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JOHNATAN DA SILVA COSTA

COMPACT STATIC PERFECT FLUID SPACE-TIMES AND QUASI-EINSTEIN MANIFOLDS WITH BOUNDARY

FORTALEZA
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Orientador: Prof. Dr. Ernani de Sousa Ribeiro Junior.

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“Há muito tempo eu era jovem, eu viajei sozinho, então acabei me perdendo no caminho: me considerei rico quando eu encontrei outro, o homem é o deleite do homem” (Hávamál, [ca. 1270], estrofe 47).

“É preciso imaginar Sísifo feliz” (Camus, 1942, p. 124).

RESUMO

O objetivo deste trabalho é estudar variedades compactas com bordo a partir de duas estruturas distintas. Na primeira parte, investigamos a geometria do espaço-tempo estático com fluido perfeito em variedades compactas com bordo. Usamos a fórmula de Reilly generalizada para estabelecer uma desigualdade geométrica para um espaço-tempo estático com fluido perfeito envolvendo a área do bordo e seu volume. Além disso, obtemos novas estimativas de bordo para este espaço. Uma das estimativas de bordo é obtida em termos da massa de Brown-York. Ademais, fornecemos um novo contra-exemplo (simplesmente conexo) para a conjectura Cosmic no-hair para dimensão arbitrária $n \geq 4$. Na segunda parte deste trabalho, voltamos nossa atenção para a geometria de variedades quasi-Einstein compactas com bordo. Estabelecemos os possíveis valores para a curvatura escalar constante de uma variedade quasi-Einstein compacta com bordo. Mostramos que uma variedade m-quasi-Einstein compacta simplesmente conexa de dimensão 3 com bordo e curvatura escalar constante deve ser isométrica, a menos de scaling, ao hemisfério redondo $\mathbb{S}^3_+$, ou ao cilindro $I \times \mathbb{S}^2$ com a métrica do produto, onde I é um intervalo fechado. Para a dimensão $n = 4$, provamos que uma variedade m-quasi-Einstein de dimensão 4 compacta simplesmente conexa M^4 com bordo e curvatura escalar constante é isométrica, a menos de scaling, ao hemisfério redondo $\mathbb{S}^4_+$, ou ao cilindro $I \times \mathbb{S}^3$ com a métrica do produto ou ao espaço produto $\mathbb{S}^2_+ \times \mathbb{S}^2$ com a métrica produto warped duplo. Outros resultados para dimensões maiores ou iguais a 5 também são discutidos.

Palavras-chave: variedades compactas com bordo; métricas estáticas; fluido perfeito; estimativas de bordo; variedades quasi-Einstein; curvatura escalar constante; resultados de rigidez.

ABSTRACT

The purpose of this work is to study compact manifolds with boundary from the point of view of two distinct structures. In the first part, we investigate the geometry of static perfect fluid space-time on compact manifolds with boundary. We use the generalized Reilly's formula to establish a geometric inequality for a static perfect fluid space-time involving the area of the boundary and its volume. Moreover, we obtain new boundary estimates for this space. One of the boundary estimates is obtained in terms of the Brown-York mass. In addition, we provide a new (simply connected) counterexample to the Cosmic no-hair conjecture for arbitrary dimension $n \geq 4$. At the second part of this work, we turn our attention to the geometry of compact quasi-Einstein manifolds with boundary. We establish the possible values for the constant scalar curvature of a compact quasi-Einstein manifold with boundary. Moreover, we show that a 3-dimensional simply connected compact m-quasi-Einstein manifold with boundary and constant scalar curvature must be isometric, up to scaling, to either the standard hemisphere $\mathbb{S}^3_+$, or the cylinder $I \times \mathbb{S}^2$ with the product metric, where I is a closed interval. For dimension $n = 4$, we prove that a 4-dimensional simply connected compact m-quasi-Einstein manifold M^4 with boundary and constant scalar curvature is isometric, up to scaling, to either the standard hemisphere $\mathbb{S}^4_+$, or the cylinder $I \times \mathbb{S}^3$ with the product metric, or the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ with the doubly warped product metric. Other results for dimension greater than or equal to 5 are also discussed.

Keywords: compact manifolds with boundary; static metrics; perfect fluid; boundary estimates; quasi-Einstein manifolds; constant scalar curvature; rigidity results.

CONTENTS

1 INTRODUCTION

The study of general relativity establishes a connection between physics and differential geometry. In a most simplified way, the theory of general relativity developed by Einstein presents the gravity as the curvature of the universe because of the matter, we refer the reader to "*On the Foundations of the General Theory of Relativity*" in [104] for a nice discussion on this topic. In this context, the Einstein field equation plays a fundamental role as an interface for studies beyond pure mathematics. To be precise, given a Riemannian manifold (M^n, g), $n \geq 3$, and a positive smooth function f on M^n, we say that $(\widehat{M}^{n+1}, \widehat{g}) = M^n \times_f \mathbb{R}$ endowed with the metric $\widehat{g} = g - f^2 dt^2$ is a static space-time. Thus, the Einstein equation over $(\widehat{M}^{n+1}, \widehat{g}) = M^n \times_f \mathbb{R}$ is given by

$$Ric_{\widehat{g}} - \frac{R_{\widehat{g}}}{2}\widehat{g} + \Lambda\widehat{g} = T, \tag{1.1}$$

where Λ is the cosmological constant, T is the stress-energy-momentum tensor, $Ric_{\widehat{g}}$ and $R_{\widehat{g}}$ stand for the Ricci tensor and the scalar curvature for the metric $\widehat{g}$, respectively. The case $T = 0$ means that we are in vacuum. Notice that, in the vacuum, $Ric_{\widehat{g}} = \left(\frac{R_{\widehat{g}}}{2} - \Lambda\right)\widehat{g}$. Manifolds satisfying such a relation, i.e., the Ricci tensor to be a multiple of the metric, are called Einstein manifolds (or Einstein metrics) (see [16]) and constitute as special solutions to the Einstein equation in the vacuum.

When the stress-energy-momentum tensor is given by $T = \mu f^2 dt^2 + \rho g$, we say that it represents a perfect fluid. The smooth functions μ and ρ are *mass-energy density* and *pressure* of the fluid (as measured in the rest frame). The fluid is called "perfect" because of the absence of heat conduction terms and stress terms corresponding to viscosity. For more details, see, e.g., [37],[38],[65],[66],[84] and [105]. If $\Lambda = 0$, solutions to equation (1.1) with perfect fluid as a matter field are called *static perfect fluid space-times*.

The perfect fluid space-times are natural generalizations of the static vacuum spaces and certain solutions of (1.1) provide models for galaxies, black holes and stars (see [58], [84], [105]). In particular, they are used in developing realistic stellar models (or models for fluid planets) and represent a homogeneous fluid filled universe that is undergoing accelerated expansion. Astronomical evidences also indicate that the universe can be modeled as a space-time containing a perfect fluid; see [37], [38], [72], [105] and the references therein.

Since we are interested in the Riemannian part of the static perfect fluid space-times, let us fix some terminology (see [37], [66] and [101]): A Riemannian manifold (M^n, g) is said to be a spatial factor of a static perfect fluid space-time if there exist smooth functions $f > 0$ and ρ on M^n satisfying

$$f\mathring{Ric} = \mathring{\nabla}^2 f \tag{1.2}$$

and

$$\Delta f = \left(\frac{n-2}{2(n-1)}R + \frac{n}{n-1}\rho\right) f, \tag{1.3}$$

where $\mathring{Ric}$, $\mathring{\nabla}^2 f$ and R stand for the traceless Ricci, traceless Hessian of f and scalar curvature of (M^n, g), respectively. When M^n has non-empty boundary ∂M, we assume in addition that $f^{-1}(0) = \partial M$. The function f is usually called lapse function or static potential in the literature. In this case, (M^n, g, f, ρ) will be called static perfect fluid space-time *(SPFST)*.

One should be emphasized that the *dominant energy condition* is said to be satisfied when $\mu \geq |\rho|$, which means that the speed of the energy flow can not be equal or greater than to the light. As was observed by Hawking and Ellis [58], the dominant energy condition holds for all known matter; see also [85, p. 347] and [105, p. 219]. In geometrical point of view, Eq. (1.2) is related to important special metrics, as for instance, vacuum static spaces (see [1], [7], [37], [101]), Miao-Tam critical metrics or V-static spaces (see [4], [5], [8], [34], [76], [116]) and Einstein-type manifolds ([16], [26], [42], [64], [59], [60]). One easily verifies from (1.2) and (1.3) that

$$\mu = \frac{R}{2}.$$

Moreover, Coutinho *et al.* [37, Proposition 2] provided a necessary and sufficient condition for a static perfect fluid space-time to have constant scalar curvature which suggests that Eqs. (1.2) and (1.3) alone do not implies that constancy of the scalar curvature. Indeed, in contrast with static spaces and V-static spaces, there are examples of static perfect fluid space-times with non-constant scalar curvature; see the examples in [9] and [73].

A classical example of SPFST with connected (non-empty) boundary is the n-dimensional hemisphere $\mathbb{S}^n_+(r)$ of radius r endowed with the standard metric $g_{\mathbb{S}^n(r)}$ and potential function $f(h) = \cos(h)$, where $h \leq \frac{\pi}{2}$ is the height function. In this case, $\partial M = \mathbb{S}^{n-1}(r)$. For the case of disconnected boundary, we have $[0, \pi] \times \mathbb{S}^{n-1}$ with product metric $g = dt^2 + (n-2)g_{\mathbb{S}^{n-1}}$ and potential function $f(t) = \sin(t)$ (the boundary is the union of two copies of $\mathbb{S}^{n-1}$).

In this thesis, we deal with compact Riemannian manifolds with boundary. Chapter 2 collects some fundamental concepts that will be used in the rest of the work. Chapter 3 is devoted to discuss some results on static perfect fluid space-times obtained in [35] by the author joint with R. Diógenes, N. Pinheiro and E. Ribeiro. Chapter 4, presents some rigidity results on quasi-Einstein manifolds according to the article [36], written by the author together with E. Ribeiro and D. Zhou. We conclude with Chapter 5 commenting about the importance of this thesis and further developments.

In Chapter 3, it will be discussed some geometric inequalities involving the area of the boundary of a SPFST in order to obtain rigidity results concerning to the hemisphere $\mathbb{S}^n_+$ with round metric as well as new obstruction results. It has been conjectured in 1984 by Boucher, Gibbons and Horowitz in [22], [23] that: *the hemisphere $\mathbb{S}^n_+$ is the only possible n-dimensional (simply connected) positive static triple with single-horizon (connected boundary) and positive scalar curvature.* This conjecture is known as *Cosmic no-hair conjecture.* It is closely related to Fischer-Marsden conjecture which asserts that the standard unit round spheres $(\mathbb{S}^n, g_{\mathbb{S}^n})$ are the only closed Riemannian manifold with scalar curvature $n(n-1)$ admitting static potential (see [1] and [101]). In the last decades some partial answers to the Cosmic no-hair conjecture were obtained. For instance, by assuming that (M^n, g) is Einstein, it suffices to apply the Obata type theorem due to Reilly [92] to conclude that the conjecture is true. Moreover, Kobayashi [65] and Lafontaine [68] proved independently that such a conjecture is also true under conformally flat condition. Qing and Yuan [89] proved the Cosmic no-hair conjecture by considering a weaker hypothesis on the Cotton tensor. In the work [55], Gibbons, Hartnoll and Pope constructed counterexamples to the Cosmic no-hair conjecture in the cases of dimension $4 \leq n \leq 8$, but their counterexamples are not simply connected.

One of the contributions presented in Chapter 3 is a new (simply connected) counterexample to the Cosmic no-hair conjecture for arbitrary dimension $n \geq 4$, which is inspired by an example of quasi-Einstein manifold obtained in [45].

Example 1.1 (Counter example to the Cosmic no-hair conjecture for $n \geq 4$)**.** *Let $M^n = \mathbb{S}^{p+1}_+ \times \mathbb{S}^q$, $q > 1$, with the product metric*

$$g = dr^2 + \sin^2(r)g_{\mathbb{S}^p} + \frac{q-1}{p+1}g_{\mathbb{S}^q},$$

where $r(x,y) = r(x)$ is the height function of $\mathbb{S}^{p+1}$. By considering the potential function $f(r) = \cos(r)$ and $r \leq \frac{\pi}{2}$, one obtains that (M^n, g) must satisfy the Eqs. (1.2) *and* (1.3) *with constant scalar curvature given by $R = (n-1)(p+1)$.*

The previous example has positive constant scalar curvature and hence, it is clearly a positive static triple. Besides, as mentioned, it is simply connected.

Geometric inequalities are classical objects of study in geometry and physics. They are useful in proving new classification results and put away some possible examples of special metrics on a given manifold. In the recent years, the Reilly's formula [92] have been shown a promising tool to gain new geometric inequalities. In [97], Ros used the Reilly's formula to prove the Alexandrov's rigidity theorem for high order mean curvatures. Besides, Miao, Tam and Xie [77] used the Reilly's formula to obtain a stability inequality for Wang-Yau energy. A similar result was obtained by Kwong and Miao [67] to the boundary of static spaces. More recently, Qiu and Xia [91] proved a generalized Reilly's formula that was used to give an alternative proof of the Alexandrov's theorem and prove a new Heintze-Karcher inequality for Riemannian manifolds with boundary and sectional curvature bounded from below. Subsequently, Xia [113] used the generalized Reilly's formula to establish a Minkowski type inequality for weighted mixed volumes in non-Euclidean space forms. Very recently, Diógenes, Pinheiro and Ribeiro [43] used the generalized Reilly's formula by Qiu and Xia to obtain a sharp integral estimates for critical metrics of the volume functional that were used to obtain a sharp boundary estimate for such metrics.

We used the generalized Reilly's formula by Qiu and Xia (Proposition 3.1) to establish a new boundary estimate for SPFST. More precisely, we get the following result.

Theorem 1.1. *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M and positive scalar curvature satisfying*

$$\frac{n-2}{2(n-1)}R + \frac{n}{n-1}\rho = -\tau, \tag{1.4}$$

where τ is a positive constant. Then we have:

$$Vol(M) \geq \frac{1}{\tau}\sqrt{\frac{R_{min} + 3\tau}{2n}}\,|\partial M|. \tag{1.5}$$

Moreover, if equality holds in (1.5), then (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.

Remark 1. *As was mentioned, the constant τ in Theorem 1.1 is positive. Indeed, supposing that $\tau \leq 0$, since $\Delta f = -\tau f$ and f is a nonnegative function with $f^{-1}(0) = \partial M$, we may use the Maximum Principle to infer that $f = 0$ in M, which leads to a contradiction.*

Remark 2. *We highlight that by assuming the dominant energy condition in Theorem 1.1, one obtains that the scalar curvature of M^n must be positive. In fact, the dominant energy condition asserts that $\frac{R}{2} \geq |\rho|$ and hence, if $R(p) = 0$ for some point $p \in M$, then $\rho(p) = 0$, which contradicts the assumption that τ is a positive constant. Moreover, by [37, Proposition 2], we have $\rho_{|\partial M} = -\frac{1}{2} R_{|\partial M}$ and consequently, $\tau = \frac{1}{n-1} R_{|\partial M}$. In particular,* (1.4) *implies that the scalar curvature is constant along the boundary.*

As a consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.1. *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M and constant positive scalar curvature R. Then we have:*

$$Vol(M) \geq \sqrt{\frac{(n-1)(n+2)}{2nR}}\, |\partial M|. \tag{1.6}$$

Moreover, if equality holds in (1.6)*, then (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.*

Before proceeding, it is fundamental to recall the definition of Brown-York mass. Let Σ be a connected hypersurface in (M^n, g) such that $(\Sigma, g_{|\Sigma})$ can be embedded in $\mathbb{R}^n$ as a convex hypersurface. Then, the Brown-York mass $\mathfrak{m}_{BY}$ of Σ with respect to g is given by

$$\mathfrak{m}_{BY}(\Sigma, g) = \int_\Sigma (H_0 - H_g) dS_g,$$

where H_0 and H_g are the mean curvature of Σ as hypersurface of $\mathbb{R}^n$ and M, respectively, and dS_g is the volume element of on Σ induced by g. In [115], motivated by the Riemannian Penrose inequality, Yuan obtained a boundary estimate for static spaces in terms of the Brown-York mass. A similar result was established for quasi-Einstein manifolds by Diógenes, Gadelha and Ribeiro [44]. In another direction, inspired by ideas outlined in [37], Andrade and Melo [2] proved recently that, under suitable conditions, the Hawking mass of Einstein-type manifolds is bounded from below by the area of the boundary.

The next result establishes a sharp boundary estimate for compact SPFST with (possibly disconnected) boundary in terms of the Brown-York mass $\mathfrak{m}_{BY}$.

Theorem 1.2. *Let* (M^n, g, f, ρ), $n \geq 3$, *be a compact static perfect fluid space-time with (possibly disconnected) boundary and positive scalar curvature satisfying the dominant energy condition. Suppose that each boundary component* $(\partial M_i, g)$ *can be isometrically embedded in* $\mathbb{R}^n$ *as a convex hypersurface. Then we have*

$$|\partial M_i| \leq c\,\mathfrak{m}_{BY}(\partial M_i, g), \tag{1.7}$$

where c *is a positive constant. Moreover, equality occurs for some component* ∂M_i *if and only if* M^n *is isometric to the standard hemisphere* $\mathbb{S}^n_+$.

A key ingredient in the proof of Theorem 1.2 is the positive mass theorem for Brown-York mass by Shi-Tam [102], which is equivalent to the (higher dimensional) positive mass theorem for ADM mass by Schoen and Yau [98], [99], [100] and Lohkamp [70]. It should be mentioned that the isometric embedding condition in Theorem 1.2 was needed to use the positive mass theorem. According to the solution of the Weyl problem, the isometrical embedding assumption can be replaced by control on sectional curvatures, as for instance, positive Gaussian curvature when $n = 3$, see, e.g., [46], [115].

As an application of Theorem 1.2 we have the following result.

Corollary 1.2. *Let* (M^n, g, f, ρ), $n \geq 3$, *be a compact static perfect fluid space-time with (possibly disconnected) boundary and positive scalar curvature. Assume the dominant energy condition and that each boundary component* $(\partial M_i, g)$ *can be isometrically embedded in* $\mathbb{R}^n$ *as a convex hypersurface. Then we have*

$$|\partial M_i| \leq \widetilde{c} \int_{\partial M_i} (R^{\partial M_i} + |\mathring{h}_i|^2) dS_g$$

for some positive constant $\widetilde{c}$, *where* $\mathring{h}_i$ *is the traceless second fundamental form of* ∂M_i *as a hypersurface of* $\mathbb{R}^n$. *Moreover, equality occurs for some connected component of the boundary if and only if* (M^n, g) *is isometric to the round hemisphere* $\mathbb{S}^n_+$.

In another direction, but also related to the Einstein metrics and static spaces, we consider the quasi-Einstein manifolds. To be precise, a complete n-dimensional Riemannian manifold (M^n, g), $n \geq 2$, possibly with boundary ∂M, is called an *m-quasi-Einstein manifold*, or simply *quasi-Einstein manifold*, if there is a smooth potential function u on

M^n satisfying the system

$$\begin{cases} \nabla^2 u = \dfrac{u}{m}(Ric - \lambda g) & \text{in } M, \\ u > 0 & \text{on } int(M), \\ u = 0 & \text{on } \partial M, \end{cases} \tag{1.8}$$

for some constants λ and $0 < m < \infty$ (see [26], [59] and [60]). When $m = 1$, we assume in addition that $\Delta u = -\lambda u$ in order to recover the *static equation*: $-(\Delta u)g + \nabla^2 u - uRic = 0$. Moreover, an m-quasi-Einstein manifold will be called *trivial* if u is constant, otherwise it will be *nontrivial*. We notice that the triviality implies that M^n is an Einstein manifold.

The study of quasi-Einstein manifolds is directly related to the existence of warped product Einstein metrics on a given manifold. As discussed by Besse [16, pg. 267], an m-quasi-Einstein manifold corresponds to a base of a warped product Einstein metric; for more details, see, e.g., [16, Corollary 9.107, pg. 267] and [14], [16], [26], [29], [31], [74], [95]. Choosing $u = e^{-\frac{f}{m}}$ in (1.8) when $\partial M = \emptyset$, an ∞-quasi-Einstein manifold is precisely a gradient Ricci soliton (M^n, g, f), see [26], [31], [39] and [95]. Despite similarity, there are examples of quasi-Einstein manifolds that are in stark contrast to the Ricci solitons. Another interesting motivation to investigate quasi-Einstein manifolds derives from the study of diffusion operators by Bakry and Émery [12], which is linked to the theory of smooth metric measure spaces; see, e.g., [15], [27], [28], [74], [95], [106], [108], [109] and the references therein. In particular, 1-quasi-Einstein manifolds are more commonly called *static spaces*. Besides being interesting on their own, as already mentioned, static spaces have connections to the positive mass theorem and general relativity (see [26, Remark 2.3] and [1], [19], [20], [65], [68], [90], [89]). Additionally, quasi-Einstein metrics have attracted interest in physics due to their relation with the geometry of a degenerate Killing horizon and horizon limit; see, e.g., [10], [11] and [110].

Explicit examples of nontrivial compact and noncompact m-quasi-Einstein manifolds can be found in, e.g., [16], [17], [18], [26], [27], [28], [60], [71], [94], [95] and [107]. In "Besse's book" [16, pg. 267-272], it was established the classification of 1 and 2-dimensional m-quasi-Einstein manifolds. Our focus is on nontrivial compact m-quasi-Einstein manifolds with (non-empty) boundary ∂M. Hence, by the work [60, Theorem 4.1], they have necessarily $\lambda > 0$. In this perspective, it is fundamental to recall some examples of compact m-quasi-Einstein manifolds with boundary and constant scalar curvature:

(i) The hemisphere $\mathbb{S}^n_+$ with the standard metric $g = dr^2 + \sin^2 r g_{\mathbb{S}^{n-1}}$ and potential function $u(r) = \cos r$, where r is a height function with $r \leq \frac{\pi}{2}$;

(ii) $\left[0, \sqrt{m/\lambda}\pi\right] \times \mathbb{S}^{n-1}$, for $\lambda > 0$, endowed with the metric $g = dt^2 + \frac{n-2}{\lambda} g_{\mathbb{S}^{n-1}}$ and potential function $u(t,x) = \sin\left(\sqrt{\lambda/m}\, t\right)$;

(iii) $\mathbb{S}^{p+1}_+ \times \mathbb{S}^q$, $q > 1$, with the doubly warped product metric

$$g = dr^2 + \sin^2 r g_{\mathbb{S}^p} + \frac{q-1}{p+m} g_{\mathbb{S}^q},$$

where $r(x,y) = h(x)$ and h is a height function on $\mathbb{S}^{p+1}_+$, potential function $u = \cos r$ with $r \leq \frac{\pi}{2}$ and $\lambda = p + m$.

He, Petersen and Wylie [60, Proposition 2.4] showed that a nontrivial compact quasi-Einstein manifold with boundary and constant Ricci curvature is isometric to Example (i). It turns out that these three quoted examples have constant scalar curvature and therefore, one question that naturally arises is to know *whether a nontrivial compact (simply connected) m-quasi-Einstein manifold with boundary and constant scalar curvature must be necessarily one of them.*

Remark 3. *For dimension $n \geq 5$, it is possible to obtain another example with constant scalar curvature, as it is presented in Chapter 4. We also highlight that Examples (ii) and (iii) above can be presented in a more general setting by replacing the round spheres $\mathbb{S}^{n-1}$ and $\mathbb{S}^q$, respectively, by an arbitrary compact Einstein manifold with positive scalar curvature. Moreover, Example (iii) was obtained recently by Diógenes, Gadelha and Ribeiro in [45] and we adapted it to obtain Example 1.1.*

In Chapter 4, our aim is to investigate compact m-quasi-Einstein manifolds, $m > 1$, (M^n, g, u, λ) with boundary and constant scalar curvature. Specifically, we classify nontrivial compact quasi-Einstein metrics with boundary and constant scalar curvature in dimension 3 and 4. Moreover, we obtain some related results for arbitrary dimensions.

A straightforward computation, by using the classical Reilly's theorem [93, Theorem B], guarantees that the hemisphere $\mathbb{S}^2_+$ is the only nontrivial 2-dimensional simply connected compact m-quasi-Einstein manifold with boundary and constant scalar curvature. In [60], He, Petersen and Wylie investigated m-quasi-Einstein manifolds with constant scalar curvature. In particular, for the specific dimension $n = 3$, they proved that an m-quasi-Einstein manifold with constant scalar curvature is rigid, i.e., it is Einstein or its universal cover is a product of Einstein manifolds (see [60, Theorem 1.3]). Other related results for compact m-quasi-Einstein manifold with boundary and constant scalar curvature were discussed in [42], [45], [59]. Nevertheless, the explicit classification of compact m-quasi-Einstein manifolds with (non-empty) boundary and constant scalar curvature was not established yet. In another direction, Petersen and Wylie [87] studied rigid gradient Ricci solitons. It is known by the works of Hamilton [56], Ivey [62], Perelman [86], Naber [82], Ni-Wallach [83] and Cao-Chen-Zhu [24] that 2 and 3-dimensional gradient shrinking Ricci solitons are rigid and, moreover, they are entirely classified. However, for dimension 4 (or higher), this is no longer true according to the example of Feldman, Ilmanen and Knopf [47]. A more recent result due to Cheng and Zhou [40], combined with Fernández-Lopéz and García-Río [48], establishes the complete classification of 4-dimensional gradient shrinking Ricci solitons with constant scalar curvature, which in turn provides a partial solution for the next problem, raised by Huai-Dong Cao during the IX Workshop on Differential Geometry (2019) in Maceió:

Conjecture 1. *Let (M^n, g, f), $n \geq 4$, be a complete n-dimensional gradient shrinking Ricci soliton. If (M, g) has constant scalar curvature, then it must be rigid, i.e., a finite quotient of $N^k \times \mathbb{R}^{n-k}$ for some Einstein manifold N with positive scalar curvature.*

The results obtained in [36] were also motivated by these results on Ricci solitons.

Inspired by the question mentioned earlier and by works due to Cheng and Zhou [40], Fernández-Lopéz and García-Río [48] and He, Petersen and Wylie [60], we will classified (explicitly) compact 3 and 4-dimensional m-quasi-Einstein manifolds with boundary and constant scalar curvature. To that end, in the same spirit of [48], we first establish the possible values for the constant scalar curvature of an n-dimensional compact m-quasi-Einstein manifold with boundary. More precisely, we have the following result.

Theorem 1.3. *Let (M^n, g, u, λ) be a nontrivial compact m-quasi-Einstein manifold with boundary, $m > 1$ and constant scalar curvature R. Then we have:*

$$R \in \left\{ \frac{n(n-1)}{m+n-1}\lambda, \frac{m+n(n-2)}{m+n-2}\lambda, \ldots, (n-1)\lambda \right\}. \tag{1.9}$$

In general, one has $R = \frac{k(m-n)+n(n-1)}{m+n-k-1}\lambda$, for some $k \in \{0, 1, \ldots, n-1\}$.

We point out that the value of the scalar curvature in (1.9) may be regarded in terms of the dimension k of the set of critical points (or equivalently, the maximum points); see the proof of Theorem 1.3 in Chapter 4. In Example (i), we see that $R = \frac{n(n-1)\lambda}{m+n-1}$ and the only critical point is the north pole, i.e., $k=0$. In Example (ii), we have $R=(n-1)\lambda$ and the set of critical points for the potential function $\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)$ is precisely $\left\{\frac{\sqrt{m}}{\sqrt{\lambda}}\frac{\pi}{2}\right\}\times\mathbb{S}^{n-1}$, which has dimension $n-1$. While in Example (iii), it holds that $R=\frac{q(m-n)+n(n-1)}{m+n-q-1}\lambda$ and the set of critical points for the potential function is $\{north\,pole\}\times\mathbb{S}^q$.

Remark 4. *It follows from the proof of Theorem 1.3 that, under a mild modification, for a (not necessarily compact with boundary) quasi-Einstein manifold with constant scalar curvature R and $m>1$, one has*

$$R\in\left\{\frac{n(n-1)}{m+n-1}\lambda,\frac{m+n(n-2)}{m+n-2}\lambda,\ldots,(n-1)\lambda,n\lambda\right\},$$

provided that the set of critical points of the potential function is non-empty.

Before discussing our next result, we recall that if an m-quasi-Einstein manifold has constant scalar curvature R and $m>1$, then

$$|\mathring{Ric}|^2=-\frac{m+n-1}{n(m-1)}(R-n\lambda)\left(R-\frac{n(n-1)}{m+n-1}\lambda\right); \tag{1.10}$$

for more details, see [59, Proposition 3.3] and [26, Lemma 3.2].

Remark 5. *Observe that in considering $R=\frac{n(n-1)}{m+n-1}\lambda$ into (1.10), i.e., the lower value of (1.9), one deduces that M^n is necessarily Einstein and therefore, it suffices to apply [60, Proposition 2.4] to conclude that M^n is isometric to the standard hemisphere $\mathbb{S}^n_+$.*

The next result address the value $R=\frac{m+n(n-2)}{m+n-2}\lambda$ for the scalar curvature on quasi-Einstein manifolds with boundary.

Proposition 1.1. *There is no compact nontrivial quasi-Einstein manifold M^n with boundary and constant scalar curvature $R=\frac{m+n(n-2)}{m+n-2}\lambda$.*

In the sequel, we shall consider the extremal value case of (1.9), namely, $R=(n-1)\lambda$. In this situation, we have the following result which can be compared with [60, Theorem 1.9].

Theorem 1.4. *Let (M^n,g,u,λ), $n\geq 3$, be a nontrivial simply connected compact m-quasi-Einstein manifold with boundary and $m>1$. Then M^n has constant scalar curvature $R=(n-1)\lambda$ if and only if it is isometric, up to scaling, to the cylinder $I\times N$ with product metric, where N is a compact λ-Einstein manifold.*

As a consequence of Theorem 1.4 and Proposition 2.4 in [60], we shall obtain an explicit classification for compact 3-dimensional m-quasi-Einstein manifolds with boundary and constant scalar curvature. To be precise, we have the following result.

Theorem 1.5. *Let (M^3, g, u, λ) be a nontrivial simply connected compact 3-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^3 has constant scalar curvature if and only if it is isometric, up to scaling, to either*

(a) the standard hemisphere $\mathbb{S}^3_+$, or

(b) the cylinder $I \times \mathbb{S}^2$ with the product metric.

From now on, we focus on dimension $n = 4$. In this scenario, it is known from Theorem 1.3 that the possible values for the constant scalar curvature R are

$$\left\{\frac{12}{m+3}\lambda, \frac{m+8}{m+2}\lambda, 2\frac{(m+2)}{(m+1)}\lambda, 3\lambda\right\}.$$

If $R = \frac{12}{m+3}\lambda$, it then follows from Remark 5 that M^4 is isometric, up to scaling, to the standard hemisphere $\mathbb{S}^4_+$. Besides, by Proposition 1.1, there is no compact 4-dimensional quasi-Einstein manifold with boundary and constant scalar curvature $R = \frac{m+8}{m+2}\lambda$. In the case $R = 3\lambda$, it suffices to invoke Theorem 1.4 to conclude that M^4 is isometric to the cylinder $I \times \mathbb{S}^3$ with product metric. Interestingly, Example (*iii*) has constant scalar curvature $R = 2\frac{(m+2)}{(m+1)}\lambda$. This fact has left open the question of whether $\mathbb{S}^2_+ \times \mathbb{S}^2$ is the unique 4-dimensional compact quasi-Einstein manifold with boundary and constant scalar curvature $R = 2\frac{(m+2)}{(m+1)}\lambda$. To answer this question, we have established the following rigidity result.

Theorem 1.6. *Let (M^4, g, u, λ) be a nontrivial simply connected compact 4-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^4 has constant scalar curvature $R = 2\frac{(m+2)}{(m+1)}\lambda$ if and only if it is isometric, up to scaling, to the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ with the doubly warped product metric.*

The proof of Theorem 1.6 is essentially inspired by the work of Cheng and Zhou [40]. As a consequence of Theorem 1.3, Remark 5, Theorem 1.4 and Theorem 1.6, we get the following classification result.

Corollary 1.3. *Let (M^4, g, u, λ) be a nontrivial simply connected compact 4-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^4 has constant scalar curvature if and only if it is isometric, up to scaling, to either* *(i) the standard hemisphere* $\mathbb{S}^4_+$, *or*

(ii) the cylinder $I \times \mathbb{S}^3$ *with the product metric, or*

(iii) the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ *with the doubly warped product metric.*

2 PRELIMINARIES

The purpose of this chapter is to establish the fundamental concepts and basic tools that will be used in the next chapters. Section 2.1 collects general features on Riemannian geometry and other related results. In Sections 2.2 and 2.3, it will be presented the general concepts of static perfect fluid space-time and quasi-Einstein manifolds, respectively.

2.1 Basic notations and auxiliary results

Let M^n be a smooth manifold and g its Riemannian metric. We denote a Riemannian manifold as (M^n, g). A Riemannian connection is an affine connection ∇ such that, for all smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

With these considerations, the Riemannian curvature tensor is the covariant 4-tensor $Rm : \mathfrak{X}^4(M) \to C^\infty(M)$ given by

$$\begin{aligned} Rm(X,Y,Z,V) &= g(R(X,Y)V, Z) \\ &= g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V, Z), \end{aligned}$$

for all $X, Y, Z, V \in \mathfrak{X}(M)$, where the covariant 3-tensor R is usually known as curvature tensor. Over a point $p \in M$, we consider a coordinate system $\{x^i\}_{i=1}^n$. Then, we may express

$$R_{ijkl} = Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right).$$

Using this coordinates, we define the Ricci tensor as the trace of the Riemannian tensor as follows

$$Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = R_{ik} = g^{jl} R_{ijkl},$$

where we are adopting the Einstein summation convention. Similarly, the scalar curvature R is given by

$$R = g^{ik} R_{ik}.$$

Given two covariant 2-tensors S and T, we define the Kulkarni-Nomizu product $\odot$ between S and T as

$$(S\odot T)_{ijkl} = S_{ik}T_{jl} + S_{jl}T_{ik} - S_{il}T_{jk} - S_{jk}T_{il}.$$

Now, we recall the Weyl tensor, denoted by W and defined by

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(Ric\odot g)_{ijkl} - \frac{R}{2(n-1)(n-2)}(g\odot g)_{ijkl}. \tag{2.1}$$

The Weyl tensor is the traceless part of the Riemannian curvature tensor and it has all properties of that tensor with the addition that it is traceless with respect to any two indices. Related to the Weyl tensor, we have the Cotton tensor C given by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \tag{2.2}$$

In dimension 3, it is well known that the Weyl tensor vanishes. Besides, for $n \geq 4$, zero Weyl tensor is equivalent to locally conformally flatness. For $n \geq 4$, we can obtain from (2.2) that

$$\nabla_l W_{ijkl} = -\frac{n-3}{n-2}C_{ijk}.$$

The Cotton tensor is skew-symmetric with respect to the first two indices and it satisfies the first Bianchi identity, i.e.,

$$C_{ijk} = -C_{jik} \quad \text{and} \quad C_{ijk} + C_{jki} + C_{kij} = 0.$$

Furthermore, similar to the Weyl tensor, the cotton tensor is trace-free with respect to any two indices. For $n = 3$, $C = 0$ if and only if the manifold is locally conformally flat.

Now, given a function $f \in C^\infty(M)$, the covariant symmetric 2-tensor Hessian of f, denoted by $\nabla^2 f$, is defined as

$$\nabla^2 f(X,Y) = Y(X(f)) - \nabla_Y X(f) = g(\nabla_X \nabla f, Y).$$

In coordinates, we denote $\nabla^2 f(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \nabla_i \nabla_j f$. Taking the trace of the Hessian, we obtain the Laplacian of f, i.e., $g^{ij}\nabla_i\nabla_j f = \Delta f$.

Proceeding, we recall the concept of *warped products*.

Definition 2.1. *Let (B^n, g_B) and (F^l, g_F) be Riemannian manifolds and $\varphi \in C^\infty(B)$ a positive smooth function. We say that the product $B \times F$ endowed with the metric*

$g = g_B + \varphi^2 g_F$ is a warped product. We denote this product as $B \times_\varphi F$, where B and F are named base and fiber of the product, respectively, and the metric g is called warped product metric.

