_1^2 + \dots + x_n^2 - t^2 = -1, t > 0\}$$

Now, let Ω be a geodesic ball of radius r_0 centered at $(0, \dots, 0, 1)$. Consider

$$f(x) := \frac{\kappa}{n-1} \left(1 - \frac{\cosh r(x)}{\cosh r_0} \right),$$

where $r(x)$ is the geodesic distance from the center $(0, \dots, 0, 1)$. As in the previous example, it is not hard to check that (Ω, g, f) is a V -static critical metric.

Finally, we present another important example by Miao and Tam [65, Corollary 3.1]. We highlight that [65, Corollary 3.2] guarantees the existence of a solution for the V -static equation in a complete spatial Ads-Schwarzschild manifold with positive mass.

Example 11 ([65]). *Let (M^n, g_{Sch}) be a complete spatial Schwarzschild manifold with positive mass, that is $M = \mathbb{R}^n \setminus B_{2m}$, where $m > 0$ is the mass parameter, and*

$$g_{Sch} = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}.$$

We consider the domain $\Omega_R = \{x \in M; 2m < r \leq |x| \leq R\}$. This domain, equipped with the Schwarzschild metric admits a potential function f that satisfies the V -static equation. In this case, f is the solution of the following ODE

$$\left(1 - \frac{2m}{r^{n-2}}\right) f'' + \left(\frac{n-1}{r} - \frac{m(n-1)}{r^{n-1}}\right) f' = -\frac{\kappa n}{n-1}.$$

In order to proceed, we establish a few properties of compact V -static manifolds without boundary. As a first step, we show that such manifolds must necessarily have positive scalar curvature (cf. [64, Theorem 7]).

Proposition 7. *Let (M^n, g, f) be a compact V -static manifold without boundary. Then (M^n, g) has positive scalar curvature.*

Proof. Since (M^n, g) has constant scalar curvature, we first assume that $R = 0$. In this case, it follows from (2.54) that $(n-1)\Delta f = -\kappa n$ and hence, by the Maximum Principle, f is constant. Now, suppose that $R < 0$. Thus, we can choose points $p, q \in M$ such that $f(q) = \min f$ and $f(p) = \max f$. Then, one sees that

$$-Rf(q) - \kappa n \leq -Rf - \kappa n \leq -Rf(p) - \kappa n,$$

and by (2.54), we have

$$0 \leq \Delta f(q) \leq \Delta f \leq \Delta f(p) \leq 0.$$

Thus, $\Delta f = 0$ and therefore, f must be constant. This finishes the proof. \square

Next, we shall show that if the potential function of a compact V -static manifold without boundary does change sign, then it is necessarily Example 5.

Proposition 8. *Let (M^n, g, f) be a compact V -static manifold without boundary. Suppose that f does not change sign. Then (M^n, g) is isometric to the standard sphere \mathbb{S}^n .*

Proof. Notice that

$$\operatorname{div}(\mathring{Ric}(\nabla f)) = (\operatorname{div}\mathring{Ric})(\nabla f) + \langle \mathring{Ric}, \mathring{\nabla}^2 f \rangle.$$

By the twice contracted second Bianchi identity ($2\operatorname{div}Ric = \nabla R$), one sees that $\operatorname{div}\mathring{Ric} = 0$. Hence, it follows from (2.55) that

$$\operatorname{div}(\mathring{Ric}(\nabla f)) = f|\mathring{Ric}|^2. \quad (2.63)$$

Upon integrating (2.63) over M^n , we apply the Stokes' theorem combined with the fact that M^n has no boundary in order to infer

$$\int_M f|\mathring{Ric}|^2 dV_g = 0.$$

Since f does not change sign, we deduce that $\mathring{Ric} \equiv 0$, and hence the manifold is Einstein. Therefore, by applying [65, Theorem 2.2], we conclude that (M^n, g) is isometric to the standard sphere \mathbb{S}^n , as stated. \square

Now, in the context of a V -static metric (M^n, g, f) , it is important to recall an auxiliary 3-tensor T introduced by Barros–Diógenes–Ribeiro [10] and inspired by the tensor D defined by Cao–Chen [24] in the study of Ricci solitons. More precisely, given an n -dimensional V -static metric (M^n, g, f) , the tensor T_{ijk} is given by

$$\begin{aligned} T_{ijk} = & \frac{n-1}{n-2} \left(R_{ik} \nabla_j f - R_{jk} \nabla_i f \right) - \frac{R}{n-2} \left(g_{ik} \nabla_j f - g_{jk} \nabla_i f \right) \\ & + \frac{1}{n-2} \left(g_{ik} R_{js} \nabla_s f - g_{jk} R_{is} \nabla_s f \right). \end{aligned} \quad (2.64)$$

Notice that T_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices.

It turns out that the tensor T can be expressed in terms of the Weyl and Cotton tensors as follows (cf. [10, Lemma 2]).

Lemma 3 ([10]). *Let (M^n, g, f) be an n -dimensional V -static metric. Then we have:*

$$fC_{ijk} = T_{ijk} + W_{ijkl} \nabla_l f. \quad (2.65)$$

The tensor T is closely related to the geometry of the level surfaces Σ_c of the potential function f , namely, $\Sigma_c = \{x \in M : f(x) = c\}$. This relation is summarized in the last two results of this chapter, originally obtained by Barros–Diógenes–Ribeiro (see [10, Proposition 1 and Lemma 3]).

Lemma 4 ([10]). *Let (M^n, g, f) be a V -static metric. Let $\Sigma = \{f = f(p)\}$ be a level set of f . If g_{ab} denotes the induced metric on Σ , then, at any point where $\nabla f \neq 0$, we have*

$$|fT|^2 = \frac{2(n-1)^2}{(n-2)^2} |\nabla f|^4 \sum_{a,b=2}^n \left| h_{ab} - \frac{H}{n-1} g_{ab} \right|^2 + \frac{n-1}{2(n-2)} |\nabla^\Sigma \sigma|^2, \quad (2.66)$$

where σ is given by (2.56), h_{ab} and H are the second fundamental form and the mean curvature of Σ , while ∇^Σ is the Riemannian connection of Σ .

Assuming that T vanishes identically, we obtain the following interesting properties, which is a central key to our analysis.

Proposition 9 ([10]). *Let (M^n, g, f) be an n -dimensional V -static metric with $T \equiv 0$. Let c be a regular value of f and $\Sigma_c = \{x \in M : f(x) = c\}$ be a level set of f . Consider $e_1 := \frac{\nabla f}{|\nabla f|}$ and choose an orthonormal frame $\{e_2, \dots, e_n\}$ tangent to Σ_c . Then the following assertions hold:*

- (1) *The second fundamental form $h_{\alpha\beta}$ of Σ_c is $h_{\alpha\beta} = \frac{H}{n-1} g_{\alpha\beta}$;*
- (2) *$|\nabla f|$ is constant on Σ_c ;*
- (3) *$R_{1\alpha} = 0$ for any $\alpha \geq 2$ and e_1 is an eigenvector of Ric;*
- (4) *The mean curvature of Σ_c is constant;*
- (5) *On Σ_c , the Ricci tensor either has a unique eigenvalue or two distinct eigenvalues with multiplicity 1 and $n-1$, respectively. Moreover, the eigenvalue with multiplicity 1 is in the direction of ∇f ;*
- (6) *$R_{1\alpha\beta\gamma} = 0$, $\alpha, \beta, \gamma \in \{2, \dots, n\}$.*

We shall present the proof of this proposition for the sake of completeness.

Proof. To begin with, note that the first two items follow directly from Lemma 4 together with (2.56). In order to prove (3), notice that, since $T \equiv 0$, (2.64) yields

$$\begin{aligned} 0 &= T(e_i, \nabla f, \nabla f) \\ &= Ric(e_i, \nabla f) |\nabla f|^2 - Ric(\nabla f, \nabla f) \langle \nabla f, e_i \rangle, \end{aligned}$$

so that

$$Ric(e_i, \nabla f) |\nabla f|^2 = Ric(\nabla f, \nabla f) \langle \nabla f, e_i \rangle.$$

So, for $a \geq 2$, we infer $R_{1a} = 0$. Moreover $Ric(e_1) = g^{ij} R_{1j} e_i = R_{11} e_1$. Hence, $e_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector for the Ricci tensor, which proves the third assertion.

