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ON CONCENTRATED LOADS AT THE BOUNDARY OF A PIEZOELECTRIC HALF-PLANE

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Abstract

Two fundamental problems of the linear theory of piezoelectricity are studied from a theoretical point of view, namely, the cases of a point force and a point charge acting on a piezoelectric half-plane. Exact expressions for all relevant electroelastic variables are obtained through a state space methodology in conjunction with the Fourier transform. Numerical results are also provided to illustrate both qualitative and quantitative behavior of the induced electromechanical fields.

1. Introduction

In this paper we are concerned with the theoretical analysis of a piezoelectric half-plane subjected to either a point force or point charge. While the study is primarily devoted to crystals of the 6 *mm* class and poled ferroelectric ceramics, a larger class of piezoelectric crystals and man-made piezoelectric materials can be accommodated within the present formulation. Moreover, solutions to equivalent problems involving non-electrified anisotropic elastic bodies or rigid dielectrics can be obtained as particular cases.

Recall that a piezoelectric material is one that deforms when subjected to an electric field and, conversely, it induces electric charge when subjected to pressure. Furthermore, the piezoelectric effect can be manifested only in materials lacking a center of symmetry. Thus, this effect is found in 20 different crystal classes possessing various degrees of material and electrical anisotropy, such as quartz, lithium niobate, cadmium sulfide, gallium arsenide, Rochelle salt, barium titanate and lead titanate. Piezoelectricity can also be induced in isotropic bulk materials like ceramics and polymers through the application of a strong DC electric field. As a result of this process, known as poling, the material becomes transversely isotropic with the symmetry axis being parallel to the direction of the poling field.

The range of applications involving piezoelectric crystals and ceramics is quite wide. Typical examples are electromechanical transducers, delay lines, medical instrumentation, detonation devices, sonar projectors, microelectronic components and "smart structures". In most of these applications the crystal or ceramic is exposed to severe mechanical and electrical loading conditions which may result into structural failure or dielectric breakdown. Proof that these problems are growing in importance is given by the appearance in the last few years of articles dealing with the effect of defects in piezoelectric and ferroelectric ceramics. Among others, we can mention the works of McMeeking (1990), Pak (1990), Shindo *et al.* (1990), Sosa (1991), Suo *et al.* (1992), Suo (1993) and references provided therein.

Although in the present article we do not deal with defects, phenomena like mechanical failure and electrical breakdown also arise due to the nature of the applied loads. A natural example is that of conducting surfaces with sharp points, where it is known that surface charge density and the corresponding electric field are inversely proportional to the radius of the conductor's tip. The case of the point charge in this article is a simplified model to such phenomenon but useful as a first approximation to predict, for example, the behavior of the elastic fields in the neighborhood of an electrode's tip. Thus this paper and the articles mentioned above share a common objective: to gain a better understanding of the behavior of dielectric materials in the presence of electroelastic raisers.

The solution to boundary-value problems involving piezoelectric materials is extremely difficult from a mathematical point of view because of material anisotropy and electroelastic coupling effects. As has been the case in many other branches of continuum mechanics, mathematical difficulties can be circumvented by resorting to a numerical approach. While it is clear that realistic configurations involving the piezoelectric effect will call for numerical solutions, it is also true that any good algorithm should be tested against exact solutions to some important problems. To the best of our knowledge these solutions are not available. Hence this paper appears to be the first attempt to provide exact results to the two fundamental problems cited before.