We say that lifting of tangent vectors to B and F are called horizontal and vertical tangent vector to the warped product, respectively.

One can obtains the Hessian of a smooth function $f \in C^\infty(M)$ by the Lie derivative as

$$\nabla^2 f = \frac{1}{2}\mathcal{L}_{\nabla f} g.$$

It turns out that the Lie derivative is an important tool to do calculations when we put it together with the Cartan's formula.

Lemma 2.1 (Cartan's formula)**.** *Let ω be a k-form and $X \in \mathfrak{X}(M)$. Then, we have that*

$$\mathcal{L}_X(\omega) = d(i_X\omega) + i_X(d\omega),$$

where i and d stand for the interior product and the exterior derivative, respectively.

As an example of application of the Lie derivative, one can compute the Hessian of a smooth function f defined on a warped product $(I \times F, dt^2 + \varphi^2(t)g_F)$, with $f(t,x) = f(t)$. By linearity and product rule for the Lie derivative, one sees that

$$\begin{aligned} 2\nabla^2 f &= \mathcal{L}_{\nabla f} g = \mathcal{L}_{\nabla f}(dt^2 + \varphi^2(t)g_F) \\ &= \mathcal{L}_{\nabla f}(dt^2) + (\mathcal{L}_{\nabla f}(\varphi^2(t)))g_N + \varphi^2(t)\mathcal{L}_{\nabla f}(g_N). \end{aligned}$$

By the properties of Lie derivative and the fact that f do not depends on the variables in the fiber F, we have $\mathcal{L}_{\nabla f}(dt^2) = (\mathcal{L}_{\nabla f} dt) \otimes dt + dt \otimes (\mathcal{L}_{\nabla f} dt)$, $\mathcal{L}_{\nabla f}(\varphi^2(t)) = 2\varphi(t)g(\nabla f, \nabla\varphi)$ and $\mathcal{L}_{\nabla f} g_F = 0$. Since $\nabla f = f'(t)\nabla t$ and dt is an exact 1-form, Lemma 2.1 allows us to conclude that $\mathcal{L}_{\nabla f} dt = f''(t)dt$. Therefore,

$$\nabla^2 f = f''(t)dt^2 + f'(t)\varphi(t)\varphi'(t)g_N. \tag{2.3}$$

It is also interesting to present the computations of the curvature tensors of a warped product in terms of the base and fiber metrics. The following proposition corresponds to [85, Corollary 43].

Proposition 2.1 ([85]). *The Ricci curvature of a warped product manifold $M = B\times_{\varphi} F$ with $m = dim(F)$, X,Y and Z,V any horizontal and vertical vectors, respectively, satisfies:*

i. $Ric(X,Y) = Ric_B(X,Y) - \frac{m}{\varphi}\nabla^2_{g_B}\varphi(X,Y)$,

ii. $Ric(X,V) = 0$,

iii. $Ric(Z,V) = Ric_F(Z,V) - (\varphi\Delta_{g_B}\varphi + (m-1)|\nabla\varphi|^2_{g_B})g_F(Z,V)$.

We will make use of (2.3) and the expressions in Proposition 2.1 to make many computations along this work. As a consequence of Proposition 2.1, we get the following result.

Proposition 2.2. *Let (M^n, g) be a warped product manifold with $g = dt^2 + \varphi^2(t)g_{_N}$, where $g_{_N}$ is an κ-Einstein metric, i.e., $Ric_N = \kappa g_N$, with $\kappa > 0$. Suppose that either $\varphi(t) = \alpha t$ or $\varphi(t) = a\sinh(\sqrt{\beta}t) + b\cosh(\sqrt{\beta}t)$, where α and β are positive constants and $a, b \in \mathbb{R}$. Then the scalar curvature R of M^n can not be a positive constant.*

Proof. We shall divide the proof into two cases. First, we assume that $\varphi(t) = \alpha t$. Thus, by Proposition 2.1, one obtains that

$$Ric(\partial_t, \partial_t) = 0 \quad \text{and} \quad Ric(V,W) = (\kappa - (n-2)\alpha^2)g_{_N}(V,W).$$

From this, we have

$$R = (n-1)\frac{\kappa - (n-2)\alpha^2}{\alpha^2 t^2}.$$

By assuming that R is a positive constant, one concludes that φ is constant, which leads to a contradiction.

Secondly, we assume that $\varphi(t) = a\sinh(\sqrt{\beta}t) + b\cosh(\sqrt{\beta}t)$. Using Proposition 2.1 again yields

$$\begin{aligned} Ric(\partial_t, \partial_t) &= -\frac{n-1}{\varphi}\varphi'' = -(n-1)\beta, \\ Ric(V,W) &= [\kappa - (\varphi\varphi'' + (n-2)(\varphi')^2)]g_{_N}(V,W) \\ &= [\kappa - \beta\varphi^2 - (n-2)(\varphi')^2]g_{_N}(V,W), \end{aligned}$$

where we have used that $\varphi'' = \beta\varphi$. By tracing, we then get

$$R = -2(n-1)\beta + (n-1)\frac{\kappa - (n-2)(\varphi')^2}{\varphi^2}. \tag{2.4}$$

Now, by assuming that R is a positive constant, one obtains that $\frac{\kappa-(n-2)(\varphi')^2}{\varphi^2}$ is also a constant. Therefore, taking the derivative, we see that

$$\frac{-2(n-2)\varphi'\varphi''\varphi^2 - 2\varphi\varphi'(\kappa-(n-2)(\varphi')^2)}{\varphi^4} = 0, \tag{2.5}$$

and since φ and φ' can only vanish in a set of measure zero and the fact that $\varphi'' = \beta\varphi$, one deduces that (2.5) is equivalent to

$$(n-2)((\varphi')^2 - \beta\varphi^2) - \kappa = 0.$$

Plugging this into (2.4), we arrive at $R = -n(n-1)\beta < 0$, which also leads to a contradiction. $\square$

Before to conclude this subsection, we need to recall some useful facts on distance function that are used in the proof of the main results in Chapter 4. Let M be a complete Riemannian manifold and N a properly immersed submanifold of M. Assume that $\pi : \nu N \to N$ is the normal bundle. There is an induced connection ∇^ν on νN and a decomposition of tangent bundle $T(\nu N)$ as

$$T(\nu N) = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{V}_\xi := \ker(d\pi)_\xi$ and $\mathcal{H}_\xi$ consists of all tangent vectors to parallel sections passing through ξ. If $\alpha : (-\delta, \delta) \to \nu N$ is a smooth curve representing $v \in T(\nu N)$, then $v^{\mathcal{H}} = (\pi \circ \alpha)'(0)$ and $v^{\mathcal{V}} = (\frac{\nabla^\nu}{\partial s}\alpha)(0) = v - \mathcal{V}^{\mathcal{H}}$. Thus, $\mathcal{H}_\xi$ and $\mathcal{V}_\xi$ are isomorphic to $T_{\pi(\xi)}N$ and $\nu_{\pi(\xi)}N$, respectively. This decomposition induces a natural Riemannian metric on $T(\nu N)$ such that π is a Riemannian submersion. With aid of this notation, we have the following lemma.

Lemma 2.2 ([13])**.** *Let $\alpha : (-\delta, \delta) \to \nu N$ be a smooth curve representing $v \in T(\nu N)$. Define*

$$J(t) := \frac{\partial}{\partial s}\bigg|_{s=0} \exp_{\pi\circ\alpha(s)}(t\alpha(s)).$$

Then $J(t)$ is a Jacobi field along the geodesic $\gamma(t) = \exp(t\alpha(0))$ and

$$J(0) = v^{\mathcal{H}}, \;\; J(1) = (d\exp)_{\alpha(0)}(v) \quad \textit{and} \quad J'(0) = v^{\mathcal{V}} + A_{\alpha(0)}v^{\mathcal{H}}.$$

Here, A_η stands for the shape operator with respect to normal vector η.

Proceeding, let UN be the unit normal bundle of N equipped with volume element $d\theta dp$, where dp denotes the volume element of N and $d\theta$ is the volume element of unit sphere $\mathbb{S}_p^{n-k-1}$ in $\nu_p N$. Thereby, we may define $\Phi : (0,+\infty)\times UN \to M\backslash N$ by $\Phi(r,\theta) = \exp(r\theta)$.

Along the normal geodesic $\gamma_\theta(r) = \exp(r\theta)$, we can choose a parallel orthonormal base $\{e_1(r),\ldots,e_n(r)\}$ such that

$$A_\theta e_i(0) = \lambda_i, \text{ for } i = 1,\ldots,k-1, \quad \text{and} \quad e_n = \partial r = \gamma_\theta'(r).$$

Hence, $J_i(r) = (d\Phi)_{(r,\theta)}(e_i)$, $i = 1,2,\cdots,n$, must satisfy

$$\begin{aligned}
&J_i''(t) + R(\gamma_\theta'(t), J_i(t))\gamma_\theta'(t) = 0, \text{ for } i = 1,\ldots,k;\\
&J_i(0) = e_i(0), \text{ for } i = 1,\ldots,k;\\
&J_i'(0) = \lambda_i e_i(0), \text{ for } i = 1,\ldots,k;\\
&J_i(0) = 0, \text{ for } i = k+1,\ldots,n;\\
&J_i'(0) = e_i(0), \text{ for } i = k+1,\ldots,n.
\end{aligned}$$

Next, we consider the following notation

$$\begin{aligned}
&J_{ij} = \langle J_i, e_j\rangle, \text{ for } i = 1,\ldots,k;\\
&K_{ij} = \langle R(\gamma_\theta, e_i)\gamma_\theta, e_j\rangle, \text{ for } i = 1,\ldots,k;\\
&\mathcal{A} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n).
\end{aligned}$$

Also consider $\mathcal{J} := (J_{ij})_{(k-1)\times(k-1)}$ and $\mathcal{K} := (K_{ij})_{(k-1)\times(k-1)}$. With these notations, one obtains that

$$\begin{cases}
\mathcal{J}'' + \mathcal{K}\mathcal{J} = 0;\\
\mathcal{J}(0) = \mathrm{diag}\left(\mathcal{I}_{k\times k}, \mathcal{O}_{(n-k-1)\times(n-k-1)}\right);\\
\mathcal{J}'(0) = \mathrm{diag}\left(\mathcal{A}, \mathcal{I}_{(n-k-1)\times(n-k-1)}\right).
\end{cases}$$

If $\gamma_\theta|_{[0,r]}$ does not contain focal points, then $\mathcal{J}$ is invertible on $(0,r)$. Next, let $\sigma(x)$ be the distance function from N. Therefore, $\sigma(\gamma_\theta(r)) = r$, provided that $r \in (0, r_\theta)$. Moreover, by denoting $\mathcal{U}_{ij}(r) := \nabla^2\sigma(e_i,e_j)(\gamma_\theta(r))$ and taking into account that $\nabla^2\sigma(J_i,J_j) = \langle J_i', J_j\rangle$, we get the following lemma.

Lemma 2.3 ([13]). *Let N be a proper submanifold in M. Then for any $\theta \in \nu N$, along the normal geodesic $\gamma_\theta(r) = \exp(r\theta)$, the Hessian of the distance function $\sigma(x) = dist(x,N)$*

satisfies

$$\begin{cases} \mathcal{U}' + \mathcal{U}^2 + \mathcal{K} = 0, \\ \mathcal{U} = \begin{pmatrix} \mathcal{A}_\theta & \\ & \frac{1}{r}\mathcal{I} \end{pmatrix} + r \begin{pmatrix} -\mathcal{A}_\theta^2 - \mathcal{K}_{11}(0) & \mathcal{V}_{12} \\ \mathcal{V}_{21} & \mathcal{V}_{22} \end{pmatrix} + O(r^2), \end{cases}$$

where $\mathcal{U} = \nabla^2\sigma|_{\{\gamma_\theta'(r)\}^\perp}$, $\mathcal{K} = \mathcal{K}_\theta = R(\gamma_\theta', \ldots)\gamma_\theta'$ *and* $\mathcal{A}_\theta$ *is the shape operator of* N *with respect to* θ. *In particular, the mean curvature* $H(\theta, r)$ *of the level sets of* σ *at* $\gamma_\theta(r)$ *satisfies*

$$H(\theta, r) = tr(\mathcal{A}_\theta) + \frac{n-k-1}{r} + O(r) \tag{2.6}$$

and

$$\nabla^2 \frac{\sigma^2}{2}(\gamma_\theta(r)) = \begin{pmatrix} r\mathcal{A}_\theta & \\ & \mathcal{I}_{(n-k)\times(n-k)} \end{pmatrix} + O(r^2). \tag{2.7}$$

Moreover, at N, *the function* σ^2 *has two eigenvalues* 0 *and* 2 *of multiplicities* m *and* $n-k$, *respectively.*

To conclude this subsection, we are going to present the proof of the following algebraic inequality.

Lemma 2.4. *Let* $a_1 \geq \ldots \geq a_n$ *be* $n \geq 2$ *real numbers. Then*

$$a_i a_j \geq \frac{b}{2(n-1)},$$

where $b = \left(\sum_{i=1}^n a_i\right)^2 - (n-1)\sum_{i=1}^n a_i^2$. *In particular, if* $b \geq 0$, *then either all* $a_i \geq 0$ *or all* $a_i \leq 0$.

Proof. The case $n = 2$ is straightforward. Now, for $n > 2$, notice that

$$\left(\sum_{i=1}^n a_i\right)^2 = \left(\sum_{i=1}^{n-1} a_i\right)^2 + 2a_n \sum_{i=1}^{n-1} a_i + a_n^2.$$

Hence, we see that

$$(n-1)\sum_{i=1}^n a_i^2 + b = \left(\sum_{i=1}^{n-1} a_i\right)^2 + 2a_n \sum_{i=1}^{n-1} a_i + a_n^2,$$

so that

$$(n-1)\sum_{i=1}^{n-1} a_i^2 + (n-2)a_n^2 + b = \left(\sum_{i=1}^{n-1} a_i\right)^2 + 2a_n \sum_{i=1}^{n-1} a_i.$$

In view of this, one obtains that

$$(n-2)\sum_{i=1}^{n-1} a_i^2 + (n-2)a_n^2 - 2a_n \sum_{i=1}^{n-1} a_i + b = \left(\sum_{i=1}^{n-1} a_i\right)^2 - \sum_{i=1}^{n-1} a_i^2,$$

which implies that

$$2\sum_{i<j\leq n-1} a_i a_j = (n-2)\sum_{i=1}^{n-1} a_i^2 + (n-2)a_n^2 - 2a_n \sum_{i=1}^{n-1} a_i + b.$$

Rearranging terms, one sees that

$$(n-2)a_n^2 - 2\left(\sum_{i=1}^{n-1} a_i\right) a_n + \left[(n-2)\sum_{i=1}^{n-1} a_i^2 + b - 2\sum_{i<j\leq n-1} a_i a_j\right] = 0.$$

Of which, we have

$$\begin{aligned}\left(\sum_{i=1}^{n-1} a_i\right)^2 &= (n-2)\left[(n-2)\sum_{i=1}^{n-1} a_i^2 + b - 2\sum_{i<j\leq n-1} a_i a_j\right] + (n-2)^2\left(a_n - \frac{1}{n-2}\sum_{i=1}^{n-1} a_i\right)^2 \\ &= (n-2)\left[(n-1)\sum_{i=1}^{n-1} a_i^2 - \left(\sum_{i=1}^{n-1} a_i\right)^2 + b\right] + (n-2)^2\left(a_n - \frac{1}{n-2}\sum_{i=1}^{n-1} a_i\right)^2.\end{aligned}$$

Consequently,

$$\left(\sum_{i=1}^{n-1} a_i\right)^2 \geq (n-2)\sum_{i=1}^{n-1} a_i^2 + \frac{n-2}{n-1}b. \tag{2.8}$$

Moreover, if equality holds in (2.8), then $a_n = \frac{1}{n-2}\sum_{i=1}^{n-1} a_i$. Now, it suffices to repeat an analogous process $n-2$ times in order to obtain the asserted inequality. □

Notice that the same conclusion is true if one assumes that $b \leq \left(\sum_{i=1}^{n} a_i\right)^2 - (n-1)\sum_{i=1}^{n} a_i^2$. As an application of Lemma 2.4, one has that $|\mathring{Ric}|^2 \leq \frac{R^2}{n(n-1)}$ implies that the eigenvalues of the Ricci tensor have the same sign. Indeed, since $|\mathring{Ric}|^2 = |Ric|^2 - \frac{R^2}{n}$, we can take an orthonormal frame $\{e_i\}_{i=1}^{n}$ which diagonalizes tensor Ric, i.e., $Ric(e_i) = \lambda_i e_i$, in order to achieve at

$$\sum_{i=1}^{n} \lambda_i^2 \leq \frac{1}{n-1}\left(\sum_{i=1}^{n} \lambda_i\right)^2.$$

Applying Lemma 2.4 with $b = 0$, one obtains that $\lambda_i \lambda_j \geq 0$, which says that all eigenvalues have the same sign.

2.2 Static perfect fluid space-time

In this section, we are going to talk about static perfect fluid space-times (SPFST). This class of manifolds arises as special solution to Einstein field equation over a static space-time $(\widehat{M}^{n+1}, \widehat{g}) = M^n \times_f \mathbb{R}$, which is endowed with the static metric $\widehat{g} = g - f^2 dt^2$, that is

$$Ric - \frac{R}{2} g + \Lambda g = \kappa T, \tag{2.9}$$

where the cosmological constant Λ is zero and stress-energy-momentum tensor T corresponds to a perfect fluid, i.e.,

$$T = \mu f^2 dt^2 + \rho g,$$

where the smooth functions μ and ρ are, respectively, mass-energy density and pressure of the fluid. The name "perfect" becomes from the absence of heat conduction terms and stress terms corresponding to viscosity. Explicitly, SPFST is a semi-Riemannian manifold $(\widehat{M}^{n+1}, \widehat{g})$ satisfying

$$Ric_{\widehat{g}} - \frac{R_{\widehat{g}}}{2} \widehat{g} = \mu f^2 dt^2 + \rho g. \tag{2.10}$$

Although the general concept refers to a Lorentzian manifold, we will consider only the spatial factor M of $\widehat{M}$ as a static perfect fluid space-time. Formally, we have the following definition (see also [101], [66]).

Definition 2.2. *A Riemannian manifold (M^n, g) is said to be a spatial factor of a static perfect fluid space-time if there exist smooth functions $f > 0$ and ρ on M^n satisfying the perfect fluid equations:*

$$f \mathring{Ric} = \mathring{\nabla}^2 f \tag{2.11}$$

and

$$\Delta f = \left(\frac{n-2}{2(n-1)} R + \frac{n}{n-1} \rho \right) f, \tag{2.12}$$

where $\mathring{Ric}$, $\mathring{\nabla}^2$ stand for the traceless Ricci and traceless Hessian tensor, respectively. When M has non-empty boundary ∂M, it will be assumed in addition that $f^{-1}(0) = \partial M$. In this case, f will be called a potential function and we denote such a space as (M^n, g, f, ρ).

The equation (2.11) is satisfied for a large class of manifolds, as for example, Riemannian manifolds (M^n, g) satisfying

$$-\Delta f g + \nabla^2 f - f Ric = \delta g$$

for some smooth function f on M and a constant δ. If $\delta = 0$, we obtain the fundamental equation of a positive static triple [1]. When $\delta = 1$, we obtain the expression of a critical metric of the volume functional (or V-static space) [76].

The motivation for Definition 2.2 comes from the warped product curvature expressions. Indeed, by items (i) and (iii) of Proposition 2.1 and taking into account that the dimension of the fiber is one, we infer that

$$R_{\widehat{g}} = R - \frac{1}{f}\Delta f - \frac{1}{f}\Delta f = R - \frac{2}{f}\Delta f.$$

Evaluating the Einstein equation (2.10) on horizontal vectors and using the above equation, we then obtain

$$Ric_M - \frac{\nabla^2 f}{f} - \frac{R}{2} g + \frac{\Delta f}{f} g = \rho g. \tag{2.13}$$

Taking the trace, one sees that

$$\Delta f = \left(\frac{n-2}{2(n-1)} R + \frac{n}{n-1} \rho \right) f, \tag{2.14}$$

which corresponds to (2.12). Furthermore, rewriting (2.13), we see that

$$\left(Ric_M - \frac{R}{n} g \right) - \frac{1}{f} \left(\nabla^2 f - \frac{\Delta f}{n} g \right) = \frac{n-2}{2n} Rg + \rho g - \frac{n-1}{n} \frac{\Delta f}{f} g$$

and hence (2.14) implies that the right-hand member is zero and so, we obtain (2.11).

The warped product curvature expressions also allows us to obtain an expression for the mass-energy density μ. In fact, evaluating (2.10) in vertical vectors and using item (iii) of Proposition 2.1, we deduce

$$f \Delta f + \frac{R}{2} f^2 - f \Delta f = \mu f^2,$$

i.e.,

$$\mu = \frac{R}{2}.$$

Thus, the dominant energy condition is equivalent to $\frac{R}{2} \geq |\rho|$. Notice that we have used in this calculations that the fiber of the warped product $\widehat{M}$ is endowed with a semi-Riemannian metric.

Static perfect fluid space-times with boundary constitutes as a natural extension of static spaces. The following proposition corresponds to Proposition 2 of [37] and establishes the connection between these classes of manifolds.

Proposition 2.3 ([37])**.** *Let (M^n, g, f, ρ) be a static perfect fluid space-time. Then the scalar curvature R of M is constant if and only if $(\frac{1}{2}R+\rho)f$ is constant.*

Proof. First of all, by using the well known formula $div(\nabla^2 f) = Ric(\nabla f) + \nabla\Delta f$, one sees that

$$div(\mathring{\nabla}^2 f) = \nabla\Delta f + Ric(\nabla f) - \frac{1}{n}\nabla\Delta f = \frac{n-1}{n}\nabla\Delta f + Ric(\nabla f). \tag{2.15}$$

On the other hand, it is easy to check that

$$div(f\mathring{Ric}) = f div(Ric) + Ric(\nabla f) - \frac{R}{n}\nabla f - \frac{f}{n}\nabla R.$$

By the twice contracted second Bianchi identity, i.e., $div(Ric) = \frac{1}{2}\nabla R$, we infer that

$$div(f\mathring{Ric}) = \frac{n-2}{2n} f\nabla R + Ric(\nabla f) - \frac{R}{n}\nabla f. \tag{2.16}$$

Now, taking into account (2.11) combined with (2.15) and (2.16), one deduces that

$$\frac{n-2}{2n} f\nabla R - \frac{R}{n} = \frac{n-1}{n}\nabla\Delta f. \tag{2.17}$$

This jointly with the Laplacian equation (2.12) yields

$$\frac{n-3}{2n} f\nabla R = \frac{1}{2}R\nabla f + \rho\nabla f + \frac{n-2}{2n} f\nabla R + f\nabla\rho,$$

which can be rewritten as

$$\frac{1}{2}R\nabla f + \rho\nabla f + f\nabla\rho = 0.$$

Of which, one deduces that

$$\frac{1}{2} f\nabla R = \nabla\left[\left(\frac{1}{2}R + \rho\right) f\right], \tag{2.18}$$

and the result follows. □

Notice that this result suggests that the static perfect fluid space-time alone does not implies the constancy of the scalar curvature, which occurs in the case of vacuum static and V-static spaces. Indeed, the works [9] and [73] present examples of non-compact static perfect fluid space-times with non-constant scalar curvature. Moreover, if the manifold has non-empty boundary, then R is constant if and only if $\rho = -\frac{R}{2}$, in virtue of $f_{|\partial M} = 0$, and so the potential function must satisfy

$$-\Delta f g + \nabla^2 f - fRic = 0,$$

which is precisely the equation of a static manifold. In addition, if we have $R = 0$, then it holds $\rho = 0$ and we achieve at

$$fRic = \nabla^2 f \text{ and } \Delta f = 0,$$

which is the vacuum static equation with null cosmological constant. This is a special case for the physics studies because one of the most important solutions of the above equation is the Schwarzschild metric [16], [103], which models black holes.

We are now able to present some examples of compact static perfect fluid space-times with boundary.

Example 2.1. *Let $\mathbb{S}^n_+$ with the metric $g = dr^2 + \sin^2(r) g_{\mathbb{S}^{n-1}}$ be the standard hemisphere, when $r \leq \frac{\pi}{2}$ is the height function. Thus $\mathbb{S}^n_+$ with potential function $f(r) = \cos(r)$ is a static perfect fluid space-time.*

First of all, we know that $\mathbb{S}^n_+$ with the standard metric is an Einstein manifold, i.e., $\mathring{Ric} = 0$. From the Lie derivative, Lemma 2.1 and using that

$$\nabla f(r) = -\sin(r)\nabla r,$$

one deduces that

$$\nabla^2 f(r) = -\cos(r)(dr^2 + \sin^2(r) g_{\mathbb{S}^{n-1}}) = -f(r)g,$$

then $\mathring{\nabla}^2 f = 0$ and $\Delta f = -\frac{R}{n-1} f$, since $R = n(n-1)$. Moreover, it is clear that $f^{-1}(0) = \partial M$. This shows that this space is in fact a static perfect fluid space-time.

The example above corresponds to the case of connected boundary. But, it is also interesting to present an example with disconnected boundary, as follows.

Example 2.2. *Let $M = [0, \pi] \times \mathbb{S}^{n-1}$ be a Riemannian product with metric $g = dt^2 + (n-2) g_{\mathbb{S}^{n-1}}$ and potential function $f(t) = \sin(t)$. Thus M is a compact oriented static perfect fluid space-time with disconnected boundary (the boundary is the union of two copies of $\mathbb{S}^{n-1}$).*

In fact, notice that $f(p)=0$ if and only if $p\in\{0\}\times\mathbb{S}^{n-1}\cup\{\pi\}\times\mathbb{S}^{n-1}$, that is a disjoint union. Next, again by the Lie derivative, Lemma 2.1 and

$$\nabla f(t)=\cos(t)\nabla t,$$

one obtains that

$$\nabla^2 f(t)=-\sin(t)dt^2=-f(t)dt^2.$$

Noticing that $Ric=(n-2)g_{\mathbb{S}^{n-1}}$, we arrive at

$$fRic-f\frac{R}{n}g=-\frac{n-1}{n}fdt^2+f\frac{n-2}{n}g_{\mathbb{S}^{n-1}}=\nabla^2 f-\frac{\Delta f}{n}g,$$

which concludes the argument. Finally, observe that $R=(n-1)$ implies $\Delta f=-\frac{R}{n-1}f$.

Since the scalar curvature, in both examples, is constant, one concludes that they are static spaces. They receive particular terminology in this theory: the hemisphere in Example 2.1 is called de Sitter solution and the cylinder in Example 2.2 is named Nariai solution of the static equation.

An interesting property of manifolds satisfying equations similar to (2.11) and (2.12) is the analyticity of their metric and potential function in harmonic coordinates. The proof is quite similar to [33, Proposition 2.8] for static metrics (see also [41, Proposition 2.3] for V-static metrics).

Proceeding, it follows from [69, Lemma 1] that $|\nabla f|$ does not vanish on the boundary. The next lemma is an alternative proof of this fact.

Proposition 2.4. *Let (M^n,g,f,ρ) be a compact static perfect fluid space-time with boundary ∂M. Then $|\nabla f|$ is a nonzero constant along ∂M.*

Proof. Since f vanishes on ∂M, one sees from Eqs. (2.11) and (2.12) that

$$X(|\nabla f|^2)=2\nabla^2 f(\nabla f,X)=0,$$

for any $X\in C^\infty(\partial M)$. Hence, $|\nabla f|$ is constant along ∂M. Now, we need to show that $|\nabla f|_{|\partial M}\neq 0$. Indeed, let p be a point in ∂M and $\gamma:[0,\varepsilon)\to M$ be a geodesic parametrized by arc length with $\gamma(0)=p$ and $\gamma'(0)\perp\partial M$. Choosing $u(t)=(f\circ\gamma)(t)$, one has

$$u''(t)=\nabla^2 f(\gamma'(t),\gamma'(t)).$$

Then, it follows from (2.11) and (2.12) that there exists a smooth function $F(t)$ so that

$$u''(t) = F(t)u(t).$$

Consequently,

$$\begin{cases} u''(t) &= F(t)u(t), \\ u'(0) &= g(\nabla f(p), \gamma'(0)), \\ u(0) &= f(p) = 0. \end{cases}$$

So, by assuming that $\nabla f(p) = 0$, one deduces that $u'(0) = 0$ and then, by using the existence and uniqueness theorem for ODE, we infer that $u = 0$ over a neighborhood of $t = 0$ in $[0, \varepsilon)$, which leads to a contradiction with the fact that $f > 0$ in the interior of M and $\gamma(t)$ lies in the interior of M for all $t \in (0, \varepsilon_0)$ with sufficient small $\varepsilon_0 > 0$. This concludes the proof of the proposition. $\square$

A direct consequence of this proposition and equation (2.18) is that

$$\rho\mid_{\partial M} = -\frac{R\mid_{\partial M}}{2}.$$

Proceeding, since f is nonnegative, one sees that $\nu = -\frac{\nabla f}{|\nabla f|}$ is the unit outward normal vector field of ∂M. In particular, the divergence theorem implies that the integral of Δf is not identically zero. In fact, observe that

$$\int_M \Delta f dV_g = \int_{\partial M} g(\nabla f, \nu) dS_g = -|\nabla f|_{|\partial M}|\partial M| \neq 0. \tag{2.19}$$

This together with the dominant energy condition $\frac{R}{2} \geq |\rho|$ also implies that $R \geq 0$, but not identically zero.

From now on, consider an orthonormal frame $\{e_i\}_{i=1}^n$ with $e_n = -\frac{\nabla f}{|\nabla f|}$. Thus, from the second fundamental formula, for $1 \leq a, b, c, d \leq n-1$, one obtains that

$$h_{ab} = -\langle \nabla_{e_a}\nu, e_b \rangle = \frac{1}{|\nabla f|}\nabla_a\nabla_b f = 0$$

and hence, ∂M is totally geodesic. Thus, by the Gauss equation, i.e.,

$$R^{\partial M}_{abcd} = R_{abcd} - h_{ad}h_{bc} + h_{ac}h_{bd}, \tag{2.20}$$

we then obtain

$$R^{\partial M}_{abcd} = R_{abcd}.$$

Moreover, we infer

$$R^{\partial M}_{ac} = R_{ac} - R_{ancn}$$

and

$$R^{\partial M} = R - 2R_{nn}. \tag{2.21}$$

We recall that Proposition 2 in [37] asserts that the scalar curvature R of a static perfect fluid space-time $(M^n,\, g,\, f, \rho)$ is constant if and only if $(\frac{1}{2}R + \rho)f$ is constant. Thus, since M is a compact Riemannian manifold with (non-empty) boundary and $f_{|\partial M} = 0$, one concludes that R is constant if and only if $(\frac{1}{2}R + \rho)f \equiv 0$. Therefore, by using that $f > 0$ in the interior of M, we infer that the scalar curvature is constant if and only if

$$\rho = -\frac{R}{2} \text{ on } M. \tag{2.22}$$

Notice furthermore that if $\rho = -\frac{R}{2}$ over M, then the scalar curvature must be constant even in the empty boundary case.

The following divergence formula was established in [37]. For sake of completeness, we present here its proof.

Lemma 2.5 ([37])**.** *Let $(M^n,\, g)$ be a Riemannian manifold and f is a smooth function satisfying* (2.11)*. Then, in the interior of M, one has*

$$div\left[\frac{1}{f}\left(\nabla|\nabla f|^2 - 2\frac{\Delta f}{n}\nabla f\right)\right] = 2f|\mathring{Ric}|^2 + \frac{n-2}{n}\langle\nabla R, \nabla f\rangle. \tag{2.23}$$

Proof. Given a $(0,2)$-tensor T on a Riemannian manifold (M^n, g), it is well known that

$$div(T(\varphi X)) = \varphi div(T)(X) + \varphi\langle\nabla X, T\rangle + T(\nabla\varphi, X),$$

for all $\varphi \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Taking $T = \mathring{Ric}$, $X = \nabla f$ and $\varphi = 1$ in the last identity, one obtains

$$div(\mathring{Ric}(\nabla f)) = div(\mathring{Ric})(\nabla f) + \langle\mathring{Ric}, \nabla^2 f\rangle.$$

Using the static perfect fluid space-time equation (3.5) and the twice contracted second Bianchi identity, that is, the identity $div(Ric) = \frac{1}{2}\nabla R$, ones consequently get

$$div(\mathring{Ric}(\nabla f)) = f|\mathring{Ric}|^2 + \frac{n-2}{2n}\langle\nabla R, \nabla f\rangle. \tag{2.24}$$

On the other hand, observe that

$$\begin{aligned} f\mathring{Ric}(\nabla f) &= \mathring{\nabla}^2 f(\nabla f) - \frac{\Delta f}{n}\nabla f \\ &= \frac{1}{2}\nabla|\nabla f|^2 - \frac{\Delta f}{n}\nabla f, \end{aligned} \tag{2.25}$$

and so

$$2\mathring{Ric}(\nabla f) = \frac{1}{f}\left(\nabla|\nabla f|^2 - \frac{2\Delta f}{n}\nabla f\right). \tag{2.26}$$

Finally, comparing (2.24) and (2.26), one concludes that

$$div\left[\frac{1}{f}\left(\nabla|\nabla f|^2 - \frac{2\Delta f}{n}\nabla f\right)\right] = 2f|\mathring{Ric}|^2 + \frac{n-2}{n}\langle\nabla R, \nabla f\rangle, \tag{2.27}$$

which finishes the proof. □

Lemma 2.5 can be seen as a Robinson-Shen type identity [101] for manifolds satisfying Equation (2.11), which includes a large class of spaces; see also [1], [20] and [21]. In the case of constant scalar curvature, for example, static space or V-static space, we deduce that the expression in the left hand side of (2.23) is necessarily nonnegative.

Next, recall the following integral formula that relates the norm of the traceless Ricci tensor and the scalar curvature of the boundary; for more details, see Lemma 3 of [37]. This formula will be used in the proof of the boundary estimates.

Lemma 2.6 ([37])**.** *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M. Then one has*

$$\int_{\partial M}|\nabla f|R^{\partial M}dS_g = 2\int_M f|\mathring{Ric}|^2 dV_g - \frac{n-2}{n}\int_M R\Delta f dV_g.$$

Proof. Upon integrating (2.24), we use Stokes' theorem to infer

$$\begin{aligned} \int_M div(\mathring{Ric}(\nabla f))dV_g &= \int_M f|\mathring{Ric}|^2 dV_g + \frac{n-2}{2n}\int_M \langle\nabla R, \nabla f\rangle dV_g \\ &= \int_M f|\mathring{Ric}|^2 + \frac{n-2}{2n}\left(\int_{\partial M} R\langle\nabla f, \nu\rangle dS_g - \int_M R\Delta f dV_g\right). \end{aligned} \tag{2.28}$$

Using again Stokes' theorem, one sees that

$$\begin{aligned} \int_M div(\mathring{Ric}(\nabla f))dV_g &= \int_{\partial M}\langle\mathring{Ric}(\nabla f), \nu\rangle dS_g \\ &= -\int_{\partial M}|\nabla f|\mathring{Ric}(\nu, \nu)dS_g. \end{aligned} \tag{2.29}$$

Combining (2.28) and (2.29), one obtains that

$$\begin{aligned} -\int_{\partial M}|\nabla f|\left(R_{nn}-\frac{R}{n}\right)dS_g &= \int_M f|\mathring{Ric}|^2 dV_g-\frac{n-2}{2n}\int_{\partial M}R|\nabla f| \\ &\quad -\frac{n-2}{2n}\int_M R\Delta f dV_g. \end{aligned}$$

Therefore, it suffices to use (2.21) to infer

$$\begin{aligned} \int_M f|\mathring{Ric}|^2 dV_g-\frac{n-2}{2n}\int_M R\Delta f dV_g &= -\int_{\partial M}\left(\frac{R-R^{\partial M}}{2}-\frac{R}{2}\right)|\nabla f|dS_g \\ &\quad +\frac{n-2}{2n}\int_{\partial M}R|\nabla f|dS_g \\ &= -\int_{\partial M}\left(\frac{n-2}{2n}R-\frac{R^{\partial M}}{2}-\frac{n-2}{2n}R\right)|\nabla f|dS_g \\ &= \frac{1}{2}\int_{\partial M}|\nabla f|R^{\partial M}dS_g, \end{aligned} \tag{2.30}$$

which concludes the proof of the lemma. □

It is very interesting to classify manifolds satisfying certain structural equation. In this direction, Coutinho et al. classified in [37, Proposition 1] all possible Einstein Riemannian manifolds (M^n,g), $n\geq 3$, with empty or connected boundary satisfying $f\mathring{Ric}=\mathring{\nabla}^2 f$ for some $f\in C^\infty(M)$. Since their result will be useful in this work, their proof will be present next.