Proceeding, we recall that the Codazzi equation gives

$$R_{1abc} = \nabla_b^\Sigma h_{ca} - \nabla_c^\Sigma h_{ba}, \quad a, b, c = 2, \dots, n. \quad (2.67)$$

By taking the trace in (2.67) with respect to indices a and c we get

$$\begin{aligned} R_{1c} &= \nabla_b^\Sigma H - g^{ac} \nabla_c^\Sigma h_{ab} \\ &= \nabla_b^\Sigma H - g^{ac} \nabla_c^\Sigma \frac{H}{n-1} g_{ba} \\ &= \frac{n-2}{n-1} \nabla_b^\Sigma H, \end{aligned}$$

where the second equality follows from the first assertion of the proposition. Now, since $R_{1b} = 0$, we conclude that H is constant on Σ , thus (4) is proved. Moving forward, since $e_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector of Ric , we may choose a frame

$$\{e_1 = \frac{\nabla f}{|\nabla f|}, e_2, \dots, e_n\}$$

diagonalizing the Ricci tensor such that $Ric(e_k) = \lambda_k e_k$ for $k = 1, \dots, n$. Using that $T \equiv 0$ once again, we infer that, for every $a, b \geq 2$,

$$\begin{aligned} 0 &= T_{a1b} = \frac{n-1}{n-2} (R_{ab} \nabla_1 f - R_{b1} \nabla_a f) - \frac{R}{n-2} (g_{ab} \nabla_1 f - g_{1b} \nabla_a f) \\ &\quad + \frac{1}{n-2} (g_{ab} R_{1s} \nabla^s f - g_{1b} R_{as} \nabla^s f) \\ &= \frac{n-1}{n-2} R_{ab} |\nabla f| - \frac{R}{n-2} |\nabla f| + \frac{1}{n-2} g_{ab} \lambda_1 |\nabla f|. \end{aligned}$$

It directly follows from this computation that

$$R_{ab} = \frac{R - \lambda_1}{n-1} g_{ab}$$

and then

$$\lambda_2 = \dots = \lambda_n = \frac{R - \lambda_1}{n-1},$$

which gives (5). Finally, the last assertion is straightforward from (2.67) together with assertions (1) and (4) of the proposition. So, we complete the proof of the proposition. \square

3 GEOMETRIC AND ANALYTICAL RESULTS FOR ρ -EINSTEIN SOLITONS

This chapter presents results from the paper *Geometric and analytical results for ρ -Einstein solitons* [38]. The primary objective of the aforementioned paper was to establish interesting properties—from both geometric and analytical perspectives—for ρ -Einstein solitons. Accordingly, this chapter is divided into two parts. First, we develop analytical results for this class of solitons, focusing on spectral properties of the drifted Laplacian operator. Second, we investigate some geometric aspects, particularly the volume growth of geodesic balls. Before proceeding to the proof of the main results, we present a brief introduction and motivation regarding our subjects of research.

For a given $\rho \in \mathbb{R}$, we say that (M^n, g, f, λ) is a gradient ρ -Einstein soliton if it satisfies the equation

$$\text{Ric} + \nabla^2 f = (\rho R + \lambda)g. \quad (3.1)$$

Following the terminology used for Ricci solitons, we say that the ρ -Einstein soliton is *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. When $\rho = \frac{1}{2(n-1)}$, it is called *Schouten soliton*; see [30]. Moreover, note that the gradient Ricci soliton equation is obtained when $\rho = 0$ in (3.1).

Gradient ρ -Einstein solitons were first introduced in [30]. According to the work of Catino, Mazzieri and Mongodi [28], these solitons arise as self-similar solutions to the Ricci-Bourguignon flow, originally introduced by Bourguignon in [20],

$$\frac{\partial}{\partial t} g_t = -2(\text{Ric}_{g_t} - \rho R g_t);$$

see also [31]. In the last years, several topological, geometric and analytical features concerning Schouten solitons have also been proven. For instance, Catino and Mazzieri [30] showed that a complete steady Schouten soliton must be Ricci flat. Moreover, they proved that 3-dimensional Schouten solitons are isometric to $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. While Borges [19] classified gradient Schouten solitons with vanishing Bach tensor. Furthermore, he [18] obtained interesting results regarding the asymptotic behavior of the potential function f and the norm of its gradient.

3.1 A Lichnerowicz-Obata theorem

As previously mentioned, the first part of this work aims to establish analytical properties of ρ -Einstein solitons. Particularly, we are interested in developing a Lichnerowicz-

Obata type theorem for these manifolds. Instead of the usual Ricci tensor, we may consider the Bakry-Émery Ricci tensor given by

$$Ric_f := Ric + \nabla^2 f$$

where $\nabla^2 f$ stands for the hessian of the potential function f . However, it is important to note that, in this case, certain topological differences arise. For instance, the Bonnet-Myers theorem does not hold under the assumption that $Ric_f \geq \delta > 0$ when f is non-constant, and the manifold may fail to be compact, as exemplified by the Gaussian shrinking soliton $(\mathbb{R}^n, g_{can}, f(x) = \frac{|x|^2}{4})$.

Combined, the works of Bakry-Émery [4], Morgan [66], Hein-Naber [57] highly advanced in the spectral properties for smooth metric measurement spaces, which led to the work of Cheng and Zhou [34], where they proved that the drifted Laplacian operator has discrete spectrum and were able to provide a Lichnerowicz-Obata type theorem for these manifolds. Motivated by their works, we investigate if this result is valid for ρ -Einstein solitons. More precisely, we have the following result.

Theorem 10. *Let (M^n, g, f, λ) be a complete gradient shrinking ρ -Einstein soliton with $\rho > 0$ and nonnegative scalar curvature. Then the following assertions hold:*

1. *The spectrum of Δ_f is discrete;*
2. *$\lambda_1(\Delta_f) \geq \lambda$;*
3. *Equality holds in assertion (2) if and only if (M^n, g, f) is $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$, i.e., the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{|x|^2}{4})$, up to scaling.*

It is worth noting that the discreteness and lower bound of the spectral gap also follow from Theorem 1. However, a key advantage of our approach lies in the rearrangement of its proof, which allows us to establish the rigidity case.

As a consequence of Theorem 10 combined with the scalar curvature estimates proved by Catino and Mazzieri [30, Corollary 5.2] and Borges [18, Theorem 1.1], we get the following corollary for Schouten solitons.

Corollary 4. *Let (M^n, g, f, λ) be a complete gradient shrinking Schouten soliton. Then the following assertions hold:*

1. *The spectrum of Δ_f is discrete;*
2. *$\lambda_1(\Delta_f) \geq \lambda$;*
3. *Equality holds in assertion (2) if and only if (M^n, g, f) is $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$, i.e., the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{|x|^2}{4})$, up to scaling.*

Now we present the proof of Theorem 10

Proof of Theorem 10. To begin with, we consider an eigenfunction of the drifted Laplacian operator Δ_f , i.e.,

$$\Delta_f u + \delta u = 0, \quad \int_M u^2 e^{-f} dV < \infty,$$

where $u \in H^1(M, \mu) \cap C^\infty(M)$. From the weighted Böchner formula

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla(\Delta_f u) \rangle + Ric_f(\nabla u, \nabla u),$$

one sees that

$$\begin{aligned} \frac{1}{2} \Delta_f |\nabla u|^2 &= |\nabla^2 u|^2 - \delta |\nabla u|^2 + (\rho R + \lambda) |\nabla u|^2 \\ &= |\nabla^2 u|^2 + (\lambda - \delta) |\nabla u|^2 + \rho R |\nabla u|^2. \end{aligned} \quad (3.2)$$

From now on, we adapt some ideas outlined by Cheng and Zhou in [34] to prove the lower bound estimate. Fix a point $p \in M$ and let $B_p(r)$ be the geodesic ball centered at p with radius r . Moreover, choose a cut-off function ϕ such that $\phi \equiv 1$ in B_r , $\phi \equiv 0$ outside B_{r+1} and $|\nabla \phi| \leq 1$. Besides, multiplying (3.2) by ϕ^2 and integrating over M , one obtains that

$$\begin{aligned} \frac{1}{2} \int_M \phi^2 \Delta_f |\nabla u|^2 d\mu &= \int_M \phi^2 |\nabla^2 u|^2 d\mu + (\lambda - \delta) \int_M \phi^2 |\nabla u|^2 d\mu \\ &\quad + \rho \int_M \phi^2 R |\nabla u|^2 d\mu, \end{aligned} \quad (3.3)$$

where $d\mu = e^{-f} dV$.

Next, by the weighted Green formula, one deduces that

$$\begin{aligned} \int_M \phi^2 \Delta_f |\nabla u|^2 d\mu &= - \int_M \langle \nabla \phi^2, \nabla |\nabla u|^2 \rangle d\mu \\ &= -4 \int_M \phi \langle \nabla_{\nabla \phi} \nabla u, \nabla u \rangle d\mu \\ &= -4 \int_M \phi \nabla^2 u(\nabla \phi, \nabla u) d\mu. \end{aligned} \quad (3.4)$$

At the same time, from Young's inequality, we get

$$\begin{aligned}
-2\phi(\nabla^2 u)(\nabla\phi, \nabla u) &= -2 \sum_{i,j=1}^n \phi(\nabla^2 u)_{ij} \nabla_i \phi \nabla_j u \\
&\leq \sum_{i,j=1}^n \left(\varepsilon \phi^2 (\nabla^2 u)_{ij}^2 + \frac{1}{\varepsilon} (\nabla_i \phi)^2 (\nabla_j u)^2 \right) \\
&= \varepsilon \phi^2 |\nabla^2 u|^2 + \frac{1}{\varepsilon} |\nabla\phi|^2 |\nabla u|^2.
\end{aligned} \tag{3.5}$$

Plugging (3.5) into (3.4), we obtain

$$\int_M \phi^2 \Delta_f |\nabla u|^2 d\mu \leq 2\varepsilon \int_M \phi^2 |\nabla^2 u|^2 d\mu + \frac{2}{\varepsilon} \int_M |\nabla\phi|^2 |\nabla u|^2 d\mu. \tag{3.6}$$

This jointly with (3.3) gives

$$\begin{aligned}
(1-\varepsilon) \int_{B_{r+1}} \phi^2 |\nabla^2 u|^2 d\mu &\leq \frac{1}{\varepsilon} \int_{B_{r+1}} |\nabla\phi|^2 |\nabla u|^2 d\mu + (\delta - \lambda) \int_{B_{r+1}} \phi^2 |\nabla u|^2 d\mu \\
&\quad - \rho \int_{B_{r+1}} \phi^2 R |\nabla u|^2 d\mu \\
&\leq \frac{1}{\varepsilon} \int_{B_{r+1}} |\nabla\phi|^2 |\nabla u|^2 d\mu + (\delta - \lambda) \int_{B_{r+1}} \phi^2 |\nabla u|^2 d\mu.
\end{aligned}$$

Since the right hand side of the above expression is integrable, letting $r \rightarrow \infty$, we obtain that $\int_M |\nabla^2 u|^2 d\mu < \infty$.