Although the present analysis is simplified by being restricted to two dimensions, it still preserves a reasonable degree of generality by retaining elastic and electric anisotropy as well as full electromechanical coupling. The solution to the problem of the half-plane subjected to a generalized point load of mechanical or electrical nature is obtained by means of a state space approach. This particular methodology, which has its roots in classical dynamics and modern control theory, has not been fully exploited in continuum mechanics. Important contributions of the state space approach to problems in elasticity and coupled thermoelasticity were given by Bahar (1972, 1975), Bahar and Hetnarski (1978) and Sosa and Behar (1992). The application of the state space methodology within the framework of electroelasticity is due to Sosa (1992), where the emphasis was placed on a general mathematical formulation towards the solution of problems in laminated structures. For the sake of completeness we repeat briefly the salient steps of this formulation. Our main objective, however, is to show how the state space approach together with the use of Fourier transforms leads to a compact and efficient description of a certain class of electromechanical problems. Typically, problems in plane elasticity are constrained to finding stresses. If, additionally, the displacements are sought, one must integrate the constitutive relations. This last step is not always a simple one. An interesting feature of the present formulation is that it allows us to obtain stresses and displacements at the same time. The same is true for the electrical variables: electric displacement and electric potential stand on equal footing. An important corollary of our results is that, although tempting for purposes of simplification, neglecting coupling effects may lead to erroneous conclusions. To illustrate this point we make a comparison of the electroelastic solution for the point force with its counterpart of non-electric anisotropic elasticity.

The paper is organized as follows: in Section 2 we introduce the general equations governing the linear theory of piezoelectricity, which are later reduced to the particular case of plane strain conditions. In Section 3 we provide a brief outline of the state space methodology to attack problems of electroelasticity, while in Section 4 we present the solutions to the fundamental problems described previously. Finally, in Section 5 we show the behavior of certain elastic and electrical variables for a particular class of material.

2. Mathematical Formulation of Piezoelectricity

The description of piezoelectricity is based on the combination of elements of elasticity and electrodynamics, where the basic variables are the stress $\boldsymbol{\sigma}$, the strain $\boldsymbol{\varepsilon}$, the electric displacement $\mathbf{D}$, and the electric field $\mathbf{E}$. The present paper is constrained to static problems where we also neglect body forces and electric charge density. Under these circumstances the field equations are given by

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \quad \operatorname{div} \mathbf{D} = 0 \tag{2.1}$$

where the second equation is Gauss's law of electrostatics. In theoretical studies it is usually necessary to introduce, in addition to the aforementioned variables, the elastic displacement $\mathbf{u}$ and the electric potential ϕ through the relations

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}), \quad \mathbf{E} = -\nabla \phi \tag{2.2}$$

where the second expression implies the assumption of the quasi-electrostatic approximation, that is $\nabla \times \mathbf{E} = \mathbf{0}$.

Formulae (2.1) and (2.2) constitute a system of 13 relations for 22 unknowns. The additional nine equations are provided by the constitutive relations reflecting electroelastic interactions. These relations can be derived formally using thermodynamic potentials and can be arranged in four different manners, depending on which basic variables are chosen as independent in the thermodynamic potential. If these are the strain and the electric field, the constitutive equations take the form (Parton and Kudryavtsev, 1988)

$$\boldsymbol{\sigma} = \mathbf{C}^{\mathrm{E}} \boldsymbol{\varepsilon} - \mathbf{e}^{\mathrm{T}} \mathbf{E}$$

$$\mathbf{D} = \mathbf{e} \boldsymbol{\varepsilon} + \boldsymbol{\epsilon}^{\varepsilon} \mathbf{E} \tag{2.3}$$

where $\mathbf{C}^{\mathrm{E}}$ is the fourth-order tensor of elastic moduli (measured at constant or zero electric field), $\mathbf{e}$ is the third-order piezoelectric tensor, $\boldsymbol{\epsilon}^{\varepsilon}$ is the second-order dielectric tensor (measured at constant or zero strain), and the superscript T denotes the transpose of a tensor. Notice that if the piezoelectric effect is absent or neglected, one can set $\mathbf{e} = \mathbf{0}$ in (2.3) which reduces the first expression to the well-known generalized Hooke's law and the second relation to the constitutive description of rigid dielectrics.