Proposition 2.5 ([37]). *Let (M^n,g,f), $n\geq 3$, be a compact Einstein manifold with positive scalar curvature and $f\in C^\infty(M)$ satisfying $f\mathring{Ric}=\mathring{\nabla}^2 f$. Then:*

(a) If ∂M is empty, then M^n is isometric to a round sphere $\mathbb{S}^n$;

(b) If ∂M is connected non-empty and $f\mid_{\partial M}$ is constant, then M^n is isometric to a geodesic ball on a sphere $\mathbb{S}^n$.

Proof. Since M^n is an Einstein manifold, it is immediate that $\mathring{Ric}\equiv 0$ and so the equality $f\mathring{Ric}=\mathring{\nabla}^2 f$ immediately implies

$$\nabla^2 f=\frac{\Delta f}{n}g. \tag{2.31}$$

Thus, ∇f is a conformal vector field. Moreover, it was calculated an expression for the gradient of the Laplacian of f in (2.17), that is,

$$\nabla\Delta f=\frac{n-2}{2(n-1)}f\nabla R-\frac{R}{n-1}\nabla f.$$

Since $n \geq 3$ and the manifold is Einstein, one knows that the scalar curvature is constant and the last expression implies

$$\nabla^2 \Delta f = -\frac{R}{n-1}\nabla^2 f = -\frac{R}{n(n-1)}\Delta f g,$$

where it was used (2.31).

Supposing now that the boundary is empty, one can makes use of [Theorem 4.1, [114]] to conclude that (M^n, g) is isometric to a round sphere, which is the conclusion of item (a). On the other hand, (2.31) and the fact that M has constant scalar curvature, one deduces that $\Delta f + \frac{R}{n-1} f$ is constant. Thus, if $\partial M \neq \emptyset$ and $f\,|_{\partial M}$ is constant, one infers that Δf is constant along ∂M. The result then follows from an Obata type theorem [93, Theorem B] due to Reilly. This finishes the proof of the proposition. □

2.3 Quasi-Einstein manifolds

In this section, we recall basic facts on m-quasi-Einstein manifolds. First of all, we remember that the fundamental equation of an m-quasi-Einstein manifold (M^n, g, u, λ), possibly with boundary, is given by

$$\nabla^2 u = \frac{u}{m}(Ric - \lambda g), \tag{2.32}$$

where $u > 0$ in the interior of M and $u = 0$ on the boundary ∂M. By tracing (2.32), one sees that

$$\Delta u = \frac{u}{m}(R - n\lambda). \tag{2.33}$$

This implies that $\Delta u = 0$ along ∂M. Besides, Propositions 2.2 and 2.3 of [59] guarantee that $|\nabla u|$ does not vanish on the boundary and it is constant on each component of ∂M. From this, we infer that $\nu = -\frac{\nabla u}{|\nabla u|}$ is the unit outward normal vector field over ∂M. In particular, by the Stokes' formula, Δu is not identically zero. Actually, observe that

$$\int_M \Delta u \, dM_g = \int_{\partial M} \langle \nabla u, \nu \rangle \, dS_g = -|\nabla u|_{|\partial M}\, |\partial M| < 0. \tag{2.34}$$

Remark 6. *It follows from (2.33) and (2.34) that if the scalar curvature R is constant, then $R < n\lambda$ (cf. [59, Corollary 4.3]).*

From now on, we consider an orthonormal frame $\{e_i\}_{i=1}^n$ with $e_1 = \nu = -\frac{\nabla u}{|\nabla u|}$. Under this coordinates, since $u = 0$ on ∂M, the second fundamental form satisfies

$$h_{ab} = -\langle \nabla_{e_a} \nu, e_b \rangle = \frac{1}{|\nabla u|} \nabla_a \nabla_b u = 0,$$

for any $2 \le a, b, c, d \le n$. Hence, ∂M is totally geodesic. Also, by the Gauss equation (2.20), one obtains that

$$R^{\partial M} = R - 2R_{11}. \tag{2.35}$$

We further recall some important features of m-quasi-Einstein manifolds (cf. [26], [42], [59]).

Lemma 2.7. *Let* (M^n, g, u, λ) *be an* m*-quasi-Einstein manifold with* $m > 1$*. Then we have:*

1.
$$\frac{1}{2}u\nabla R = -(m-1)Ric(\nabla u) - (R-(n-1)\lambda)\nabla u;$$

2.
$$\frac{u^2}{m}(R-\lambda n) + (m-1)|\nabla u|^2 = -\lambda u^2 + \mu,$$

where μ *is a constant;*

3.
$$\begin{aligned} \frac{1}{2}\Delta R &= -\frac{m+2}{2u}\langle \nabla u, \nabla R\rangle - \frac{m-1}{m}\left|Ric - \frac{R}{n}g\right|^2 \\ &\quad - \frac{(n+m-1)}{mn}(R-n\lambda)\left(R - \frac{n(n-1)}{n+m-1}\lambda\right); \end{aligned}$$

4.
$$u\left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) = mR_{ijkl}\nabla_l u + \lambda\left(\nabla_i u g_{jk} - \nabla_j u g_{ik}\right) - \left(\nabla_i u R_{jk} - \nabla_j u R_{ik}\right).$$

We highlight that Eq. (2) of Lemma 2.7 determines a type of "*integrability condition*". Besides, Equation (4) of Lemma 2.7 was obtained by Diógenes and Gadelha in [42, Lemma 1].

From assertion (1) of Lemma 2.7, if an m-quasi-Einstein manifold M^n has constant scalar curvature and $m > 1$, then

$$Ric(\nabla u) = \frac{(n-1)\lambda - R}{m-1}\nabla u. \tag{2.36}$$

Consequently, the traceless Ricci tensor $\mathring{Ric}$ must satisfy

$$\mathring{Ric}(\nabla u) = \frac{n(n-1)\lambda - (m+n-1)R}{n(m-1)}\nabla u. \tag{2.37}$$

Furthermore, Eq. (3) of Lemma 2.7 together with the assumption that the scalar curvature R is constant imply that

$$|\mathring{Ric}|^2 = -\frac{m+n-1}{n(m-1)}(R-n\lambda)\left(R-\frac{n(n-1)\lambda}{m+n-1}\right). \tag{2.38}$$

We now set the covariant 2-tensor P by

$$P = Ric - \frac{(n-1)\lambda - R}{m-1}g. \tag{2.39}$$

In this perspective, by assuming that M has constant scalar curvature, we have from (2.36) that $P(\nabla u) = 0$. Furthermore, by using the orthonormal frame $\{e_i\}_{i=1}^n$ that diagonalizes the Ricci tensor, one observes that $P(e_i) = \mu_i e_i$. In [59], it was introduced the 4-tensor Q related to P as follows

$$Q = Rm + \frac{1}{m}P \odot g + \frac{(n-m)\lambda - R}{2m(m-1)}g \odot g, \tag{2.40}$$

where $\odot$ stands for the Kulkarni-Nomizu product[1] and Rm is the Riemann tensor. For covariant 2-tensors S and T, the Kulkarni-Nomizu product is given by

$$(S \odot T)_{ijkl} = S_{ik}T_{jl} + S_{jl}T_{ik} - S_{il}T_{jk} - S_{jk}T_{il}. \tag{2.41}$$

With these tools, one deduces the following result.

Proposition 2.6. *Let (M^n, g, u, λ) be an m-quasi-Einstein manifold. Then we have:*

$$u(\nabla_i P_{jk} - \nabla_j P_{ik}) = mQ_{ijkl}\nabla_l u + \frac{1}{2}(g \odot g)_{ijkl}P_{sl}\nabla_s u.$$

Proof. We start by rewriting the expression (4) of Lemma 2.7 in terms of the tensor $P = Ric - \frac{(n-1)\lambda - R}{m-1}g$ in order to obtain

$$\begin{aligned} u\left(\nabla_i P_{jk} - \nabla_j P_{ik}\right) &+ u\left(\nabla_i \rho g_{jk} - \nabla_j \rho g_{ik}\right) \\ &= mR_{ijkl}\nabla_l u + (\lambda - \varrho)\left(\nabla_i u g_{jk} - \nabla_j u g_{ik}\right) - \left(\nabla_i u P_{jk} - \nabla_j u P_{ik}\right), \end{aligned}$$

where

$$\varrho = \frac{(n-1)\lambda - R}{m-1}. \tag{2.42}$$

[1] Our definition of Kulkarni-Nomizu product differs from [59] by a constant 1/2 and sign.

Moreover, by assertion (1) of Lemma 2.7, one sees that $\frac{u}{2}\nabla\varrho = P(\nabla u)$ (see also [59, Proposition 5.2]) and hence,

$$\begin{aligned} u\Big(\nabla_i P_{jk} - \nabla_j P_{ik}\Big) &+ 2\Big(P_{il}\nabla_l u g_{jk} - P_{jl}\nabla_l u g_{ik}\Big) \\ &= mR_{ijkl}\nabla_l u + (\lambda - \varrho)\Big(\nabla_i u g_{jk} - \nabla_j u g_{ik}\Big) - \Big(\nabla_i u P_{jk} - \nabla_j u P_{ik}\Big). \end{aligned} \tag{2.43}$$

On the other hand, it follows from (2.41) that

$$(g \odot g)_{ijkl}\nabla_l u = 2(g_{ik}\nabla_j u - g_{jk}\nabla_i u),$$

$$(g \odot g)_{ijkl}P_{sl}\nabla_s u = 2(P_{js}\nabla_s u g_{ik} - P_{is}\nabla_s u g_{jk})$$

and

$$(P \odot g)_{ijkl}\nabla_l u = (P_{ik}\nabla_j u - P_{jk}\nabla_i u) + (P_{jl}\nabla_l u g_{ik} - P_{il}\nabla_l u g_{jk}).$$

Substituting these expressions into (2.43) yields

$$\begin{aligned} u(\nabla_i P_{jk} - \nabla_j P_{ik}) &= mR_{ijkl}\nabla_l u + (\varrho - \lambda)(g_{ik}\nabla_j u - g_{jk}\nabla_i u) + (P_{ik}\nabla_j u - P_{jk}\nabla_i u) \\ &\quad +2(P_{jl}\nabla_l u g_{ik} - P_{il}\nabla_l u g_{jk}) \\ &= mR_{ijkl}\nabla_l u + \frac{(\varrho - \lambda)}{2}(g \odot g)_{ijkl}\nabla_l u + (P \odot g)_{ijkl}\nabla_l u \\ &\quad +\frac{1}{2}(g \odot g)_{ijkl}P_{sl}\nabla_s u \\ &= mQ_{ijkl}\nabla_l u + \frac{1}{2}(g \odot g)_{ijkl}P_{sl}\nabla_s u, \end{aligned}$$

where the last equality follows from (2.40). □

As a consequence of Proposition 2.6, we deduce the following identities that were first proved by He, Petersen and Wylie in [60, Proposition 3.7]. Taking into account that our convention for the Kulkarni-Nomizu product (2.41) and $Ric(X,Y) = \operatorname{tr} Rm(X,\cdot,Y,\cdot)$ differ from [60], for the reader's convenience, we are going to present a proof here.

Proposition 2.7 ([60]). *Let (M^n, g, u, λ) be an m-quasi-Einstein manifold with constant scalar curvature and $m > 1$. Then we have:*

1.

$$\frac{u}{m}(\nabla_i P_{jk} - \nabla_j P_{ik}) = \frac{u}{m}(\nabla_i R_{jk} - \nabla_j R_{ik}) = Q_{ijkl}\nabla_l u,$$

2.
$$\frac{u}{m}\nabla_i P_{jk}\nabla_i u = \left(\frac{u}{m}\right)^2\Big((\lambda-\varrho)\,P_{jk} - P_{ik}P_{ij}\Big) + Q_{ijkl}\nabla_l u\nabla_i u,$$

where $\varrho = \frac{(n-1)\lambda-R}{m-1}$.

Proof. Initially, since M^n has constant scalar curvature and $P = Ric - \frac{(n-1)\lambda-R}{m-1}g$, ones sees that $P(\nabla u) = 0$ and therefore, the first assertion follows directly from Proposition 2.6.

We now deal with the second one. By using again that $P(\nabla u) = 0$, one observes that

$$0 = \nabla_j\,(P_{ik}\nabla_i u) = (\nabla_j P_{ik})\nabla_i u + P_{ik}\nabla_j\nabla_i u.$$

This jointly with (2.32) yields

$$\nabla_j P_{ik}\nabla_i u = -P_{ik}\nabla_j\nabla_i u = -P_{ik}\left(\frac{u}{m}\,(P_{ji}-(\lambda-\varrho)g_{ji})\right) = -\frac{u}{m}\Big(P_{ik}P_{ji}-(\lambda-\varrho)P_{jk}\Big).$$

Thereby, it suffices to use the first assertion in order to infer

$$\frac{u}{m}\nabla_i P_{jk}\nabla_i u = \left(\frac{u}{m}\right)^2\Big((\lambda-\varrho)\,P_{jk} - P_{ik}P_{ij}\Big) + Q_{ijkl}\nabla_l u\nabla_i u,$$

as desired. □

On an m-quasi-Einstein manifold, we may express the Cotton tensor in terms of the Weyl tensor and an auxiliary 3-tensor T_{ijk} as follows (see [42, Lemma 2]).

Lemma 2.8 ([42])**.** *Let (M^n, g, u, λ) be an m-quasi-Einstein manifold. Then it holds*

$$uC_{ijk} = mW_{ijkl}\nabla_l u + T_{ijk}, \tag{2.44}$$

where the 3-tensor T_{ijk} is given by

$$\begin{aligned} T_{ijk} \;\; = \;\; & \frac{m+n-2}{n-2}(R_{ik}\nabla_j u - R_{jk}\nabla_i u) + \frac{m}{n-2}(R_{jl}\nabla_l u g_{ik} - R_{il}\nabla_l u g_{jk}) \\ & + \frac{(n-1)(n-2)\lambda + mR}{(n-1)(n-2)}(\nabla_i u g_{jk} - \nabla_j u g_{ik}) - \frac{u}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \end{aligned}$$

We highlight that the tensor T_{ijk} has the same symmetric properties of the Cotton tensor and it is motivated by ideas outlined by Cao and Chen in [25]; see also [32] and [89]. Besides, it is convenient to express the tensor T_{ijk} in terms of the traceless Ricci tensor

$$\begin{aligned} T_{ijk} &= \frac{m+n-2}{n-2}(\mathring{R}_{ik}\nabla_j u - \mathring{R}_{jk}\nabla_i u) + \frac{m}{n-2}(\mathring{R}_{jl}\nabla_l u g_{ik} - \mathring{R}_{il}\nabla_l u g_{jk}) \\ &+ \frac{n(n-1)\lambda - (m+n-1)R}{n(n-1)}(\nabla_i u g_{jk} - \nabla_j u g_{ik}) \\ &- \frac{u}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \end{aligned} \tag{2.45}$$

Now, it is convenient to recall the following terminology (see [60]).

Definition 2.3. *An m-quasi-Einstein manifold (M^n, g, u, λ) is said to be rigid if it is Einstein or its universal cover is a product of Einstein manifolds.*

In [60], it was established the following result for m-quasi-Einstein manifolds.

Proposition 2.8 ([60])**.** *Suppose (M^n, g, u, λ) is a nontrivial m-quasi-Einstein manifold which is also Einstein, then up to multiples of the potential function u or the metric g, it is isometric to one of the examples in Table 1.*

M	g	u	λ	ϱ	μ
$[-\frac{\pi}{2}, \frac{\pi}{2}]$	dr^2	$u(r) = \cos(r)$	m	0	$m-1$
$[0, \infty)$	dr^2	$u(r) = r$	0	0	$m-1$
$[0, \infty)$	dr^2	$u(r) = \sinh(r)$	$-m$	0	$m-1$
$(-\infty, \infty)$	dr^2	$u(r) = e^r$	$-m$	0	0
$(-\infty, \infty)$	dr^2	$u(r) = \cosh(r)$	$-m$	0	$1-m$
$\mathbb{S}^n_+$	$dr^2 + \sin^2(r) g_{\mathbb{S}^{n-1}}$	$u(r) = \cos(r)$	$m+n-1$	$n-1$	$m-1$
$[0, \infty) \times F$	$dr^2 + g_F$	$u(r) = r$	0	0	$m-1$
$[0, \infty) \times N$	$dr^2 + \cosh^2(r) g_N$	$u(r) = \sinh(r)$	$-(m+n-1)$	$-(n-1)$	$m-1$
$(-\infty, \infty) \times F$	$dr^2 + e^{2r}(r) g_N$	$u(r) = e^r$	$-(m+n-1)$	$-(n-1)$	0
$\mathbb{H}^n$	$dr^2 + \sinh^2(r) g_N$	$u(r) = \cosh(r)$	$-(m+n-1)$	$-(n-1)$	$1-m$

TABLE 1: Nontrivial m-quasi-Einstein manifolds that are also Einstein. Here, ϱ is given as in (2.42), F is Ricci flat and N has Ricci curvature $-(n-2)$.

Using Proposition 2.8, He, Petersen and Wylie proved the following result for rigid quasi-Einstein manifolds.

Proposition 2.9 ([60])**.** *A non-trivial complete rigid m-quasi-Einstein manifold (M^n, g, u, λ) is one of the examples in Table 1, or its universal cover splits off as*

$$\widetilde{M} = (M_1, g_1) \times (M_2, g_2) \quad \textit{with} \quad u(x,y) = u(y),$$

where (M_1, g_1, λ) is a trivial quasi-Einstein manifold and (M_2, g_2, u) is one of the examples in Table 1.

Remark 7. *It is known that the universal covering of a quasi-Einstein manifold with $\lambda > 0$ is compact and hence, its fundamental group $\pi_1(M)$ is finite. The proof of this fact is quite similar to [49], [111] and it can be carry out by combining the arguments in the proof of [59, Theorem 4.1] (see also [88]) and [96, Remark 6.9].*

Before proceeding, we recall that a non-constant function $f : M \to \mathbb{R}$ of class at least C^2 is said to be *transnormal* if

$$|\nabla f|^2 = b(f) \tag{2.46}$$

for some C^2 function b on the range of f in $\mathbb{R}$. In addition, f is said to be *isoparametric* if there exists a continuous function a on the range of f in $\mathbb{R}$ such that

$$\Delta f = a(f). \tag{2.47}$$

In particular, (2.46) implies that the level set hypersurfaces of f (i.e., $M_t = f^{-1}(t)$, where t is a regular value of f) are parallel, and the integral curves of ∇f are the shortest geodesics connecting the level sets. Besides, (2.47) guarantees that such hypersurfaces have constant mean curvatures. The preimage of the maximum (respectively, minimum) of an isoparametric (or transnormal) function f is called the *focal variety* of f. We refer the reader to [53], [54], [79] and [107] for more details.

By considering that (M^n, g, u, λ) is an m-quasi-Einstein manifold with constant scalar curvature, one deduces from (2.33) that the potential function u is isoparametric. In view of this, one easily verifies from assertion (2) of Lemma 2.7, for $m > 1$, that

$$|\nabla u|^2 = \frac{\mu}{m-1} - \frac{R + (m-n)\lambda}{m(m-1)} u^2. \tag{2.48}$$

Consequently, the potential function u is transnormal, namely,

$$b(u) = \frac{\mu}{m-1} - \frac{R + (m-n)\lambda}{m(m-1)} u^2. \tag{2.49}$$

Concerning the regularity of the potential function, for an m-quasi-Einstein manifold (M^n, g, u, λ), it is known that u and g are real analytic in harmonic coordinates (cf. Proposition 2.4 in [59]). In particular, the critical level sets of u have zero measure.

A central object in our approach is the set of maximum points of u given by

$$MAX(u) = \{p \in M : u(p) = u_{\max}\}.$$

Remark 8. *In the compact case with $m>1$, notice that every point in $MAX(u)$, which clearly is an interior point, must be a critical point. Moreover, the fact that u is a transnormal function and* (2.48) *allow us to deduce that the critical points of u have the same value. Thereby, $MAX(u)=Crit(u)$ for nontrivial compact m-quasi-Einstein manifolds.*

To conclude this section, we are going to describe an example of m-quasi-Einstein manifold for dimension $n\geq 5$, on $\left[0,\frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right]\times\mathbb{S}^p\times\mathbb{S}^q$ (see also [51]).

Example 2.3. *Let $\lambda>0$ be an arbitrary constant and consider $M^n=\left[0,\frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right]\times\mathbb{S}^p\times\mathbb{S}^q$, $p,q>1$, endowed with the metric*

$$g=dt^2+\frac{p-1}{\lambda}g_{\mathbb{S}^p}+\frac{q-1}{\lambda}g_{\mathbb{S}^q}.$$

This space is an m-quasi-Einstein manifold with potential function $u(t)=\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)$ and constant scalar curvature $R=(n-1)\lambda$. Indeed, we first notice that

$$Ric=(p-1)g_{\mathbb{S}^p}+(q-1)g_{\mathbb{S}^q}\quad\text{and}\quad\nabla u=u'\nabla t=\frac{\sqrt{\lambda}}{\sqrt{m}}\cos\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)\nabla t.$$

Thereby, since $u=u(t)$ and the warping function is constant, we deduce from (2.3) *that*

$$\nabla^2u=-\frac{\lambda}{m}\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)dt^2.\tag{2.50}$$

On the other hand, one observes that

$$\begin{aligned}\frac{u}{m}(Ric-\lambda g)=&\frac{1}{m}\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)\left[(p-1)g_{\mathbb{S}^p}+(q-1)g_{\mathbb{S}^q}-(\lambda dt^2+(p-1)g_{\mathbb{S}^p}+(q-1)g_{\mathbb{S}^q})\right]\\=&-\frac{\lambda}{m}\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)dt^2.\end{aligned}$$

Plugging this into (2.50) gives (2.32).

In conclusion, $u=0$ if and only if either $t=0$ or $t=\frac{\sqrt{m}}{\sqrt{\lambda}}\pi$ and consequently, the boundary consists of two disjoint copies of $\mathbb{S}^p\times\mathbb{S}^q$.

3 GEOMETRY OF STATIC PERFECT FLUID SPACE-TIME

In this chapter, we discuss the results obtained in *Geometry of static perfect fluid space-time*, a joint work with R. Diógenes, N. Pinheiro and E. Ribeiro Jr. [35]. The central discussion of the aforementioned paper is to obtain geometric inequalities for compact static perfect fluid space-times (SPFST) with boundary ∂M. Furthermore, we present a new example of SPFST with connected boundary which is counter-example to the cosmic no-hair conjecture for arbitrary dimension $n \geq 4$. This chapter is divided in three sections: In Sections 3.1 and 3.2, we state Theorems 1 and 2 of the article aforementioned and their corollaries, respectively; Section 3.3 discuss a counter-example for the cosmic no-hair conjecture, moreover we present an examples of a static perfect fluid space-time with non-constant scalar curvature obtained in [9].

In [35], it was established new geometric inequalities in order to estimate the area of the boundary of a compact SPFST. Moreover, the case of equality was also discussed. In this sense, our first section presents Theorem 1 of [35] and its corollary, which is related to the isoperimetric inequality. To do so, we make use of the generalized Reilly's formula by Qiu and Xia [91] in order to obtain a new boundary estimate for SPFST. More precisely, we have the following result.

Theorem 3.1. *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M and positive scalar curvature satisfying*

$$\frac{n-2}{2(n-1)}R + \frac{n}{n-1}\rho = -\tau, \tag{3.1}$$

where τ is a positive constant. Then we have:

$$Vol(M) \geq \frac{1}{\tau}\sqrt{\frac{R_{min}+3\tau}{2n}}\,|\partial M|. \tag{3.2}$$

Moreover, if equality holds in (3.2), then (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.

As was mentioned in the statement of Theorem 3.1, the constant τ is positive. Indeed, supposing that $\tau \leq 0$, since $\Delta f = -\tau f$ and f is a nonnegative function with $f^{-1}(0) = \partial M$, we may use the Maximum Principle to infer that $f = 0$ in M, which leads to a contradiction.

We highlight that, by assuming the dominant energy condition in Theorem 3.1, one obtains that the scalar curvature of M^n must be positive. In fact, the dominant energy condition asserts $\frac{R}{2} \geq |\rho|$ and hence, if $R(p) = 0$ for some point $p \in M$, then $\rho(p) = 0$, which contradicts the assumption that τ is a positive constant. Moreover, from equation (2.18) and Proposition 2.4, we have $\rho\,|_{\partial M} = -\frac{1}{2}R\,|_{\partial M}$ and consequently, $\tau = \frac{1}{n-1}R\,|_{\partial M}$. In particular, (3.1) implies that the scalar curvature is constant along the boundary.

As a consequence of Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M and constant positive scalar curvature R. Then we have:*

$$Vol(M) \geq \sqrt{\frac{(n-1)(n+2)}{2nR}}\,|\partial M|. \tag{3.3}$$

Moreover, if equality holds in (3.3), then (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.

At the second part of this chapter, we deal with the Brown-York mass. The study of static spaces are related with the mass concept, as, for instance, ADM (Arnowitt-Deser-Misner) mass [20]. So, it is interesting to find a result over SPFST involving some mass concept. The next result establishes a sharp boundary estimate for compact static perfect fluid space-time with (possibly disconnected) boundary in terms of the Brown-York mass $\mathfrak{m}_{BY}$. Now, let us recall the definition of Brown-York mass.

Definition 3.1. *Let Σ be a connected hypersurface in (M^n, g) such that $(\Sigma, g\,|_{\Sigma})$ can be embedded in $\mathbb{R}^n$ as a convex hypersurface. Then, the Brown-York mass $\mathfrak{m}_{BY}$ of Σ with respect to g is given by*

$$\mathfrak{m}_{BY}(\Sigma, g) = \int_{\Sigma} (H_0 - H_g)\,dS_g,$$

where H_0 and H_g are the mean curvature of Σ as hypersurface of $\mathbb{R}^n$ and M, respectively, and dS_g is the volume element of on Σ induced by g.

In [115], motivated by the Riemannian Penrose inequality, Yuan obtained a boundary estimate for static spaces in terms of the Brown-York mass. A similar result was established for quasi-Einstein manifolds by Diógenes, Gadelha and Ribeiro [44]. In another direction, inspired by ideas outlined in [37], Andrade and Melo [2] proved recently that, under suitable conditions, the Hawking mass of Einstein-type manifolds is bounded from below by the area of the boundary. Then, we have the following result.

Theorem 3.2. *Let (M^n, g, f, ρ), $n \geq 3$, be a compact static perfect fluid space-time with (possibly disconnected) boundary and positive scalar curvature satisfying the dominant* *energy condition. Suppose that each boundary component* $(\partial M_i, g)$ *can be isometrically embedded in* $\mathbb{R}^n$ *as a convex hypersurface. Then we have*

$$|\partial M_i| \leq c\,\mathfrak{m}_{BY}(\partial M_i, g), \tag{3.4}$$

where c *is a positive constant. Moreover, equality occurs for some component* ∂M_i *if and only if* M^n *is isometric to the standard hemisphere* $\mathbb{S}^n_+$.

A key ingredient in the proof of Theorem 3.2 is the positive mass theorem for Brown-York mass by Shi-Tam [102], which is equivalent to the (higher dimensional) positive mass theorem for ADM mass by Schoen and Yau [98], [99], [100], and Lohkamp [70]. It should be mentioned that the isometric embedding condition in Theorem 3.2 was needed to use the positive mass theorem. According to the solution of the Weyl problem, the isometrical embedding assumption can be replaced by control on sectional curvatures, as for instance, positive Gaussian curvature when $n = 3$, see, e.g., [46], [115].

As an application of Theorem 3.2, we also obtain an integral estimate for the area of each connected component ∂M_i of the boundary in terms of its scalar curvature $R^{\partial M_i}$ and the norm of its second fundamental form as a hypersurface of $\mathbb{R}^n$. The result is stated as follows.

Corollary 3.2. *Let* (M^n, g, f, ρ), $n \geq 3$, *be a compact static perfect fluid space-time with (possibly disconnected) boundary and positive scalar curvature. Assume the dominant energy condition and that each boundary component* $(\partial M_i, g)$ *can be isometrically embedded in* $\mathbb{R}^n$ *as a convex hypersurface. Then we have*

$$|\partial M_i| \leq \widetilde{c} \int_{\partial M_i} (R^{\partial M_i} + |\mathring{h}_i|^2) dS_g$$

for some positive constant $\widetilde{c}$, *where* $\mathring{h}_i$ *is the traceless second fundamental form of* ∂M_i *as a hypersurface of* $\mathbb{R}^n$. *Moreover, equality occurs for some connected component of the boundary if and only if* (M^n, g) *is isometric to the round hemisphere* $\mathbb{S}^n_+$.

In Section 3.3, we are going to present some examples, one of them with non-constant scalar curvature, which corresponds to [9, Example 2]. These examples are relevant in the theory due to their properties. The principal result in such a section is the new example developed in [35], which is a new counter-example to the cosmic no-hair conjecture for arbitrary dimension $n \geq 4$. Such a conjecture was proposed in 1984 by

Boucher, Gibbons and Horowitz in [22], [23] and can be state as: *the hemisphere* $\mathbb{S}^n_+$ *is the only possible* n*-dimensional (simply connected) positive static triple with single-horizon (connected boundary) and positive scalar curvature.* More precisely, we have the following example.

Example 3.1. *Let* $M^n = \mathbb{S}^{p+1}_+ \times \mathbb{S}^q$, $q > 1$, *with the product metric*

$$g = dr^2 + \sin^2(r) g_{\mathbb{S}^p} + \frac{q-1}{p+1} g_{\mathbb{S}^q},$$

where $r(x,y) = r(x)$ *is the height function of* $\mathbb{S}^{p+1}$. *Moreover, we consider the potential function* $f(r) = \cos(r)$ *with* $r \leq \frac{\pi}{2}$. *Thus,* (M^n, g) *satisfies* (2.11) *and* (2.12). *In particular, since it has constant scalar curvature, then it is a static space.*

3.1 Volume comparison estimate using the generalized Reilly's formula

First of all, let us recall the definition of a static perfect fluid space-time.

Definition 3.2. *A Riemannian manifold* (M^n, g) *is said to be a spatial factor of a static perfect fluid space-time (SPFST) if there exist smooth functions* $f > 0$ *and* ρ *on* M^n *satisfying the perfect fluid equations:*

$$f\mathring{Ric} = \mathring{\nabla}^2 f \tag{3.5}$$

and

$$\Delta f = \left(\frac{n-2}{2(n-1)} R + \frac{n}{n-1}\rho \right) f, \tag{3.6}$$

where $\mathring{Ric}$, $\mathring{\nabla}^2$ *stand for the traceless Ricci and traceless Hessian tensor, respectively. When* M *has non-empty boundary* ∂M, *it will be assumed in addition that* $f^{-1}(0) = \partial M$. f *will be called a potential function and denote a such space as* (M^n, g, f, ρ).

As it was mentioned in Chapter 1, Definition 3.2 approaches a SPFST as a spatial factor, i.e., the base of a Lorentzian warped product with a static metric.

Before present our first theorem, we need to discuss some key results. We recall the following useful generalized Reilly's formula, obtained previously by Qiu and Xia [91].

Proposition 3.1 ([91])**.** *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. Given two functions f and u on M^n and a constant κ, we have*

$$\int_M f\left((\Delta u+\kappa nu)^2-|\nabla^2 u+\kappa ug|^2\right)dV_g=(n-1)\kappa\int_M(\Delta f+n\kappa f)u^2\,dV_g$$
$$+\int_M\left(\nabla^2 f-(\Delta f)g-2(n-1)\kappa fg+fRic\right)(\nabla u,\nabla u)\,dV_g$$
$$+\int_{\partial M} f\left[2\left(\frac{\partial u}{\partial\nu}\right)\Delta_{\partial M}u+H\left(\frac{\partial u}{\partial\nu}\right)^2+h(\nabla_{\partial M}u,\nabla_{\partial M}u)+2(n-1)\kappa\left(\frac{\partial u}{\partial\nu}\right)u\right]dS_g$$
$$+\int_{\partial M}\frac{\partial f}{\partial\nu}\left(|\nabla_{\partial M}u|^2-(n-1)\kappa u^2\right)dS_g,$$

where H and h stand for the mean curvature and second fundamental form of ∂M, respectively.

We point out that, by considering $\kappa=0$ and $f=1$ in Proposition 3.1, we recover the classical Reilly's formula. For sake of completeness, we present the proof of Proposition 3.1 here.