Also, by Young's inequality, one concludes that

$$\begin{aligned}
|2\phi(\nabla^2 u)(\nabla\phi, \nabla u)| &= \left| 2 \sum_{i,j=1}^n \phi(\nabla^2 u)_{ij} \nabla_i \phi \nabla_j u \right| \\
&\leq \sum_{i,j=1}^n \varepsilon \phi^2 (\nabla^2 u)_{ij}^2 |\nabla_i \phi| + \frac{1}{\varepsilon} |\nabla_i \phi| (\nabla_j u)^2 \\
&\leq \varepsilon \phi^2 |\nabla^2 u|^2 |\nabla\phi| + \frac{1}{\varepsilon} |\nabla\phi| |\nabla u|^2,
\end{aligned}$$

which combined with (3.4) yields

$$\begin{aligned}
\left| \int_M \phi^2 \Delta_f |\nabla u|^2 d\mu \right| &\leq 2\varepsilon \int_M \phi^2 |\nabla^2 u|^2 |\nabla\phi| d\mu + \frac{2}{\varepsilon} \int_M |\nabla\phi| |\nabla u|^2 d\mu \\
&\leq 2\varepsilon \int_{B_{r+1} \setminus B_r} |\nabla^2 u|^2 d\mu + \frac{2}{\varepsilon} \int_{B_{r+1} \setminus B_r} |\nabla u|^2 d\mu.
\end{aligned} \tag{3.7}$$

Letting $r \rightarrow \infty$, and noting that the right hand side of (3.7) is integrable, we conclude that it converges to zero. Thus,

$$\int_M \Delta_f |\nabla u|^2 d\mu = 0.$$

Therefore, by using (3.3), we arrive at

$$0 = \int_M |\nabla^2 u|^2 d\mu + (\lambda - \delta) \int_M |\nabla u|^2 d\mu + \rho \int_M R |\nabla u|^2 d\mu. \quad (3.8)$$

Taking into account that u is not constant, we conclude that $\delta \geq \lambda$, which proves the asserted lower bound.

Now, we need to analyze the equality case, i.e., $\delta = \lambda$. To do so, we claim that if equality holds, then $R \equiv 0$. We argue by contradiction by supposing that $\delta = \lambda$ and $R \not\equiv 0$. In this situation, since $R \geq 0$, there exists a point $x \in M$ such that $R(x) > 0$. Hence, from (3.8), we have $\nabla^2 u \equiv 0$, which implies that ∇u is a nontrivial parallel vector field. Therefore, M^n must be isometric to a product manifold $M^n = \Sigma^{n-1} \times \mathbb{R}$, for some complete manifold Σ^{n-1} . In particular, the function u is constant on the level set $\Sigma \times \{t\}$, with $t \in \mathbb{R}$, and $|\nabla u|$ only depends on \mathbb{R} . Moreover, the scalar curvature of M^n only depends on the scalar curvature of Σ^{n-1} . In another words, we have

$$R|\nabla u|^2(p, q) = R_\Sigma(p)|\nabla u|^2(q),$$

where $(p, q) \in \Sigma^{n-1} \times \mathbb{R}$, R stands for the scalar curvature of M^n and R_Σ is the scalar curvature of Σ^{n-1} . Besides, taking into account that $R(x) = R_\Sigma(p)$, where $x = (p, q) \in \Sigma^{n-1} \times \mathbb{R}$, one sees that $R_\Sigma(p) > 0$. At the same time, since $\rho R \geq 0$ and $\delta = \lambda$, we have from (3.8) that

$$0 = R|\nabla u|^2(p, s) = R_\Sigma(p)|\nabla u|^2(s), \quad \forall s \in \mathbb{R}. \quad (3.9)$$

Therefore, $|\nabla u|^2(s) = 0$, $\forall s \in \mathbb{R}$, which contradicts the fact that ∇u is a nontrivial vector field and then, such a point x does not exist. Consequently, we have $R \equiv 0$.

To conclude, since $R \equiv 0$, we obtain that (M^n, g, f, λ) must satisfy $\text{Ric} + \nabla^2 f = \lambda g$, then it is a gradient shrinking Ricci soliton with zero scalar curvature, it follows from [33, 74] that (M^n, g, f, λ) is the Gaussian shrinking soliton up to scaling. This concludes the proof of the theorem. \square

3.2 Volume growth estimates for geodesic balls

A precise understanding of the volume growth rate is a key geometric property from which various other features of the underlying Riemannian manifold can be derived. A classical

theorem due to Calabi [22] and Yau [79] asserts that the geodesic balls of complete noncompact manifolds with nonnegative Ricci curvature have at least linear volume growth, i.e.,

$$\text{Vol}(B_p(r)) \geq cr,$$

where r is the distance function and c is a constant. While the classical Bishop volume comparison theorem [16] guarantees that the geodesic balls of complete noncompact Riemannian manifolds with nonnegative Ricci curvature have at most polynomial volume growth. In the last years, similar results were obtained for gradient Ricci solitons in [23],[32],[67],[68] and quasi-Einstein manifolds in, e.g., [11],[12],[36]. Recently, Borges [18] adapted some ideas developed by Cao and Zhou [23] in order to prove volume growth estimates for Schouten solitons. To be precise, he showed that given a complete noncompact shrinking Schouten soliton, there are positive constants C_1, C_2 and r_0 , depending only on λ and n , such that

$$C_1 r^{\frac{n}{2} - \frac{(n-2)\theta}{4(n-1)\lambda}} \leq \text{Vol}(B_q(r)) \leq C_2 r^{n - \frac{(n-2)\delta}{2(n-1)\lambda}},$$

for any $r > r_0$, where $q \in M$, $\delta = \inf_{p \in M} R(p)$ and $\theta = \sup_{p \in M} R(p)$. The proof of this result relies on the behavior of the potential function f . We highlight that it is not known whether the same approach is valid for ρ -Einstein solitons in general. In the same context, Munteanu and Wang [69, Theorem 1.4] showed that any n -dimensional smooth metric measure space $(M^n, g, e^{-f} dv)$ with $\text{Ric}_f \geq \frac{1}{2}$ and $|\nabla f|^2 \leq f$ must satisfy

$$\text{Vol}(B_p(r)) \leq c(n)r^n,$$

for every $r > 0$, where $c(n)$ is a constant depending only on the dimension of the manifold. While Cheng, Ribeiro and Zhou [36, Theorem 5] obtained the precise value of the constant $c(n)$ for gradient shrinking Ricci solitons. Similar estimates are interesting for ρ -Einstein solitons. Here, by adapting some techniques outlined in [36], we establish the following volume growth estimate for geodesic balls of complete noncompact ρ -Einstein solitons.

Theorem 11. *Let (M^n, g, f) be a complete noncompact n -dimensional gradient shrinking ρ -Einstein soliton with $\rho > 0$ and $R \geq 0$. Then, for all $r > 0$, the volume of the geodesic ball $B_p(r)$ must satisfy*

$$\text{Vol}(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\Phi} r^{n-1} dr d\theta, \quad (3.10)$$

where

$$\Phi = -\frac{\lambda r^2}{6} + f(\theta, r) + f(p) - \frac{2}{r} \int_0^r f(\theta, s) ds.$$

Moreover, equality holds in (3.10) for all $r > 0$ if and only if (M^n, g, f) is $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda|x-p|^2}{2})$, i.e., the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{|x|^2}{4})$, up to scaling and translation.

Remark 8. We point out that Theorem 10 and Theorem 11 are also true if we assume $\rho < 0$ and $R \leq 0$, with a minor difference in the rigidity part of Theorem 10. In particular, the proof of this fact is essentially the same.