Substitution of (2.2) into (2.3) and further use of (2.1) yields a system of four partial differential equations coupling the three components of $\mathbf{u}$ with the potential ϕ. Such a system of equations, which must be furnished with mechanical and electrical boundary conditions is, in general, very difficult to solve in an exact manner. A substantial simplification can be achieved, however, if the mathematical description is based on a two-dimensional model. This article is constrained to the half-plane $-\infty < x < \infty$, $z \geqslant 0$ where the z axis coincides with the six fold axis of symmetry in the case of a 6 *mm* crystal class, or with the poling axis in the case of poled ferroelectric ceramics. Plane strain conditions are assumed in this plane, that is $\varepsilon_{yy} = \varepsilon_{yx} = \varepsilon_{yz} = 0$. From the electrical point of view we also assume that $E_y = 0$. Accordingly, (2.1) reduces to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{2.4}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0. \tag{2.5}$$

Moreover, the strain–displacement and electric field–electric potential relations become

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad 2\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{2.6}$$

$$E_x = -\frac{\partial \phi}{\partial x}, \quad E_z = -\frac{\partial \phi}{\partial z} \tag{2.7}$$

where u and w are the components of the elastic displacement in the x and z directions, respectively. Finally, the constitutive relations can be written in matrix form as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix} = \begin{Vmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{44} \end{Vmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \end{Bmatrix} - \begin{Vmatrix} 0 & e_{31} \\ 0 & e_{33} \\ e_{15} & 0 \end{Vmatrix} \begin{Bmatrix} E_x \\ E_z \end{Bmatrix} \tag{2.8}$$

$$\begin{Bmatrix} D_x \\ D_z \end{Bmatrix} = \begin{Vmatrix} 0 & 0 & e_{15} \\ e_{31} & e_{33} & 0 \end{Vmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ 2\varepsilon_{xz} \end{Bmatrix} + \begin{Vmatrix} \epsilon_{11} & 0 \\ 0 & \epsilon_{33} \end{Vmatrix} \begin{Bmatrix} E_x \\ E_z \end{Bmatrix} \tag{2.9}$$

where for convenience we have dropped the superscripts in the constants $\mathbf{C}$ and $\boldsymbol{\epsilon}$. The structure of (2.8) and (2.9) is not exclusive of the crystals and ceramics described before. For example, identical structures for the material matrices are obtained (when reduced to two dimensions) for some crystals belonging to the orthorhombic, tetragonal and hexagonal systems.

Equations (2.4)–(2.9) can now be combined to yield three coupled partial differential equations to be solved for u, w and ϕ. While simpler in structure compared to its three-dimensional counterpart, this system of equations is still difficult to solve in closed form. In the next section we show that the state space approach is a powerful tool to provide a simple and systematic solution to a certain class of problems.

3. State Space Formulation for Electroelasticity

The basic idea behind the state space formulation is to describe a given physical system in terms of the minimum possible number of variables. In the particular case under consideration this can be achieved by eliminating σ_{xx} and D_x from (2.4)–(2.9) which results into a new system of partial differential equations without mixed derivatives, that is

$$
\begin{aligned}
\frac{\partial u}{\partial z} &= -\frac{\partial w}{\partial x}+\frac{1}{C_{44}}\sigma_{zx}-\frac{e_{15}}{C_{44}}\frac{\partial \phi}{\partial x}\\
\frac{\partial w}{\partial z} &= -\frac{\alpha}{\gamma}\frac{\partial u}{\partial x}+\frac{\epsilon_{33}}{\gamma}\sigma_{zz}+\frac{e_{33}}{\gamma}D_z\\
\frac{\partial \sigma_{zz}}{\partial z} &= -\frac{\partial \sigma_{zx}}{\partial x}\\
\frac{\partial \sigma_{zx}}{\partial z} &= \left(\frac{C_{13}^2}{C_{33}}-C_{11}-\frac{\beta^2}{\gamma C_{33}}\right)\frac{\partial^2 u}{\partial x^2}+\left(\frac{\beta e_{33}}{\gamma C_{33}}-\frac{C_{13}}{C_{33}}\right)\frac{\partial \sigma_{zz}}{\partial x}-\frac{\beta}{\gamma}\frac{\partial D_z}{\partial x}\\
\frac{\partial \phi}{\partial z} &= -\frac{\beta}{\gamma}\frac{\partial u}{\partial x}+\frac{e_{33}}{\gamma}\sigma_{zz}-\frac{C_{33}}{\gamma}D_z\\
\frac{\partial D_z}{\partial z} &= -\frac{e_{15}}{C_{44}}\frac{\partial \sigma_{zx}}{\partial x}+\frac{\kappa}{C_{44}}\frac{\partial^2 \phi}{\partial x^2}
\end{aligned}
\tag{3.1}
$$

where

$$\alpha = C_{13}\epsilon_{33}+e_{31}e_{33},\quad \beta = C_{13}e_{33}-C_{33}e_{31},\quad \gamma = C_{33}\epsilon_{33}+e_{33}^2,\quad \kappa = e_{15}^2+C_{44}\epsilon_{11}.$$