Proof. To begin with, observe that

$$\frac{1}{2}\Delta(f|\nabla u|^2)=\frac{1}{2}(\Delta f)|\nabla u|^2+\frac{1}{2}f\Delta|\nabla u|^2+\langle\nabla f,\nabla|\nabla u|^2\rangle \tag{3.7}$$

Moreover, using the Stokes' theorem, one has

$$\begin{aligned}&-\frac{1}{2}\int_M(\Delta f)|\nabla u|^2\,dV_g+\frac{1}{2}\int_{\partial M}|\nabla u|^2\frac{\partial f}{\partial\nu}\,dS_g-\int_{\partial M}\langle\nabla u,\nabla f\rangle\frac{\partial u}{\partial\nu}\,dS_g\\&+\int_M\nabla^2 f(\nabla u,\nabla u)\,dV_g+\int_M\langle\nabla u,\nabla f\rangle\Delta u\,dV_g\\&-\int_M\langle\nabla\langle\nabla f,\nabla u\rangle,\nabla u\rangle\,dV_g\\&=0,\end{aligned} \tag{3.8}$$

where we have used that $\langle\nabla f,\nabla|\nabla u|^2\rangle=2\nabla^2 u(\nabla u,\nabla f)$. Integrating (3.7) over M and using again the Stokes' theorem, one obtains that

$$\begin{aligned}\int_M\langle\nabla f,\nabla|\nabla u|^2\rangle\,dV_g &= \frac{1}{2}\int_M\Delta(f|\nabla u|^2)\,dV_g-\frac{1}{2}\int_M(\Delta f)|\nabla u|^2\,dV_g-\frac{1}{2}\int_M f\Delta|\nabla u|^2 dV_g\\&=\frac{1}{2}\int_{\partial M}|\nabla u|^2\frac{\partial f}{\partial\nu}\,dS_g-\frac{1}{2}\int_M\left[(\Delta f)|\nabla u|^2-\langle\nabla f,\nabla|\nabla u|^2\rangle\right]dV_g.\end{aligned}$$

Of which, we may use (3.8) to deduce

$$\begin{aligned}\int_M\langle\nabla f,\nabla|\nabla u|^2\rangle\,dV_g &= -\frac{3}{2}\int_M(\Delta f)|\nabla u|^2\,dV_g+\frac{3}{2}\int_{\partial M}\frac{\partial f}{\partial\nu}|\nabla u|^2\,dS_g\\&\quad-\int_{\partial M}\langle\nabla f,\nabla u\rangle\frac{\partial u}{\partial\nu}\,dS_g+\int_M\nabla^2 f(\nabla u,\nabla u)\,dV_g\\&\quad+\int_M\langle\nabla f,\nabla u\rangle\Delta u\,dV_g.\end{aligned} \tag{3.9}$$

Now, by using the Böchner's formula, i.e.,

$$\frac{1}{2}\Delta|\nabla u|^2 = \langle\nabla\Delta u, \nabla u\rangle + Ric(\nabla u, \nabla u) + |\nabla^2 u|^2,$$

together with

$$\frac{1}{2}\Delta(f|\nabla u|^2) = \frac{f}{2}\Delta|\nabla u|^2 + \langle\nabla f, \nabla|\nabla u|^2\rangle + \frac{1}{2}|\nabla u|^2\Delta f,$$

we then get

$$\begin{aligned}\int_M \langle\nabla f, \nabla|\nabla u|^2\rangle \, dV_g &= -\int_M f\left[Ric(\nabla u, \nabla u) + \langle\nabla\Delta u, \nabla u\rangle + |\nabla^2 u|^2\right] dV_g \\ &\quad + \frac{1}{2}\int_M \frac{\partial}{\partial\nu}(f|\nabla u|^2)\, dV_g - \frac{1}{2}\int_M |\nabla u|^2\Delta f \, dV_g.\end{aligned}$$

Comparing the last expression with (3.9), one sees

$$\begin{aligned}&\frac{1}{2}\int_{\partial M}\frac{\partial}{\partial\nu}(f|\nabla u|^2)\, dS_g - \int_M f\left(|\nabla^2 u|^2 + Ric(\nabla u, \nabla u) + \langle\nabla\Delta u, \nabla u\rangle\right) dV_g \\ &= -\int_M (\Delta f)|\nabla u|^2 \, dV_g + \frac{3}{2}\int_{\partial M}\frac{\partial f}{\partial\nu}|\nabla u|^2 \, dS_g - \int_{\partial M}\langle\nabla f, \nabla u\rangle\frac{\partial u}{\partial\nu}\, dS_g \\ &\quad + \int_M \nabla^2 f(\nabla u, \nabla u)\, dV_g + \int_M \langle\nabla f, \nabla u\rangle\Delta u \, dV_g.\end{aligned} \tag{3.10}$$

In another direction, upon integrating by parts, we achieve

$$\int_M f\langle\nabla\Delta u, \nabla u\rangle \, dV_g = -\int_M f(\Delta u)^2 \, dV_g - \int_M \langle\nabla f, \nabla u\rangle\Delta u \, dV_g + \int_{\partial M} f(\Delta u)\frac{\partial u}{\partial\nu}\, dS_g.$$

On the other hand, on the boundary, we have

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla u|^2 = \left\langle\nabla_{\partial M} u, \nabla_{\partial M}\left(\frac{\partial u}{\partial\nu}\right)\right\rangle - \mathrm{III}(\nabla_{\partial M} u, \nabla_{\partial M} u) + \frac{\partial u}{\partial\nu}\left(\Delta u - \Delta_{\partial M} u - H\frac{\partial u}{\partial\nu}\right).$$

Also, notice that

$$\begin{aligned}\frac{\partial f}{\partial\nu}|\nabla u|^2 - \langle\nabla f, \nabla u\rangle\frac{\partial u}{\partial\nu} &= \frac{\partial f}{\partial\nu}|\nabla_{\partial M} u|^2 + \frac{\partial f}{\partial\nu}\left(\frac{\partial u}{\partial\nu}\right)^2 - \frac{\partial f}{\partial\nu}\left(\frac{\partial u}{\partial\nu}\right)^2 - \langle\nabla_{\partial M} f, \nabla_{\partial M} u\rangle\frac{\partial u}{\partial\nu} \\ &= \frac{\partial f}{\partial\nu}|\nabla_{\partial M} u|^2 - \langle\nabla_{\partial M} f, \nabla_{\partial M} u\rangle\frac{\partial u}{\partial\nu}\end{aligned}$$

and

$$div_{\partial M}\left(f\nabla_{\partial M} u\frac{\partial u}{\partial\nu}\right) = \langle\nabla_{\partial M} f, \nabla_{\partial M} u\rangle\frac{\partial u}{\partial\nu} + f\Delta_{\partial M} u\frac{\partial u}{\partial\nu} + f\left\langle\nabla_{\partial M} u, \nabla_{\partial M}\left(\frac{\partial u}{\partial\nu}\right)\right\rangle.$$

Integrating the last equality on the boundary of M, one obtains that

$$-\int_{\partial M}\langle\nabla_{\partial M} f, \nabla_{\partial M} u\rangle\frac{\partial u}{\partial\nu}\, dS_g = \int_{\partial M} f\Delta_{\partial M} u\frac{\partial u}{\partial\nu}\, dS_g + \int_{\partial M} f\left\langle\nabla_{\partial M} u, \nabla_{\partial M}\left(\frac{\partial u}{\partial\nu}\right)\right\rangle dS_g.$$

Putting together all of this conclusions into (3.10), one sees that

$$\int_{\partial M} f\left[-\mathrm{III}(\nabla_{\partial M}u,\nabla_{\partial M}u)+\frac{\partial u}{\partial\nu}\left(-2\Delta_{\partial M}u-H\frac{\partial u}{\partial\nu}\right)\right]dS_g-\int_{\partial M}\frac{\partial f}{\partial\nu}|\nabla_{\partial M}u|^2\,dS_g$$
$$=\int_M f\left(|\nabla^2u|^2-(\Delta u)^2\right)dV_g+\int_M\left(-(\Delta f)g+\nabla^2 f+fRic\right)(\nabla u,\nabla u)\,dV_g.$$

To conclude, it suffices to use the fact that

$$\int_M f\left(|\nabla^2u|^2-(\Delta u)^2\right)dV_g=\int_M f\left[|\nabla^2u+\kappa ug|^2-(\Delta u+n\kappa u)^2\right]dV_g$$
$$+(n-1)\kappa\left[\int_{\partial M}\left(2fu\frac{\partial u}{\partial\nu}-u^2\frac{\partial f}{\partial\nu}\right)dS_g+\int_M\left((\Delta f)u^2-2f|\nabla u|^2\right)dV_g\right]$$
$$+n(n-1)\kappa^2\int_M u^2f\,dV_g.$$

So, the proof is finished. □

As an application, we are going to establish a key lemma that will be fundamental in the proof of the main result of this section.

Lemma 3.1. *Let (M^n,g) be a compact manifold with boundary ∂M. We assume that f is a smooth function on M^n satisfying*

$$f\mathring{Ric}=\mathring{\nabla}^2 f\quad \text{and}\quad f\,|_{\partial M}=0.$$

Then we have:

$$\begin{aligned}\int_{\partial M}\frac{\partial f}{\partial\nu}\left(|\nabla_{\partial M}\eta|^2-(n-1)\kappa\eta^2\right)dS_g &= \frac{1}{n}\int_M[(n-1)\Delta f+Rf]g(\nabla u,\nabla u)dV_g\\ &-(n-1)\kappa\int_M(\Delta f+n\kappa f)u^2dV_g\\ &-\int_M f|\nabla^2u+\kappa ug|^2dV_g\\ &-2\int_M f[Ric-(n-1)\kappa g](\nabla u,\nabla u)dV_g,\end{aligned}\tag{3.11}$$

where η is any function on ∂M and u is a solution of

$$\begin{cases}\Delta u+n\kappa u=0 & in\ \ M,\\ u=\eta & on\ \ \partial M.\end{cases}\tag{3.12}$$

Proof. By Proposition 3.1, equation (3.12) and the fact that $f=0$ on ∂M, one sees that

$$\begin{aligned}\int_{\partial M}\frac{\partial f}{\partial\nu}\left(|\nabla_{\partial M}\eta|^2-(n-1)\kappa\eta^2\right)dS_g &= -\int_M f|\nabla^2u+\kappa ug|^2dV_g\\ &-(n-1)\kappa\int_M(\Delta f+n\kappa f)u^2dV_g\\ &-\int_M\left(\nabla^2 f-(\Delta f)g+fRic\right)(\nabla u,\nabla u)\,dV_g\\ &+2(n-1)\kappa\int_M f|\nabla u|^2\,dV_g.\end{aligned}$$

Now, using $f\mathring{Ric} = \mathring{\nabla}^2 f$, we get

$$
\begin{aligned}
\int_{\partial M} \frac{\partial f}{\partial \nu}\left(|\nabla_{\partial M}\eta|^2 - (n-1)\kappa\eta^2\right) dS_g \;=\; & -\int_M f|\nabla^2 u + \kappa u g|^2 dV_g \\
& -(n-1)\kappa \int_M (\Delta f + n\kappa f)u^2 dV_g \\
& -\int_M \left(2fRic - \frac{R}{n} fg - \frac{n-1}{n}\Delta f g\right)(\nabla u, \nabla u)\, dV_g \\
& +2(n-1)\kappa \int_M f|\nabla u|^2\, dV_g \\
=\; & \frac{1}{n}\int_M [(n-1)\Delta f + Rf] g(\nabla u, \nabla u) dV_g \\
& -(n-1)\kappa \int_M (\Delta f + n\kappa f)u^2 dV_g \\
& -\int_M f|\nabla^2 u + \kappa u g|^2 dV_g \\
& -2\int_M f[Ric - (n-1)\kappa g](\nabla u, \nabla u) dV_g,
\end{aligned}
$$

as we wanted to prove. □

We are now ready to discuss our first theorem of this chapter.

Theorem 3.3 (Theorem 3.1)**.** *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M and positive scalar curvature satisfying*

$$
\frac{n-2}{2(n-1)} R + \frac{n}{n-1}\rho = -\tau, \tag{3.13}
$$

where τ is a positive constant. Then we have:

$$
Vol(M) \geq \frac{1}{\tau}\sqrt{\frac{R_{min} + 3\tau}{2n}}\,|\partial M|. \tag{3.14}
$$

Moreover, if equality holds in (3.14)*, then (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.*

Proof. To begin with, we have from (3.6) that

$$
\Delta f = -\tau f,
$$

where $\tau = -\frac{(n-2)}{2(n-1)}R - \frac{n}{n-1}\rho$. In particular, it follows from equation (2.19) that $\tau > 0$. Now, choosing $u = f$ and $\kappa = \frac{\tau}{n}$, one obtains that

$$
\begin{cases}
\Delta u + n\kappa u = 0 & \text{in } M, \\
u = 0 & \text{on } \partial M.
\end{cases}
$$

Hence, by using Lemma 3.1, one sees that

$$\begin{aligned}\int_M f\Big|\nabla^2 f+\frac{\tau}{n}fg\Big|^2 dV_g \;&-\; \frac{1}{n}\int_M Rf|\nabla f|^2 dV_g-\Big(\frac{n-1}{n}\Big)\int_M(\Delta f)|\nabla f|^2 dV_g\\ &+\; 2\int_M f\Big[Ric-\Big(\frac{n-1}{n}\Big)\tau g\Big](\nabla f,\nabla f)dV_g=0. \qquad (3.15)\end{aligned}$$

In another direction, by the classical Bochner's formula

$$\frac{1}{2}\Delta|\nabla f|^2=Ric(\nabla f,\nabla f)+|\nabla^2 f|^2+\langle\nabla f,\,\nabla\Delta f\rangle,$$

one deduces that

$$2fRic(\nabla f,\nabla f)=f\Delta|\nabla f|^2-2f|\nabla^2 f|^2+2\tau f|\nabla f|^2.$$

Upon integrating, one sees that

$$\begin{aligned}2\int_M f\Big[Ric-\Big(\frac{n-1}{n}\Big)\tau g\Big](\nabla f,\nabla f)dV_g \;&=\; \int_M f\Delta|\nabla f|^2 dV_g-2\int_M f|\nabla^2 f|^2 dV_g\\ &\quad+\frac{2\tau}{n}\int_M f|\nabla f|^2 dV_g. \qquad (3.16)\end{aligned}$$

Also, by Green's identity and the fact that $f^{-1}(0)=\partial M$ and, by Proposition 2.4, $|\nabla f|_{|\partial M}$ is a nonzero constant along the boundary, it is easy to check that

$$\int_M f\Delta|\nabla f|^2 dV_g=\int_M|\nabla f|^2\Delta f dV_g+|\nabla f|^3_{|\partial M}|\partial M|. \qquad (3.17)$$

Substituting (3.17) into (3.16) yields

$$\begin{aligned}2\int_M f\Big[Ric-\Big(\frac{n-1}{n}\Big)\tau g\Big](\nabla f,\nabla f)\,dV_g \;&=\; \int_M|\nabla f|^2\Delta f dV_g+|\nabla f|^3_{|\partial M}|\partial M|\\ &\quad-2\int_M f|\mathring{\nabla}^2 f|^2\,dV_g-\frac{2}{n}\int_M f(\Delta f)^2\,dV_g\\ &\quad+\frac{2\tau}{n}\int_M f|\nabla f|^2\,dV_g,\end{aligned}$$

where we have used that $|\nabla^2 f|^2=|\mathring{\nabla}^2 f|^2+\frac{(\Delta f)^2}{n}$. Using (3.15), we then obtain

$$\begin{aligned}\int_M f|\mathring{\nabla}^2 f|^2 dV_g \;&=\; -\frac{1}{n}\int_M Rf|\nabla f|^2 dV_g+\frac{1}{n}\int_M(\Delta f)|\nabla f|^2 dV_g\\ &\quad+|\nabla f|^3_{|\partial M}|\partial M|-\frac{2}{n}\int_M f(\Delta f)^2 dV_g+\frac{2\tau}{n}\int_M f|\nabla f|^2 dV_g. \qquad (3.18)\end{aligned}$$

Proceeding, on integrating by parts, one deduces that

$$\int_M f(\Delta f)^2\,dV_g=-\tau\int_M f^2\Delta f\,dV_g=2\tau\int_M f|\nabla f|^2\,dV_g. \qquad (3.19)$$

Plugging (3.19) together with (3.6) and the value of τ into (3.18) gives

$$\begin{aligned}\int_M f|\mathring{\nabla}^2 f|^2 dV_g &= -\frac{1}{n}\int_M fR|\nabla f|^2 dV_g + |\nabla f|^3_{|\partial M}|\partial M| \\ &\quad -\frac{3\tau}{n}\int_M f|\nabla f|^2 dV_g. \end{aligned} \tag{3.20}$$

Since $-\Delta f > 0$ (in the interior of M), by Hölder's inequality, one sees that

$$\begin{aligned}\Big(\int_M f\Delta f\, dV_g\Big)^2 &\leq \int_M f^2\,(-\Delta f)\, dV_g \int_M (-\Delta f)\, dV_g \\ &= \int_M f^2\,(-\Delta f)\, dV_g\, |\nabla f|_{|\partial M}|\partial M|,\end{aligned}$$

where we used the Stokes' formula. With aid of (3.19), we infer

$$\begin{aligned}|\nabla f|_{|\partial M}|\partial M|\int_M f|\nabla f|^2 dV_g &= \frac{1}{2}|\nabla f|_{|\partial M}|\partial M|\int_M f^2(-\Delta f)dV_g \\ &\geq \frac{1}{2}\Big(\int_M f\Delta f dV_g\Big)^2. \end{aligned} \tag{3.21}$$

Again, by Hölder's inequality, we get

$$\begin{aligned}\Big(\int_M \Delta f dV_g\Big)^2 &\leq \Big(\int_M \tau dV_g\Big)\Big(\int_M f\big(-\Delta f\big)dV_g\Big) \\ &= \tau Vol(M)\Big(\int_M f\big(-\Delta f\big)dV_g\Big),\end{aligned}$$

so that

$$\tau^2 Vol(M)^2\Big(\int_M f\Delta f dV_g\Big)^2 \geq \Big(\int_M \Delta f dV_g\Big)^4 = |\nabla f|^4_{|\partial M}|\partial M|^4. \tag{3.22}$$

Combining (3.22) and (3.21), one obtains that

$$2\tau^2 Vol(M)^2\int_M f|\nabla f|^2 dV_g \geq |\nabla f|^3_{|\partial M}|\partial M|^3.$$

Substituting this into (3.20), one sees that

$$\begin{aligned}0 &\leq \int_M f|\mathring{\nabla}^2 f|^2 dV_g \\ &= -\frac{1}{n}\int_M Rf|\nabla f|^2 dV_g + |\nabla f|^3_{|\partial M}|\partial M| - \frac{3\tau}{n}\int_M f|\nabla f|^2 dV_g \\ &\leq -\frac{R_{min}+3\tau}{n}\int_M f|\nabla f|^2 dV_g + |\nabla f|^3_{|\partial M}|\partial M| \\ &\leq -\frac{(R_{min}+3\tau)|\nabla f|^3_{|\partial M}|\partial M|^3}{2n\tau^2 Vol(M)^2} + |\nabla f|^3_{|\partial M}|\partial M|.\end{aligned}$$

From this, it follows that

$$Vol(M) \geq \frac{1}{\tau}\sqrt{\frac{R_{min}+3\tau}{2n}}\,|\partial M|, \tag{3.23}$$

as asserted.

To conclude, if equality holds in (3.23), then

$$\nabla^2 f = -\frac{\tau}{n} f g.$$

Moreover, since $f = 0$ on the boundary ∂M (and f is constant on ∂M), we may use Theorem B (II) of [93] to deduce that (M^n, g) has constant sectional curvature $\frac{\tau}{n}$ and this forces (M^n, g) to be an Einstein manifold. Thus, it suffices to invoke Proposition 1 of [37] together with the fact that the boundary is totally geodesic in order to conclude that (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$. This finishes the proof of the theorem. □

As it was pointed out, static perfect fluid space-times generalize vacuum static spaces. It is already know that, in the context of compact SPFST with boundary, constant scalar curvature implies that the manifold must satisfy the static equation; see Proposition 2.3. Thus, we have a direct corollary of Theorem 3.3 for the case of positive static triple.

Corollary 3.3 (Corollary 3.1)**.** *Let (M^n, g, f, ρ) be a compact oriented static perfect fluid space-time with boundary ∂M and constant positive scalar curvature R. Then we have:*

$$Vol(M) \geq \sqrt{\frac{(n-1)(n+2)}{2nR}}\,|\partial M|. \tag{3.24}$$

Moreover, if equality holds in (3.24)*, then (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.*

Proof. The condition that (M^n, g, f, ρ) has positive constant scalar curvature R implies from Proposition 2.3 that $\rho = -\frac{R}{2}$ and so, $\tau = \frac{R}{n-1}$ is constant. Therefore, Corollary 3.1 follows from Theorem 3.1. □

3.2 Boundary area estimate in terms of the Brown-York mass

Before to present our estimate for the area of the boundary in terms of the Brown-York mass, we shall establish a lemma that will be fundamental in the proof of the result. To do so, similar to [44] and [115], we set the conformal metric $\overline{g} = v^{\frac{4}{n-2}} g$ with

$$v = (1+\alpha f)^{-\frac{n-2}{2}} \quad \text{and} \quad \alpha^{-1} = \max_M \left(f^2 + \frac{n(n-1)}{R_g} |\nabla f|^2 \right)^{\frac{1}{2}}. \tag{3.25}$$

With aid of these notations, we have the following lemma.

Lemma 3.2. *Let (M^n, g, f, ρ), $n \geq 3$, be a compact static perfect fluid space-time with positive scalar curvature and satisfying the dominant energy condition. Then the scalar curvature $R_{\overline{g}}$ with respect to the conformal metric $\overline{g}$ is nonnegative. Moreover, $R_{\overline{g}} = 0$ if and only if $\Delta_g f = -\frac{R_g}{n-1} f$ and $f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2$ is constant on M^n.*

Proof. Initially, by using the conformal metric defined above, a direct computations yields

$$\begin{aligned} \Delta_g v &= \Delta[(1+\alpha f)^{-\frac{n-2}{2}}] \\ &= div\left[-\frac{n-2}{2}(1+\alpha f)^{-\frac{n}{2}}\alpha\nabla f\right] \\ &= -\frac{n-2}{2}\alpha(1+\alpha f)^{-\frac{n}{2}}\Delta f + \frac{n(n-2)}{4}\alpha^2(1+\alpha f)^{-\frac{n+2}{2}}|\nabla f|^2. \end{aligned} \tag{3.26}$$

On the other hand, it follows from Eq. (3.6) and the dominant energy condition that

$$\begin{aligned} \Delta f &= \left(\frac{n-2}{2(n-1)}R_g + \frac{n}{n-1}\rho\right) f \\ &\geq \left(\frac{n-2}{2(n-1)}R_g - \frac{n}{2(n-1)}R_g\right) f \\ &= -\frac{R_g}{n-1} f. \end{aligned}$$

This combined with (3.26) gives

$$\begin{aligned} \Delta_g v \leq\ & \frac{n-2}{2(n-1)}\alpha(1+\alpha f)^{-\frac{n}{2}} R_g f \\ & + \frac{n(n-2)}{4}\alpha^2(1+\alpha f)^{-\frac{n+2}{2}}|\nabla f|^2. \end{aligned}$$

Rearranging the terms, one sees that

$$\Delta_g v \leq \frac{n(n-2)}{4}\alpha(1+\alpha f)^{-\frac{n+2}{2}}\left[\frac{2R_g}{n(n-1)}(1+\alpha f)f + \alpha|\nabla f|^2\right]. \tag{3.27}$$

In another direction, it is well known, by the formulae for conformal metric, that

$$R_{\overline{g}} = v^{-\frac{n+2}{n-2}}\left(R_g v - 4\frac{n-1}{n-2}\Delta_g v\right),$$

where $R_{\overline{g}}$ is the scalar curvature with respect to the conformal metric $\overline{g}$. From this, one obtains that

$$\begin{aligned}
\Delta_g v &= \frac{n-2}{4(n-1)}R_g v - \frac{n-2}{4(n-1)}v^{\frac{n+2}{n-2}}R_{\overline{g}}\\
&= \frac{n-2}{4(n-1)}R_g(1+\alpha f)^{-\frac{n-2}{2}} - \frac{n-2}{4(n-1)}((1+\alpha f)^{-\frac{n-2}{2}})^{\frac{n+2}{n-2}}R_{\overline{g}}\\
&= \frac{n-2}{4(n-1)}R_g(1+\alpha f)^{-\frac{n-2}{2}} - \frac{n-2}{4(n-1)}(1+\alpha f)^{-\frac{n+2}{2}}R_{\overline{g}}\\
&= \frac{n-2}{4(n-1)}(1+\alpha f)^{-\frac{n+2}{2}}\left[(1+\alpha f)^2 R_g - R_{\overline{g}}\right].
\end{aligned}$$

Substituting this into (3.27) and rearranging terms, one deduces that

$$R_g(1+\alpha f)^2 - R_{\overline{g}} \leq n(n-1)\alpha\left[\frac{2R_g}{n(n-1)}(1+\alpha f)f + \alpha|\nabla f|^2\right].$$

Of which, we obtain

$$\begin{aligned}
R_{\overline{g}} &\geq R_g(1+\alpha f)^2 - n(n-1)\alpha\left[\frac{2R_g}{n(n-1)}(1+\alpha f)f + \alpha|\nabla f|^2\right]\\
&= R_g\left[(1+\alpha f)^2 - 2\alpha(1+\alpha f)f - \frac{n(n-1)}{R_g}\alpha^2|\nabla f|^2\right]\\
&= R_g\left[1-\alpha^2\left(f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2\right)\right]. \qquad (3.28)
\end{aligned}$$

Therefore, by using (3.28) and the value chosen for α in (3.25), one concludes that $R_{\overline{g}} \geq 0$, as asserted.

Finally, it follows from (3.28) that $R_{\overline{g}} = 0$ if and only if $\Delta f = -\frac{R_g}{n-1}f$ and $f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2$ is constant over M, which in particular implies $\rho = -\frac{R_g}{2}$ and consequently, R_g is constant. So, the proof is finished. $\square$

We are now ready to discuss the central result of this section.

Theorem 3.4 (Theorem 3.2)**.** *Let (M^n, g, f, ρ), $n \geq 3$, be a compact static perfect fluid space-time with (possibly disconnected) boundary and positive scalar curvature satisfying the dominant energy condition. Suppose that each boundary component $(\partial M_i, g)$ can be isometrically embedded in $\mathbb{R}^n$ as a convex hypersurface. Then we have*

$$|\partial M_i| \leq c\,\mathfrak{m}_{BY}(\partial M_i, g), \qquad (3.29)$$

where c is a positive constant. Moreover, equality occurs for some component ∂M_i if and only if M^n is isometric to the standard hemisphere $\mathbb{S}^n_+$.

Proof. In the first part of the proof, we shall follow the arguments of [44, 46, 115]. First of all, we have from (3.25) that

$$v=(1+\alpha f)^{-\frac{n-2}{2}} \quad \text{and} \quad \alpha^{-1}=\max_M\left(f^2+\frac{n(n-1)}{R_g}|\nabla f|^2\right)^{\frac{1}{2}}. \tag{3.30}$$

We now claim that the mean curvature $H^i_{\overline{g}}$ of ∂M_i with respect to the conformal metric $\overline{g}=v^{\frac{4}{n-2}}g$ is strictly positive. Indeed, since $f|_{\partial M}=0$, it follows from (3.30) that $v\,|_{\partial M}=1$. Hence, $\overline{g}=g$ over the boundary and $(\partial M_i,\overline{g})$ is isometric to $(\partial M_i,\, g)$, which by assumption can be isometrically embedded in $\mathbb{R}^n$ as a convex hypersurface with mean curvature H^i_0, induced by the Euclidian metric. Besides, taking into account that mean curvature of ∂M_i with respect to the conformal metric $\overline{g}$ is given by

$$H^i_{\overline{g}}=v^{-\frac{2}{n-2}}\left(H^i_g+2\frac{n-1}{n-2}\partial_\nu(\log(v))\right), \tag{3.31}$$

one obtains that

$$\begin{aligned} H^i_{\overline{g}} &= \frac{2(n-1)}{n-2}\left\langle\nabla(1+\alpha f)^{-\frac{n-2}{2}},\nu\right\rangle \\ &= (n-1)\alpha|\nabla f|_{|\partial M_i}, \end{aligned} \tag{3.32}$$

where we have used that $H^i_g=0$ and $\nu=-\frac{\nabla f}{|\nabla f|}$. This proves that $H^i_{\overline{g}}>0$, as claimed.

Proceeding, from (3.32) we get that

$$\begin{aligned} \mathfrak{m}_{BY}(\partial M_i,\overline{g}) &= \int_{\partial M_i}(H^i_0-H^i_{\overline{g}})dS_g \\ &= \mathfrak{m}_{BY}(\partial M_i,g)-(n-1)\alpha|\nabla f|_{|\partial M_i}|\partial M_i|. \end{aligned} \tag{3.33}$$

Hence, by Lemma 3.2, we deduce that $R_{\overline{g}}\geq 0$. Moreover, since $H^i_{\overline{g}}>0$, we may invoke the Positive Mass Theorem for the Brown-York mass (see, e.g., [102] and [115]) to conclude that $\mathfrak{m}_{BY}(\partial M_i,\overline{g})\geq 0$. Consequently,

$$\begin{aligned} |\partial M_i| &\leq \frac{1}{(n-1)\alpha|\nabla f|_{|\partial M_i}}\mathfrak{m}_{BY}(\partial M_i,g) \\ &= \frac{1}{(n-1)\alpha|\nabla f|_{|\partial M_i}}\int_{\partial M_i}H^i_0 dS_g, \end{aligned} \tag{3.34}$$

which proves (3.29).

Next, if equality holds in (3.34) for some component ∂M_{i_0}, then we necessarily have

$$\mathfrak{m}_{BY}(\partial M_{i_0},\overline{g})=0.$$

Hence, by invoking the equality case in the Positive Mass Theorem for the Brown-York mass, one deduces that the conformal metric $\overline{g}$ is flat and therefore, $(M^n, \overline{g})$ is isometric to a bounded domain in $\mathbb{R}^n$. Besides, $R_{\overline{g}} = 0$ and then it follows from Lemma 3.2 that $\Delta_g f = -\frac{R_g}{n-1} f$ and $\frac{n(n-1)}{R_g}|\nabla f|^2 + f^2$ is constant. In particular, $\rho = -\frac{R_g}{2}$. But, in this case, R_g must be constant over M. Thus, one obtains that

$$0 = \nabla \left[\frac{n(n-1)}{R_g}|\nabla f|^2 + f^2 \right] = \frac{2n(n-1)}{R_g}\nabla^2 f(\nabla f) + 2f\nabla f$$

so that

$$\nabla|\nabla f|^2 - \frac{2\Delta_g f}{n}\nabla f = 0.$$

Now, it suffices to apply Lemma 2.5 to conclude that $|\mathring{Ric}|^2 = 0$ in M. Hence, we may use Proposition 1 of [37] and the fact that ∂M is totally geodesic in order to conclude that (M^n, g) is isometric to $\mathbb{S}^n_+$.

On the other hand, if (M^n, g) is isometric to $\mathbb{S}^n_+$, one deduces that $\partial M = \mathbb{S}^{n-1}$. Thereby, the Brown-York mass of $\mathbb{S}^{n-1}$ is given by

$$\mathfrak{m}_{BY}(\mathbb{S}^{n-1}) = \int_{\mathbb{S}^{n-1}} (n-1) dS_{g_{\mathbb{S}^{n-1}}} = (n-1)\omega_{n-1},$$

where ω_{n-1} is the volume of the round $(n-1)$-dimensional sphere. At the same time, since $\mathbb{S}^n_+$ has constant scalar curvature $R_g = n(n-1)$, it follows from (3.6) that $\Delta_g f = -\frac{R_g}{n-1} f = -nf$ and consequently, $f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2$ is constant on M. Indeed, a direct computation using (3.5) yields

$$\begin{aligned}
\nabla\left(f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2\right) &= 2f\nabla f + 2\nabla^2 f(\nabla f) \\
&= -\frac{2\Delta f}{n}\nabla f + 2\nabla^2 f(\nabla f) \\
&= 2\mathring{\nabla}^2 f(\nabla f) = 2f\mathring{Ric}(\nabla f) = 0.
\end{aligned}$$

Hence, one obtains that

$$\alpha^{-2} = \left(f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2\right)\Big|_{\partial M} = |\nabla f|^2_{|\partial M},$$

so that

$$\alpha = \frac{1}{|\nabla f|_{|\partial M}}.$$

Of which, we arrive at

$$\frac{\mathfrak{m}_{BY}(\partial M, g)}{\alpha(n-1)|\nabla f|_{|\partial M}} = \omega_{n-1} = |\partial M|,$$

which is the equality in (3.34). So, the proof is completed. $\square$

As a consequence of the ideas used in the proof of Theorem 3.4, we have the following corollary.

Corollary 3.4 (Corollary 3.2)**.** *Let* (M^n, g, f, ρ), $n \geq 3$, *be a compact static perfect fluid space-time with (possibly disconnected) boundary and positive scalar curvature. Assume the dominant energy condition and that each boundary component* $(\partial M_i, g)$ *can be isometrically embedded in* $\mathbb{R}^n$ *as a convex hypersurface. Then we have*

$$|\partial M_i| \leq \widetilde{c} \int_{\partial M_i} (R^{\partial M_i} + |\mathring{h}_i|^2) dS_g$$

for some positive constant $\widetilde{c}$, *where* $\mathring{h}_i$ *is the traceless second fundamental form of* ∂M_i *as a hypersurface of* $\mathbb{R}^n$. *Moreover, equality occurs for some connected component of the boundary if and only if* (M^n, g) *is isometric to the round hemisphere* $\mathbb{S}^n_+$.

Proof. Initially, we assume that $(\partial M_i, \overline{g})$ and $(\partial M_i, g)$ are isometric. In this case, one has $R^{\partial M_i}_{\overline{g}} = R^{\partial M_i}_g$. By using the Gauss' equation for ∂M_i as an embedded hypersurface of $\mathbb{R}^n$, one obtains that

$$R^{\partial M_i}_g = (H^i_0)^2 - |h_i|^2 = \frac{n-2}{n-1}(H^i_0)^2 - |\mathring{h}_i|^2, \tag{3.35}$$

where h_i and $\mathring{h}_i$ stand for the second fundamental form and traceless second fundamental form of ∂M_i, respectively. Now, we use (3.34) and the Hölder's inequality in order to infer

$$\begin{aligned} |\partial M_i| &\leq \frac{1}{(n-1)^2\alpha^2 |\nabla f|^2_{|\partial M_i}} \int_{\partial M_i} (H^i_0)^2 dS_g \\ &= \frac{1}{(n-1)(n-2)\alpha^2 |\nabla f|^2_{|\partial M_i}} \int_{\partial M_i} (R^{\partial M_i}_g + |\mathring{h}_i|^2) dS_g. \end{aligned} \tag{3.36}$$

Clearly, if equality holds in (3.36), then (3.34) also becomes an equality. Hence, one concludes that (M^n, g) is isometric to the round hemisphere $\mathbb{S}^n_+$.

On the other hand, if M is isometric to $\mathbb{S}^n_+$ with standard metric, then $R^{\partial M}_g = (n-2)(n-1)$ and $\mathring{h}_i = 0$. Furthermore, one has

$$\Big(f^2 + \frac{n(n-1)}{R_g}|\nabla f|^2\Big)\,|_{\partial M} = |\nabla f|^2_{|\partial M}$$

and consequently, we obtain

$$\frac{1}{(n-1)(n-2)\alpha^2 |\nabla f|^2_{|\partial M_i}} \int_{\partial M_i} (R^{\partial M_i}_g + |\mathring{h}_i|^2) dS_g = \omega_{n-1} = |\partial M|,$$

which gives the equality in (3.36). This concludes the proof of the corollary. $\square$

3.3 New examples of SPFST

As was mentioned in Chapter 2, the classical examples of SPFST are the standard hemisphere $\mathbb{S}^n_+$ (connected boundary) with standard metric and the product $[0,\pi]\times\mathbb{S}^{n-1}$ (disconnected boundary) with metric $g = dt^2 + (n-2)g_{\mathbb{S}^{n-1}}$. In what follows, we will discuss the new example of simply connected SPFST with boundary and constant scalar curvature stated in Example 1.1, see also [45, Example 2] for more details.

Example 3.2 (Example 3.1)**.** *Let $M^n = \mathbb{S}^{p+1}_+ \times \mathbb{S}^q$, $q > 1$, with the product metric*

$$g = dr^2 + \sin^2(r)g_{\mathbb{S}^p} + \frac{q-1}{p+1}g_{\mathbb{S}^q},$$

where $r(x,y) = r(x)$ is the height function of $\mathbb{S}^{p+1}$. Moreover, we consider the potential function $f(r) = \cos(r)$ with $r \leq \frac{\pi}{2}$. Thus, (M^n, g) satisfies (2.11) *and* (2.12)*. In particular, since it has constant scalar curvature, then it is a static space.*

To check such an example, we first observe that

$$\nabla f = -\sin(r)\nabla r.$$

From this, one can make use of Cartan's formula (Lemma 2.1) to deduces that

$$\nabla^2 f = \frac{1}{2}\mathcal{L}_{\nabla f}g = -\cos(r)dr^2 - \cos(r)\sin^2(r)g_{\mathbb{S}^p} = -f(dr^2 + \sin^2(r)g_{\mathbb{S}^p}).$$

Consequently,

$$\nabla^2 f = -fg_{\mathbb{S}^{p+1}_+}.$$

In particular, one sees that

$$\Delta f = g^{ij}\nabla_i\nabla_j f = -(p+1)f.$$

Next, since $g = g_{\mathbb{S}^{p+1}_+} + \frac{q-1}{p+1}g_{\mathbb{S}^q}$ is a product metric, we may write

$$Ric = pg_{\mathbb{S}^{p+1}_+} + (q-1)g_{\mathbb{S}^q}.$$

Thereby, the scalar curvature is constant and given by

$$R = (p+q)(p+1) = (n-1)(p+1).$$

Of which, we arrive at

$$-(\Delta f)g + \nabla^2 f - fRic = (p+1)fg - fg_{\mathbb{S}^{p+1}_+} - f(pg_{\mathbb{S}^{p+1}_+} + (q-1)g_{\mathbb{S}^q}) = 0,$$

which proves that $\mathbb{S}^{p+1}_{+} \times \mathbb{S}^{q}$ *is a static manifold with connected boundary* $\partial M = \mathbb{S}^{p} \times \mathbb{S}^{q}$ *and* $f(r) = \cos(r)$ *vanishes on the boundary. Furthermore, one easily verifies that* $\mathbb{S}^{p+1}_{+} \times \mathbb{S}^{q}$ *is simply connected.*

Remark 9. *As previously mentioned, Example 3.2 is a simply connected static space with positive scalar curvature and connected boundary. Therefore, it is a counterexample to the Cosmic no-hair conjecture for arbitrary dimension* $n \geq 4$. *Remember that such a conjecture says that: the hemisphere* $\mathbb{S}^{n}_{+}$ *is the only possible* n*-dimensional (simply connected) positive static triple with single-horizon (connected boundary) and positive scalar curvature (see [22], [23]).*

Reasoning as in Example 3.2, we also obtain the following example of positive static triple.