Proof of Theorem 11. Here, we adapt some ideas used by Cheng, Ribeiro and Zhou in [36], see also [59]. To begin with, multiplying (2.24) by r^2 and integrating from ε to r , we get

$$\int_{\varepsilon}^r t^2 w' dt + \frac{1}{n-1} \int_{\varepsilon}^r t^2 w^2 dt + \int_{\varepsilon}^r t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt \leq 0.$$

This implies

$$\int_{\varepsilon}^r (t^2 w)' dt \leq \int_{\varepsilon}^r \left(2tw - \frac{1}{n-1} t^2 w^2\right) dt - \int_{\varepsilon}^r t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt.$$

Now, letting $\varepsilon \rightarrow 0$, it follows that

$$\begin{aligned} r^2 w &\leq \int_0^r \left(2tw - \frac{1}{n-1} t^2 w^2\right) dt - \int_0^r t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt \\ &= \int_0^r \left(-\frac{1}{n-1} (tw - (n-1))^2 + (n-1)\right) dt - \int_0^r t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt \\ &\leq \int_0^r (n-1) - t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt. \end{aligned}$$

Thus, one deduces that

$$\left(\log\left(\frac{J(\theta, r)}{r^{n-1}}\right)\right)' \leq -\frac{1}{r^2} \int_0^r t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt. \quad (3.11)$$

Again, by integrating (3.11) from ε to r , we arrive at

$$\int_{\varepsilon}^r \left(\log\frac{J(\theta, r)}{s^{n-1}}\right)' ds \leq -\int_{\varepsilon}^r \left[\frac{1}{s^2} \int_0^s t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt\right] ds.$$

Since $\lim_{r \rightarrow 0} \frac{J(\theta, r)}{r} = 1$, by letting $\varepsilon \rightarrow 0$ and integrating by parts, we obtain

$$\log\left(\frac{J(\theta, r)}{r^{n-1}}\right) \leq \frac{1}{r} \int_0^r t^2 Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt - \int_0^r t Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt. \quad (3.12)$$

Hence, combining (3.11) and (3.12), we get

$$\left(r \log\frac{J(\theta, r)}{r^{n-1}}\right)' \leq -\int_0^r t Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) dt. \quad (3.13)$$

On the other hand, since M^n is a gradient ρ -Einstein soliton, we have

$$Ric\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \rho R + \lambda - f''(t),$$

where $f(t) = f(\gamma(t))$, $0 \leq t \leq r$, and γ is a minimizing geodesic with $\gamma(0) = p$. Thus, (3.13) becomes

$$\begin{aligned} \left(r \log \frac{J(\theta, r)}{r^{n-1}} \right)' &\leq - \int_0^r t(\rho R + \lambda - f''(t)) dt \\ &\leq - \frac{\lambda r^2}{2} + r f'(r) - f(r) + f(0) - \int_0^r t \rho R dt. \end{aligned}$$

From this, it follows that

$$\begin{aligned} \left(r \log \frac{J(\theta, r)}{r^{n-1}} \right) &\leq - \frac{\lambda r^3}{6} + \int_0^r t f'(t) dt - \int_0^r f(t) dt + r f(0) - \int_0^r \int_0^s t \rho R dt ds \\ &= - \frac{\lambda r^3}{6} + \int_0^r (t f)'(t) dt - 2 \int_0^r f(t) dt + r f(0) - \int_0^r \int_0^s t \rho R dt ds \quad (3.14) \\ &= - \frac{\lambda r^3}{6} + r f(r) - 2 \int_0^r f(t) dt + r f(0) - \int_0^r \int_0^s t \rho R dt ds. \end{aligned}$$

Thus, by using the hypothesis, one deduces that

$$\left(r \log \frac{J(\theta, r)}{r^{n-1}} \right) \leq - \frac{\lambda r^3}{6} + r f(r) + r f(0) - 2 \int_0^r f(t) dt, \quad (3.15)$$

which yields

$$J(\theta, r) \leq e^{(-\frac{\lambda r^2}{6} + f(r) + f(p) - \frac{2}{r} \int_0^r f(t) dt)} r^{n-1}.$$

Consequently,

$$\begin{aligned} \text{Vol}(B_p(r)) &= \int_{\mathbb{S}^{n-1}} \int_0^{\min\{r, \rho(\theta)\}} J(\theta, r) dr d\theta \\ &\leq \int_{\mathbb{S}^{n-1}} \int_0^{\min\{r, \rho(\theta)\}} e^{\Phi} r^{n-1} dr d\theta \\ &\leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\Phi} r^{n-1} dr d\theta, \end{aligned} \quad (3.16)$$

where

$$\Phi = - \frac{\lambda r^2}{6} + f(\theta, r) + f(p) - \frac{2}{r} \int_0^r f(\theta, s) ds$$

and $\rho(\theta)$ stands for the cut-locus radius in the direction of θ . Finally, if equality holds in (3.16), then it must also hold in (3.15). Therefore,

$$- \int_0^r \int_0^s t \rho R dt ds = 0,$$

for all $0 < r < \rho(\theta)$. Taking into account that $\rho R \geq 0$, for each point $p \in M$ there exists an open neighborhood U_p such that $R \equiv 0$ in U_p and hence, since p is arbitrary, one deduces that $R \equiv 0$

in M . Similar to the proof of the previous theorem, we conclude that (M^n, g, f, λ) must satisfy $Ric + \nabla^2 f = \lambda g$ with zero scalar curvature. Consequently, M^n is the Gaussian shrinking soliton (see [33, 74]). So, the proof is completed. \square

In general, as in the case of smooth metric measure spaces, it is also very important to obtain weighted volume growth estimates for geodesic balls for ρ -Einstein solitons. This is due to the fact that the existence of the potential function f provides us useful analytical information associated with the drifted Laplacian operator. In [1, Lemma 1], Agila and Gomes proved an estimate for the weighted volume of geodesic balls under suitable conditions involving the scalar curvature and the potential function. However, one can see that the integral bound derived in their work diverges as the radius of the geodesic ball tends to infinity. Therefore, it is interesting to obtain new (refined) volume growth estimates for such manifolds. As a consequence of the proof of Theorem 11, we have the following result.

Theorem 12. *Let (M^n, g, f) be a complete noncompact n -dimensional gradient ρ -Einstein soliton. Then we have*

$$Vol_f(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^\Psi r^{n-1} dr d\theta, \quad (3.17)$$

where

$$\Psi = -\frac{\lambda r^2}{6} + f(p) - \frac{2}{r} \int_0^r f(\theta, s) ds - \frac{1}{r} \int_0^r \int_0^s t \rho R dt ds.$$

Proof of Theorem 12. Initially, by using the volume form in polar coordinates

$$dV_{exp_p(r\theta)} = J(\theta, r) dr d\theta$$

for $\theta \in S_p M$ and a fixed point p , we may denote the weighted volume form as

$$dV_f = e^{-f(r, \theta)} J(r, \theta) dr d\theta.$$

It follows from the proof of Theorem 11 that

$$r \log \frac{J(\theta, r)}{r^{n-1}} \leq -\frac{\lambda r^3}{6} + r f(r) - 2 \int_0^r f(t) dt + r f(0) - \int_0^r \int_0^s t \rho R dt ds. \quad (3.18)$$

Consequently,

$$J(\theta, r) \leq e^\Psi r^{n-1},$$

where

$$\Psi = \left(-\frac{\lambda r^2}{6} + f(r) + f(p) - \frac{2}{r} \int_0^r f(t) dt - \frac{1}{r} \int_0^r \int_0^s t \rho R dt ds \right).$$

Equivalently,

$$J(\theta, r) e^{-f(r, \theta)} \leq e^\Psi r^{n-1}. \quad (3.19)$$

Hence, by integration, one sees that

$$\begin{aligned} \text{Vol}_f(B_p(r)) &= \int_{\mathbb{S}^{n-1}} \int_0^{\min\{r, \rho(\theta)\}} J(\theta, r) e^{-f(r, \theta)} dr d\theta \\ &\leq \int_{\mathbb{S}^{n-1}} \int_0^{\min\{r, \rho(\theta)\}} e^\Psi r^{n-1} dr d\theta \\ &\leq \int_{\mathbb{S}^{n-1}} \int_0^r e^\Psi r^{n-1} dr d\theta. \end{aligned} \quad (3.20)$$

This is precisely the statement of our theorem. Thus, the proof is completed. \square

Combining Theorem 12 and the potential function estimates obtained by Munteanu and Wang [70, Theorem 0.4], we get the following corollary.

Corollary 5. *Let (M^n, g, f) be a complete noncompact n -dimensional gradient shrinking ρ -Einstein soliton with $\rho R \geq \delta > -\lambda$. Then there exists $r_0 > 0$ such that, for all $r \geq r_0$,*

$$\text{Vol}_f(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\frac{\delta}{2}} r^{n-1} dr d\theta. \quad (3.21)$$

In other words, Corollary 5 asserts that the weighted volume of a complete noncompact gradient shrinking ρ -Einstein soliton with $\rho R \geq \delta > -\lambda$ is less than or equal to the weighted volume of the Gaussian shrinking soliton.

Remark 9. *We highlight that under the hypothesis of Corollary 5, it follows that*

$$\text{Ric} + \nabla^2 f \geq (\delta + \lambda)g,$$

which is a positive constant. In this situation, Wei and Wylie [78, Theorem 3.1] proved a similar estimate.

Proof of Corollary 5. To prove the corollary, it suffices to estimate the function Ψ . Indeed, since $\rho R \geq \delta$, we have

$$\text{Ric}_f \geq (\delta + \lambda)g.$$

So, by [70, Theorem 0.4], we get the following estimate

$$f(x) \geq \frac{\delta + \lambda}{2} r(x)^2 - ar(x),$$

for some positive constant a . This therefore implies that

$$-\frac{2}{r} \int_0^r f(r, \theta) dr d\theta \leq -\frac{(\delta + \lambda)r^2}{3} + ar. \quad (3.22)$$

On the other hand, our assumption implies

$$-\frac{1}{r} \int_0^r \int_0^s t \rho R dt ds \leq -\frac{\delta}{r} \int_0^r \int_0^s t dt ds = -\frac{\delta r^2}{6}. \quad (3.23)$$

Plugging (3.22) and (3.23) into the expression of Ψ in Theorem 12, we conclude that there exists a constant $c > 0$ such that

$$\Psi \leq -cr^2 + ar. \quad (3.24)$$

Finally, taking r_0 large enough, it follows that

$$\text{Vol}_f(B_p(r)) \leq \int_{\mathbb{S}^{n-1}} \int_0^r e^{\frac{r^2}{2}} r^{n-1} dr d\theta,$$

as asserted. □

4 CRITICAL METRICS OF THE VOLUME FUNCTIONAL ON COMPLETE MANIFOLDS

This chapter contains the results obtained in the article *Critical metrics of the volume functional on complete manifolds*, a joint work with R. Diógenes and E. Ribeiro Jr. [39]. Our central aim was to establish new rigidity theorems for V -static metrics on complete manifolds without boundary. In the first part of the aforementioned article, we address the problem of classifying these metrics with parallel Ricci tensor. In the second part, we investigate the relation between complete V -static metrics with the Bach tensor, which led to a classification under the Bach-flat condition.