The solutions to boundary-value problems of elasticity and electrostatics can be approached independently by the methodology described in this section by simply setting $e_{31} = e_{33} = e_{15} = 0$ in (3.1). Since the coupling phenomenon is the main issue of the paper, we shall not pursue these particular cases any further.

The next step towards mathematical simplification consists of using the Fourier transform to reduce (3.1) to a system of ordinary differential equations. Letting the Fourier transform of a function $f(x)$ be defined as

$$FT\{f(x)\} = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx \tag{3.2}$$

and, furthermore, assuming that u, $\partial u/\partial x$, w, σ_{zz}, σ_{xz}, ϕ, $\partial\phi/\partial x$, and D_z tend to zero as $|x| \to \infty$, (3.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{Bmatrix} \hat{u} \\ \hat{w} \\ \hat{\sigma}_{zz} \\ \hat{\sigma}_{xz} \\ \hat{\phi} \\ \hat{D}_z \end{Bmatrix} = \begin{Vmatrix} 0 & \mathrm{i}\xi & 0 & a_{14} & a_{15}\mathrm{i}\xi & 0 \\ a_{21}\mathrm{i}\xi & 0 & a_{23} & 0 & 0 & a_{26} \\ 0 & 0 & 0 & \mathrm{i}\xi & 0 & 0 \\ a_{41}\xi^2 & 0 & a_{43}\mathrm{i}\xi & 0 & 0 & a_{46}\mathrm{i}\xi \\ a_{51}\mathrm{i}\xi & 0 & a_{53} & 0 & 0 & a_{56} \\ 0 & 0 & 0 & a_{64}\mathrm{i}\xi & a_{65}\xi^2 & 0 \end{Vmatrix} \begin{Bmatrix} \hat{u} \\ \hat{w} \\ \hat{\sigma}_{zz} \\ \hat{\sigma}_{xz} \\ \hat{\phi} \\ \hat{D}_z \end{Bmatrix} \tag{3.3}$$

where the coefficients a_{ij} are given by

$$a_{14} = \frac{1}{C_{44}}, \quad a_{15} = \frac{e_{15}}{C_{44}}, \quad a_{21} = \frac{\alpha}{\gamma}, \quad a_{23} = \frac{\epsilon_{33}}{\gamma}, \quad a_{26} = \frac{e_{33}}{\gamma}$$

$$a_{41} = C_{11} + \frac{\beta^2}{\gamma C_{33}} - \frac{C_{13}^2}{C_{33}}, \quad a_{43} = \frac{C_{13}}{C_{33}} - \frac{\beta e_{33}}{\gamma C_{33}}, \quad a_{46} = \frac{\beta}{\gamma}$$

$$a_{51} = \frac{\beta}{\gamma}, \quad a_{53} = \frac{e_{33}}{\gamma}, \quad a_{56} = -\frac{C_{33}}{\gamma}, \quad a_{64} = \frac{e_{15}}{C_{44}}, \quad a_{65} = -\frac{\kappa}{C_{44}}. \tag{3.4}$$

The column matrices appearing in (3.3) suggest that we collect the six independent electromechanical variables into a state vector, namely $\mathbf{S}(x,z) = \{u, w, \sigma_{zz}, \sigma_{xz}, \phi, D_z\}^{\mathrm{T}}$ which after use of the Fourier transform becomes $\hat{\mathbf{S}}(\xi,z) = \{\hat{u}, \hat{w}, \hat{\sigma}_{zz}, \hat{\sigma}_{xz}, \hat{\phi}, \hat{D}_z\}^{\mathrm{T}}$. As a result, we can now write (3.3) compactly as