Example 3.3. *We consider* $M = [0,\pi] \times \mathbb{S}^{p} \times \mathbb{S}^{q}$ *endowed with the metric*

$$g = dt^2 + (p-1)g_{\mathbb{S}^p} + (q-1)g_{\mathbb{S}^q}.$$

Suppose that $f(t) = \sin(t)$. *Hence, we deduce that* (M, g, f) *is a positive static triple with disconnected boundary consisting of two copies of* $\mathbb{S}^{p} \times \mathbb{S}^{q}$, *i.e.,*

$$\partial M = (\{0\} \times \mathbb{S}^{p} \times \mathbb{S}^{q}) \cup (\{\pi\} \times \mathbb{S}^{p} \times \mathbb{S}^{q}).$$

From Proposition 2.5 (cf. [37, Proposition 2]), we know that Eqs. (3.5) and (3.6) do not guarantee that a SPFST has constant scalar curvature. All of the examples presented in Chapter 2 are compact with constant scalar curvature, so it is interesting to seek for examples of SPFST with non-constant scalar curvature. In [73], Massod-ul-Alam discuss an spherically symmetric example of non-trivial non-compact SPFST due to Wyman [112]. In [9], Barboza, Leandro and Pina obtained a complete characterization of semi-Riemannian non-compact conformally flat static perfect fluid space-time symmetric with respect to a given group of translations. Explicitly, given a pseudo-Euclidian metric

$$\delta = \sum_{i=1}^{n} \varepsilon_i dx_i^2,$$

considering canonical coordinates $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$, $n \geq 3$, and coefficients $\varepsilon_i = \pm 1$ with $\varepsilon_j = 1$ for some j, $i, j \in \{1, \ldots, n\}$. Take a linear map $\xi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\xi(x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$ was chosen arbitrarily. In the case we are deal now, the static metric with $\mathbb{R}^n$ as their spatial factor is given by

$$\widehat{g} = \frac{1}{\varphi^2(x)}\delta - f^2(x)dt^2,$$

in which one sees that $(\mathbb{R}^n, \frac{1}{\varphi^2(x)}\delta)$ is locally conformally flat, where $\varphi, f : (a,b) \subset \mathbb{R} \to \mathbb{R}_+$ such that we adopt $\varphi = \varphi \circ \xi$, $f = f \circ \xi : M \subset \mathbb{R}^n \to \mathbb{R}_+$, where $M = \{x \in \mathbb{R}^n : a < \xi(x) < b\} = \xi^{-1}(a,b)$ is open in $\mathbb{R}^n$, satisfying (3.5) and (3.6). Such a manifold is a SPFST symmetric with respect to the additive group $G = \{x \in \mathbb{R}^n : \xi(x) = 0\}$ of translations in $\mathbb{R}^n$.

The next example corresponds to [9, Example 2] and it is a non-compact non-trivial SPFST with non-constant scalar curvature.

Example 3.4 ([9], Example 2)**.** *Taking* $\varphi(\xi) = e^{\xi}$, *we may take* $f(\xi) = e^{(-1+\sqrt{n-1})\xi}$. *Therefore, since* φ *is always positive,* $\widehat{M} = \mathbb{R}^n \times \mathbb{R}$ *with metric tensor*

$$\widehat{g} = \frac{\delta}{e^{2\xi}} - e^{2(-1+\sqrt{n-1})\xi}dt^2.$$

Then, the mass-energy density and pressure of a metric with this expression are given by

$$\mu(\xi) = -\frac{|\alpha|^2(n-1)(n-2)}{2}e^{2\xi}$$

and

$$\rho(\xi) = |\alpha|^2\left(\frac{n-1}{n}\right)\left[\frac{(n-2)^2}{2} - (-1-\sqrt{n-1})(n-1-\sqrt{n-1})\right]e^{2\xi},$$

respectively.

4 RIGIDITY OF COMPACT QUASI-EINSTEIN MANIFOLDS WITH BOUNDARY

This chapter is based in the paper *Rigidity of compact quasi-Einstein manifolds with boundary*, written by the author together with Ernani Ribeiro Jr. and Detang Zhou [36]. It is divided into four sections, each one corresponding to the main results of the aforementioned paper.

For the convenience of the reader, we will recall here the definition of quasi-Einstein manifolds.

Definition 4.1. *A complete n-dimensional Riemannian manifold (M^n, g), $n \geq 2$, possibly with boundary ∂M, is called an m-quasi-Einstein manifold, or simply quasi-Einstein manifold, if there is a smooth potential function u on M^n satisfying the system*

$$\nabla^2 u = \frac{u}{m}(Ric - \lambda g), \tag{4.1}$$

where $u > 0$ in the interior of M and $u = 0$ on the boundary ∂M. By tracing (4.1)*, one sees that*

$$\Delta u = \frac{u}{m}(R - n\lambda). \tag{4.2}$$

for some constants λ and $0 < m < \infty$ (see [26], [59] and [60]).

This chapter is mainly motivated by the uniqueness problem for compact quasi-Einstein manifolds with boundary and constant scalar curvature. In 2014, He, Petersen and Wylie [60, Proposition 2.4] (see Proposition 2.8) proved that the only example of a compact quasi-Einstein manifold with boundary of dimension $n \geq 2$ with constant Ricci curvature is the round hemisphere $\mathbb{S}^n_+$. In other words, they classified such manifolds under constant Ricci curvature condition. One problem that naturally arises is to *classify all nontrivial compact (simply connected) quasi-Einstein manifolds with boundary and constant scalar curvature.*

In this chapter, we address the above problem. Our approach is inspired in some results on gradient Ricci solitons. To be precise, Fernández-Lopéz and García-Río [48, Theorem 1] proved that the possible values for the constant scalar curvature R of an n-dimensional complete gradient Ricci soliton are

$$\{0, \lambda, \ldots, (n-1)\lambda, n\lambda\}.$$

In particular, they proved in [48, Theorem 10] that does not exist a complete gradient shrinking ($\lambda>0$) Ricci soliton with scalar curvature $R=\lambda$ and they stated that every four-dimensional complete gradient shrinking Ricci soliton with $R\neq 2\lambda$ is rigid [48, Theorem 4], i.e., it is isometric to a quotient of $N^{n-k}\times\mathbb{R}^k$, where N is an Einstein manifold and $f=\frac{\lambda}{2}|x|^2$ in the Euclidian factor. Very recently, Cheng and Zhou [40] proved that four-dimensional complete noncompact gradient shrinking Ricci solitons with constant scalar curvature $R=2\lambda$ are isometric to a quotient of $\mathbb{S}^2\times\mathbb{R}^2$, which completes the classification of noncompact complete gradient shrinking Ricci solitons with constant scalar curvature in dimension 4. It is important to point out that the concept of rigid quasi-Einstein metrics was investigated by He, Petersen and Wylie in [60]. Furthermore, Case, Shu and Wei [26] proved that compact quasi-Einstein manifolds with constant scalar curvature must satisfy $\frac{n(n-1)\lambda}{m+n-1}\leq R\leq n\lambda$.

These above results inspired our first theorem in this chapter, which it will be proved in Section 4.1.

Theorem 4.1. *Let (M^n,g,u,λ) be a nontrivial compact m-quasi-Einstein manifold with boundary, $m>1$ and constant scalar curvature R. Then we have:*

$$R\in\left\{\frac{n(n-1)}{m+n-1}\lambda,\frac{m+n(n-2)}{m+n-2}\lambda,\ldots,(n-1)\lambda\right\}. \tag{4.3}$$

In general, one has $R=\frac{k(m-n)+n(n-1)}{m+n-k-1}\lambda$, for some $k\in\{0,1,\ldots,n-1\}$.

As we will see in the proof of Theorem 4.1, the integer $k\in\{0,1,\ldots,n-1\}$ is the dimension of the set of critical points $Crit(u)$ of the potential function u, or equivalently, the dimension of the set $MAX(u)$ of points in M which attains the maximum value. The reason why the value $n\lambda$ do not appears in (4.3) follows from Remark 6.

Proceeding, taking into account the possible values for the constant scalar curvature, a natural way to proceed is to seek for examples of compact quasi-Einstein manifolds with these scalar curvature values. It is well known that substituting the value $R=\frac{n(n-1)}{m+n-1}\lambda$ into Eq. (2.38), one sees that such a quasi-Einstein manifold is Einstein and thus, we may apply [60, Proposition 2.4] (see Proposition 2.8) to infer that it is isometric, up to scaling, to the round hemisphere $\mathbb{S}^n_+$. So, the first value in (4.3) is classified.

In the same spirit, Section 4.2 explores the extremal value assumed by R in Theorem 4.1, namely, $R=(n-1)\lambda$. More precisely, we have the following result.

Theorem 4.2. *Let (M^n, g, u, λ), $n \geq 3$, be a nontrivial simply connected compact m-quasi-Einstein manifold with boundary and $m > 1$. Then M^n has constant scalar curvature $R = (n-1)\lambda$ if and only if it is isometric, up to scaling, to the cylinder $I \times N$ with product metric, where N is a compact λ-Einstein manifold.*

As a consequence of Theorem 4.1, Theorem 4.2 and Proposition 4.1, we obtain a classification result for dimension 3 that will be proved in Section 4.3. To be precise, we have the following theorem.

Theorem 4.3. *Let (M^3, g, u, λ) be a nontrivial simply connected compact 3-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^3 has constant scalar curvature if and only if it is isometric, up to scaling, to either*

(a) the standard hemisphere $\mathbb{S}^3_+$, or

(b) the cylinder $I \times \mathbb{S}^2$ with the product metric.

To finish this chapter, Section 4.4 deal with rigidity results in dimension 4, similarly to the case of gradient shrinking Ricci solitons in [40]. Notice that, from Theorem 4.1, the possible values for the scalar curvature in dimension 4 are

$$\left\{\frac{12}{m+3}\lambda, \frac{m+8}{m+2}\lambda, \frac{2(m+2)}{m+1}\lambda, 3\lambda\right\}. \tag{4.4}$$

As we commented before, the value $R = \frac{12}{m+3}\lambda$ implies that the manifold is isometric to the round hemisphere $\mathbb{S}^4_+$. From Proposition 4.1, does not exist a compact quasi-Einstein manifold with boundary with scalar curvature given by $R = \frac{m+8}{m+2}\lambda$. Furthermore, from Theorem 4.3, $R = 3\lambda$ gives us that (M^4, g) is isometric to $I \times \mathbb{S}^3$, where we may use that Einstein manifolds in dimension 3 have constant sectional curvature to infer that the fiber is isometric, up to scaling, to $\mathbb{S}^3$. Interestingly, the example of $\mathbb{S}^2_+ \times \mathbb{S}^2$ with metric

$$g = dr^2 + \sin^2(r) g_{\mathbb{S}^1} + \frac{1}{\lambda} g_{\mathbb{S}^2}$$

and potential function $u = \cos(r)$, $r \leq \frac{\pi}{2}$, has constant scalar curvature $R = \frac{2(m+2)}{m+1}\lambda$. The only value in (4.4) which remains to study to complete the classification in dimension 4 is $R = \frac{2(m+2)}{m+1}\lambda$. The next result deals with this remaining case.

Theorem 4.4. *Let (M^4, g, u, λ) be a nontrivial simply connected compact 4-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^4 has constant scalar curvature $R = 2\frac{(m+2)}{(m+1)}\lambda$ if and only if it is isometric, up to scaling, to the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ with the doubly warped product metric.*

As a consequence, we get the classification in dimension 4.

Corollary 4.1. *Let* (M^4, g, u, λ) *be a nontrivial simply connected compact* 4*-dimensional* m*-quasi-Einstein manifold with boundary and* $m > 1$*. Then* M^4 *has constant scalar curvature if and only if it is isometric, up to scaling, to either*

(i) the standard hemisphere $\mathbb{S}^4_+$*, or*

(ii) the cylinder $I \times \mathbb{S}^3$ *with the product metric, or*

(iii) the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ *with the doubly warped product metric.*

4.1 Possible values for the constant scalar curvature

In this section, we prove the possible values for the constant scalar curvature depending on λ on a compact quasi-Einstein manifold with boundary.

Before to proceed, we remark that each connected component of the maximum point set $MAX(u) = \{p \in M : u(p) = u_{max}\}$ of a potential function on a compact quasi-Einstein manifold with boundary and constant scalar curvature is a smooth manifold. This fact follows from [107, Theorem A, item (a)] on transnormal functions. Originally, this result was established by Wang for transnormal functions over complete Riemannian manifolds, i.e., without boundary, but his result holds analogously for compact manifolds with boundary making a minor adaptation.

We are now ready to prove Theorem 4.1.

Theorem 4.5 (Theorem 4.1)**.** *Let* (M^n, g, u, λ) *be a nontrivial compact* m*-quasi-Einstein manifold with boundary,* $m > 1$ *and constant scalar curvature* R*. Then we have:*

$$R \in \left\{ \frac{n(n-1)}{m+n-1}\lambda, \frac{m+n(n-2)}{m+n-2}\lambda, \ldots, (n-1)\lambda \right\}. \tag{4.5}$$

In general, one has $R = \frac{k(m-n)+n(n-1)}{m+n-k-1}\lambda$*, for some* $k \in \{0, 1, \ldots, n-1\}$*.*

Proof. In the first part of the proof, we shall follow Proposition 3.13 of [60]. To begin with, denoting $\alpha = \frac{R+(m-n)\lambda}{m(m-1)}$ and $\widetilde{\mu} = \frac{\mu}{m-1}$, one sees from (2.48) that

$$\frac{|\nabla u|^2}{\widetilde{\mu} - \alpha u^2} = 1 \tag{4.6}$$

defines a distance function $r = \frac{1}{\sqrt{\alpha}} \arccos\left(\frac{u}{\sqrt{\widetilde{\mu}\alpha^{-1}}}\right)$. Notice that $\widetilde{\mu} - \alpha u^2 \geq 0$ and it is zero only over $Crit(u)$, but Eq. (4.6) holds over M by continuity. In particular, we can recover the potential function by taking $u(r)=\sqrt{\tilde{\mu}\alpha^{-1}}\cos(\sqrt{\alpha}r)$. From Remark 8, the set of critical points for u coincides with the set of maximum values, namely, $Crit(u)=MAX(u)$. Hence, we may correspond $MAX(u)=r^{-1}(0)$. So, following the argument in [107, Lemma 7] with the appropriate adaptation and using that u is zero on each boundary component, we deduce that each connected component of $MAX(u)$ is a smooth submanifold. Thereby, it follows from Lemma 2.3 that

$$\Delta r=\operatorname{tr}(\mathcal{A}_\theta)+\frac{n-k-1}{r}+O(r), \tag{4.7}$$

where k is the dimension of a connected component N of $MAX(u)$ and A_θ stands for the second fundamental form with respect to θ. By (4.1), without loss of generality, we may multiply the potential function u by a constant β so that βu is a potential function for the same metric and constant λ as u. In view of this, we can assume that $u(r)=\cos(\sqrt{\alpha}r)$ and consequently, we deduce

$$\nabla_i\nabla_j u=-\sqrt{\alpha}\sin(\sqrt{\alpha}r)\nabla_i\nabla_j r-\alpha\cos(\sqrt{\alpha}r)\nabla_i r\nabla_j r$$

and

$$\Delta u=-\sqrt{\alpha}\sin(\sqrt{\alpha}r)\Delta r-\alpha\cos(\sqrt{\alpha}r)|\nabla r|^2. \tag{4.8}$$

Taking into account the Taylor expansions, around $r=0$,

$$\sin(\sqrt{\alpha}r)=\sqrt{\alpha}r+O(r^3)\quad\text{and}\quad\cos(\sqrt{\alpha}r)=1+O(r^2),$$

we obtain from (4.7) and (4.8) that

$$\begin{aligned}\Delta u &= (-\alpha r+O(r^3))\left(\operatorname{Tr}(\mathcal{A}_\theta)+\frac{n-k-1}{r}+O(r)\right)+(-\alpha+O(r^2))\\ &= -\alpha(n-k)+O(r).\end{aligned} \tag{4.9}$$

It is known from (2.39) that $P=Ric-\frac{(n-1)\lambda-R}{m-1}g$. In particular, by setting $\varrho=\frac{(n-1)\lambda-R}{m-1}$, we may write the trace $\Delta u=\frac{u}{m}(R-n\lambda)$ of the fundamental equation (4.1) in terms of P and ϱ, at the connected component N of $MAX(u)$, as

$$\Delta u=\frac{1}{m}(Tr(P)-n(\lambda-\varrho)), \tag{4.10}$$

where we have used that $u\,|_N=1$. Then, since $\alpha=\frac{\lambda-\varrho}{m}$, we combine (4.9), restricted to N, and (4.10) in order to infer

$$Tr(P)=k(\lambda-\varrho).$$

In particular, the last equation implies that the dimension of each connected component of $MAX(u)$ is the same because $Tr(P)$ is constant in M and $\lambda-\varrho>0$.

We now claim that tangent and normal vector fields to N are the eigenvectors corresponding to $\lambda-\varrho$ and 0, respectively. Indeed, given a point $p\in N$ and $X\in\mathfrak{X}(N)$ a tangent vector at p, since $\nabla u\,|_N=0$, we have

$$\nabla^2 u(X)(p)=\nabla_X\nabla u(p)=0,$$

where we have used the fact that $\nabla_X\nabla u(p)$ only hinges upon on the value of $X(p)$ and ∇u along of a curve through p with X as a tangent vector at p. Hence, by using (4.1), we obtain

$$0=\nabla_X\nabla u(p)=\frac{u}{m}\left(P(X)-(\lambda-\varrho)X\right).$$

Consequently, $P(X)=(\lambda-\varrho)X$, for all $X\in\mathfrak{X}(N)$ and therefore, the tangent vectors to N corresponds to the eigenvalue $\lambda-\varrho$ for P. Besides, it follows from assertion (2) of Proposition 2.7 that, at $Crit(u)$,

$$P\circ(P-(\lambda-\varrho)I)=0.$$

Thus, the only possible eigenvalues for P at N are $\lambda-\varrho$ and 0. Moreover, since $Tr(P)=k(\lambda-\varrho)$ and $k=dim(N)$, one concludes that normal vectors to N correspond to the eigenvalue 0.

Proceeding, one concludes that

$$P\,|_N=\begin{pmatrix}(\lambda-\varrho)I_k & 0\\ 0 & [\mathbf{0}]_{n-k}\end{pmatrix}$$

is the $n\times n$ matrix of the tensor P at the manifold N. In terms of the Ricci tensor, we have

$$Ric\,|_N=\begin{pmatrix}\lambda I_k & 0\\ 0 & \frac{(n-1)\lambda-R}{m-1}I_{n-k}\end{pmatrix}. \tag{4.11}$$

In particular, taking the trace in (4.11), we see that

$$R=\frac{k(m-n)+n(n-1)}{m+n-k-1}\lambda,$$

for some $k\in\{0,1\ldots,n-1\}$, where we also have used that $R<n\lambda$ (see Remark 6). So, the proof is finished. $\square$

As we discussed briefly in the beginning of this chapter, the lower value for the scalar curvature in (4.5) is classified. Indeed, recall that if an m-quasi-Einstein manifold has constant scalar curvature R and $m>1$, then

$$|\mathring{Ric}|^2=-\frac{m+n-1}{n(m-1)}(R-n\lambda)\left(R-\frac{n(n-1)}{m+n-1}\lambda\right); \tag{4.12}$$

for more details, see [59, Proposition 3.3] and [26, Lemma 3.2] (see also Lemma 2.7). Considering $R=\frac{n(n-1)}{m+n-1}\lambda$ into (4.12), i.e., the lower value of (4.5), one deduces that M^n is necessarily Einstein and therefore, it suffices to apply Proposition 2.8 to conclude that M^n is isometric to the standard hemisphere $\mathbb{S}^n_+$.

Remark 10. *It is known that the universal covering of a quasi-Einstein manifold with $\lambda>0$ is compact and hence, its fundamental group $\pi_1(M)$ is finite. The proof of this fact is quite similar to [49] and [111], and it can be carry out by combining the arguments in the proof of [59, Theorem 4.1] (see also [88]) and [96, Remark 6.9].*

Next, we deal with the value $R=\frac{m+n(n-2)}{m+n-2}\lambda$. To do so, we follow essentially the idea of [48, Theorem 10] on the nonexistence of complete gradient shrinking Ricci solitons with $R=\lambda$ making use of [54, Theorem 2.2] on the connectedness of $MAX(u)$ if $Codim(MAX(u))\geq 2$. To be precise, we have the following proposition.

Proposition 4.1. *There is no compact nontrivial quasi-Einstein manifold M^n, $n\geq 3$, with boundary and constant scalar curvature $R=\frac{m+n(n-2)}{m+n-2}\lambda$.*

Proof. We argue by contradiction, assuming that a compact nontrivial quasi-Einstein manifold M^n with boundary has constant scalar curvature $R=\frac{m+n(n-2)}{m+n-2}\lambda$, which corresponds to the case $k=1$ in Theorem 4.5. Hence, by the work of Wang [107] (see also [53, Theorem 1.1] and [75, Theorem 6.1]), one obtains that $MAX(u)$ is a focal variety of the isoparametric function u of dimension one and connected. So $MAX(u)$ is totally geodesic. This therefore implies that $MAX(u)=\mathbb{S}^1$ and consequently, M is homotopic to $\mathbb{S}^1$ (see [79]), which leads to a contradiction with the fact that M^n has finite fundamental group (see Remark 10). Thus, the proof is completed. □

4.2 Rigidity for the extremal value case of the scalar curvature

In this section, our purpose is to present the proof of Theorem 4.3, that is the rigidity result for the case $R=(n-1)\lambda$. To do so, we need to prove some auxiliary lemmas.

The first lemma uses the tensor T defined in equation (2.45).

Lemma 4.1. *Let (M^n, g, u, λ) be an m-quasi-Einstein manifold with constant scalar curvature. Then we have:*

$$\begin{aligned}\mathring{R}_{ik}T_{ijk}\nabla_j u &= \frac{m+n-2}{n-2}|\mathring{Ric}|^2|\nabla u|^2 - \frac{2m+n-2}{n-2}\mathring{Ric}^2(\nabla u,\nabla u)\\ &\quad + \frac{(n(n-1)\lambda-(m+n-1)R)^2}{n^2(n-1)(m-1)}|\nabla u|^2\\ &= \frac{n-2}{2(m+n-2)}|T|^2,\end{aligned} \tag{4.13}$$

where $\mathring{Ric}^2_{ij} = \mathring{R}_{ik}\mathring{R}_{kj}$.

Proof. By using that the scalar curvature R is constant and Eq. (2.45), one obtains that

$$\begin{aligned}\mathring{R}_{ik}T_{ijk} &= \frac{m+n-2}{n-2}(|\mathring{Ric}|^2\nabla_j u - \mathring{R}_{ik}\mathring{R}_{jk}\nabla_i u) - \frac{m}{n-2}\mathring{R}_{ik}\mathring{R}_{il}\nabla_l u g_{jk}\\ &\quad + \frac{n(n-1)\lambda-(m+n-1)R}{n(n-1)}\mathring{R}_{ik}\nabla_i u g_{jk}\\ &= \frac{m+n-2}{n-2}(|\mathring{Ric}|^2\nabla_j u - \mathring{R}_{ik}\mathring{R}_{jk}\nabla_i u) - \frac{m}{n-2}\mathring{R}_{ij}\mathring{R}_{il}\nabla_l u\\ &\quad + \frac{n(n-1)\lambda-(m+n-1)R}{n(n-1)}\mathring{R}_{ij}\nabla_i u.\end{aligned}$$

Applying this for $\nabla_j u$, we see that

$$\begin{aligned}\mathring{R}_{ik}T_{ijk}\nabla_j u &= \frac{m+n-2}{n-2}|\mathring{Ric}|^2|\nabla u|^2 - \frac{m+n-2}{n-2}\nabla_j u\mathring{R}_{ik}\mathring{R}_{jk}\nabla_i u\\ &\quad - \frac{m}{n-2}\nabla_j u\mathring{R}_{ij}\mathring{R}_{il}\nabla_l u + \frac{n(n-1)\lambda-(m+n-1)R}{n(n-1)}\mathring{Ric}(\nabla u,\nabla u)\\ &= \frac{m+n-2}{n-2}|\mathring{Ric}|^2|\nabla u|^2 - \frac{2m+n-2}{n-2}\mathring{Ric}^2(\nabla u,\nabla u)\\ &\quad + \frac{n(n-1)\lambda-(m+n-1)R}{n(n-1)}\mathring{Ric}(\nabla u,\nabla u).\end{aligned}$$

So, it suffices to use (2.37) in the last term of the above equality in order to infer the first equality in (4.13).

Finally, since T is trace-free in any two indices and skew-symmetric in their first two indices, we get

$$\begin{aligned}\mathring{R}_{ik}T_{ijk}\nabla_j u &= \frac{1}{2}(\mathring{R}_{ik}T_{ijk}\nabla_j u - \mathring{R}_{ik}T_{jik}\nabla_j u)\\ &= \frac{1}{2}T_{ijk}(\mathring{R}_{ik}\nabla_j u - \mathring{R}_{jk}\nabla_i u)\\ &= \frac{n-2}{2(m+n-2)}|T|^2,\end{aligned}$$

where in the last equality we have used (2.45). This finishes the proof of the lemma. □

As a consequence of Lemma 4.1, by considering the orthonormal frame $\{e_i\}_{i=1}^n$ with $e_1 = -\frac{\nabla u}{|\nabla u|}$ so that $\mathring{Ric}(e_i) = \xi_i e_i$, we obtain the following result.

Corollary 4.2. *Let (M^n, g, u, λ) be an m-quasi-Einstein manifold with constant scalar curvature and $m > 1$. Then T is identically zero if and only if the Ricci tensor has at most two different eigenvalues, one of them has multiplicity at least $n-1$ and its eigenspace corresponds to the orthogonal complement of ∇u.*

Proof. Taking into account that $\xi_1 = \frac{n(n-1)\lambda-(m+n-1)R}{n(m-1)}$, one deduces from (4.13) that

$$\begin{aligned}\frac{n-2}{2(m+n-2)}|T|^2 &= \left[\frac{m+n-2}{n-2}\sum_{i=1}^{n}\xi_i^2 + \frac{m-1}{n-1}\xi_1^2\right]|\nabla u|^2 - \frac{2m+n-2}{n-2}\xi_1^2|\nabla u|^2\\ &= \frac{m+n-2}{n-2}\left[\sum_{i=1}^{n}\xi_i^2 - \frac{n}{n-1}\xi_1^2\right]|\nabla u|^2\end{aligned}$$

on the regular points of the potential function u. Moreover, since $Tr(\mathring{Ric}) = \sum_{i=1}^n \xi_i = 0$, we infer

$$\frac{n-2}{2(m+n-2)}|T|^2 = \frac{m+n-2}{n-2}\left[\sum_{i=2}^{n}\xi_i^2 - \frac{1}{n-1}\left(\sum_{i=2}^{n}\xi_i\right)^2\right]|\nabla u|^2.$$

By the Cauchy-Schwarz inequality, we conclude that $T \equiv 0$ if and only if the Ricci tensor has at most two different eigenvalues with $\lambda_2 = \ldots = \lambda_n$ at regular points of u, for eigenvalues of the Ricci given by $\lambda_i = \xi_i + \frac{R}{n}$. To conclude the proof, it suffices to recall that u is real analytical in harmonic coordinates and consequently, the set of critical points of u has zero measure in M. □

In the sequel, we shall consider the extremal value case of (4.5), namely, $R = (n-1)\lambda$. In this situation, we have the following result which can be compared with [60, Theorem 1.9].

Theorem 4.6 (Theorem 4.2)**.** *Let $\left(M^n, g, u, \lambda\right)$, $n \geq 3$, be a nontrivial simply connected compact m-quasi-Einstein manifold with boundary and $m > 1$. Then M^n has constant scalar curvature $R = (n-1)\lambda$ if and only if it is isometric, up to scaling, to the cylinder $\left[0, \frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right] \times N$ with product metric, where N is a compact λ-Einstein manifold.*

Proof. First of all, since $R = (n-1)\lambda$, it follows from (2.36) that the eigenvalue λ_1 associated to the eigenvector ∇u for the Ricci tensor is zero. We now need to show that all non-zero eigenvalues of the Ricci tensor are equals to λ. Before to do so, we first claim that

$$|\mathring{Ric}|^2 = \frac{R^2}{n(n-1)}. \tag{4.14}$$

Indeed, since R is constant, one deduces from assertion (3) in Lemma 2.7 (see also [26, Lemma 3.2]) that

$$(m-1)|\mathring{Ric}|^2 = -\frac{m+n-1}{n}(R-n\lambda)\left(R-\frac{n(n-1)}{m+n-1}\lambda\right).$$

Whence, for $R=(n-1)\lambda$, we see that

$$(m-1)|\mathring{Ric}|^2 \quad = \quad -R^2\frac{(m+n-1)}{n}\left(1-\frac{n}{n-1}\right)\left(1-\frac{n}{m+n-1}\right), \tag{4.15}$$

and consequently,

$$|\mathring{Ric}|^2 = \frac{R^2}{n(n-1)},$$

as claimed.

Letting λ_i, $i\neq 1$, the possible non-zero eigenvalues of the Ricci tensor, one deduces that

$$\sum_{i=2}^{n}(\lambda_i-\lambda)^2 = |Ric|^2 - 2\lambda R + (n-1)\lambda^2 = |\mathring{Ric}|^2 - \frac{R^2}{n(n-1)},$$

where we have used that $|\mathring{Ric}|^2 = |Ric|^2 - \frac{R^2}{n}$ and $R=(n-1)\lambda$. Therefore, one obtains from (4.14) that $\lambda_i=\lambda$, for $i=2,\ldots,n$, i.e., the eigenvalues of the Ricci are all constants with $\lambda_2=\ldots=\lambda_n=\lambda$. Thereby, Corollary 4.2 guarantees that $T\equiv 0$. In particular, since the Ricci tensor is parallel, then the Cotton tensor (2.2) also vanishes and then, by Lemma 2.8, we have $W_{ijkl}\nabla_l u=0$. Now, we are in the position to invoke Theorem 1.2 of [59] to infer that the metric splits off as $g=dt^2+\varphi^2(t)\widetilde{g}_N$, where $\widetilde{g}_N$ is κ-Einstein with nonnegative Ricci curvature and $u=u(t)$.

In view of (2.36), we get

$$Ric(\nabla u,\nabla u) = \frac{(n-1)\lambda-R}{m-1}(u')^2 = 0$$

and hence, we may apply Proposition 2.1 to infer

$$0 = Ric(\nabla u,\nabla u) = (u')^2 Ric(\partial t,\partial t) = -(u')^2\frac{(n-1)}{\varphi}\varphi''.$$

Since u is analytical in harmonic coordinates (and u is not constant), we conclude that $\varphi''(t)/\varphi(t) = 0$, which implies that $\varphi(t) = c$ or $\varphi(t) = ct$, for some positive constant c. However, according to Proposition 2.2, the second case can not hold.

Proceeding, since $g = dt^2 + c^2\widetilde{g}_N$ and $\widetilde{g}_N$ is a κ-Einstein metric, we may use again Proposition 2.1 to deduce

$$Ric(Z,V) = \kappa\widetilde{g}_N(Z,V), \text{ for all } Z,V \in \mathfrak{X}(N).$$

Consequently, the scalar curvature is $R = \frac{\kappa}{c^2}(n-1)$ and moreover, $\lambda = \frac{\kappa}{c^2}$ and (N^{n-1}, g_N) is λ-Einstein manifold, where $g_N = c^2\widetilde{g}_N$.

Finally, observe that, by (2.3) and the fact that $Ric = \lambda g_N$, the potential function $u = u(t)$ satisfies

$$u''(t)dt^2 = \nabla^2 u = \frac{u}{m}(Ric - \lambda g) = -\lambda\frac{u}{m}dt^2$$

and $u\mid_{\partial M} = 0$. Hence, without loss of generality, we can consider the solution $u(t) = \sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)$. Thereby, we conclude that M^n is isometric, up to scaling, to the cylinder $\left[0, \frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right] \times N$, where N is a compact λ-Einstein manifold. So, the proof of Theorem 4.6 is finished. $\square$

To conclude this section, we are going to describe the example of m-quasi-Einstein manifold on $\left[0, \frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right] \times \mathbb{S}^p \times \mathbb{S}^q$ (see also [51]).

Example 4.1. *Let $\lambda > 0$ be an arbitrary constant and consider $M^n = \left[0, \frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right] \times \mathbb{S}^p \times \mathbb{S}^q$, $p, q > 1$, endowed with the metric*

$$g = dt^2 + \frac{p-1}{\lambda}g_{\mathbb{S}^p} + \frac{q-1}{\lambda}g_{\mathbb{S}^q}.$$

This space is an m-quasi-Einstein manifold with potential function $u(t) = \sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)$ and constant scalar curvature $R = (n-1)\lambda$. Indeed, we first notice that

$$Ric = (p-1)g_{\mathbb{S}^p} + (q-1)g_{\mathbb{S}^q} \quad \text{and} \quad \nabla u = u'\nabla t = \frac{\sqrt{\lambda}}{\sqrt{m}}\cos\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)\nabla t.$$

Thereby, since $u = u(t)$ and the warping function is constant, we deduce from (2.3) that

$$\nabla^2 u = -\frac{\lambda}{m}\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)dt^2. \tag{4.16}$$

On the other hand, one observes that

$$\begin{aligned}\frac{u}{m}(Ric-\lambda g)=&\frac{1}{m}\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)\left[(p-1)g_{\mathbb{S}^p}+(q-1)g_{\mathbb{S}^q}-(\lambda dt^2+(p-1)g_{\mathbb{S}^p}+(q-1)g_{\mathbb{S}^q})\right]\\ =&-\frac{\lambda}{m}\sin\left(\frac{\sqrt{\lambda}}{\sqrt{m}}t\right)dt^2.\end{aligned}$$

Plugging this into (4.16) gives (4.1)*.*

In conclusion, $u=0$ *if and only if either* $t=0$ *or* $t=\frac{\sqrt{m}}{\sqrt{\lambda}}\pi$ *and consequently, the boundary consists of two disjoint copies of* $\mathbb{S}^p\times\mathbb{S}^q$*.*

4.3 Classification in dimension 3

We now present the proof of Theorem 4.3, which establishes the explicit classification of compact 3-dimensional m-quasi-Einstein manifolds with boundary and constant scalar curvature. To be precise, we have the following result.