We first recall a fundamental result due to Miao and Tam [65, Theorem 2.2], who showed that if a V -static metric ($\kappa \neq 0$) on a connected, complete manifold without boundary is Einstein, then (M^n, g) must be isometric to one of the following:

- the standard sphere \mathbb{S}^n ,
- the Euclidean space \mathbb{R}^n ,
- the hyperbolic space \mathbb{H}^n , or
- a warped product space $(\mathbb{R} \times \Sigma^{n-1}, dt^2 + \cosh^2 t g_0)$, where (Σ^{n-1}, g_0) is complete, Einstein with $Ric_{g_0} = -(n-2)g_0$, and the solution is $f(t, x) = A \sinh t + \frac{1}{n-1}$, with $A > 0$.

This result was originally proved for $\kappa = 1$, but, up to a rescaling of f , it also holds to general V -static metrics ($\kappa \neq 0$). We note that the case of connected, compact manifolds with boundary was treated in [65, Theorem 1.1].

Since manifolds with parallel Ricci tensor have harmonic curvature (although the converse does not generally hold; see [45]), it is natural to ask whether the Einstein condition assumed in [65, Theorem 2.2] can be relaxed to the weaker assumption of parallel Ricci tensor, that is, $\nabla Ric = 0$. This question was answered affirmatively in the case of compact manifolds with boundary by Baltazar–Ribeiro [5], who proved that if a critical metric of the volume functional on an n -dimensional compact, connected manifold with boundary has parallel Ricci tensor, then the manifold is isometric to a geodesic ball in a simply connected space form \mathbb{R}^n , \mathbb{H}^n , or \mathbb{S}^n . Notably, this result excludes the *Nariai space*—the manifold $I \times \mathbb{S}^n$ with the product metric—as a V -static space with $\kappa \neq 0$, even though it is a static vacuum space (i.e., $\kappa = 0$); see [2, 71].

4.1 V-static metrics with parallel Ricci tensor

In this section, we address the question aforementioned in the setting of complete manifolds without boundary. Before stating our first main theorem, we need to recall the following lemma, first established in [10, Lemma 1]. As only the case $\kappa = 1$ was addressed, we include the proof here

Lemma 5 ([10]). *Let (M^n, g, f) be an n -dimensional V -static metric. Then we have:*

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl} \nabla_l f + \frac{R}{n-1} (\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}).$$

Proof. Computing the derivative $\nabla_i(fR_{jk})$ by using (2.51) we infer

$$(\nabla_i f)R_{jk} + f(\nabla_i R_{jk}) = \nabla_i \nabla_j \nabla_k f - (\nabla_i \Delta f)g_{jk}. \quad (4.1)$$

Now, (2.54) yields $\nabla_i \Delta f = -\frac{R}{n-1} \nabla_i f$, where we used that the scalar curvature is constant (see Proposition 6). Hence, combining this with (4.1) we deduce

$$f \nabla_i R_{jk} - (\nabla_i f)R_{jk} + \nabla_i \nabla_j \nabla_k f + \frac{R}{n-1} \nabla_i f g_{jk}. \quad (4.2)$$

The result now follows directly from the Ricci identity. \square

Now, we focus on proving a rigidity result for complete V -static metrics without boundary. More precisely, we prove the following theorem.

Theorem 13. *Let (M^n, g, f) be an n -dimensional connected, complete V -static metric with $\kappa \neq 0$ and parallel Ricci tensor. Then (M^n, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^n , or*
- (2) *the Euclidean space \mathbb{R}^n , or*
- (3) *the hyperbolic space \mathbb{H}^n when $\nabla f(p) = 0$ for some $p \in M$, or*
- (4) *a warped product space $(\mathbb{R} \times \Sigma^{n-1}, dt^2 + \cosh^2 t g_0)$, where (Σ^{n-1}, g_0) is complete, Einstein with $\text{Ric}_{g_0} = -(n-2)g_0$, $f(t, x) = \kappa (A \sinh t + \frac{1}{n-1})$, and $A > 0$ is constant.*

Remark 10. *The proof of Theorem 13 is partly inspired by [5]. However, since M^n has no boundary, it is necessary to derive new differential identities to circumvent the use of integration arguments, yielding a more direct and self-contained proof; see also Corollary 6.*

Proof of Theorem 13. Since M^n has parallel Ricci tensor, we have $\nabla_i R_{jk} = 0$ for all indices $1 \leq i, j, k \leq n$. Hence, by using the Ricci identity, one obtains that

$$\begin{aligned} 0 &= f(\nabla_l \nabla_k R_{lj}) R_{kj} = f(\nabla_k \nabla_l R_{lj} + R_{lkli} R_{ij} + R_{lkji} R_{li}) R_{kj} \\ &= f R_{ki} R_{ij} R_{kj} + f R_{lkji} R_{li} R_{kj}. \end{aligned}$$

Rearranging the indices, we get

$$f R_{ij} R_{ik} R_{kj} = f R_{ijkl} R_{jl} R_{ik}. \quad (4.3)$$

On the other hand, by using (2.53), one sees that

$$\begin{aligned} f R_{ijkl} R_{jl} R_{ik} &= R_{ijkl} R_{jl} (-(\Delta f + \kappa) g_{ik} + \nabla_i \nabla_k f) \\ &= -(\Delta f + \kappa) |Ric|^2 + R_{ijkl} R_{jl} \nabla_i \nabla_k f \\ &= -(\Delta f + \kappa) |Ric|^2 + \nabla_i (R_{ijkl} R_{jl} \nabla_k f) - \nabla_i R_{ijkl} R_{jl} \nabla_k f, \end{aligned}$$

where we used that $\nabla_i R_{jl} = 0$. It follows from the Bianchi identity that

$$(div Rm)_{jkl} = \nabla_k R_{jl} - \nabla_l R_{jk}. \quad (4.4)$$

Hence, our assumption implies $div Rm = 0$. Consequently,

$$f R_{ijkl} R_{jl} R_{ik} = -(\Delta f + \kappa) |Ric|^2 + \nabla_i (R_{ijkl} R_{jl} \nabla_k f). \quad (4.5)$$

Notice that

$$\nabla_i (\nabla_j f R_{ik} R_{jk}) = \nabla_i \nabla_j f R_{ik} R_{jk},$$

and using (2.53), we have

$$\nabla_i (\nabla_j f R_{ik} R_{jk}) = (\Delta f + \kappa) |Ric|^2 + f R_{ij} R_{ik} R_{jk}.$$

This jointly with (4.5) gives

$$\nabla_i (R_{ijkl} R_{jl} \nabla_k f) = \nabla_i (\nabla_j f R_{ik} R_{jk}). \quad (4.6)$$

From Lemma 5 and our assumption, one sees that

$$\begin{aligned} R_{ijkl}\nabla_k f R_{jl} &= \left(\frac{R}{n-1} (\nabla_i f g_{jl} - \nabla_j f g_{il}) - (\nabla_i f R_{jl} - \nabla_j f R_{il}) \right) R_{jl} \\ &= \frac{R^2}{n-1} \nabla_i f - \frac{R}{n-1} \nabla_j f R_{ij} - \nabla_i f |Ric|^2 + \nabla_j f R_{il} R_{jl}, \end{aligned}$$

so that

$$\begin{aligned} \nabla_i (R_{ijkl}\nabla_k f R_{jl}) &= \frac{R^2}{n-1} \Delta f - \frac{R}{n-1} \nabla_i \nabla_j f R_{ij} - \Delta f |Ric|^2 \\ &\quad + \nabla_i (\nabla_j f R_{ik} R_{jk}), \end{aligned} \tag{4.7}$$

where we used again the Ricci parallel condition. By combining (4.7) and (4.6), one obtains that

$$0 = \frac{R^2}{n-1} \Delta f - \frac{R}{n-1} \nabla_i \nabla_j f R_{ij} - \Delta f |Ric|^2. \tag{4.8}$$

Now, it follows from (2.53) that

$$\nabla_i \nabla_j f R_{ij} = f |Ric|^2 + (\Delta f + \kappa) R,$$

which combined with (4.8) yields

$$\begin{aligned} 0 &= \frac{R^2}{n-1} \Delta f - \frac{fR}{n-1} |Ric|^2 - \frac{R^2(\Delta f + \kappa)}{n-1} - \Delta f |Ric|^2 \\ &= \frac{\kappa n}{n-1} \left(|Ric|^2 - \frac{R^2}{n} \right), \end{aligned} \tag{4.9}$$

where in the second equality we used (2.54). Since $\kappa \neq 0$, one obtains that

$$\left| Ric - \frac{R}{n} g \right|^2 = |Ric|^2 - \frac{R^2}{n} = 0,$$

and, therefore, (M^n, g) is an Einstein manifold. Thus, it suffices to apply [65, Theorem 2.2] to conclude that (M^n, g) is isometric to either the standard sphere \mathbb{S}^n , or the Euclidean space \mathbb{R}^n , or the hyperbolic space \mathbb{H}^n , or a warped product space $(\mathbb{R} \times \Sigma^{n-1}, dt^2 + \cosh^2 t g_0)$, where (Σ^{n-1}, g_0) is complete, Einstein with $Ric = -(n-2)g_0$, $f(t, x) = \kappa \left(A \sinh t + \frac{1}{n-1} \right)$, and $A > 0$ is constant. So, the proof is completed. \square

Another advantage of this approach is that the proof of Theorem 13 yields a new and more direct proof of [5, Corollary 1]. More precisely, we get the following corollary.