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}z}(\xi,z) = \mathbf{A}(\xi)\hat{\mathbf{S}}(\xi,z) \tag{3.5}$$

where $\mathbf{A}$ is the 6×6 matrix appearing in (3.3) whose only feature is having zeros in its main diagonal. The solution to (3.5) is easily found to be

$$\hat{\mathbf{S}}(\xi,z) = \exp[z\mathbf{A}(\xi)]\hat{\mathbf{S}}(\xi,0) \tag{3.6}$$

where the exponential matrix $\exp[z\mathbf{A}(\xi)]$ is the transfer matrix that maps the "initial" transformed state vector $\hat{\mathbf{S}}(\xi,0)$ into the field. As shown by SOSA (1992), an explicit expression for the transfer matrix can be found by means of series expansions and use of the Cayley–Hamilton theorem. Thus

$$\exp[z\mathbf{A}] \overset{\mathrm{def}}{=} \mathbf{B}(\xi,z) = a_0\mathbf{1} + a_1\mathbf{A} + a_2\mathbf{A}^2 + a_3\mathbf{A}^3 + a_4\mathbf{A}^4 + a_5\mathbf{A}^5 \tag{3.7}$$

where the coefficients $a_0,\ldots,a_5$ are expressed in terms of the eigenvalues λ_k ($k = 1,\ldots,6$) associated with the matrix $\mathbf{A}$. We can show that these coefficients are given by (SOSA, 1992)

$$a_j = \frac{1}{2}\sum_{k=1}^{3} \Lambda_{jk}[\mathrm{e}^{\lambda_k z} + (-1)^j \mathrm{e}^{-\lambda_k z}], \quad j = 0,\ldots,5 \tag{3.8}$$

where the Λ_{jk} are also expressed in terms of the eigenvalues, namely

$$\Lambda_{0k} = \prod_{j=k+1}^{k+2} \frac{\lambda_j^2}{d_k}, \quad \Lambda_{1k} = \prod_{j=k+1}^{k+2} \frac{\lambda_j^2}{\lambda_k d_k}, \quad \Lambda_{2k} = \sum_{j=k+1}^{k+2} \frac{\lambda_j^2}{d_k},$$

$$\Lambda_{3k} = \sum_{j=k+1}^{k+2} \frac{\lambda_j^2}{\lambda_k d_k}, \quad \Lambda_{4k} = \frac{1}{d_k}, \quad \Lambda_{5k} = \frac{1}{\lambda_k d_k},$$

$$d_k = (\lambda_{k+1}^2 - \lambda_k^2)(\lambda_{k+2}^2 - \lambda_k^2), \quad k = 1,2,3. \tag{3.9}$$

From the previous expressions it should be clear that the most important step towards the complete characterization of the transfer matrix **B** is the determination of the eigenvalues of the matrix **A**, which satisfy the characteristic equation

$$\lambda^6 + p\xi^2\lambda^4 + q\xi^4\lambda^2 + r\xi^6 = 0 \tag{3.10}$$

where the constants p, q and r are given by

$$p = \frac{1}{\gamma C_{44}}\left[\alpha C_{44} + \frac{\beta}{C_{33}}(2e_{15}C_{33} - e_{33}C_{44} - \beta) + \frac{\gamma}{C_{33}}(C_{13}C_{44} + C_{13}^2 - C_{11}C_{33}) - \kappa C_{33}\right]$$

$$q = \frac{1}{\gamma C_{44}}\left[\frac{(C_{11}C_{33} - C_{13}^2)}{C_{33}}\left(C_{44}\epsilon_{33} + 2e_{15}e_{33} - \frac{e_{15}^2 C_{33}}{C_{44}}\right) + \frac{\kappa}{C_{44}}(C_{11}C_{33} - C_{13}^2 - C_{13}C_{44})\right.$$

$$+ \frac{C_{13}}{C_{33}}(\alpha C_{44} + \beta e_{15}) - \frac{\kappa}{\gamma}(\alpha C_{33} + \beta e_{33}) + \frac{\alpha\beta}{\gamma}\left(e_{15} - \frac{e_{33}C_{44}}{C_{33}}\right)$$