Theorem 4.7 (Theorem 4.3)**.** *Let* (M^3,g,u,λ) *be a nontrivial simply connected compact 3-dimensional* m*-quasi-Einstein manifold with boundary and* $m>1$*. Then* M^3 *has constant scalar curvature if and only if it is isometric, up to scaling, to either*

(a) the standard hemisphere $\mathbb{S}^3_+$*, or*

(b) the cylinder $\left[0,\frac{\sqrt{m}}{\sqrt{\lambda}}\pi\right]\times\mathbb{S}^2$ *with the product metric.*

Proof. To begin with, since (M^3,g) has constant scalar curvature and, by assertion (1) of Lemma 2.7 (see also Eq. (2.36)), ∇u is an eigenvector of Ric in this case, consider an orthonormal frame $\{e_i\}_{i=1}^3$ that diagonalizes the Ricci curvature Ric so that $e_1=-\frac{\nabla u}{|\nabla u|}$ and λ_i are the eigenvalues associated to e_i, for $i=1,2,3$. Thus, under this coordinates, one obtains that

$$\mathring{Ric}=\begin{pmatrix}\xi_1 & 0 & 0\\ 0 & \xi_2 & 0\\ 0 & 0 & \xi_3\end{pmatrix},$$

where

$$\begin{cases}\xi_1+\xi_2+\xi_3=0,\\ \xi_1^2+\xi_2^2+\xi_3^2=|\mathring{Ric}|^2,\end{cases}\tag{4.17}$$

and $\xi_i = \lambda_i - \frac{R}{3}$ are the eigenvalues of the traceless Ricci tensor $\mathring{Ric}$. A straightforward computation using (2.37) yields

$$\xi_1 = \frac{6\lambda-(m+2)R}{3(m-1)}. \tag{4.18}$$

In another direction, since R is constant, it follows from (3) of Lemma 2.7 that

$$|\mathring{Ric}|^2 = -\frac{1}{3(m-1)}(R-3\lambda)((m+2)R-6\lambda).$$

This combined with (4.17) gives

$$\xi_1^2+\xi_2^2+\xi_3^2 = -\frac{1}{3(m-1)}(R-3\lambda)((m+2)R-6\lambda).$$

By using (4.18), one sees that

$$\begin{aligned}
\xi_2^2+\xi_3^2 &= -\frac{1}{3(m-1)}(R-3\lambda)((m+2)R-6\lambda)-\left(\frac{(m+2)R-6\lambda}{3(m-1)}\right)^2 \\
&= -\frac{1}{3(m-1)}((m+2)R-6\lambda)\left((R-3\lambda)+\frac{(m+2)R-6\lambda}{3(m-1)}\right) \\
&= -\frac{1}{3(m-1)}((m+2)R-6\lambda)\left(\frac{(4m-1)R}{3(m-1)}-\frac{3(3m-1)\lambda}{3(m-1)}\right) \\
&= -\frac{1}{(3(m-1))^2}((m+2)R-6\lambda)((4m-1)R-3(3m-1)\lambda).
\end{aligned}$$

Next, since $(\xi_2+\xi_3)^2=\xi_1^2$, which in turn implies $\xi_2^2+\xi_3^2+2\xi_2\xi_3=\xi_1^2$, one obtains that

$$\begin{aligned}
2\xi_2\xi_3 &= \xi_1^2-(\xi_2^2+\xi_3^2) \\
&= \left(\frac{(m+2)R-6\lambda}{3(m-1)}\right)^2+\frac{1}{(3(m-1))^2}((m+2)R-6\lambda)((4m-1)R-3(3m-1)\lambda) \\
&= \frac{1}{(3(m-1))^2}((m+2)R-6\lambda)[(4m-1)R-3(3m-1)\lambda+(m+2)R-6\lambda] \\
&= \frac{1}{9(m-1)^2}((m+2)R-6\lambda)((5m+1)R-3(3m+1)\lambda),
\end{aligned} \tag{4.19}$$

which guarantees that $\xi_2\xi_3$ is constant.

We now assume that $\xi_2\xi_3=0$. Thereby, by the analyticity of g, one observes that $\xi_2=0$ or $\xi_3=0$. Considering $\xi_3=0$, one deduces from (4.17) that $\xi_2=-\xi_1$. Hence, by (4.18), the eigenvalues of Ricci curvature are constant. This then implies that the Ricci tensor is parallel. In particular, the Cotton tensor C_{ijk} also vanishes. Now, since $W=0$ in dimension 3, one obtains from Lemma 2.8 that $T\equiv 0$. Besides, it follows from Corollary 4.2 that at least two eigenvalues of the Ricci tensor are equals. This forces $\mathring{Ric}=0$ and then, (M^3,g) is Einstein. So, it suffices to apply Proposition 2.4 of [60] to conclude that (M^3,g) is isometric to the standard hemisphere $\mathbb{S}^3_+$.

On the other hand, by assuming that $\xi_2\xi_3\neq 0$, one deduces that

$$\xi_2=\frac{1}{2\xi_3}\left[\frac{1}{9(m-1)^2}((m+2)R-6\lambda)((5m+1)R-3(3m+1)\lambda)\right]=\frac{\zeta}{2\xi_3}.$$

In particular, by (4.17), one has $-\xi_1=\xi_2+\xi_3=\frac{\zeta}{2\xi_3}+\xi_3$ and hence, (4.18) guarantees that

$$2\xi_3^2+\zeta=2\xi_3\frac{(m+2)R-6\lambda}{3(m-1)}. \tag{4.20}$$

Computing the discriminant for ξ_3, we infer

$$\Delta \;=\; -\left(\frac{6}{3(m-1)}\right)^2 m((m+2)R-6\lambda)(R-2\lambda).$$

Then, solving the polynomial (4.20), one sees that

$$\xi_3=\frac{((m+2)R-6\lambda)\pm 3\sqrt{m((m+2)R-6\lambda)(2\lambda-R)}}{6(m-1)}.$$

Notice that the eigenvalue ξ_2 satisfies an expression equivalent to (4.20) and by (4.17), one deduces that

$$\xi_2=\frac{((m+2)R-6\lambda)\mp 3\sqrt{m((m+2)R-6\lambda)(2\lambda-R)}}{6(m-1)}.$$

Therefore, ξ_2 and ξ_3 are constants.

Analogous to the previous case, we observe that the Ricci tensor is parallel and hence, the Cotton tensor vanishes. Thereby, it follows from Corollary 4.2 that $\xi_2=\xi_3\neq 0$, but it holds if and only if $R=2\lambda$. At this point, it suffices to invoke Theorem 4.6 to conclude that (M^3,g) is isometric, up to scaling, to the cylinder $[0,\frac{\sqrt{m}}{\sqrt{\lambda}}\pi]\times N$. Moreover, we deduce from (2.21) and Killing-Hopf theorem that $N=\mathbb{S}^2$. Thus, the proof of Theorem 3.4 is concluded. □

4.4 Classification in dimension 4

We divide this section into two subsections: the first one establishes some key lemmas, for arbitrary dimension $n\geq 3$, that will play a crucial role in the proof of Theorem 4.4, while the second subsection collects the proofs of the classification results.

4.4.1 Some key lemmas

To begin with, we shall prove the following lemma, which provides a useful expression for $u\Delta(Ric)$.

Lemma 4.2. *Let (M^n, g) be an n-dimensional Riemannian manifold satisfying (4.1). Then we have:*

$$
\begin{aligned}
u(\Delta R_{ik}) &= \nabla_i R_{sk}\nabla_s u + m\nabla_k R_{is}\nabla_s u + \frac{u}{2}\nabla_i\nabla_k R + \frac{1}{2}\nabla_i u\nabla_k R \\
&\quad + \frac{(m+1)}{m} uR_{is}R_{sk} + 2uR_{jiks}R_{js} - (m+2)\nabla_s R_{ik}\nabla_s u \\
&\quad - \frac{u}{m}\left(R-(m+n-2)\lambda\right)R_{ik} + \frac{\lambda u}{m}\left(R-(n-1)\lambda\right)g_{ik}.
\end{aligned}
$$

Proof. Firstly, it follows from assertion (4) of Lemma 2.7 that

$$
u\nabla_j R_{ik} = u\nabla_i R_{jk} + mR_{jikl}\nabla_l u + \lambda\left(\nabla_j u g_{ik} - \nabla_i u g_{jk}\right) - \left(\nabla_j u R_{ik} - \nabla_i u R_{jk}\right).
$$

This jointly with the fact that $\nabla_j(u\nabla_j R_{ik}) = \nabla_j u\nabla_j R_{ik} + u\Delta R_{ik}$ gives

$$
\begin{aligned}
u\Delta R_{ik} &= \nabla_j(u\nabla_j R_{ik}) - \nabla_j u\nabla_j R_{ik} \\
&= \nabla_j\left(u\nabla_i R_{jk} + mR_{jikl}\nabla_l u + \lambda\left(\nabla_j u g_{ik} - \nabla_i u g_{jk}\right) - \left(\nabla_j u R_{ik} - \nabla_i u R_{jk}\right)\right) \\
&\quad - \nabla_j u\nabla_j R_{ik} \\
&= \nabla_j u\nabla_i R_{jk} + u\nabla_j\nabla_i R_{jk} + m\nabla_j R_{jikl}\nabla_l u + mR_{jikl}\nabla_j\nabla_l u + \lambda\Delta u g_{ik} \\
&\quad - \lambda\nabla_k\nabla_i u - \Delta u R_{ik} - \nabla_j u\nabla_j R_{ik} + \nabla_j\nabla_i u R_{jk} + \nabla_i u\nabla_j R_{jk} - \nabla_j u\nabla_j R_{ik}.
\end{aligned}
$$

Next, by using the twice contracted second Bianchi identity ($\nabla_j R_{jk} = \frac{1}{2}\nabla_k R$) and the first contracted second Bianchi identity ($\nabla_j R_{jikl} = \nabla_k R_{il} - \nabla_l R_{ik}$), one sees that

$$
\begin{aligned}
u\Delta R_{ik} &= -\nabla_j u\nabla_j R_{ik} + \nabla_j u\nabla_i R_{jk} + u\nabla_j\nabla_i R_{jk} + m\left(\nabla_k R_{il} - \nabla_l R_{ik}\right)\nabla_l u \\
&\quad + mR_{jikl}\nabla_j\nabla_l u + \lambda\Delta u g_{ik} - \lambda\nabla_k\nabla_i u - \Delta u R_{ik} - \nabla_j u\nabla_j R_{ik} \\
&\quad + \nabla_j\nabla_i u R_{jk} + \frac{1}{2}\nabla_i u\nabla_k R \\
&= -\nabla_j u\nabla_j R_{ik} + \nabla_j u\nabla_i R_{jk} + \frac{u}{2}\nabla_i\nabla_k R + uR_{is}R_{sk} + uR_{jiks}R_{js} \\
&\quad + m\left(\nabla_k R_{il} - \nabla_l R_{ik}\right)\nabla_l u + mR_{jikl}\nabla_j\nabla_l u + \lambda\Delta u g_{ik} - \lambda\nabla_k\nabla_i u \\
&\quad - \Delta u R_{ik} - \nabla_j u\nabla_j R_{ik} + \nabla_j\nabla_i u R_{jk} + \frac{1}{2}\nabla_i u\nabla_k R,
\end{aligned}
\tag{4.21}
$$

where in the last equality we have used the Ricci identity, i.e.,

$$
\nabla_j\nabla_i R_{jk} = \nabla_i\nabla_j R_{jk} + R_{jijs}R_{sk} + R_{jiks}R_{js}.
$$

Plugging (4.1) and (2.12) into (4.21) yields

$$
\begin{aligned}
u\Delta R_{ik} &= -\nabla_j u\nabla_j R_{ik}+\nabla_j u\nabla_i R_{jk}+\frac{u}{2}\nabla_i\nabla_k R+uR_{is}R_{sk}+uR_{jiks}R_{js}\\
&\quad +m\left(\nabla_k R_{il}-\nabla_l R_{ik}\right)\nabla_l u+uR_{jikl}\left(R_{jl}-\lambda g_{jl}\right)+\frac{\lambda u}{m}\left(R-\lambda n\right)g_{ik}\\
&\quad -\frac{\lambda u}{m}\left(R_{ki}-\lambda g_{ki}\right)-\frac{u}{m}\left(R-\lambda n\right)R_{ik}-\nabla_j u\nabla_j R_{ik}\\
&\quad +\frac{u}{m}\left(R_{ji}-\lambda g_{ji}\right)R_{jk}+\frac{1}{2}\nabla_i u\nabla_k R\\
&= \nabla_i R_{jk}\nabla_j u+m\nabla_k R_{il}\nabla_l u+\frac{u}{2}\nabla_i\nabla_k R+\frac{1}{2}\nabla_i u\nabla_k R+\frac{(m+1)}{m}uR_{is}R_{sk}\\
&\quad +2uR_{jiks}R_{js}-(m+2)\nabla_j R_{ik}\nabla_j u+\left(\lambda u-\frac{\lambda u}{m}-\frac{u}{m}\left(R-\lambda n\right)-\frac{\lambda u}{m}\right)R_{ik}\\
&\quad +\frac{\lambda u}{m}\left(R-(n-1)\lambda\right)g_{ik}.
\end{aligned}
$$

Rearranging terms, one concludes that

$$
\begin{aligned}
u\Delta R_{ik} &= \nabla_i R_{sk}\nabla_s u+m\nabla_k R_{is}\nabla_s u+\frac{u}{2}\nabla_i\nabla_k R+\frac{1}{2}\nabla_i u\nabla_k R\\
&\quad +\frac{(m+1)}{m}uR_{is}R_{sk}+2uR_{jiks}R_{js}-(m+2)\nabla_s R_{ik}\nabla_s u\\
&\quad -\frac{u}{m}\left(R-(m+n-2)\lambda\right)R_{ik}+\frac{\lambda u}{m}\left(R-(n-1)\lambda\right)g_{ik},
\end{aligned}
$$

as we wanted to prove. □

As an application of Lemma 4.2, we are able to obtain a key expression for $\Delta(Ric^3)_{ik}=\Delta\left(R_{ij}R_{jl}R_{lk}\right)$.

Lemma 4.3. *Let (M^n,g) be an n-dimensional Riemannian manifold satisfying (4.1). Then we have:*

$$
\begin{aligned}
u\Delta(Ric^3)_{ik} &+ (m+2)\nabla_s u\nabla_s(Ric^3)_{ik}\\
&= \nabla_i R_{sj}\nabla_s uR_{jl}R_{lk}+\nabla_j R_{sl}\nabla_s uR_{ij}R_{lk}+\nabla_l R_{sk}\nabla_s uR_{ij}R_{jl}\\
&\quad +2u\left(\nabla_s R_{ij}\nabla_s R_{jl}R_{lk}+\nabla_s R_{ij}R_{jl}\nabla_s R_{lk}+R_{ij}\nabla_s R_{jl}\nabla_s R_{lk}\right)\\
&\quad +m\left(\nabla_j R_{is}\nabla_s uR_{jl}R_{lk}+\nabla_l R_{js}\nabla_s uR_{ij}R_{lk}+\nabla_k R_{ls}\nabla_s uR_{ij}R_{jl}\right)\\
&\quad +\frac{u}{2}\left(\nabla_i\nabla_j RR_{jl}R_{lk}+\nabla_j\nabla_l RR_{ij}R_{lk}+\nabla_l\nabla_k RR_{ij}R_{jl}\right)\\
&\quad +\frac{1}{2}\left(\nabla_i u\nabla_j RR_{jl}R_{lk}+\nabla_j u\nabla_l RR_{ij}R_{lk}+\nabla_l u\nabla_k RR_{ij}R_{jl}\right)\\
&\quad +\frac{(m+1)}{m}u\left(R_{is}R_{sj}R_{jl}R_{lk}+R_{js}R_{sl}R_{ij}R_{lk}+R_{ls}R_{sk}R_{ij}R_{jl}\right)\\
&\quad +2u\left(R_{dijs}R_{ds}R_{jl}R_{lk}+R_{djls}R_{ds}R_{ij}R_{lk}+R_{dlks}R_{ds}R_{ij}R_{jl}\right)\\
&\quad -3\frac{u}{m}\left(R-(m+n-2)\lambda\right)(Ric^3)_{ik}+3\frac{\lambda u}{m}\left(R-(n-1)\lambda\right)R_{il}R_{lk}.
\end{aligned}
$$

Proof. One easily verifies that

$$\begin{aligned} u\Delta(Ric^3)_{ik} &= u\Delta(R_{ij}R_{jl}R_{lk}) \\ &= (u\Delta R_{ij})R_{jl}R_{lk} + R_{ij}(u\Delta R_{jl})R_{lk} + R_{ij}R_{jl}(u\Delta R_{lk}) \\ &\quad +2u\left(\nabla_s R_{ij}\nabla_s R_{jl}R_{lk} + \nabla_s R_{ij}R_{jl}\nabla_s R_{lk} + R_{ij}\nabla_s R_{jl}\nabla_s R_{lk}\right). \end{aligned} \tag{4.22}$$

Next, it follows from Lemma 4.2 that

$$\begin{aligned} u\left(\Delta R_{ij}\right)R_{jl}R_{lk} &= \nabla_i R_{sj}\nabla_s u R_{jl}R_{lk} + m\nabla_j R_{is}\nabla_s u R_{jl}R_{lk} + \frac{u}{2}\nabla_i\nabla_j R R_{jl}R_{lk} \\ &\quad +\frac{1}{2}\nabla_i u\nabla_j R R_{jl}R_{lk} + \frac{(m+1)}{m}uR_{is}R_{sj}R_{jl}R_{lk} + 2uR_{dijs}R_{ds}R_{jl}R_{lk} \\ &\quad -(m+2)\nabla_s R_{ij}\nabla_s u R_{jl}R_{lk} - \frac{u}{m}\left(R-(m+n-2)\lambda\right)R_{ij}R_{jl}R_{lk} \\ &\quad +\frac{\lambda u}{m}\left(R-(n-1)\lambda\right)R_{il}R_{lk}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} R_{ij}\left(u\Delta R_{jl}\right)R_{lk} &= \nabla_j R_{sl}\nabla_s u R_{ij}R_{lk} + m\nabla_l R_{js}\nabla_s u R_{ij}R_{lk} + \frac{u}{2}\nabla_j\nabla_l R R_{ij}R_{lk} \\ &\quad +\frac{1}{2}\nabla_j u\nabla_l R R_{ij}R_{lk} + \frac{(m+1)}{m}uR_{js}R_{sl}R_{ij}R_{lk} + 2uR_{djls}R_{ds}R_{ij}R_{lk} \\ &\quad -(m+2)\nabla_s R_{jl}\nabla_s u R_{ij}R_{lk} - \frac{u}{m}\left(R-(m+n-2)\lambda\right)R_{jl}R_{ij}R_{lk} \\ &\quad +\frac{\lambda u}{m}\left(R-(n-1)\lambda\right)R_{il}R_{lk} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} R_{ij}R_{jl}\left(u\Delta R_{lk}\right) &= \nabla_l R_{sk}\nabla_s u R_{ij}R_{jl} + m\nabla_k R_{ls}\nabla_s u R_{ij}R_{jl} + \frac{u}{2}\nabla_l\nabla_k R R_{ij}R_{jl} \\ &\quad +\frac{1}{2}\nabla_l u\nabla_k R R_{ij}R_{jl} + \frac{(m+1)}{m}uR_{ls}R_{sk}R_{ij}R_{jl} + 2uR_{dlks}R_{ds}R_{ij}R_{jl} \\ &\quad -(m+2)\nabla_s R_{lk}\nabla_s u R_{ij}R_{jl} - \frac{u}{m}\left(R-(m+n-2)\lambda\right)R_{ij}R_{jl}R_{lk} \\ &\quad +\frac{\lambda u}{m}\left(R-(n-1)\lambda\right)R_{ij}R_{jk}. \end{aligned} \tag{4.25}$$

Therefore, inserting (4.23), (4.24) and (4.25) into (4.22) yields the asserted result. □

As a consequence of Lemma 4.3, we deduce the following corollary.

Corollary 4.3. *Let (M^n, g) be an n-dimensional Riemannian manifold satisfying (4.1) with constant scalar curvature. Then we have:*

$$\begin{aligned} u\Delta\left(Tr(Ric^3)\right) &+ (m+2)\nabla_s u\nabla_s(Tr(Ric^3)) \\ &= 3(m+1)\nabla_i R_{sj}R_{jl}R_{il}\nabla_s u + \frac{3(m+1)u}{m}Ric^2_{ij}Ric^2_{ij} + 6uR_{ds}R_{dijs}R_{jl}R_{il} \\ &\quad -\frac{3u}{m}\left(R-(m+n-2)\lambda\right)Tr(Ric^3) \\ &\quad +\frac{3\lambda u}{m}\left(R-(n-1)\lambda\right)|Ric|^2 + 6u\nabla_s R_{ij}\nabla_s R_{jl}R_{il}, \end{aligned}$$

where $Tr(Ric^3) = R_{ij}R_{jl}R_{li}$ *and* $Ric^2_{ij} = R_{ik}R_{kj}$.

Proof. By using that M^n has constant scalar curvature into Lemma 4.3, one deduces that

$$\begin{aligned}
u\Delta Ric^3_{ik} &= \Big(\nabla_i R_{sj}R_{jl}R_{lk} + \nabla_j R_{sl}R_{ij}R_{lk} + \nabla_l R_{sk}R_{ij}R_{jl}\Big)\nabla_s u \\
&\quad + m\Big(\nabla_j R_{is}R_{jl}R_{lk} + \nabla_l R_{js}R_{ij}R_{lk} + \nabla_k R_{ls}R_{ij}R_{jl}\Big)\nabla_s u \\
&\quad + \frac{m+1}{m}u\Big(Ric^2_{ij}R_{jl}R_{lk} + Ric^2_{jl}R_{ij}R_{lk} + Ric^2_{lk}R_{ij}R_{jl}\Big) \\
&\quad + 2uR_{ds}\Big(R_{dijs}R_{jl}R_{lk} + R_{djls}R_{ij}R_{lk} + R_{dlks}R_{ij}R_{jl}\Big) \\
&\quad - (m+2)\nabla_s\Big(R_{ij}R_{jl}R_{lk}\Big)\nabla_s u - \frac{3u}{m}[R-(m+n-2)\lambda]R_{ij}R_{jl}R_{lk} \\
&\quad + 2u(\nabla_s R_{ij}\nabla_s R_{jl}R_{lk} + \nabla_s R_{ij}R_{jl}\nabla_s R_{lk} + R_{ij}\nabla_s R_{jl}\nabla_s R_{lk}) \\
&\quad + \frac{3\lambda u}{m}[R-(n-1)\lambda]R_{is}R_{sk}.
\end{aligned}$$

Besides, tracing the above expression, one sees that

$$\begin{aligned}
u\Delta Tr(Ric^3) &= \Big(\nabla_i R_{sj}R_{jl}R_{li} + \nabla_j R_{sl}R_{ij}R_{li} + \nabla_l R_{si}R_{ij}R_{jl}\Big)\nabla_s u \\
&\quad + m[\nabla_j R_{is}R_{jl}R_{li} + \nabla_l R_{js}R_{ij}R_{li} + \nabla_i R_{ls}R_{ij}R_{jl}]\nabla_s u \\
&\quad + \frac{m+1}{m}u[Ric^2_{ij}R_{jl}R_{li} + Ric^2_{jl}R_{ij}R_{li} + Ric^2_{li}R_{ij}R_{jl}] \\
&\quad + 2uR_{ds}[R_{dijs}R_{jl}R_{li} + R_{djls}R_{ij}R_{li} + R_{dlis}R_{ij}R_{jl}] \\
&\quad - (m+2)\nabla_s[R_{ij}R_{jl}R_{li}]\nabla_s u - \frac{3u}{m}[R-(m+n-2)\lambda]R_{ij}R_{jl}R_{li} \\
&\quad + 2u(\nabla_s R_{ij}\nabla_s R_{jl}R_{li} + \nabla_s R_{ij}R_{jl}\nabla_s R_{li} + R_{ij}\nabla_s R_{jl}\nabla_s R_{li}) \\
&\quad + \frac{3\lambda u}{m}[R-(n-1)\lambda]R_{is}R_{si} \\
&= (m+1)[\nabla_i R_{sj}R_{jl}R_{li} + \nabla_j R_{sl}R_{ij}R_{il} + \nabla_l R_{is}R_{ij}R_{jl}]\nabla_s u \\
&\quad + \frac{3(m+1)u}{m}Ric^2_{ij}Ric^2_{ij} + 6uR_{ds}R_{dijs}R_{jl}R_{il} \\
&\quad - (m+2)\nabla_s(Tr(Ric^3))\nabla_s u - \frac{3u}{m}\left(R-(m+n-2)\lambda\right)Tr(Ric^3) \\
&\quad + \frac{3\lambda u}{m}\left(R-(n-1)\lambda\right)|Ric|^2 + 6u\nabla_s R_{ij}\nabla_s R_{jl}R_{il}.
\end{aligned}$$

The result then follows from the fact that $\nabla_i R_{sj}R_{jl}R_{li} = \nabla_j R_{sl}R_{ij}R_{il} = \nabla_l R_{is}R_{ij}R_{jl}$. $\square$

To proceed, it is essential to ensure an expression for $u\Delta\Big(Tr(P^3)\Big)$.

Lemma 4.4. *Let (M^n, g) be an n-dimensional Riemannian manifold satisfying (4.1) with constant scalar curvature and $m > 1$. Then we have:*

$$\begin{aligned}
u\Delta Tr(P^3) &= 3(m+1)\Big(\nabla_i P_{sj}P_{jl}P_{il}\nabla_s u + 2\varrho\nabla_i P_{sj}P_{ij}\nabla_s u\Big)\\
&\quad +6u\Big(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho\nabla_s P_{ij}\nabla_s P_{ij}\Big)\\
&\quad +6u\Big(P_{ds}R_{dijs}P_{jl}P_{il} + 2\varrho P_{ds}R_{dijs}P_{ij}\Big) - (m+2)\nabla_s(Tr(P^3))\nabla_s u\\
&\quad +\frac{3(m+1)u}{m}Tr(P^4) + \frac{3u}{m}\left(3(m+1)\varrho + (m-1)\lambda\right)Tr(P^3)\\
&\quad +\frac{3\varrho u}{m}\left((m+3)\varrho + 2(m-1)\lambda\right)|P|^2\\
&\quad +\frac{3\varrho^2 u}{m}\left((m+1)\varrho + (m-1)\lambda\right)Tr(P)\\
&\quad +6\varrho^3 u\left((m+n-1)\varrho - (n-1)\lambda\right),
\end{aligned}$$

Proof. Initially, we compute Ric^3_{ik} in terms of $P = Ric - \varrho g$, where $\varrho = \frac{(n-1)\lambda - R}{m-1}$. Indeed, it is easy to check that

$$\begin{aligned}
Ric^3_{ik} &= R_{ij}R_{jl}R_{lk}\\
&= (P_{ij} + \varrho g_{ij})(P_{jl} + \varrho g_{jl})(P_{lk} + \varrho g_{lk})\\
&= P_{ij}P_{jl}P_{lk} + P_{ij}P_{jl}\varrho g_{lk} + P_{ij}\varrho g_{jl}P_{lk} + P_{ij}\varrho g_{jl}\varrho g_{lk}\\
&\quad +\varrho g_{ij}P_{jl}P_{lk} + \varrho g_{ij}P_{jl}\varrho g_{lk} + \varrho g_{ij}\varrho g_{jl}P_{lk} + \varrho g_{ij}\varrho g_{jl}\varrho g_{lk}\\
&= P^3_{ik} + 3\varrho P^2_{ik} + 3\varrho^2 P_{ik} + \varrho^3 g_{ik}.
\end{aligned}$$

Whence, it follows that

$$Tr(Ric^3) = Ric^3_{ii} = Tr(P^3) + 3\varrho|P|^2 + 3\varrho^2 Tr(P) + n\varrho^3. \tag{4.26}$$

Next, notice that

$$Tr(P) = \frac{R(m+n-1) - n(n-1)\lambda}{m-1}$$

and moreover, by Proposition 3.3 in [60] (see also (3) in Lemma 2.7), since M^n has constant scalar curvature, one sees that $|P|^2 = (\lambda - \varrho)Tr(P)$. Besides, $Tr(P)$ and $|P|^2$ are also constants. Thereby, we have

$$u\Delta(Tr(Ric^3)) = u\Delta(Tr(P^3)). \tag{4.27}$$

We now need to obtain an expression for $\nabla_i R_{sj} R_{jl} R_{il} \nabla_s u$ in terms of P. Indeed, one observes that

$$\begin{aligned}
\nabla_i R_{sj} R_{jl} R_{il} \nabla_s u &= [\nabla_i (P_{sj} + \varrho g_{sj})](P_{jl} + \varrho g_{jl})(P_{il} + \varrho g_{il}) \nabla_s u \\
&= \nabla_i P_{sj} P_{jl} P_{il} \nabla_s u + \nabla_i P_{sj} P_{jl} \varrho g_{il} \nabla_s u + \nabla_i P_{sj} \varrho g_{jl} P_{il} \nabla_s u \\
&\quad + \nabla_i P_{sj} \varrho g_{jl} \varrho g_{il} \nabla_s u \\
&= \nabla_i P_{sj} P_{jl} P_{il} \nabla_s u + \varrho \nabla_i P_{sj} P_{ji} \nabla_s u + \varrho \nabla_i P_{sj} P_{ij} \nabla_s u \\
&\quad + \varrho^2 \nabla_i P_{si} \nabla_s u \\
&= \nabla_i P_{sj} P_{jl} P_{il} \nabla_s u + 2\varrho \nabla_i P_{sj} P_{ij} \nabla_s u, \qquad (4.28)
\end{aligned}$$

where we have used that $\nabla_i P_{si} = 0$, which follows from the fact that M has constant scalar curvature and the twice contracted second Bianchi identity. Next, we compute

$$\begin{aligned}
Ric^2_{ij} Ric^2_{ij} &= R_{ik} R_{kj} R_{jl} R_{li} \\
&= \left(P_{ik} P_{kj} + 2\varrho P_{ij} + \varrho^2 g_{ij}\right)\left(P_{il} P_{lj} + 2\varrho P_{ij} + \varrho^2 g_{ij}\right) \\
&= P_{ik} P_{kj} P_{il} P_{lj} + 4\varrho P_{ik} P_{kj} P_{ji} + 6\varrho^2 P_{ij} P_{ij} + 4\varrho^3 Tr(P) + \varrho^4 n \\
&= Tr(P^4) + 4\varrho Tr(P^3) + 6\varrho^2 |P|^2 + 4\varrho^3 Tr(P) + n\varrho^4 \qquad (4.29)
\end{aligned}$$

and

$$\begin{aligned}
R_{ds} R_{dijs} R_{jl} R_{il} &= (P_{ds} + \varrho g_{ds}) R_{dijs} (P_{jl} + \varrho g_{jl})(P_{il} + \varrho g_{il}) \\
&= (P_{ds} P_{jl} P_{il} + 2\varrho P_{ds} P_{ij} + \varrho^2 P_{ds} g_{ij} + \varrho g_{ds} P_{jl} P_{il} \\
&\quad + 2\varrho^2 g_{ds} P_{ij} + \varrho^3 g_{ds} g_{ij}) R_{dijs} \\
&= P_{ds} R_{dijs} P_{jl} P_{il} + 2\varrho P_{ds} P_{ji} R_{dijs} - \varrho^2 P_{ds}(P_{ds} + \varrho g_{ds}) \\
&\quad - \varrho (P_{ij} + \varrho g_{ij}) P_{jl} P_{il} - 2\varrho^2 P_{ij}(P_{ij} + \varrho g_{ij}) - \varrho^3 R \\
&= P_{ds} R_{dijs} P_{jl} P_{il} + 2\varrho P_{ds} R_{dijs} P_{ij} - 4\varrho^2 |P|^2 - 3\varrho^3 Tr(P) \\
&\quad - \varrho Tr(P^3) - \varrho^3 R. \qquad (4.30)
\end{aligned}$$

At the same time, observe that

$$\begin{aligned}
\nabla_s R_{ij} \nabla_s R_{jl} R_{il} &= \nabla_s (P_{ij} + \varrho g_{ij}) \nabla_s (P_{jl} + \varrho g_{jl})(P_{il} + \varrho g_{il}) \\
&= \nabla_s P_{ij} \nabla_s P_{jl} P_{il} + \varrho \nabla_s P_{ij} \nabla_s P_{ij}. \qquad (4.31)
\end{aligned}$$

Moreover, as already mentioned, constant scalar curvature implies that $|P|$ and $Tr(P)$ are also constants. Therefore, one deduces that

$$\nabla_s (Tr(Ric^3)) \nabla_s u = \nabla_s (Tr(P^3)) \nabla_s u. \qquad (4.32)$$

Thereby, using (4.27), jointly with (4.26), (4.28), (4.29), (4.30), (4.31) and (4.32) into Corollary 4.3, one obtains that

$$\begin{aligned}
u\Delta Tr(P^3) &= u\Delta Tr(Ric^3)\\
&= 3(m+1)\left(\nabla_i P_{sj}P_{jl}P_{il}\nabla_s u + 2\varrho\nabla_i P_{sj}P_{ij}\nabla_s u\right)\\
&\quad +\frac{3(m+1)u}{m}\left(Tr(P^4)+4\varrho Tr(P^3)+6\varrho^2|P|^2+4\varrho^3 Tr(P)+n\varrho^4\right)\\
&\quad +6u\left(P_{ds}R_{dijs}P_{jl}P_{il}+2\varrho P_{ds}R_{dijs}P_{ij}-4\varrho^2|P|^2-3\varrho^3 Tr(P)-\varrho Tr(P^3)-\varrho^3 R\right)\\
&\quad -(m+2)\nabla_s(Tr(P^3))\nabla_s u\\
&\quad -\frac{3u}{m}\left(R-(m+n-2)\lambda\right)\left(Tr(P^3)+3\varrho|P|^2+3\varrho^2 Tr(P)+n\varrho^3\right)\\
&\quad +\frac{3\lambda u}{m}\left(R-(n-1)\lambda\right)\left(|P|^2+2\varrho Tr(P)+n\varrho^2\right)\\
&\quad +6u\left(\nabla_s P_{ij}\nabla_s P_{jl}P_{il}+\varrho\nabla_s P_{ij}\nabla_s P_{ij}\right),
\end{aligned}$$

where we also used that $|Ric|^2=|P+\varrho g|^2=|P|^2+2\varrho Tr(P)+n\varrho^2$. Consequently, taking into account that $\varrho^3 R=-(m-1)\varrho^4+\varrho^3(n-1)\lambda$ and $R-(m+n-2)\lambda=-(m-1)(\varrho+\lambda)$, we have

$$\begin{aligned}
u\Delta Tr(P^3) &= 3(m+1)\left(\nabla_i P_{sj}P_{jl}P_{il}\nabla_s u+2\varrho\nabla_i P_{sj}P_{ij}\nabla_s u\right)\\
&\quad +6u\left(\nabla_s P_{ij}\nabla_s P_{jl}P_{il}+\varrho\nabla_s P_{ij}\nabla_s P_{ij}\right)\\
&\quad +6u\left(P_{ds}R_{dijs}P_{jl}P_{il}+2\varrho P_{ds}R_{dijs}P_{ij}\right)-(m+2)\nabla_s(Tr(P^3))\nabla_s u\\
&\quad +\frac{3(m+1)u}{m}Tr(P^4)+\left(\frac{12(m+1)\varrho u}{m}-6\varrho u+\frac{3u}{m}(m-1)(\varrho+\lambda)\right)Tr(P^3)\\
&\quad +\left(\frac{18(m+1)\varrho^2 u}{m}-24\varrho^2 u+\frac{9\varrho u}{m}(m-1)(\varrho+\lambda)-\frac{3(m-1)\lambda\varrho u}{m}\right)|P|^2\\
&\quad +\left(\frac{12(m+1)\varrho^3 u}{m}-18\varrho^3 u+\frac{9\varrho^2 u}{m}(m-1)(\varrho+\lambda)-\frac{6(m-1)\lambda\varrho^2 u}{m}\right)Tr(P)\\
&\quad +\left(\frac{3(m+1)n\varrho^4 u}{m}-6u\varrho^3(-(m-1)\varrho+(n-1)\lambda)+\frac{3n\varrho^3 u}{m}(m-1)(\varrho+\lambda)\right.\\
&\quad \left.-\frac{3(m-1)n\lambda\varrho^3 u}{m}\right).
\end{aligned}$$

Simplifying the last four terms in the right hand side of the above expression, we achieve

$$\begin{aligned} u\Delta Tr(P^3) &= 3(m+1)\Big(\nabla_i P_{sj}P_{jl}P_{il}\nabla_s u + 2\varrho\nabla_i P_{sj}P_{ij}\nabla_s u\Big) \\ &\quad +6u\Big(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho\nabla_s P_{ij}\nabla_s P_{ij}\Big) \\ &\quad +6u\Big(P_{ds}R_{dijs}P_{jl}P_{il} + 2\varrho P_{ds}R_{dijs}P_{ij}\Big) - (m+2)\nabla_s(Tr(P^3))\nabla_s u \\ &\quad +\frac{3(m+1)u}{m}Tr(P^4) + \frac{3u}{m}\left(3(m+1)\varrho + (m-1)\lambda\right)Tr(P^3) \\ &\quad +\frac{3\varrho u}{m}\left((m+3)\varrho + 2(m-1)\lambda\right)|P|^2 \\ &\quad +\frac{3\varrho^2 u}{m}\left((m+1)\varrho + (m-1)\lambda\right)Tr(P) \\ &\quad +6\varrho^3 u\left((m+n-1)\varrho - (n-1)\lambda\right), \end{aligned}$$

which finishes the proof of the lemma. □

4.4.2 Rigidity results

From now on, we shall adapt the approach outlined by Cheng and Zhou in [40]. To do so, we first establish the following proposition.