Corollary 6. *Let (M^n, g, f) be an n -dimensional connected, compact V -static metric with $\kappa \neq 0$, parallel Ricci tensor and smooth boundary ∂M . Then (M^n, g) is isometric to a geodesic ball in a simply connected space form \mathbb{R}^n , \mathbb{H}^n , or \mathbb{S}^n .*

Proof. By following the same steps as in the proof of Theorem 13 up to equation (4.9), we conclude that (M^n, g) is an Einstein manifold. It then suffices to apply Theorem 1.1 in [65] to deduce that (M^n, g) is isometric to a geodesic ball in a simply connected space form \mathbb{R}^n , \mathbb{H}^n , or \mathbb{S}^n . \square

4.2 Bach-flat V -static metrics

In this section, we discuss results concerning Bach-flat V -static metrics. Specifically, we present the proofs of Theorem 6, Corollary 3, and Theorem 7.

We begin by establishing the following key lemma, which is analogous to [10, Lemma 5] and [24, Lemma 4.1]. However, unlike the case of Ricci solitons, in our setting there is no ‘‘integrability condition’’ analogous to Hamilton’s equation for Ricci solitons (see Eq. (4.13) in [37]), which poses an obstacle to determining the asymptotic behavior of the potential function f . Therefore, we shall assume here that f is proper.

Lemma 6. *Let (M^n, g, f) , $n \geq 4$, be a n -dimensional V -static metric and f be a proper function. Then we have:*

$$\int_M f^2 B(\nabla f, \nabla f) dV_g = -\frac{1}{2(n-1)} \int_M f^2 |T|^2 dV_g, \quad (4.10)$$

where B is the Bach tensor.

Proof. First of all, we combine (2.11) and (2.8) to infer

$$B_{ij} = \frac{1}{n-2} (\nabla_k C_{kij} + R_{kl} W_{ikjl}). \quad (4.11)$$

Consequently,

$$\begin{aligned} f^2 B_{ij} &= \frac{1}{n-2} (f^2 \nabla_k C_{kij} + f^2 R_{kl} W_{ikjl}) \\ &= \frac{1}{n-2} (\nabla_k (f^2 C_{kij}) - 2f C_{kij} \nabla_k f + f^2 R_{kl} W_{ikjl}). \end{aligned}$$

Now, using Lemma 3 and (2.8) one obtains that

$$\begin{aligned}
f^2 B_{ij} &= \frac{1}{n-2} \left(\nabla_k [f(W_{kijl} \nabla_l f + T_{kij})] - 2f C_{kij} \nabla_k f + f^2 R_{kl} W_{ikjl} \right) \\
&= \frac{1}{n-2} \left(\nabla_k (f T_{kij}) + f (\nabla_k W_{kijl}) \nabla_l f + f W_{kijl} \nabla_k \nabla_l f \right. \\
&\quad \left. + W_{kijl} \nabla_k f \nabla_l f - 2f C_{kij} \nabla_k f + f^2 R_{kl} W_{ikjl} \right) \\
&= \frac{1}{n-2} \left(\nabla_k (f T_{kij}) + \frac{n-3}{n-2} f C_{jki} \nabla_k f + f (\nabla_k \nabla_l f - f R_{kl}) W_{kijl} \right. \\
&\quad \left. + W_{kijl} \nabla_k f \nabla_l f + 2f C_{ikj} \nabla_k f \right).
\end{aligned}$$

Using (2.53) and the fact that the Weyl tensor is trace-free in any pair of indices, we obtain

$$f^2 B_{ij} = \frac{1}{n-2} \left(\nabla_k (f T_{kij}) + \frac{n-3}{n-2} f C_{jki} \nabla_k f + 2f C_{ikj} \nabla_k f + W_{kijl} \nabla_k f \nabla_l f \right).$$

Hence,

$$f^2 B(\nabla f, \nabla f) = \frac{1}{n-2} \nabla_k (f T_{kij}) \nabla_i f \nabla_j f. \quad (4.12)$$

Proceeding, we analyze the right hand side of (4.12). Observe that

$$\begin{aligned}
\nabla_k (f T_{kij}) \nabla_i f \nabla_j f &= \nabla_k (f T_{kij} \nabla_i f \nabla_j f) - f T_{kij} \nabla_k \nabla_i f \nabla_j f - f T_{kij} \nabla_i f \nabla_k \nabla_j f \\
&= \nabla_k (f T_{kij} \nabla_i f \nabla_j f) - f T_{kij} (f R_{ki} + (\kappa + \Delta f) g_{ik}) \nabla_j f \\
&\quad - f T_{kij} \nabla_i f (f R_{kj} + (\kappa + \Delta f) g_{kj}) \\
&= \nabla_k (f T_{kij} \nabla_i f \nabla_j f) - f^2 T_{kij} R_{ki} \nabla_j f - f^2 T_{kij} R_{kj} \nabla_i f,
\end{aligned}$$

where we have used (2.53) and the fact that T is trace-free. Rearranging the indices and using the antisymmetry of T , we get

$$(n-2) f^2 B(\nabla f, \nabla f) = \nabla_k (f T_{kij} \nabla_i f \nabla_j f) - \frac{1}{2} f^2 T_{kij} (R_{kj} \nabla_i f - R_{ik} \nabla_j f).$$

So, it follows from (2.64) that

$$(n-2) f^2 B(\nabla f, \nabla f) = \nabla_k (f T_{kij} \nabla_i f \nabla_j f) - \frac{n-2}{2(n-1)} f^2 |T|^2. \quad (4.13)$$

Notice that if M^n is compact, the result follows immediately from the divergence theorem. Otherwise, if M^n is not compact, we consider the set

$$\Omega_r = \{p \in M : |f(p)| \leq r\}.$$

Taking into account that f is a proper function, we apply the divergence theorem for Ω_r in order to infer

$$\int_{\Omega_r} \nabla_k (f T_{kij} \nabla_i f \nabla_j f) d\Omega = \int_{\partial\Omega_r} f T_{kij} \nabla_i f \nabla_j f N_k dS, \quad (4.14)$$

where N is the unitary normal vector field over $\partial\Omega_r$, i.e., $N = \frac{\nabla f}{|\nabla f|}$ or $N = -\frac{\nabla f}{|\nabla f|}$. But, using again (2.64), one concludes that

$$\int_{\partial\Omega_r} f T_{kij} \nabla_i f \nabla_j f N_k dS = \pm \int_{\partial\Omega_r} f T_{kij} \nabla_i f \nabla_j f \frac{\nabla_k f}{|\nabla f|} dS = 0. \quad (4.15)$$

Therefore, on integrating (4.13) over Ω_r and using (4.14) and (4.15), we obtain

$$\int_{\Omega_r} f^2 B(\nabla f, \nabla f) d\Omega = -\frac{1}{2(n-1)} \int_{\Omega_r} f^2 |T|^2 d\Omega. \quad (4.16)$$

Finally, since f is a proper function, Ω_r is an exhaustion of M^n and hence, letting $r \rightarrow \infty$, we get

$$\int_M f^2 B(\nabla f, \nabla f) dV_g = -\frac{1}{2(n-1)} \int_M f^2 |T|^2 dV_g, \quad (4.17)$$

as asserted. \square

Before proceeding, we need to establish an additional structural property of V -static metrics. The following proposition, inspired by [10, Lemma 4] and [29, Theorem 1.2], highlights a key geometric implication of the condition $T \equiv 0$ in the study of V -static metrics.

Proposition 10. *Let (M^n, g, f) , $n \geq 3$, be a complete, noncompact V -static metric with $\kappa \neq 0$. Suppose that the tensor T vanishes identically. Then, in a neighborhood of every regular level set of f , the manifold (M^n, g) is locally a warped product with $(n-1)$ -dimensional Einstein fibers.*

Proof. Since $\Sigma_c = \{p \in M; f(p) = c\}$ is the regular level set of the potential function f and $T = 0$ on M , assertion (2) of Proposition 9 ensures that $|\nabla f|$ is constant along Σ_c . Consequently, in a neighborhood U of Σ_c containing no critical points of f , the function f depends only on the signed distance r to the hypersurface Σ_c . Therefore, after a suitable change of variables, the metric g can be locally expressed in the form

$$g = dr^2 + g_{\alpha\beta}(r, \theta) d\theta^\alpha \otimes d\theta^\beta,$$

where $\{\theta^2, \dots, \theta^n\}$ is a local coordinate system on Σ_c , and r is defined on a maximal interval $r \in (r_{min}, r_{max})$ with $r_{min} \in [-\infty, 0)$ and $r_{max} \in (0, \infty]$.