$$\left. + \frac{\beta^2}{\gamma C_{33}}(\epsilon_{33}C_{44} + e_{15}e_{33})\right]$$

$$r = \frac{\kappa}{\gamma^2 C_{44}}\left[(C_{13}^2 - C_{11}C_{33})\left(\epsilon_{33} + \frac{e_{33}^2}{C_{33}}\right) - \alpha C_{13} - \frac{\beta e_{33}C_{13}}{C_{33}}\right]. \tag{3.11}$$

The structure of (3.10) allows us to find its roots in closed form. Moreover, it turns out that for every physically admissible transversely isotropic piezoelectric material the eigenvalues can always be expressed in the following form [see also Parton and Kudryavtsev (1988) for similar results in a different setting]:

$$\lambda_{1,4} = \pm a|\xi|, \quad \lambda_{2,5} = \pm(b + \mathrm{i}c)|\xi|, \quad \lambda_{3,6} = \pm(b - \mathrm{i}c)|\xi| \tag{3.12}$$

where a, b and c are real numbers depending on the components of **C**, **e** and ϵ.

Knowledge of the eigenvalues and, therefore, of the coefficients a_j, together with the various powers of **A** provides the complete determination of the exponential matrix. Hence the solution in the transformed domain becomes

$$\hat{\mathbf{S}}(\xi, z) = \mathbf{B}(\xi, z)\hat{\mathbf{S}}(\xi, 0) \tag{3.13}$$

where it must be kept in mind that **B** depends on the material properties. This means that the transfer matrix is calculated once and for all for a particular material while different boundary-value problems will require the determination of the initial state vector. Finally, (3.13) must be inverted to find the physical variables. Finding the inverse Fourier transform of (3.13) depends strongly on the problem under consideration. In the next section we show that for the two fundamental problems of this article the inversions can be done in closed form. To conclude this section we want to point out that some interesting properties of the matrix **B** were given by SOSA (1992), among which the most important is that its determinant is equal to unity.

4. GENERALIZED POINT LOAD ON THE PIEZOELECTRIC HALF-PLANE

The response of the piezoelectric half-plane subjected on its boundary to a compressive point force P or a positive point charge Q can be studied in a unified manner by introducing a generalized point load G, which can be taken independently as of either mechanical or electrical nature. As mentioned previously, for a given material the initial state vector must be fully determined for each boundary-value problem. Within the realm of piezoelectricity, elastic and electric boundary conditions must be prescribed at $z = 0$. For the two problems under consideration it is convenient to specify the traction and the normal component of electric displacement (associated with surface charge). Hence the boundary conditions in the physical and transformed domain when $G = P$ are given by

$$\begin{aligned} \sigma_{zz}(x,0) &= -P\delta(x) \Rightarrow \hat{\sigma}_{zz}(\xi,0) = -P \\ \sigma_{xz}(x,0) &= 0 \Rightarrow \hat{\sigma}_{xz}(\xi,0) = 0 \\ D_z(x,0) &= 0 \Rightarrow \hat{D}_z(\xi,0) = 0 \end{aligned} \tag{4.1}$$

and when $G = Q$ they become

$$\begin{aligned} \sigma_{zz}(x,0) &= 0 \Rightarrow \hat{\sigma}_{zz}(\xi,0) = 0 \\ \sigma_{xz}(x,0) &= 0 \Rightarrow \hat{\sigma}_{xz}(\xi,0) = 0 \\ D_z(x,0) &= Q\delta(x) \Rightarrow \hat{D}_z(\xi,0) = Q. \end{aligned} \tag{4.2}$$