Proposition 4.2. *Let (M^4, g, u, λ) be an m-quasi-Einstein manifold with $m > 1$ and constant scalar curvature $R = \frac{2(m+2)\lambda}{m+1}$. Then we have*

$$\begin{aligned} u\Delta Tr(P^3) + (m+2)\langle\nabla(Tr(P^3)), \nabla u\rangle &= 6u\lambda Tr(P^3) + 6\frac{\lambda^2}{m+1}u|P|^2 \\ &\quad +6u\Big(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho\nabla_s P_{ij}\nabla_s P_{ij}\Big) \\ &\quad +6u\Big(P_{ds}R_{dijs}P_{jl}P_{il} + 2\varrho P_{ds}R_{dijs}P_{ij}\Big) \\ &\quad +12\varrho^4 m^2(m+1)u. \end{aligned} \tag{4.33}$$

Proof. Initially, let μ_i be the eigenvalues of P defined in (2.39) with respect to the adopted orthonormal frame $\{e_i\}_{i=1}^4$ so that $e_1 = -\frac{\nabla u}{|\nabla u|}$. In particular, it follows from (2.36) that $\mu_1 = 0$. Consequently,

$$Tr(P) = \mu_2 + \mu_3 + \mu_4 \quad \text{and} \quad |P|^2 = \mu_2^2 + \mu_3^2 + \mu_4^2,$$

where $P = Ric - \frac{3\lambda - R}{m-1}g$. Thus, for $R = \frac{2(m+2)}{m+1}\lambda$, it follows from (2.39) that

$$Tr(P) = \frac{(m+n-1)R - n(n-1)\lambda}{m-1} = \frac{(m+3)R - 12\lambda}{m-1} = \frac{2m}{m+1}\lambda, \tag{4.34}$$

which implies that $Tr(P)$ is a positive constant.

Next, by Proposition 3.3 in [60], one sees that $|P|^2=(\lambda-\varrho)Tr(P)$, where $\varrho=\frac{3\lambda-R}{m-1}$. This combined with (4.34) yields

$$|P|^2 = \frac{(m-4)\lambda+R}{m-1}Tr(P)=\frac{m}{m+1}\lambda Tr(P)=\frac{1}{2}(Tr(P))^2. \tag{4.35}$$

On the other hand, by simplifying the last three terms in the right hand side of Lemma 4.4, taking into account that $\varrho=\frac{\lambda}{m+1}$, $Tr(P)=\frac{2m}{m+1}\lambda$, $2|P|^2=(Tr(P))^2$ and $n=4$, one deduces that

$$\begin{aligned} u\Delta Tr(P^3) &= 3(m+1)\left(\nabla_i P_{sj}P_{jl}P_{il}\nabla_s u+2\varrho\nabla_i P_{sj}P_{ij}\nabla_s u\right)\\ &\quad +6u\left(\nabla_s P_{ij}\nabla_s P_{jl}P_{il}+\varrho\nabla_s P_{ij}\nabla_s P_{ij}\right)\\ &\quad +6u\left(P_{ds}R_{dijs}P_{jl}P_{il}+2\varrho P_{ds}R_{dijs}P_{ij}\right)-(m+2)\nabla_s(Tr(P^3))\nabla_s u\\ &\quad +\frac{3(m+1)u}{m}Tr(P^4)+\frac{3u}{m}\left(3(m+1)\varrho+(m-1)\lambda\right)Tr(P^3)\\ &\quad +12\varrho^4 m^2(m+1)u. \end{aligned} \tag{4.36}$$

At the same time, since $P_{sj}\nabla_s u=P(\nabla u)=0$, we have from (4.1) that

$$\begin{aligned} 0 &= \nabla_i(P_{sj}\nabla_s u)\\ &= \nabla_i P_{sj}\nabla_s u+\frac{u}{m}P_{sj}(R_{is}-\lambda g_{is})\\ &= \nabla_i P_{sj}\nabla_s u+\frac{u}{m}P^2_{ij}-\frac{(\lambda-\varrho)}{m}uP_{ij} \end{aligned}$$

so that

$$\nabla_i P_{sj}\nabla_s u=-\frac{u}{m}P^2_{ij}+\frac{(\lambda-\varrho)}{m}uP_{ij}. \tag{4.37}$$

Hence, the first term in the right hand side of (4.36) becomes

$$\begin{aligned} I &= 3(m+1)\left(\nabla_i P_{sj}P_{jl}P_{il}\nabla_s u+2\varrho\nabla_i P_{sj}P_{ij}\nabla_s u\right)\\ &= 3(m+1)\left(-\frac{u}{m}P^2_{ij}+\frac{(\lambda-\varrho)}{m}uP_{ij}\right)\left(P_{jl}P_{il}+2\varrho P_{ij}\right)\\ &= 3(m+1)\left(-\frac{u}{m}(Tr(P^4))+\frac{(\lambda-3\varrho)}{m}uTr(P^3)+2\frac{\varrho(\lambda-\varrho)}{m}u|P|^2\right). \end{aligned} \tag{4.38}$$

Substituting this into (4.36) and rearranging terms, one concludes that

$$\begin{aligned} u\Delta Tr(P^3)+(m+2)\langle\nabla(Tr(P^3)),\nabla u\rangle &= 6u\lambda Tr(P^3)+6\frac{\lambda^2}{m+1}u|P|^2\\ &\quad +6u\left(\nabla_s P_{ij}\nabla_s P_{jl}P_{il}+\varrho\nabla_s P_{ij}\nabla_s P_{ij}\right)\\ &\quad +6u\left(P_{ds}R_{dijs}P_{jl}P_{il}+2\varrho P_{ds}R_{dijs}P_{ij}\right)\\ &\quad +12\varrho^4 m^2(m+1)u. \end{aligned} \tag{4.39}$$

This concludes the proof of the proposition. □

In order to proceed, we need to prove the following result.

Proposition 4.3. *Let (M^4, g, u, λ) be an m-quasi-Einstein manifold with $m>1$ and constant scalar curvature $R=\frac{2(m+2)\lambda}{m+1}$. Then we have:*

$$\begin{aligned} uL_{m+2}(Tr(P^3)) &= 8(m+1)\varrho u Tr(P^3)+6u\nabla_s P_{ij}\nabla_s P_{jl}P_{il}-3m\varrho u|\nabla P|^2 \\ &\quad -16m^3(m+1)\varrho^4 u \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} uL_{m+2}(Tr(P^3)) &\geq 8(m+1)\varrho u Tr(P^3)-3m\varrho u|\nabla P|^2 \\ &\quad -16m^3(m+1)\varrho^4 u, \end{aligned} \tag{4.41}$$

where $uL_{m+2}(f)=u\Delta f+(m+2)\langle\nabla f,\nabla u\rangle$ and $\varrho=\frac{\lambda}{m+1}$.

Proof. First of all, observe that our assumption is equivalent to $R=2(m+2)\varrho$, where $\varrho=\frac{\lambda}{m+1}$. Moreover, one sees that

$$Tr(P)=2m\varrho \quad \text{and} \quad |P|^2=2m^2\varrho^2=\frac{1}{2}(Tr(P))^2. \tag{4.42}$$

Now, we need to compute $uL_{m+2}(|P|^2)$. To do so, since $Ric=P+\varrho g$, we notice from Lemma 4.2 that

$$\begin{aligned} u(\Delta P_{ik}) &= \nabla_i P_{sk}\nabla_s u+m\nabla_k P_{is}\nabla_s u+\frac{m+1}{m}u(P_{is}+\varrho g_{is})(P_{sk}+\varrho g_{sk}) \\ &\quad +2uR_{jiks}(P_{js}+\varrho g_{js})-(m+2)\nabla_s P_{ik}\nabla_s u \\ &\quad +\frac{u}{m}(m-1)(m+2)\varrho(P_{ik}+\varrho g_{ik})-\frac{u}{m}(m-1)(m+1)\varrho^2 g_{ik}, \end{aligned}$$

where we have used that $n=4$, $R-(m+n-2)\lambda=-(m-1)(m+2)\varrho$ and $\lambda(R-(n-1)\lambda)=-(m-1)(m+1)\varrho^2$. Next, expanding the expression in the right hand side and rearranging terms, we have

$$\begin{aligned} uL_{m+2}(P_{ik}) &= \nabla_i P_{sk}\nabla_s u+m\nabla_k P_{is}\nabla_s u+\frac{m+1}{m}uP_{ik}^2+\frac{2(m+1)\varrho u}{m}P_{ik} \\ &\quad +\frac{(m+1)\varrho^2 u}{m}g_{ik}+2uR_{jiks}P_{js}-2\varrho uP_{ik}-2\varrho^2 ug_{ik} \\ &\quad +\frac{(m-1)(m+2)\varrho u}{m}P_{ik}+\frac{(m-1)\varrho^2 u}{m}g_{ik} \\ &= \nabla_i P_{sk}\nabla_s u+m\nabla_k P_{is}\nabla_s u+\frac{m+1}{m}uP_{ik}^2+(m+1)\varrho uP_{ik}+2uR_{jiks}P_{js}. \end{aligned} \tag{4.43}$$

Proceeding, we use that $\lambda=(m+1)\varrho$ and Eq. (4.37) to infer

$$\nabla_i P_{sk}\nabla_s u=-\frac{u}{m}(P^2_{ik}-m\varrho P_{ik}).$$

Consequently,

$$\nabla_i P_{sk}\nabla_s u+m\nabla_k P_{is}\nabla_s u=-\frac{(m+1)u}{m}(P^2_{ik}-m\varrho P_{ik}).$$

This allow us to rewrite (4.43) as

$$\begin{aligned} uL_{m+2}(P_{ik}) &= -\frac{(m+1)u}{m}P^2_{ik}+(m+1)\varrho uP_{ik}+\frac{(m+1)u}{m}P^2_{ik}+(m+1)\varrho uP_{ik}+2uR_{jiks}P_{js} \\ &= 2(m+1)\varrho uP_{ik}+2uR_{jiks}P_{js}. \end{aligned}$$

At the same time, by using that $uL_{m+2}(P_{ik})=u\Delta P_{ik}+(m+2)\langle\nabla P_{ik},\nabla u\rangle$, we infer

$$\begin{aligned} uL_{m+2}(|P|^2) &= uL_{m+2}(P_{ik}P_{ik}) \\ &= u\Delta(P_{ik}P_{ik})+(m+2)\langle\nabla(P_{ik}P_{ik}),\nabla u\rangle \\ &= u(2P_{ik}\Delta P_{ik}+2|\nabla P|^2)+2(m+2)P_{ik}\langle\nabla P_{ik},\nabla u\rangle \\ &= 2u|\nabla P|^2+2uP_{ik}L_{m+2}(P_{ik}) \\ &= 2u|\nabla P|^2+4(m+1)\varrho u|P|^2+4uP_{ik}R_{jiks}P_{js}. \end{aligned}$$

Besides, since $|P|^2$ is constant, then $uL_{m+2}(|P|^2)=0$ and hence, we have

$$uP_{ik}R_{jiks}P_{js}=-\frac{u}{2}|\nabla P|^2-(m+1)\varrho u|P|^2. \tag{4.44}$$

On the other hand, it follows from (4.33) that

$$\begin{aligned} uL_{m+2}(Tr(P^3)) &= 6(m+1)\varrho uTr(P^3)+6(m+1)\varrho^2u|P|^2 \\ &\quad +6u(\nabla_sP_{ij}\nabla_sP_{jl}P_{il}+\varrho\nabla_sP_{ij}\nabla_sP_{ij}) \\ &\quad +6u(P_{ds}R_{dijs}P_{jl}P_{il}+2\varrho P_{ds}R_{dijs}P_{ij}) \\ &\quad +12m^2(m+1)\varrho^4u. \end{aligned} \tag{4.45}$$

To proceed, we need to deal with the terms that depend of the Riemannian curvature. Thereby, fix a point $p\in M$ and assume $P_{ij}=\mu_i\delta_{ij}$ at p, that is, μ_i, $i=1,2,3,4$ are the eigenvalues of the tensor P at p and recall that $\mu_1=0$. Hence, one easily verifies that

$$P_{ds}R_{dijs}P_{jl}P_{il}=\sum_{j=2}^{4}\sum_{d=2}^{4}\mu_dR_{djjd}\mu_j^2.$$

Denoting $K_{dj} = R_{djdj}$, it follows that

$$\begin{aligned}
P_{ds}R_{dijs}P_{jl}P_{il} &= -\mu_2 K_{23}\mu_3^2 - \mu_2 K_{24}\mu_4^2 - \mu_3 K_{32}\mu_2^2 - \mu_3 K_{34}\mu_4^2 - \mu_4 K_{42}\mu_2^2 - \mu_4 K_{43}\mu_3^2 \\
&= -K_{23}\mu_2\mu_3(\mu_3+\mu_2) - K_{24}\mu_2\mu_4(\mu_2+\mu_4) - K_{34}\mu_3\mu_4(\mu_3+\mu_4) \\
&= -K_{23}\mu_2\mu_3(Tr(P)-\mu_4) - K_{24}\mu_2\mu_4(Tr(P)-\mu_3) - K_{43}\mu_4\mu_3(Tr(P)-\mu_2) \\
&= -Tr(P)(K_{23}\mu_2\mu_3 + K_{34}\mu_3\mu_4 + K_{24}\mu_2\mu_4) + (K_{23}+K_{34}+K_{24})\mu_2\mu_3\mu_4.
\end{aligned} \tag{4.46}$$

Moreover, notice that

$$R_{22}+R_{33}+R_{44} = R - R_{11} = R - \varrho = Tr(P) + 3\varrho$$

and

$$K_{12}+K_{13}+K_{14} = R_{11} = \varrho,$$

which therefore implies that

$$Tr(P)+3\varrho = R - R_{11} = R_{22}+R_{33}+R_{44} = 2(K_{23}+K_{34}+K_{24}) + R_{11}.$$

Besides, $K_{23}+K_{34}+K_{24} = \frac{1}{2}(Tr(P)+2\varrho) = (m+1)\varrho$. In view of this, we may rewrite (4.46) as

$$P_{ds}R_{dijs}P_{jl}P_{il} = -2m\varrho(K_{23}\mu_2\mu_3 + K_{34}\mu_3\mu_4 + K_{24}\mu_2\mu_4) + (m+1)\varrho\mu_2\mu_3\mu_4.$$

Similarly, one easily verifies that

$$P_{ds}R_{dijs}P_{ij} = \sum_{d=2}^{4}\sum_{j=2}^{4}\mu_d R_{djjd}\mu_j = -2(K_{23}\mu_2\mu_3 + K_{24}\mu_2\mu_4 + K_{34}\mu_3\mu_4). \tag{4.47}$$

Hence, Eq. (4.45) becomes

$$\begin{aligned}
uL_{m+2}(Tr(P^3)) &= 6(m+1)\varrho u Tr(P^3) + 6(m+1)\varrho^2 u|P|^2 + 6u(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho|\nabla P|^2) \\
&\quad -12(m+2)\varrho u(K_{23}\mu_2\mu_3 + K_{24}\mu_2\mu_4 + K_{34}\mu_3\mu_4) + 6(m+1)\varrho u\mu_2\mu_3\mu_4 \\
&\quad +12m^2(m+1)\varrho^4 u \\
&= 6(m+1)\varrho u Tr(P^3) + 12m^2(m+1)\varrho^4 u + 6u(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho|\nabla P|^2) \\
&\quad -12(m+2)\varrho u(K_{23}\mu_2\mu_3 + K_{24}\mu_2\mu_4 + K_{34}\mu_3\mu_4) + 6(m+1)\varrho u\mu_2\mu_3\mu_4 \\
&\quad +12m^2(m+1)\varrho^4 u \\
&= 6(m+1)\varrho u Tr(P^3) + 6u(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho|\nabla P|^2) \\
&\quad -12(m+2)\varrho u(K_{23}\mu_2\mu_3 + K_{24}\mu_2\mu_4 + K_{34}\mu_3\mu_4) \\
&\quad +6(m+1)\varrho u\mu_2\mu_3\mu_4 + 24m^2(m+1)\varrho^4 u,
\end{aligned}$$

where we used that $|P|^2 = 2m^2\varrho^2$. Besides, by combining (4.44) and (4.47), we arrive at

$$u(K_{23}\mu_2\mu_3 + K_{24}\mu_2\mu_4 + K_{34}\mu_3\mu_4) = \frac{u|\nabla P|^2}{4} + m^2(m+1)\varrho^3 u.$$

Consequently,

$$\begin{aligned} uL_{m+2}(Tr(P^3)) &= 6(m+1)\varrho u Tr(P^3) + 6u(\nabla_s P_{ij}\nabla_s P_{jl}P_{il} + \varrho|\nabla P|^2) \\ &\quad -3(m+2)\varrho u|\nabla P|^2 - 12m^2(m+2)(m+1)\varrho^4 u \\ &\quad +6(m+1)\varrho u\mu_2\mu_3\mu_4 + 24m^2(m+1)\varrho^4 u \\ &= 6(m+1)\varrho u Tr(P^3) + 6u\nabla_s P_{ij}\nabla_s P_{jl}P_{il} - 3m\varrho u|\nabla P|^2 \\ &\quad +6(m+1)\varrho u\mu_2\mu_3\mu_4 - 12m^3(m+1)\varrho^4 u. \end{aligned} \tag{4.48}$$

At the same time, similar to [40, pg. 11], by letting $\alpha = \mu_2$, $\beta = \mu_3$ and $\kappa = \mu_4$ in the following algebraic identity

$$(\alpha+\beta+\kappa)^3 = 3(\alpha+\beta+\kappa)(\alpha^2+\beta^2+\kappa^2) - 2(\alpha^3+\beta^3+\kappa^3) + 6\alpha\beta\kappa,$$

we obtain

$$(Tr(P))^3 = 3|P|^2 Tr(P) - 2Tr(P^3) + 6\mu_2\mu_3\mu_4.$$

Of which,

$$3\mu_2\mu_3\mu_4 = Tr(P^3) - 2m^3\varrho^3. \tag{4.49}$$

This substituted into (4.48) yields

$$\begin{aligned} uL_{m+2}(Tr(P^3)) &= 8(m+1)\varrho u Tr(P^3) + 6u\nabla_s P_{ij}\nabla_s P_{jl}P_{il} - 3m\varrho u|\nabla P|^2 \\ &\quad -16m^3(m+1)\varrho^4 u, \end{aligned}$$

which proves (4.40).

Finally, for the fixed orthonormal frame, by using (4.42) and Lemma 2.4, one deduces that $\mu_i \geq 0$, for all i. Hence, $\nabla_s P_{ij}\nabla_s P_{jl}P_{il} = |\nabla P_{ii}|^2\mu_i \geq 0$ and this proves the second assertion (4.41). $\square$

Now, we establish the following essential lemma.

Lemma 4.5. *Let (M^4, g, u, λ) be an m-quasi-Einstein manifold with $m > 1$ and constant scalar curvature $R = \frac{2(m+2)\lambda}{m+1}$. Then the following inequality holds*

$$L_{m+2}\left[|\nabla u|^2\left(Tr(P^3) - 2m^3\varrho^3\right)\right] \geq 2(9m+7)\varrho|\nabla u|^2\left(Tr(P^3) - 2m^3\varrho^3\right),$$

where $\varrho = \frac{\lambda}{m+1}$.

Proof. Initially, we consider the level set $\Sigma = \Sigma(t) = u^{-1}(t)$, $0 \leq t < u_{max}$, and an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ for M^4 that diagonalizes the tensor P so that $e_1 = -\frac{\nabla u}{|\nabla u|}$ and $\{e_2, e_3, e_4\}$ is a frame over $\Sigma(t)$. Moreover, we assume $\alpha, \beta, \gamma, \eta \in \{2,3,4\}$ and $i, j, k \in \{1,2,3,4\}$. Thereby, it follows from the Gauss-equation that

$$R^{\Sigma}_{\alpha\beta\gamma\eta} = R_{\alpha\beta\gamma\eta} + h_{\alpha\gamma}h_{\beta\eta} - h_{\alpha\eta}h_{\beta\gamma},$$

which implies that

$$R^{\Sigma}_{\alpha\gamma} = R_{\alpha\gamma} - R_{\alpha 1\gamma 1} + Hh_{\alpha\gamma} - h_{\alpha\beta}h_{\beta\gamma}, \tag{4.50}$$

where h and H stand for the second fundamental form and the mean curvature, respectively. Besides, taking into account that $\varrho = \frac{\lambda}{m+1}$ as well as

$$Ric(\nabla u) = \varrho\nabla u,\ P = Ric - \varrho g,\ R = 2(m+2)\varrho,\ Tr(P) = 2m\varrho \quad \text{and} \quad |P|^2 = 2m^2\varrho^2,$$

one deduces that

$$R^{\Sigma} = R - 2\varrho + H^2 - |A|^2 = 2(m+1)\varrho + H^2 - |A|^2, \tag{4.51}$$

where $|A|^2$ is the norm of the second fundamental form.

Next, we are going to compute $h_{\alpha\beta}$ and H. Indeed, by using (4.1) in terms of P, i.e., $\nabla^2 u = \frac{u}{m}(P - m\varrho g)$, the second fundamental form is given by

$$h_{\alpha\beta} = \frac{\nabla_\alpha\nabla_\beta u}{|\nabla u|} = \frac{(P_{\alpha\beta} - m\varrho g_{\alpha\beta})}{m\sqrt{b(u)}}u, \tag{4.52}$$

where $b(u) = |\nabla u|^2$. Furthermore, our assumption on the scalar curvature implies that $P_{11} = 0$ and hence,

$$H = \frac{Tr(P) - 3m\varrho}{m\sqrt{b(u)}}u = -\frac{\varrho u}{\sqrt{b(u)}}. \tag{4.53}$$

In particular, we have from (4.52) that

$$|A|^2 = \frac{|P|^2 - 2m\varrho Tr(P) + 3m^2\varrho^2}{m^2 b(u)}u^2 = \frac{\varrho^2 u^2}{b(u)}. \tag{4.54}$$

Substituting (4.53) and (4.54) into (4.51) yields $R^{\Sigma} = 2(m+1)\varrho$.

Proceeding, we are going to deal with the Riemannian curvature tensor of Σ. In fact, since Σ has dimension 3, its curvature tensor can be expressed as

$$R^{\Sigma}_{\alpha\beta\gamma\eta} = (R^{\Sigma}_{\alpha\gamma}g_{\beta\eta} + R^{\Sigma}_{\beta\eta}g_{\alpha\gamma} - R^{\Sigma}_{\alpha\eta}g_{\beta\gamma} - R^{\Sigma}_{\beta\gamma}g_{\alpha\eta}) - \frac{R^{\Sigma}}{2}(g_{\alpha\gamma}g_{\beta\eta} - g_{\alpha\eta}g_{\beta\gamma}).$$

This jointly with (4.50) gives

$$\begin{aligned}
R^{\Sigma}_{\alpha\beta\alpha\beta} &= R^{\Sigma}_{\alpha\alpha} + R^{\Sigma}_{\beta\beta} - \frac{R^{\Sigma}}{2} \\
&= R_{\alpha\alpha} - R_{\alpha 1\alpha 1} + Hh_{\alpha\alpha} - h^2_{\alpha\alpha} + R_{\beta\beta} - R_{\beta 1\beta 1} + Hh_{\beta\beta} - h^2_{\beta\beta} - (m+1)\varrho \\
&= \mu_\alpha + \mu_\beta + 2\varrho - R_{\alpha 1\alpha 1} - R_{\beta 1\beta 1} + H(h_{\alpha\alpha} + h_{\beta\beta}) - h^2_{\alpha\alpha} - h^2_{\beta\beta} - (m+1)\varrho,
\end{aligned}$$

where $\mu_\alpha = P(e_\alpha)$ and $h_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Consequently, for fixed $\alpha \neq \beta$ again, by using the Gauss equation, Eqs. (4.52) and (4.53), we then obtain

$$\begin{aligned}
R_{\alpha\beta\alpha\beta} &= R^{\Sigma}_{\alpha\beta\alpha\beta} - h_{\alpha\alpha}h_{\beta\beta} + h^2_{\alpha\beta} \\
&= \mu_\alpha + \mu_\beta + 2\varrho - R_{\alpha 1\alpha 1} - R_{\beta 1\beta 1} + H(h_{\alpha\alpha} + h_{\beta\beta}) - h^2_{\alpha\alpha} \\
&\quad - h^2_{\beta\beta} - (m+1)\varrho - h_{\alpha\alpha}h_{\beta\beta} \\
&= \mu_\alpha + \mu_\beta + 2\varrho - R_{\alpha 1\alpha 1} - R_{\beta 1\beta 1} - \frac{\varrho(\mu_\alpha - m\varrho + \mu_\beta - m\varrho)u^2}{mb(u)} \\
&\quad - \frac{(\mu_\alpha - m\varrho)^2 u^2}{m^2 b(u)} - \frac{(\mu_\beta - m\varrho)^2 u^2}{m^2 b(u)} - (m+1)\varrho - \frac{(\mu_\beta - m\varrho)(\mu_\alpha - m\varrho)u^2}{m^2 b(u)} \\
&= \mu_\alpha + \mu_\beta + 2\varrho - R_{\alpha 1\alpha 1} - R_{\beta 1\beta 1} - (m+1)\varrho - \frac{m\varrho(\mu_\alpha + \mu_\beta - 2m\varrho)u^2}{m^2 b(u)} \\
&\quad - \frac{[\mu^2_\alpha - 2m\varrho(\mu_\alpha + \mu_\beta) + \mu^2_\beta + 2m^2\varrho^2]u^2}{m^2 b(u)} - \frac{[\mu_\beta\mu_\alpha - m\varrho(\mu_\alpha + \mu_\beta) + m^2\varrho^2]u^2}{m^2 b(u)},
\end{aligned}$$

which can be simplifying as

$$\begin{aligned}
R_{\alpha\beta\alpha\beta} &= \mu_\alpha + \mu_\beta - \frac{\varrho(\mu_\alpha + \mu_\beta)u^2}{mb(u)} + \frac{2\varrho(\mu_\alpha + \mu_\beta)u^2}{mb(u)} + \frac{\varrho(\mu_\alpha + \mu_\beta)u^2}{mb(u)} \\
&\quad + \frac{2\varrho^2 u^2}{b(u)} - \frac{2\varrho^2 u^2}{b(u)} - \frac{\varrho^2 u^2}{b(u)} + 2\varrho - \frac{(\mu^2_\alpha + \mu^2_\beta)u^2}{m^2 b(u)} \\
&\quad - \frac{\mu_\alpha\mu_\beta u^2}{m^2 b(u)} - R_{\alpha 1\alpha 1} - R_{\beta 1\beta 1} - (m+1)\varrho \\
&= \frac{(\mu_\alpha + \mu_\beta)(mb(u) + 2\varrho u^2)}{mb(u)} + \frac{\varrho(2b(u) - \varrho u^2)}{b(u)} \\
&\quad - \frac{(\mu^2_\alpha + \mu^2_\beta)u^2}{m^2 b(u)} - \frac{\mu_\alpha\mu_\beta u^2}{m^2 b(u)} - R_{\alpha 1\alpha 1} - R_{\beta 1\beta 1} - (m+1)\varrho.
\end{aligned}$$

Next, multiplying the previous expression by $\mu_\alpha\mu_\beta$ and summing over α and β, $\alpha \neq \beta$, we deduce that

$$\begin{aligned}
\sum_{\alpha\neq\beta}^{4} R_{\alpha\beta\alpha\beta}\mu_\alpha\mu_\beta &= \frac{mb(u) + 2\varrho u^2}{mb(u)} \sum_{\alpha\neq\beta}^{4} (\mu_\alpha + \mu_\beta)\mu_\alpha\mu_\beta + \frac{\varrho(2b(u) - \varrho u^2)}{b(u)} \sum_{\alpha\neq\beta}^{4} \mu_\alpha\mu_\beta \\
&\quad - \frac{2u^2}{m^2 b(u)} \sum_{\alpha\neq\beta}^{4} \mu^3_\alpha\mu_\beta - \frac{u^2}{m^2 b(u)} \sum_{\alpha\neq\beta}^{4} \mu^2_\alpha\mu^2_\beta \\
&\quad - 2\sum_{\alpha\neq\beta}^{4} R_{\alpha 1\alpha 1}\mu_\alpha\mu_\beta - (m+1)\varrho \sum_{\alpha\neq\beta}^{4} \mu_\alpha\mu_\beta. \qquad (4.55)
\end{aligned}$$

At the same time, we have to obtain expressions for each sum in (4.55). To do so, we first observe that

$$\sum_{\alpha\neq\beta}^{4}\mu_\alpha = Tr(P) = 2m\varrho \quad \text{and} \quad \sum_{\alpha\neq\beta}^{4}\mu_\alpha^2 = |P|^2 = 2m^2\varrho^2, \tag{4.56}$$

which implies that

$$\sum_{\alpha\neq\beta}^{4}\mu_\alpha\mu_\beta = \sum_{\alpha=2}^{4}\sum_{\beta\neq\alpha}\mu_\alpha\mu_\beta = \sum_{\alpha=2}^{4}\mu_\alpha(Tr(P)-\mu_\alpha) = (Tr(P))^2-|P|^2 = 2m^2\varrho^2,$$

$$\begin{aligned}\sum_{\alpha\neq\beta}^{4}(\mu_\alpha+\mu_\beta)\mu_\alpha\mu_\beta &= 2\sum_{\alpha=2}^{4}\sum_{\beta\neq\alpha}\mu_\alpha^2\mu_\beta = 2\sum_{\alpha=2}^{4}\mu_\alpha^2(Tr(P)-\mu_\alpha)\\ &= 2(Tr(P))|P|^2-2\sum_{\alpha=2}^{4}\mu_\alpha^3 = 8m^3\varrho^3-2\sum_{\alpha=2}^{4}\mu_\alpha^3,\end{aligned}$$

$$\sum_{\alpha\neq\beta}^{4}\mu_\alpha^3\mu_\beta = \sum_{\alpha=2}^{4}\sum_{\beta\neq\alpha}\mu_\alpha^3\mu_\beta = \sum_{\alpha=2}^{4}\mu_\alpha^3(Tr(P)-\mu_\alpha) = \sum_{\alpha=2}^{4}2m\varrho\mu_\alpha^3-\sum_{\alpha=2}^{4}\mu_\alpha^4$$

and

$$\sum_{\alpha\neq\beta}^{4}\mu_\alpha^2\mu_\beta^2 = \sum_{\alpha=2}^{4}\sum_{\beta\neq\alpha}\mu_\alpha^2\mu_\beta^2 = \sum_{\alpha=2}^{4}\mu_\alpha^2(|P|^2-\mu_\alpha^2) = 4m^4\varrho^4-\sum_{\alpha=2}^{4}\mu_\alpha^4.$$

We also need to obtain an expression for $R_{\alpha1\alpha1}$. From Eq. (4) of Lemma 2.7, one deduces that

$$\begin{aligned}u(\nabla_iP_{jk}-\nabla_jP_{ik})\nabla_ju &= mR_{ijkl}\nabla_lu\nabla_ju+m\varrho(\nabla_iug_{jk}-\nabla_jug_{ik})\nabla_ju\\ &\quad -(\nabla_iuP_{jk}-\nabla_juP_{ik})\nabla_ju,\end{aligned}$$

where we have used that $\lambda=(m+1)\varrho$. This combined with the fact that $P_{jk}\nabla_ju=0$ and

$$\nabla_iP_{jk}\nabla_ju = \nabla_i(P_{jk}\nabla_ju)-P_{jk}\nabla_i\nabla_ju = -\frac{u}{m}P_{jk}(P_{ij}-m\varrho g_{ij})$$

allow us to infer

$$\begin{aligned}R_{ijkl}\nabla_lu\nabla_ju &= -\varrho(\nabla_iu\nabla_ku-|\nabla u|^2g_{ik})-\frac{|\nabla u|^2}{m}P_{ik}\\ &\quad -\frac{u^2}{m^2}P_{jk}(P_{ij}-m\varrho g_{ij})-\frac{u}{m}\nabla_jP_{ik}\nabla_ju.\end{aligned}$$

By taking $i=k=\alpha$ and multiplying the last expression by $\frac{|\nabla u|^2}{|\nabla u|^2}$, we obtain

$$\begin{aligned}R_{\alpha1\alpha1}|\nabla u|^2 &= \varrho|\nabla u|^2-\frac{|\nabla u|^2}{m}\mu_\alpha-\frac{u^2}{m^2}P_{j\alpha}(P_{\alpha j}-m\varrho g_{\alpha j})-\frac{u}{m}\nabla_{\nabla u}P_{\alpha\alpha}\\ &= \frac{(m\varrho-\mu_\alpha)|\nabla u|^2}{m}-\frac{u^2}{m^2}\mu_\alpha^2+\frac{\varrho u^2}{m}\mu_\alpha-\frac{u}{m}\nabla_{\nabla u}P_{\alpha\alpha}.\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{\alpha\neq\beta}^{4} R_{\alpha 1\alpha 1}\mu_\alpha\mu_\beta &= \sum_{\alpha=2}^{4}\sum_{\beta\neq\alpha} R_{\alpha 1\alpha 1}\mu_\alpha\mu_\beta = \sum_{\alpha=2}^{4} R_{\alpha 1\alpha 1}\mu_\alpha(Tr(P)-\mu_\alpha)\\
&= \frac{1}{|\nabla u|^2}\sum_{\alpha=2}^{4}\Bigg[\frac{(m\varrho-\mu_\alpha)b(u)+\varrho\mu_\alpha u^2}{m}-\frac{u^2}{m^2}\mu_\alpha^2\\
&\quad -\frac{u}{m}\nabla_{\nabla u}P_{\alpha\alpha}\Bigg]\mu_\alpha(2m\varrho-\mu_\alpha)\\
&= \frac{1}{|\nabla u|^2}\sum_{\alpha=2}^{4}\frac{(2m^2\varrho^2\mu_\alpha-3m\varrho\mu_\alpha^2+\mu_\alpha^3)b(u)}{m}\\
&\quad +\frac{1}{|\nabla u|^2}\sum_{\alpha=2}^{4}\frac{(2m\varrho^2\mu_\alpha^2-\varrho\mu_\alpha^3)u^2}{m}-\frac{u^2}{m^2|\nabla u|^2}\sum_{\alpha=2}^{4}(2m\varrho\mu_\alpha^3-\mu_\alpha^4)\\
&\quad -\frac{u}{m|\nabla u|^2}\sum_{\alpha=2}^{4}\nabla_{\nabla u}P_{\alpha\alpha}\left(2m\varrho\mu_\alpha-\mu_\alpha^2\right).
\end{aligned}$$

In order to conclude this step, observe that

$$\nabla_{\nabla u}Tr(P^3)=3\sum_{\alpha=2}^{4}(\nabla_{\nabla u}P_{\alpha\alpha})\mu_\alpha^2 \quad\text{and}\quad 0=\nabla_{\nabla u}|P|^2=2\sum_{\alpha=2}^{4}(\nabla_{\nabla u}P_{\alpha\alpha})\mu_\alpha,$$

which combined with (4.56) gives

$$\begin{aligned}
\sum_{\alpha\neq\beta}^{4} R_{\alpha 1\alpha 1}\mu_\alpha\mu_\beta &= \frac{4m^3\varrho^3-6m^3\varrho^3}{m}+\frac{1}{m}\sum_{\alpha=2}^{4}\mu_\alpha^3+\frac{4m^3\varrho^4u^2}{mb(u)}-\frac{\varrho u^2}{mb(u)}\sum_{\alpha=2}^{4}\mu_\alpha^3\\
&\quad -\frac{u^2}{m^2b(u)}\sum_{\alpha=2}^{4}(2m\varrho\mu_\alpha^3-\mu_\alpha^4)+\frac{\nabla u(Tr(P^3))u}{3mb(u)}\\
&= -2m^2\varrho^3+\frac{4m^2\varrho^4u^2}{b(u)}+\frac{\nabla u(Tr(P^3))u}{3mb(u)}+\frac{b(u)-3\varrho u^2}{mb(u)}\sum_{\alpha=2}^{4}\mu_\alpha^3\\
&\quad +\frac{u^2}{m^2b(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4.
\end{aligned}$$