Recall that Proposition 9 asserts that Σ_c is a totally umbilical hypersurface with constant mean curvature. Hence, we have

$$\partial_r g_{\alpha\beta} = -2h_{\alpha\beta} = \phi(r)g_{\alpha\beta},$$

with $\phi(r) = -\frac{2}{n-1}H(r)$ and $\partial_r = \frac{\nabla f}{|\nabla f|}$; see also Eq. (2.24) in [10]. From this, it follows that

$$g_{\alpha\beta}(r, \theta) = e^{\Phi(r)} g_{\alpha\beta}(0, \theta), \text{ where } \Phi(r) = \int_0^r \phi(r) dr.$$

Therefore, in U , the metric g takes the form of a warped product

$$g = dr^2 + \varphi(r)^2 g_{\Sigma_c},$$

where φ is a smooth function on U , $r \in (r_{min}, r_{max})$ and g_{Σ_c} is the metric of the regular level set Σ_c . To conclude, since (U, g, \bar{f}) , where $\bar{f} = f|_U$, is a V -static warped metric, it suffices to apply [65, Proposition 3.1 (i)] to deduce that the fiber is Einstein $Ric_{\Sigma_c} = (n-2)k_0 g_{\Sigma_c}$, where k_0 is constant; see also the proof of Lemma 4 in [10]. This completes the proof of the proposition. \square

We are now ready to prove **Theorem 6**, which follows as a special case of the following more general result.

Theorem 14. *Let (M^n, g, f) , $n \geq 4$, be a complete, simply connected V -static metric with $\kappa \neq 0$ and f be a proper function. If $B(\nabla f, \nabla f) = 0$, then (M^n, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^n , or*
- (2) *the Euclidean space \mathbb{R}^n , or*
- (3) *the hyperbolic space \mathbb{H}^n or*
- (4) *$\mathbb{R} \times_{\varphi} \Sigma_c$,*

where Σ_c is a regular level set of the potential function, which is an Einstein manifold with $Ric_{\Sigma_c} = \lambda g_{\Sigma_c}$, and the warping function φ is a solution of the ODE

$$\varphi \left(\frac{R}{n-1} \varphi + 2\varphi'' \right) + (n-2)(\varphi')^2 = \lambda. \quad (4.18)$$

Proof. By analyticity [41, Proposition 2.1], it suffices to prove the result on $\Omega = \{p \in M : |\nabla f| \neq 0\}$, which exists by Sard's Theorem and the fact that f is nontrivial. In the first part of the proof, we shall adapt some ideas outlined in [26],[27]. First, pick a regular value c of the

potential function f and let Σ_c be the level surface, that is, $\Sigma_c = f^{-1}(c)$. Let $I \in \mathbb{R}$ be an open interval containing c such that f has no critical points in the open neighborhood $U_I = f^{-1}(I)$ of Σ_c . Since (M^n, g) satisfies $B(\nabla f, \nabla f) = 0$ and f is proper, Lemma 6 guarantees that (M^n, g) has vanishing T tensor¹. Therefore, Proposition 10 guarantees that U_I admits a local warped product structure

$$g = dr^2 + \varphi^2(r)g_{\Sigma_c}, \quad (4.19)$$

where

$$r(x) = \int_{f=c}^f \frac{df}{|\nabla f|}$$

is the signed distance from the regular hypersurface Σ_c , which is a totally umbilical Einstein hypersurface. In particular, since M^n is simply connected, then Σ_c is a two-sided embedded hypersurface.

In the second part of the proof, we aim to extend the local warped product structure to a global one. To this end, following [26] and [61], we divide the argument into a sequence of claims.

Claim 1. The interval in which the warped structure is defined can be extended as long as the warping function does not vanish.

Let $I_{\max} \supseteq I$ be the maximal interval for which the warped structure (4.19) holds. Observe that if $|\nabla f|$ never vanishes, the claim follows immediately. Otherwise, assume that $|\nabla f|(\bar{r}) = 0$ for some \bar{r} , so that either $(\bar{r} - \varepsilon, \bar{r}) \subset I_{\max}$ or $(\bar{r}, \bar{r} + \varepsilon) \subset I_{\max}$, that is, \bar{r} is an endpoint of the interval I_{\max} . Thus, since Ω is dense, by continuity and smoothness, $\Sigma_{\bar{c}} = f^{-1}(\bar{c}) = r^{-1}(\bar{r})$ is a totally umbilical submanifold and the warped structure can be extended to $(\bar{r} - \varepsilon, \bar{r} + \varepsilon) \subset I_{\max}$ for $\varepsilon > 0$ sufficient small.

We now have three cases: either φ never vanishes, in which case $I_{\max} = \mathbb{R}$; or it vanishes at exactly one point, implying that $I_{\max} = (a, \infty)$ with $\lim_{r \rightarrow a} \varphi(r) = 0$; or it vanishes at precisely two points, so that $I_{\max} = (a, b)$ and $\lim_{r \rightarrow a} \varphi(r) = \lim_{r \rightarrow b} \varphi(r) = 0$. Therefore, by a suitable shifting in r , one sees that (M^n, g) is isometric to one of the following possibilities:

- (i) $\mathbb{R} \times_{\varphi} \Sigma_c$; or
- (ii) $[0, \infty) \times_{\varphi} \Sigma_c$; or

¹ Obviously, the result also holds if (M^n, g) is Bach-flat.

(iii) $[0, a] \times_{\varphi} \Sigma_c$.

Claim 2. If (M^n, g) is isometric to either $[0, \infty) \times_{\varphi} \Sigma_c$ or $[0, a] \times_{\varphi} \Sigma_c$, then Σ_c is isometric to \mathbb{S}^{n-1} .

We address here the first case; the second is analogous. Suppose that (M^n, g) is isometric to $[0, \infty) \times_{\varphi} \Sigma_c$, where the warped product metric extends smoothly to $\{r = 0\}$. Since $\varphi(0) = 0$, smoothness of the metric requires that the slice $\{0\} \times \Sigma_c$ collapses to a single point, denoted by $p \in M$. Consequently, the exponential map at p ensures that Σ_c is diffeomorphic to \mathbb{S}^{n-1} . Moreover, in a neighborhood of p , the metric can be expressed in polar coordinates as

$$g = dr^2 + r^2 h_r(\theta), \quad (4.20)$$

where h_r is a smooth one-parameter family of metrics on \mathbb{S}^{n-1} , and h_0 is the standard round metric. So, taking into account (4.19), we now need to show that $h_0 = g_{\Sigma_c}$. Indeed, following [27, Claim 4] (see also [13, Lemma 9.114]), the Taylor expansion of φ has the form

$$\varphi(r) = r + ar^3 + O(r^5),$$

so that

$$\varphi^2(r) = r^2 + 2ar^4 + O(r^6). \quad (4.21)$$

In conjunction with (4.20) and (4.19) yields

$$\varphi^2(r)g_{\Sigma_c} = r^2 h_r(\theta).$$

Consequently,

$$r^2 g_{\Sigma_c} + 2ar^2 g_{\Sigma_c} + O(r^6)g_{\Sigma_c} = r^2 h_r(\theta).$$

Dividing the above equation by r^2 for $r > 0$ and letting $r \rightarrow 0$, one sees that

$$h_0 = \lim_{r \rightarrow 0} h_r = g_{\Sigma_c}.$$

Therefore, the metric on Σ_c must be the round metric, and hence Σ_c is isometric to \mathbb{S}^{n-1} , which completes the proof of Claim 2.

To proceed, observe that if a manifold (M^n, g) is expressed as a warped product of an interval and an Einstein manifold, then the following ODE holds

$$R = \frac{n-1}{\varphi^2}(\lambda - 2\varphi\varphi'') - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2, \quad (4.22)$$

where λ is the Einstein constant and φ denotes the warping function. We now distinguish three cases, which will be analyzed separately.

Case 1: (M^n, g) is isometric to $[0, a] \times_{\varphi} \mathbb{S}^{n-1}$.

In this case, we have $\lambda = n - 2$ and (4.22) becomes

$$\frac{R}{n-1}\varphi^2 = (n-2)(1 - (\varphi')^2) - 2\varphi\varphi'', \quad (4.23)$$

with $\varphi(0) = 0$. In particular, for the metric to extend smoothly, we have $\varphi'(0) = 1$. We then solve this initial value problem, which depends only on the sign of the scalar curvature R as follows.

(i) If $R > 0$, then

$$\varphi(r) = \sqrt{\frac{n(n-1)}{R}} \sin\left(\sqrt{\frac{R}{n(n-1)}}r\right)$$

and

$$[0, a] = \left[0, \sqrt{\frac{n(n-1)}{R}}\pi\right].$$

Hence, (M^n, g) is isometric to the standard sphere \mathbb{S}^n .

(ii) If $R = 0$, then $\varphi(r) = r$. As observed in the proof of Claim 1, for such an isometry to occur, the warping function would have to vanish at exactly two points; therefore, this case cannot arise.

(iii) If $R < 0$, then

$$\varphi(r) = \sqrt{-\frac{n(n-1)}{R}} \sinh\left(\sqrt{-\frac{R}{n(n-1)}}r\right).$$

As in case (ii), this situation cannot occur since φ vanishes at only one point.