In (4.1) and (4.2), P and Q are force and charge per unit of length, respectively, while $\delta(x)$ is the Dirac-delta function. Since $\hat{\mathbf{S}}(\xi,0)$ has six components and the previous equations provide only three conditions, additional relations must be established to obtain the rest of the components of the state vector on the boundary. For the half-plane it is natural to require that the state vector be bounded for large values of the depth z, that is

$$\hat{\mathbf{S}}(\xi,z) \to 0 \quad \text{as} \quad z \to \infty. \tag{4.3}$$

Making use of (3.13) together with (4.1), (4.2) and (4.3) yields

$$\hat{u}(\xi,0) = k_1\frac{\mathrm{i}G}{\xi}, \quad \hat{w}(\xi,0) = k_2\frac{G}{|\xi|}, \quad \hat{\phi}(\xi,0) = k_3\frac{G}{|\xi|} \tag{4.4}$$

where k_1, k_2 and k_3 are real numbers depending on the material properties and the nature of the applied load. Their magnitudes and units for different materials and loads are given in Tables 1 and 2 in the Appendix. It is interesting to note that if

$G = P$ and the solid is not of piezoelectric nature but anisotropic, then $k_3 = 0$. If, in addition, the solid is isotropic, then the constants k_1 and k_2 are expressed in terms of the modulus of elasticity E and Poisson's ratio ν, namely, $k_1 = -(1-\nu)/E$ and $k_2 = 2/E$. Once $\hat{\mathbf{S}}(\xi, 0)$ is known in its entirety, use of (3.13) determines the transformed state vector anywhere in the medium. The calculations yielding the electromechanical variables are lengthy but straightforward and can be performed with a symbolic manipulator computer program like *Mathematica*. It is found that

$$\hat{u}(\xi, z) = \frac{G\mathrm{i}}{\xi} F_1, \quad \hat{w}(\xi, z) = \frac{G}{|\xi|} F_2, \quad \hat{\sigma}_{zz}(\xi, z) = GF_3, \quad \hat{\sigma}_{xz}(\xi, z) = \frac{G\xi\mathrm{i}}{|\xi|} F_4,$$

$$\hat{\phi}(\xi, z) = \frac{G}{|\xi|} F_5, \quad \hat{D}_z(\xi, z) = GF_6,$$

$$F_k = \Phi_k\, \mathrm{e}^{-a|\xi|z} + [\Omega_k \cos(c|\xi|z) + \Psi_k \sin(c|\xi|z)]\, \mathrm{e}^{-b|\xi|z} \tag{4.5}$$

where Φ_k, Ω_k and Ψ_k ($k = 1, \ldots, 6$) are real constants reflecting the electric and elastic characteristics of the material. Their actual expressions are quite lengthy and are omitted because they do not introduce any relevant information in the final results. Their values and corresponding units (according to the nature of the load) for three different materials are also provided in Tables 1 and 2. To obtain the physical components of the state vector we must invert (4.5) using the inverse of (3.2). Since the mathematical structures of $\hat{w}$ and $\hat{\phi}$ on one hand, and of $\hat{\sigma}_{zz}$ and $\hat{D}_z$ on the other, are exactly the same except for their coefficients, only the inverse Fourier transforms of four variables need to be found. Thus, multiplying each of the relations in (4.5) by $\mathrm{e}^{-\mathrm{i}\xi x}/2\pi$ and integrating between $\mp\infty$ yields

$$u(x, z) = \frac{G}{\pi}\left[\Phi_1 \int_0^\infty \frac{\sin(\xi x)}{\xi}\, \mathrm{e}^{-a\xi z}\, \mathrm{d}\xi + \Omega_1 \int_0^\infty \frac{\cos(c\xi z)\sin(\xi x)}{\xi}\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi + \Psi_1 \int_0^\infty \frac{\sin(c\xi z)\sin(\xi x)}{\xi}\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi\right] \tag{4.6}$$

$$w(x, z) = \frac{G}{\pi}\left[\Phi_2 \int_0^\infty \frac{\cos(\xi x)}{\xi}\, \mathrm{e}^{-a\xi z}\, \mathrm{d}\xi + \Omega_2 \int_0^\infty \frac{\cos(c\xi z)\cos(\xi x)}{\xi}\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi + \Psi_2 \int_0^\infty \frac{\sin(c\xi z)\cos(\xi x)}{\xi}\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi\right] \tag{4.7}$$