Returning to Eq. (4.55), we then have

$$\begin{aligned}
\sum_{\alpha\neq\beta}^{4} R_{\alpha\beta\alpha\beta}\mu_\alpha\mu_\beta &= \frac{mb(u)+2\varrho u^2}{mb(u)}\left(8m^3\varrho^3-2\sum_{\alpha=2}^{4}\mu_\alpha^3\right)+\frac{\varrho(2b(u)-\varrho u^2)}{b(u)}\cdot 2m^2\varrho^2\\
&\quad -\frac{2u^2}{m^2b(u)}\left(\sum_{\alpha=2}^{4}2m\varrho\mu_\alpha^3-\sum_{\alpha=2}^{4}\mu_\alpha^4\right)-\frac{u^2}{m^2b(u)}\left(4m^4\varrho^4-\sum_{\alpha=2}^{4}\mu_\alpha^4\right)\\
&\quad -2m^2(m+1)\varrho^3+4m^2\varrho^3-\frac{8m^2\varrho^4u^2}{b(u)}-\frac{2\nabla u(Tr(P^3))u}{3mb(u)}\\
&\quad -\frac{2b(u)-6\varrho u^2}{mb(u)}\sum_{\alpha=2}^{4}\mu_\alpha^3-\frac{2u^2}{m^2b(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4.
\end{aligned}$$

Simplifying terms, we infer

$$\begin{aligned}
\sum_{\alpha\neq\beta}^{4} R_{\alpha\beta\alpha\beta}\mu_\alpha\mu_\beta &= \frac{8m^3\varrho^3 b(u)+16m^2\varrho^4u^2+4m^2\varrho^3b(u)-2m^2\varrho^4u^2-2m^2(m-1)\varrho^3b(u)}{b(u)}\\
&\quad -\frac{4m^2\varrho^4u^2}{b(u)}-\frac{8m^2\varrho^4u^2}{b(u)}-\frac{2\nabla u(Tr(P^3))u}{3mb(u)}\\
&\quad -\frac{2mb(u)+4\varrho u^2+4\varrho u^2+2b(u)-6\varrho u^2}{mb(u)}\sum_{\alpha=2}^{4}\mu_\alpha^3+\frac{2u^2+u^2-2u^2}{m^2b(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4\\
&= \frac{6m^2(m+1)\varrho^3b(u)+2m^2\varrho^4u^2}{b(u)}-\frac{2\nabla u(Tr(P^3))u}{3mb(u)}\\
&\quad -\frac{2(m+1)b(u)+2\varrho u^2}{mb(u)}Tr(P^3)+\frac{u^2}{m^2b(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4\\
&= \frac{2m^2\varrho^3[3(m+1)b(u)+\varrho u^2]}{b(u)}-\frac{2\nabla u(Tr(P^3))u}{3mb(u)}\\
&\quad -\frac{2[(m+1)b(u)+\varrho u^2]}{mb(u)}Tr(P^3)+\frac{u^2}{m^2b(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4.
\end{aligned}$$

On the other hand, it follows from (4.44) that

$$2u|\nabla P|^2+4(m+1)\varrho u|P|^2+4uP_{ik}R_{jikl}P_{jl}=0$$

and hence,

$$u|\nabla P|^2=-2(m+1)\varrho u|P|^2+2uP_{ik}R_{ijkl}P_{jl}.$$

Plugging this fact into (4.41) yields

$$\begin{aligned}
uL_{m+2}(Tr(P^3)) &\geq 8(m+1)\varrho uTr(P^3)-3m\varrho u|\nabla P|^2-16m^3(m+1)\varrho^4u\\
&= 8(m+1)\varrho uTr(P^3)+6m(m+1)\varrho^2u|P|^2-6m\varrho uP_{ik}R_{ijkl}P_{jl}\\
&\quad -16m^3(m+1)\varrho^4u\\
&= 8(m+1)\varrho uTr(P^3)-4m^3(m+1)\varrho^4u-\frac{12m^3\varrho^4u[3(m+1)b(u)+\varrho u^2]}{b(u)}\\
&\quad +\frac{4\varrho\nabla u(Tr(P^3))u^2}{b(u)}+\frac{12\varrho u[(m+1)b(u)+\varrho u^2]}{b(u)}Tr(P^3)-\frac{6\varrho u^3}{mb(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4\\
&= \frac{4\varrho u[5(m+1)b(u)+3\varrho u^2]}{b(u)}Tr(P^3)-\frac{6\varrho u^3}{mb(u)}\sum_{\alpha=2}^{4}\mu_\alpha^4\\
&\quad +\frac{4\varrho\nabla u(Tr(P^3))u^2}{b(u)}-\frac{4m^3\varrho^4u[10(m+1)b(u)+3\varrho u^2]}{b(u)}. \qquad (4.57)
\end{aligned}$$

From (4.56), it is known that μ_2, μ_3, μ_4 and $Tr(P)$ satisfy the hypothesis of Corollary A.1 in [40] and therefore,

$$\sum_{\alpha=2}^{4}\mu_\alpha^4=-\frac{10m^4\varrho^4}{3}+\frac{8m\varrho}{3}Tr(P^3).$$

Substituting the above equality into (4.57), we infer

$$\begin{aligned}
uL_{m+2}(Tr(P^3)) \;&\geq\; \frac{4\varrho u[5(m+1)b(u)+3\varrho u^2]}{b(u)}Tr(P^3)+\frac{20m^3\varrho^5u^3}{b(u)}-\frac{16\varrho^2u^3}{b(u)}Tr(P^3)\\
&\quad+\frac{4\varrho\nabla u(Tr(P^3))u^2}{b(u)}-\frac{4m^3\varrho^4u[10(m+1)b(u)+3\varrho u^2]}{b(u)}\\
&= \frac{4\varrho u[5(m+1)b(u)-\varrho u^2]}{b(u)}Tr(P^3)+\frac{4\varrho\nabla u(Tr(P^3))u^2}{b(u)}\\
&\quad-\frac{4m^3\varrho^4u[10(m+1)b(u)-2\varrho u^2]}{b(u)}\\
&= \frac{4\varrho u[5(m+1)b(u)-\varrho u^2]}{b(u)}(Tr(P^3)-2m^3\varrho^3)+\frac{4\varrho\nabla u(Tr(P^3))u^2}{b(u)}.
\end{aligned} \tag{4.58}$$

Finally, we recall that the potential function of a quasi-Einstein manifold is transnormal satisfying

$$b(u)=|\nabla u|^2=\frac{\mu}{m-1}-\frac{R+(m-n)\lambda}{m(m-1)}u^2=\varrho(u_{max}^2-u^2).$$

Hence,

$$\begin{aligned}
uL_{m+2}\left[b(u)\left(Tr(P^3)-2m^3\varrho^3\right)\right] \;&=\; ub(u)L_{m+2}(Tr(P^3))+2u\langle\nabla b(u),\nabla(Tr(P^3))\rangle\\
&\quad+(Tr(P^3)-2m^3\varrho^3)uL_{m+2}(b(u))\\
&= ub(u)L_{m+2}(Tr(P^3))-4\varrho u^2\nabla u(Tr(P^3))\\
&\quad+(Tr(P^3)-2m^3\varrho^3)(-2\varrho u^2\Delta u-2\varrho u|\nabla u|^2\\
&\quad-(m+2)2u\varrho|\nabla u|^2)\\
&= ub(u)L_{m+2}(Tr(P^3))-4\varrho u^2\nabla u(Tr(P^3))\\
&\quad-2\varrho u\left(-2\varrho u^2+(m+3)b(u)\right)(Tr(P^3)-2m^3\varrho^3),
\end{aligned} \tag{4.59}$$

where we have used that $\Delta u=-2\varrho u$ and

$$L_a(f)=u^{-a}div(u^a\nabla f)=\Delta f+au^{-1}\langle\nabla u,\nabla f\rangle, \text{ for } a\neq 0 \text{ and } f\in C^\infty(M). \tag{4.60}$$

Comparing (4.58) with (4.59) gives

$$uL_{m+2}\left[|\nabla u|^2\left(Tr(P^3)-2m^3\varrho^3\right)\right]\geq 2(9m+7)\varrho u|\nabla u|^2\left(Tr(P^3)-2m^3\varrho^3\right),$$

as we wanted to prove. $\square$

We are now ready to present the proof of Theorem 4.4.

Theorem 4.8 (Theorem 4.4)**.** *Let (M^4, g, u, λ) be a nontrivial simply connected compact 4-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^4 has constant scalar curvature $R = 2\frac{(m+2)}{(m+1)}\lambda$ if and only if it is isometric, up to scaling, to the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ with the doubly warped product metric.*

Proof. We already know that $Tr(P) = 2m\varrho$ and $|P|^2 = 2m^2\varrho^2$, i.e.,

$$|P|^2 = \frac{1}{2}(Tr(P))^2. \tag{4.61}$$

Hence, since $\mu_1 = 0$, by Lemma 2.4, the eigenvalues μ_α, $\alpha = 1, 2, 3, 4$, of P are all nonnegative.

We now set the function

$$h := |\nabla u|^2 (Tr(P^3) - 2m^3\varrho^3).$$

In particular, from (4.49) and the fact that μ_α, $\alpha = 1, 2, 3, 4$, are all nonnegative, one sees that h is a nonnegative function. Besides, since M is compact with boundary ∂M, by performing integration by parts, we deduce

$$\begin{aligned} \int_M L_{m+2}(h) dV_{m+2} &= \int_M u^{-(m+2)} div(u^{m+2}\nabla h) dV_{m+2} = \int_M div(u^{m+2}\nabla h) dV \\ &= -\int_{\partial M} u^{m+2} \left\langle \nabla h, \frac{\nabla u}{|\nabla u|} \right\rangle dS = 0, \end{aligned} \tag{4.62}$$

where we have used the fact that u vanishes on ∂M, $dV_{m+2} = u^{m+2} dV$ is the weighted measure and the second order operator L_a, $a \in \mathbb{R}$, is given by Eq. (4.60)

On the other hand, it follows from Lemma 4.5 that

$$2(9m+7)\varrho h - L_{m+2}(h) \leq 0. \tag{4.63}$$

So, upon integrating (4.63) over M, we use (4.62) in order to infer

$$2(9m+7)\varrho \int_M h \, dV_{m+2} \leq 0.$$

Of which, one obtains that

$$h = |\nabla u|^2 (Tr(P^3) - 2m^3\varrho^3) = 0.$$

Now, taking into account that $|\nabla u|^2$ is zero only over the 2-dimensional submanifold $MAX(u)$, which has measure zero, one concludes that $Tr(P^3) - 2m^2\varrho^2 \equiv 0$ on M. This jointly with Eq. (4.49) then implies $\mu_2\mu_3\mu_4 = 0$, namely, at least one of μ_2, μ_3 and μ_4 is zero. Assume $\mu_2 = 0$. Thereby, by using (4.61), one deduces that $\mu_1 = \mu_2 = 0$ and $\mu_3 = \mu_4 = m\varrho$.

Returning to the Ricci tensor, we then conclude that the Ricci tensor has exactly two distinct eigenvalues, each one with multiplicity two, namely,

$$\lambda_1 = \lambda_2 = \frac{\lambda}{m+1} \text{ and } \lambda_3 = \lambda_4 = \lambda,$$

where $Ric(e_i) = \lambda_i$, for $i = 1, 2, 3, 4$. In particular, the Ricci tensor Ric is parallel. Then, by the first contracted second Bianchi identity ($\nabla_l R_{ijkl} = \nabla_j R_{ik} - \nabla_i R_{jk}$), one obtains that the curvature tensor is harmonic. Now, we are in the position to apply [60, Corollary 1.14] to conclude that M^4 is rigid. Hence, it suffices to use Proposition 2.9 to deduce that M^4 is covered by the product $\mathbb{S}^2_+ \times \mathbb{S}^2$. Finally, since M^4 is simply connected, we may use Theorem 54.6 in [81] to conclude that the covering map is a bijective local isometry and therefore, a global isometry. Thus, M^4 is isometric, up to scaling, to the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$. This finishes the proof of the theorem. □

As a consequence of Theorem 4.5, Remark 5, Theorem 4.6 and Theorem 4.8, we get the following classification result.

Corollary 4.4. *Let (M^4, g, u, λ) be a nontrivial simply connected compact 4-dimensional m-quasi-Einstein manifold with boundary and $m > 1$. Then M^4 has constant scalar curvature if and only if it is isometric, up to scaling, to either*

(i) the standard hemisphere $\mathbb{S}^4_+$, or

(ii) the cylinder $I \times \mathbb{S}^3$ with the product metric, or

(iii) the product space $\mathbb{S}^2_+ \times \mathbb{S}^2$ with the doubly warped product metric.

Proof. The result follows from Theorem 4.5, Remark 5 (and Proposition 4.1), Theorem 4.6 and Theorem 4.8. □

5 CONCLUSION

Geometric inequalities are interesting tools to obtain results in Differential Geometry. In the first part of this thesis, we deal with geometric inequalities over static perfect fluid space-times with boundary. In this sense, we obtained some estimates involving the area of the boundary whose equality case was achieved by the round hemisphere $\mathbb{S}^n_+$. To do so, we have used the fact that the scalar curvature is not necessarily constant. We have obtained an inequality involving $Area(\partial M)$ by using the generalized Reilly's formula. Moreover, we established another inequality involving the Brown-York mass $\mathfrak{m}_{BY}$ of the boundary. In addition, we obtained a new simply connected example of static perfect fluid space-time with connected boundary. In particular, it is a counter-example to the Cosmic no-hair conjecture for dimension $n \geq 4$. Since the conjecture remains open in the case $n = 3$, it is important to address this problem in forthcoming works. Moreover, it is also important to investigate similar results to other kind of special solutions of the Einstein field equation.

In the second part of this work, we investigate classification results for compact quasi-Einstein manifolds with boundary and constant scalar curvature. We showed the possible values for the scalar curvature and we obtained the complete classification in dimension 3 and 4 under the constant scalar curvature. In particular, they are rigidy. Our classification is based in the dimension of the space of critical points of the potential function, which is a smooth submanifold of the ambient manifold, and we saw that there is a bijective correspondence between the possible values of R and the dimension of $Crit(u)$. At least in dimensions 3 and 4, also for $R = \frac{n(n-1)\lambda}{m+n-1}$ and $R = (n-1)\lambda$ in general dimension $n > 2$, the constant value of R determines the geometry of a compact quasi-Einstein manifold with boundary. The problem of classification of quasi-Einstein manifolds with constant scalar curvature remains open for dimension $n \geq 5$.

REFERENCES

[1] AMBROZIO, L. On static three-manifolds with positive scalar curvature. **Journal of Differential Geometry**, v. 107, p. 1–45, 2017.

[2] ANDRADE, M.; DE MELO, A. P. Some characterizations of compact Einstein-type manifolds. **Letters in Mathematical Physics**, v. 114, n. 35, 2024.

[3] BALTAZAR, H.; BATISTA, R.; RIBEIRO JR, E. Geometric inequalities for critical metrics of the volume functional. **Annali di Matematica Pura ed Applicata**, v. 201, p. 1463–1480, 2022.

[4] BALTAZAR, H.; DIÓGENES, R.; RIBEIRO JR, E. Isoperimetric inequality and Weitzenboöck type formula for critical metrics of the volume. **Israel Journal of Mathematics**, v. 234, p. 309–329, 2019.

[5] BARROS, A.; DIÓGENES, R.; RIBEIRO JR., E. Bach-Flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. **Journal of Geometric Analysis**, v. 25, p. 2698–2715, 2015.

[6] BARBOSA, E.; FREITAS, A.; LIMA, L. The generalized Pohozaev-Schoen identity and some geometric applications. **Communications in Analysis and Geometry**, v. 28, n. 2, p. 223–242, 2020.

[7] BARROS, A.; DA SILVA, A. Rigidity for critical metrics of the volume functional. **Mathematische Nachrichten**, v. 291, p. 709–719, 2019.

[8] BATISTA, R.; DIÓGENES, R.; RANIERI, M.; RIBEIRO JR, E. Critical metrics of the volume functional on compact three-manifolds with smooth boundary. **Journal of Geometric Analysis**, v. 27, p. 1530–1547, 2017.

[9] BARBOZA, M.; LEANDRO, B.; PINA, R. Invariant Solutions for the Einstein Field Equation. **Journal of Mathematical Physics**, v. 59, n. 6, p. 062501, 2018.

[10] BAHUAUD, E.; GUNASEKARAN, S.; KUNDURI, K.; WOOLGAR, E. Static near-horizon geometries and rigidity of quasi-Einstein manifolds. **Letters in Mathematical Physics**, v. 112, n. 116, 2022.

[11] BAHUAUD, E.; GUNASEKARAN, S.; KUNDURI, K.; WOOLGAR, E. Rigidity of quasi-Einstein metrics: The incompressible case. **Letters in Mathematical Physics**, v. 114, n. 8, 2024.

[12] BAKRY, D.; ÉMERY, M. Diffusions Hypercontractives. Séminaire de probabilités, XIX, 1983/84. **Lecture Notes in Mathematics**, v. 1123, p. 177–206, 1985.

[13] BALLMANN, W. **Riccati equation and volume estimates**. 2016. Notas de aula. `https://people.mpim-bonn.mpg.de/hwbllmnn/notes.html`.

[14] BARROS, A.; BATISTA, R.; RIBEIRO JR, E. Bounds on volume growth of geodesic balls for Einstein warped products. **Proceedings of the American Mathematical Society**, v. 143, p. 4415–4422, 2015.

[15] BATISTA, R.; RANIERI, M.; RIBEIRO JR, E. Remarks on complete noncompact Einstein warped products. **Communications in Analysis and Geometry**, v. 28, n. 3, p. 547–563, 2020.

[16] BESSE, A. L. **Einstein manifolds**. Berlin: Springer-Verlag, 1987.

[17] BÖHM, C. Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. **Inventiones Mathematicae**, v. 134, n. 1, p. 145–176, 1998.

[18] BÖHM, C. Non-compact cohomogeneity one Einstein manifolds. **Bulletin de la Société Mathématique de France**, v. 127, p. 135–177, 1999.

[19] BORGHINI, S.; CHRUŚCIEL, P.; MAZZIERI, L. On the uniqueness of Schwarzchild-de Sitter spacetime. **Archive for Rational Mechanics and Analysis**, v. 247, n. 22, 2023.

[20] BORGHINI, S.; MAZZIERI, L. On the mass of static metrics with positive cosmological constant - I. **Classical and Quantum Gravity**, v. 35, n. 12, 2018.

[21] BORGHINI, S.; CHRUŚCIEL, P.; MAZZIERI, L. On the uniqueness of Schwarzchild-de Sitter spacetime. **Archive for Rational Mechanics and Analysis**, v. 247, n. 22, p. 1250110, 2023.

[22] BOUCHER, W.; GIBBONS, G.; HOROWITZ, G. Uniqueness theorem for anti-de Sitter spacetime. **Physical Review D.**, v. 30, n. 12, p. 2447-2451, 1984.

[23] BOUCHER, W.; GIBBONS, G. **Cosmic baldness, in The very early universe**. Cambridge: Cambridge University Press, p. 273-278, 1983.

[24] CAO, H. -D.; CHEN, B. -L.; ZHU, X. -P. Recent developments on Hamilton's Ricci flow. **Surveys in differential geometry**, v. 12, Geometric flows, Int. Press, Somerville, MA, p. 47–112, 2008.

[25] CAO, H. -D.; CHEN, Q. On Bach-flat gradient shrinking Ricci solitons. **Duke Mathematical Journal**, v. 162, p. 1149–1169, 2013.

[26] CASE, J.; SHU, Y. -S.; WEI, G. Rigidity of quasi-Einstein metrics. **Differential Geometry and its Applications**, v. 29, p. 93–100, 2011.

[27] CASE, J. Smooth metric measure spaces, quasi-Einstein metrics, and tractors. **Central European Journal of Mathematics**, v. 10, p. 1733–1762, 2012.

[28] CASE, J. Smooth metric measure spaces and quasi-Einstein metrics. **International Journal of Mathematics**, v. 23, n. 10, 2012.

[29] CATINO, G. A note on four-dimensional (anti-)self-dual quasi-Einstein manifolds. **Differential Geometry and its Applications**, v. 30, n. 6, p. 660–664, 2012.

[30] CATINO, G. Rigidity of positively curved shrinking Ricci solitons in dimension four. **Geometric Flows**, v. 4, n. 1, p. 1–8, 2019.

[31] CATINO, G.; MANTEGAZZA, C.; MAZZIERI, L.; RIMOLDI, M. Locally conformally flat quasi-Einstein manifolds. **Journal für die reine und angewandte Mathematik (Crelle's Journal)**, v. 675, p. 181–189, 2013.

[32] CHEN, Q.; HE, C. On Bach flat warped product Einstein manifold. **Pacific Journal of Mathematics**, v. 265, n. 2, p. 313–326, 2013.

[33] CORVINO, J. Scalar Curvature Deformation and a Gluing Construction for the Einstein Constraint Equations. **Communications in Mathematical Physics**, v. 214, p. 137–189, 2000.

[34] CORVINO, J.; EICHMAIR, M.; MIAO, P. Deformation of scalar curvature and volume. **Mathematische Annalen**, v. 357, p. 551-584, 2013.

[35] COSTA, J.; DIÓGENES, R.; PINHEIRO, N.; RIBEIRO JR, E. Geometry of static perfect fluid space-time. **Classical and Quantum Gravity**, v. 40, n. 20, 2023.

[36] COSTA, J.; RIBEIRO JR, E.; ZHOU, D. **Rigidity of compact quasi-Einstein manifolds with boundary**. ArXiv:2401.16929 [math.DG], 2024.

[37] COUTINHO, F.; DIÓGENES, R.; LEANDRO, B.; RIBEIRO JR, E. Static perfect fluid space-time on compact manifolds. **Classical and Quantum Gravity**, v. 37, n. 1, 2020.

[38] COUTINHO, F.; LEANDRO, B.; REIS, H. On the fluid ball conjecture. **Annals of Global Analysis and Geometry**, v. 60, p. 455–468, 2021.

[39] CHENG, X.; RIBEIRO JR, E.; ZHOU, D. Volume Growth Estimates for Ricci Solitons and Quasi-Einstein Manifolds. **Journal of Geometric Analysis**, v. 32, n. 62, 2022.

[40] CHENG, X.; ZHOU, D. Rigidity of four-dimensional gradient shrinking Ricci solitons. **Journal für die reine und angewandte Mathematik (Crelles Journal)**, v. 2023, n. 802, p. 255–274, 2023.

[41] DIÓGENES, R. **Métricas críticas do funcional volume, volume mínimo e curvatura mínima em variedades de dimensão 4**. 2015. Tese (Doutorado em Matemática) - Centro de Ciências, Universidade Federal do Ceará, Fortaleza, 2015.

[42] DIÓGENES, R.; GADELHA, T. Compact quasi-Einstein manifolds with boundary. **Mathematische Nachrichten**, v. 295, p. 1690–1708, 2022.

[43] DIÓGENES, R.; PINHEIRO, N.; RIBEIRO JR, E. **Integral and boundary estimates for critical metrics of the volume functional**. Aceito para publicação na Israel Journal of Mathematics. ArXiv:2207.12344 [math.DG], 2023.

[44] DIÓGENES, R.; GADELHA, T.; RIBEIRO JR, E. Geometry of Compact quasi-Einstein Manifolds. **Manuscripta Mathematica**, v. 169, p. 167–183, 2022.

[45] DIÓGENES, R.; GADELHA, T.; RIBEIRO Jr, E. Remarks on quasi-Einstein manifolds with boundary. **Proceedings of the American Mathematical Society**, v. 150, p. 351–363, 2022.

[46] FANG, Y.; YUAN, W. Brown-York mass and positive scalar curvature II: Besse's conjecture and related problems. **Annals of Global Analysis and Geometry**, v. 56, n. 1, p. 1–15, 2019.

[47] FELDMAN, M.; ILMANEN, T.; KNOPF, D. Rotationally symmetric shrinking and expanding gradient Kahler-Ricci solitons. **Journal of Differential Geometry**, v. 65, n. 2, p. 169–209, 2003.

[48] FERNÁNDEZ-LÓPEZ, M.; GARCÍA-RÍO, E. On gradient Ricci solitons with constant scalar curvature. **Proceedings of the American Mathematical Society**, v. 144, n. 1, p. 369–378, 2016.

[49] FERNÁNDEZ-LÓPEZ, M.; GARCÍA-RÍO, E. A remark on compact Ricci solitons. **Mathematische Annalen**, v. 340, p. 893–896, 2008.

[50] FREITAS, A. New gap results on n-dimensional static manifolds. **Mathematische Nachrichten**, v. 294, n. 9, p. 1715–1723, 2021.

[51] GADELHA, T. **Variedades quasi-Einstein compactas com bordo**. 2021. Tese (Doutorado em Matemática) - Centro de Ciências, Universidade Federal do Ceará, Fortaleza, 2021.

[52] GALLOWAY, G. J. A note on the fundamental group of a compact minimal hypersurface. **Pacific Journal of Mathematics**, v. 126, n. 2, p. 243-–251, 1987.

[53] GE, J.; TANG, Z. Geometry of isoparametric hypersurfaces in Riemannian manifolds. **Asian Journal of Mathematics**, v. 18, n. 1, p. 117–126, 2014.

[54] GE, J.; TANG, Z. Isoparametric functions and exotic spheres. **Journal für die reine und angewandte Mathematik (Crelle's Journal)**, v. 2013, n. 683, p. 161–180, 2013.

[55] GIBBONS, G. W.; HARTNOLL, S. A.; POPE, C. N. Bohm and Einstein-Sasaki metrics, black holes, and cosmological event horizons. **Physical Review D**, v. 67, n. 8, 2003.

[56] HAMILTON, R. The formation of singularities in the Ricci flow. **Surveys in Differential Geometry**, v. 2, p. 7–136, 1995.

[57] HANG, F.; WANG, X. Vanishing sectional curvature on the boundary and a conjecture of Schroeder and Strake. **Pacific Journal of Mathematics**, v. 232, n. 2, p. 283-–287, 2007.

[58] HAWKING, S. W.; ELLIS, G. F. R. **The large scale structure of spacetime**. Cambridge: Cambridge University Press, 1973.

[59] HE, C.; PETERSEN, P.; WYLIE, W. On the classification of warped product Einstein metrics. **Communications in Analysis and Geometry**. v. 20, n. 2, p. 271–311, 2012.

[60] HE, C.; PETERSEN, P.; WYLIE, W. Warped product Einstein metrics over spaces with constant scalar curvature. **Asian Journal of Mathematics**, v. 18, n. 1, p. 159–190, 2014.

[61] HIJAZI, O.; MONTIEL, S.; RAULOT, S. Uniqueness of the de Sitter space-time among static vacua with positive cosmological constant. **Annals of Global Analysis and Geometry**, v. 47, p. 167–178, 2015.

[62] IVEY, T. New examples of complete Ricci solitons. **Proceedings of the American Mathematical Society**, v. 122, n. 1, p. 241–245, 1994.

[63] KHURI, M.; WOOLGAR, E.; WYLIE, W. New restrictions on the topology of extreme black holes. **Letters in Mathematical Physics**, v. 109, p. 661—673, 2019.

[64] KIM, D. -S.; KIM, Y. H. Compact Einstein warped product spaces with nonpositive scalar curvature. **Proceedings of the American Mathematical Society**, v. 131, n. 8, p. 2573–2576, 2003.

[65] KOBAYASHI, O. A differential equation arising from scalar curvature function. **Journal of the Mathematical Society of Japan**, v. 34, n. 4, p. 665–75, 1982.

[66] KOBAYASHI, O.; OBATA, M. Conformally-flatness and static space-time. In: HANO, J. and MORIMOTO, A. and MURAKAMI, S. and OKAMOTO, K. and

OZEKI, H. (Org.). **Manifolds and Lie Groups**. Progress in Mathematics, v. 14. Boston, MA: Birkh¨auser, 1981. p. 197–206.

[67] KWONG, K.; MIAO, P. A functional inequality on the boundary of static manifolds. **Asian Journal of Mathematics**, v. 21, n. 4, p. 687–696, 2017.

[68] LAFONTAINE, J. Sur la géomérie d'une généralisation de l'équation différentielle d'Obata. **Journal de Mathématiques Pures et Appliquées**, v. 62, p. 63-–72, 1983.

[69] LEANDRO, B.; PINA, H.; RIBEIRO JR, E. Volume growth for geodesic balls of static vacuum space on 3-manifolds. **Annali di Matematica Pura ed Applicata**, v. 199, p. 863–873, 2020.

[70] LOHKAMP, J. **The higher dimensional positive mass theorem II**. ArXiv:1612.07505 [math.DG], 2016.

[71] LÜ, H; PAGE, D.; POPE, C. New inhomogeneous Einstein metrics on sphere bundles over Einstein-Kähler manifolds. **Physics Letters B**, v. 593, p. 218–226, 2004.

[72] MARTIN, D.; VISSER, M. Algorithmic construction of static perfect fluid spheres. **Physical Review D**, v. 69, n. 10, p. 104028, 2004.

[73] MASOOD-UL-ALAM, A. K. M. On spherical symmetry of static perfect fluid spacetimes and the positive-mass theorem. **Classical and Quantum Gravity**, v. 4, n. 3, p. 625–633, 1987.

[74] MASTROLIA, P.; RIMOLDI, M. Some triviality results for quasi-Einstein manifolds and Einstein warped products. **Geometriae Dedicata**, v. 169, p. 225–237, 2014.

[75] MAZZEO, R.; PACARD, F. Foliations by constant mean curvature tubes. **Communications in Analysis and Geometry**, v. 13, n. 4, p. 633–670, 2005.

[76] MIAO, P.; TAM, L. -F. Einstein and conformally flat critical metrics of the volume functional. **Transactions of the American Mathematical Society**, v. 363, n. 6, p. 2907–2937, 2011.

[77] MIAO, P.; TAM, L. -F.; XIE, N. -Q. Critical points of Wang-Yau quasi-local energy. **Annales Henri Poincaré**, v. 12, n. 5, p. 987—1017, 2011.

[78] MIAO, P.; Tam, L. -F. On the volume functional of compact manifolds with boundary with constant scalar curvature. **Calculus of Variations and Partial Differential Equations**, v. 36, p. 141–171, 2009.

[79] MIYAOKA, R. Transnormal functions on a Riemannian manifold. **Differential Geometry and its Applications**, v. 31, p. 130–139, 2013.

[80] MORREY, C. B. **Multiple Integrals in the Calculus of Variations**. Berlin: Springer-Verlag, 1996.

[81] MUNKRES, J. **Topology**. 2ª ed. England: Prentice Hall, Inc., 2000.

[82] NABER, A. Noncompact shrinking four solitons with nonnegative curvature. **Journal für die reine und angewandte Mathematik (Crelle's Journal)**, v. 2010, n. 645, p. 125–153, 2010.

[83] NI, L.; WALLACH, N. On a classification of gradient shrinking solitons. **Mathematical Research Letters**, v. 15, n. 5, p. 941–955, 2008.

[84] OLIYNYK, T. Future stability of the FLRW fluid solutions in the presence of a positive cosmological constant. **Communications in Mathematical Physics**, v. 346, p. 293-312, 2016.

[85] O'NEILL, B. **Semi-Riemannian Geometry with Applications to Relativity**. Cambridge: Cambridge Academic Press, 1983.

[86] PERELMAN, G. **Ricci flow with surgery on three manifolds**, ArXiv:math/0303109 [math.DG], 2003.

[87] PETERSEN, P.; WYLIE, W. Rigidity of gradient Ricci solitons. **Pacific Journal of Mathematics**, v. 241, p. 329–345, 2009.

[88] QIAN, Z. Estimates for weighted volumes and applications. **Quarterly Journal of Mathematics**, v. 48, n. 2, p. 235–242, 1997.

[89] QING, J.; YUAN, W. A note on vacuum static spaces and related problems. **Journal of Geometry and Physics**, v. 74 (2013) 18–27.

[90] QING, J.; YUAN, W. On scalar curvature rigidity of vacuum static spaces. Mathematische Annalen 365 no. 3-4 (2016) 1257–1277.

[91] QIU, G.; XIA, C. A generalization of Reilly's formula and its applications to a new Heintze-Karcher type inequality. **International Mathematics Research Notices**, v. 2015, n. 17, p. 7608-7619, 2015.

[92] REILLY, R. Applications of the Hessian operator in a Riemannian manifold. **Indiana University Mathematics Journal**, v. 26, n. 3, p. 459–472, 1977.

[93] REILLY, R. C. Geometric applications of the solvability of Neumann problems on a Riemannian manifold. **Archive for Rational Mechanics and Analysis**, v. 75, n. 1, p. 23–29, 1980.

[94] RIBEIRO JR, E.; TENENBLAT, K. Noncompact quasi-Einstein manifolds conformal to a Euclidean space. **Mathematische Nachrichten**, v. 294, p. 132–144, 2021.

[95] RIMOLDI, M. A remark on Einstein warped products. Pacific Journal of Mathematics 252 (2011) 207–218.

[96] RIMOLDI, M. **Rigidity results for Lichnerowicz Bakry-Emery Ricci tensors**. 2011. Tese (Doutorado em Matemática) - Universita degli Studi di Milano, Milão, 2011.

[97] ROS, A. Compact hypersurfaces with constant higher order mean curvatures. **Revista Matemática Iberoamericana**, v. 3, n. 3, p. 447–453, 1987.

[98] SCHOEN, R.; YAU, S. -T. On the proof of positive mass conjecture in general relativity. **Communications in Mathematical Physics**, v. 65, p. 45–76, 1979.

[99] SCHOEN, R.; YAU, S. -T. Proof of the positive mass theorem II. **Communications in Mathematical Physics**, v. 79, p. 231—260, 1981.

[100] SCHOEN, R.; YAU, S. -T. Positive scalar curvature and minimal hypersurface singularities. **Surveys in Differential Geometry**, v. 24, p. 441–480, 2021.

[101] SHEN, Y. A note on Fischer–Marsden's conjecture. **Proceedings of the American Mathematical Society**, v. 125, n. 3, p. 901–905, 1997.

[102] SHI, Y.; TAM, L. -F. Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. **J. Differential Geom.**, v. 62, n. 1, p. 79—125, 2002.

[103] SANTOS, J. P.; LEANDRO, B. Reduction of the n-dimensional static vacuum Einstein equation and generalized Schwarzschild solutions. **Journal of Mathematical Analysis and Applications**, v. 469, n. 2, p. 882–896, 2019.

[104] PRINCETON UNIVERSITY PRESS. The Collected Papers of Albert Einstein. Digital Einstein Papers Home. Disponível em: `https://einsteinpapers.press.princeton.edu/`. Acesso em: 24 de maio de 2023.

[105] WALD, R. M. **General Relativity**. Chicago, IL: The University of Chicago Press, 1984.

[106] WANG, L. On noncompact τ-quasi-Einstein metrics. **Pacific Journal of Mathematics**, v. 254, p. 449–464, 2011.

[107] WANG, Q. M. Isoparametric functions on Riemannian manifolds. I. **Mathematische Annalen**, v. 277, n. 4, p. 639–646, 1987.

[108] WEI, G.; WYLIE, W. Comparison geometry for the Bakry-Emery Ricci tensor. **Journal of Differential Geometry**, v. 83, p. 337–405, 2009.

[109] WEI, G.; WYLIE, W. Comparison geometry for the smooth metric measure spaces. **Proceedings of the 4th International Congress of Chinese Mathematicians**, v. 2, p. 191–202, 2007.

[110] WYLIE, W. Rigidity of compact static near-horizon geometries with negative cosmological constant. **Letters in Mathematical Physics**, v. 113, n. 29, 2023.

[111] WYLIE, W. Complete shrinking Ricci solitons have finite fundamental group. **Proceedings of the American Mathematical Society**, v. 136, n. 5, p. 1803–1806, 2008.

[112] WYMAN, M. Radially Symmetric Distributions of Matter. **Physical Review**, v. 75, n. 12, 1949.

[113] XIA, C. A Minkowski type inequality in space forms. **Calculus of Variations and Partial Differential Equations**, v. 55, n. 96, p. 55–96, 2016.

[114] YANO, K. **Integral Formulas in Riemannian Geometry**. New York: Dekker, 1970.

[115] YUAN, W. Brown-York mass and positive scalar curvature I - First eigenvalue problem and its applications. **Proceedings of the American Mathematical Society**, v. 151, p. 313–326, 2023.

[116] YUAN, W. Volume comparison with respect to scalar curvature. **Analysis & Partial Differential Equations**, v. 16, n. 1, p. 1–34, 2023.