Case 2: (M^n, g) is isometric to $[0, \infty) \times_{\varphi} \mathbb{S}^{n-1}$.

Similar to the previous case, the initial data are given by $\varphi(0) = 0$ and $\varphi'(0) = 1$, and the corresponding ODE coincides with (4.23). Therefore, the possible solutions $\varphi(r)$ are the same as in Case 1.

- (i) If $R > 0$, one observes that φ vanishes at two points. However, from the proof of Claim 1, we know that in the isometric case the warping function must vanish at exactly one point. Therefore, the case $R > 0$ cannot occur.
- (ii) If $R = 0$, one deduces that $\varphi(r) = r$. Therefore, (M^n, g) is isometric to the Euclidean space \mathbb{R}^n .
- (iii) If $R < 0$, then

$$\varphi(r) = \sqrt{-\frac{n(n-1)}{R}} \sinh \left(\sqrt{-\frac{R}{n(n-1)}} r \right).$$

Thus, (M^n, g) is isometric to the hyperbolic space \mathbb{H}^n .

Case 3: (M^n, g) is isometric to $\mathbb{R} \times_{\varphi} \Sigma_c$.

In this case, it follows from (4.22) that

$$\varphi \left(\frac{R}{n-1} \varphi + 2\varphi'' \right) + (n-2)(\varphi')^2 = \lambda. \quad (4.24)$$

This analysis covers all possible cases, and the theorem is thus proved. \square

Remark 11. We note that φ satisfying

$$(\varphi')^2 = \frac{\lambda}{n-2} - \frac{R}{n-1} \varphi^2$$

is a solution of (4.24). In particular, differentiating this identity gives

$$\varphi'' = -\frac{R}{n-1} \varphi,$$

which provides a direct solution of (4.24). However, since no initial data are available in Case 3, we cannot assert the uniqueness of such a solution. Nevertheless, we have the following:

- If $R \geq 0$ and $\lambda \leq 0$, then (4.24) implies $\varphi'' \leq 0$. Because φ is a positive and weakly concave function on \mathbb{R} , it must be constant. Substituting this into (4.24) then forces $R = \lambda = 0$. Consequently, by [65, Theorem 2.2], M^n is isometric to the Euclidean space \mathbb{R}^n .

We now turn our attention to the lower dimensional cases. We begin by presenting the classification for four-dimensional, complete V -static metrics with vanishing Bach tensor. In fact, we have the following corollary

Corollary 7. *Let (M^4, g, f) be a complete, simply connected four-dimensional Bach-flat V -static metric with $\kappa \neq 0$ and f be a proper function. Then (M^4, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^4 , or*
- (2) *the Euclidean space \mathbb{R}^4 , or*
- (3) *the hyperbolic space \mathbb{H}^4 , or*
- (4) *the warped product $\mathbb{R} \times_{\varphi} \mathbb{S}^3$, where φ is a solution of the ODE*

$$\varphi \left(\varphi'' + \frac{R}{6} \varphi \right) + (\varphi')^2 = 1.$$

Proof. First, notice that items (1)–(3) of Corollary 3 follow immediately from Theorem 6, so we are left to analyze the last case. Since the function φ is not determined by an initial value problem, we are unable to explicitly solve it; however, we can still make progress in understanding the structure of Σ_c .

Observe that, under the assumption that M^4 is simply connected, the level set Σ_c is necessarily simply connected as well. Therefore, Σ_c is a compact, simply connected, 3-dimensional Einstein manifold. Thus, it follows from the Killing–Hopf theorem that Σ_c is isometric to \mathbb{S}^3 . Finally, by substituting $n = 4$ and $\lambda = 2$ into the general ODE (4.18), we obtain

$$\varphi \left(\varphi'' + \frac{R}{6} \varphi \right) + (\varphi')^2 = 1,$$

which characterizes the warping function φ , and completes the proof. \square

We now present the proof of our last theorem. It focuses on investigating the Bach-flatness condition for complete three-dimensional V -static metrics without boundary. We have the following result.

Theorem 15. *Let (M^3, g, f) be a complete, simply connected V -static metric with $\kappa \neq 0$ and f be a proper function. If $\operatorname{div} B(\nabla f) = 0$, then (M^3, g) is isometric to either*

- (1) *the standard sphere \mathbb{S}^3 , or*
- (2) *the Euclidean space \mathbb{R}^3 , or*
- (3) *the hyperbolic space \mathbb{H}^3 , or*
- (4) *the warped product $\mathbb{R} \times_{\varphi} \mathbb{S}^2$, where φ is a solution of the ODE*

$$\varphi \left(2\varphi'' + \frac{R}{2} \varphi \right) + (\varphi')^2 = 1. \tag{4.25}$$

Proof. The first part of the proof aims to assert that in this setting, (M^3, g) must have vanishing T tensor. We begin by recalling that the Cotton tensor can be written as $C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}$,

where A is the Schouten tensor defined previously. Now, using this form for the Cotton tensor, the expression for the Bach tensor in dimension 3 becomes

$$B_{ij} = \nabla_k(\nabla_k A_{ij} - \nabla_i A_{jk}). \quad (4.26)$$

By taking the derivative on both sides and rearranging the indices we arrive at

$$\nabla_i B_{ij} = (\nabla_i \nabla_k - \nabla_k \nabla_i) \nabla_k A_{ij}. \quad (4.27)$$

Using the Ricci identity we find that

$$\begin{aligned} \nabla_i B_{ij} &= -R_{il} \nabla_l A_{ij} + R_{kl} \nabla_k A_{lj} + R_{ikjl} \nabla_k A_{il} \\ &= R_{ikjl} \nabla_k A_{il}. \end{aligned}$$

Now, remembering that in dimension 3 the Weyl tensor vanishes identically, i.e., $W \equiv 0$, we get

$$R_{ikjl} \nabla_j f = (A_{ij} g_{kl} + A_{kl} g_{ij} - A_{il} g_{kj} - A_{kj} g_{il}) \nabla_j f. \quad (4.28)$$

Therefore

$$\begin{aligned} \nabla_i B_{ij} \nabla_j f &= \nabla_k A_{ik} A_{ij} \nabla_j f + \nabla_k A_{jl} A_{kl} \nabla_j f - \nabla_j A_{il} A_{il} \nabla_j f \\ &\quad - g_{il} A_{kj} \nabla_j f \nabla_k A_{il} \\ &= \nabla_k A_{ik} A_{ij} \nabla_j f + C_{kjl} A_{kl} \nabla_j f + \nabla_j A_{kl} A_{kl} \nabla_j f \\ &\quad - \nabla_j A_{il} A_{il} \nabla_j f - A_{jk} g_{il} \nabla_j f C_{kil} - \nabla_i A_{ki} A_{jk} \nabla_j f, \end{aligned}$$

where we only used the expression of the Cotton tensor in terms of the Schouten tensor. Now, recall that the Cotton C tensor is trace free in any two indices and, since we have constant scalar curvature, the Schouten tensor A is divergence free. Hence, by rearranging the indices, we get

$$\nabla_i B_{ij} \nabla_j f = C_{kji} A_{ki} \nabla_j f. \quad (4.29)$$

In order to proceed, notice that the Cotton tensor C is skew-symmetric in the first two indices. Thus, combining (4.29) with (2.9), we get

$$\begin{aligned} \nabla_i B_{ij} \nabla_j f &= -R_{ik} C_{jki} \nabla_j f \\ &= -\frac{1}{2} (R_{ik} C_{jki} \nabla_j f + R_{ij} C_{kji} \nabla_k f), \end{aligned}$$

where in the last term of the right-hand side we simply interchanged the indices j and k , which has no effect on the final result. Finally using the equation for the T tensor (2.64) and the fact that the Cotton tensor is trace-free we conclude that

$$\operatorname{div} B(\nabla f) = \frac{1}{4} C_{kji} T_{kji} \quad (4.30)$$

Applying Lemma 3 together with the fact that $W = 0$ we conclude that

$$\operatorname{div} B(\nabla f) = \frac{f}{4} |C|^2. \quad (4.31)$$

Thus, assuming that $\operatorname{div} B(\nabla f) = 0$ – in particular, when M^3 is Bach-flat – we conclude from Lemma 3 that $T \equiv 0$. It then follows from Proposition 10 that (M^3, g) is locally a warped product. Consequently, repeating the arguments from the second part of the proof of Theorem 6, we deduce that M^3 is isometric to one of the following spaces: the standard sphere \mathbb{S}^3 , the Euclidean space \mathbb{R}^3 , the hyperbolic space \mathbb{H}^3 , or $\mathbb{R} \times_{\varphi} \Sigma_c$, where Σ_c is a regular level set of the potential function and the warping function φ satisfies

$$\varphi \left(\frac{R}{2} \varphi + 2\varphi'' \right) + (\varphi')^2 = \lambda. \quad (4.32)$$

In the last case, by [65, Proposition 3.1 (i)], we have that Σ_c is Einstein manifold, i.e., $\operatorname{Ric}_{\Sigma_c} = \lambda g_{\Sigma_c}$ with λ constant. Therefore, Σ_c has constant sectional curvature (see [13, Proposition 1.120]). By using the Killing–Hopf theorem and the fact that f is proper, one concludes that the level set Σ_c is isometric to \mathbb{S}^2 . This completes the proof of the theorem. \square

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