$$\sigma_{zz}(x, z) = \frac{G}{\pi}\left[\Phi_3 \int_0^\infty \cos(\xi x)\, \mathrm{e}^{-a\xi z}\, \mathrm{d}\xi + \Omega_3 \int_0^\infty \cos(c\xi z)\cos(\xi x)\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi + \Psi_3 \int_0^\infty \sin(c\xi z)\cos(\xi x)\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi\right] \tag{4.8}$$

$$\sigma_{xz}(x, z) = \frac{G}{\pi}\left[\Phi_4 \int_0^\infty \sin(\xi x)\, \mathrm{e}^{-a\xi z}\, \mathrm{d}\xi + \Omega_4 \int_0^\infty \cos(c\xi z)\sin(\xi x)\, \mathrm{e}^{-b\xi z}\, \mathrm{d}\xi\right.$$

$$+\Psi_4 \int_0^\infty \sin(c\xi z)\sin(\xi x)\,e^{-b\xi z}\,d\xi\Bigg]. \quad (4.9)$$

The expression for $\phi(x, z)$ is identical to the expression for $w(x, z)$ with new coefficients Φ_5, Ω_5 and Ψ_5. Similarly, the expression for $D_z(x, z)$ is identical to the expression for σ_{zz} but with coefficients Φ_6, Ω_6 and Ψ_6. We note that the integrals appearing in (4.6)–(4.9) represent the Laplace transform of trigonometric functions which can be solved in a simple manner with the exception of two of them. Indeed, the integrals in the first two terms of the expression for w (and ϕ) are divergent and, therefore, cannot be found in standard mathematical tables. It can be shown, however, that these integrals do converge in the sense of generalized functions or distributions. From ZEMANIAN (1965) we find that

$$\int_0^\infty \frac{\cos\xi x}{\xi}\,e^{-a\xi z}\,d\xi = -\tfrac{1}{2}\ln(x^2+a^2z^2)$$

$$\int_0^\infty \frac{\cos(c\xi z)\cos(\xi x)}{\xi}\,e^{-b\xi z}\,d\xi = -\tfrac{1}{4}\ln[x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4]. \quad (4.10)$$

In passing we note that SOSA and BAHAR (1992) have shown that the same solutions to the divergent integrals can be obtained using classical methods of differentiation and integration.

Using the results provided by (4.10) together with the solutions corresponding to the rest of the integrals appearing in (4.6)–(4.9) we obtain the final expressions for the electromechanical fields. The elastic variables are given by

$$u(x,z) = \frac{G}{\pi}\left\{\Phi_1\tan^{-1}\left(\frac{x}{az}\right)+\frac{\Omega_1}{2}\left[\tan^{-1}\left(\frac{x+cz}{bz}\right)+\tan^{-1}\left(\frac{x-cz}{bz}\right)\right]\right.$$
$$\left.+\frac{\Psi_1}{4}\ln\left[\frac{b^2z^2+(x+cz)^2}{b^2z^2+(x-cz)^2}\right]\right\} \quad (4.11)$$

$$w(x,z) = \frac{G}{\pi}\left\{-\frac{\Phi_2}{2}\ln(x^2+a^2z^2)-\frac{\Omega_2}{4}\ln[x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4]\right.$$
$$\left.+\frac{\Psi_2}{2}\tan^{-1}\left[\frac{2bcz^2}{x^2+(b^2-c^2)z^2}\right]\right\} \quad (4.12)$$

$$\sigma_{zz}(x,z) = \frac{G}{\pi}\left[\Phi_3\left(\frac{az}{x^2+a^2z^2}\right)+\frac{(\Omega_3 b-\Psi_3 c)x^2z+(\Omega_3 b+\Psi_3 c)(b^2+c^2)z^3}{x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4}\right] \quad (4.13)$$

$$\sigma_{xz}(x,z) = \frac{G}{\pi}\left\{\Phi_4\frac{x}{x^2+a^2z^2}+\frac{\Omega_4x^3+[\Omega_4(b^2-c^2)+2\Psi_4bc]xz^2}{x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4}\right\}. \quad (4.14)$$

The electrical variables are given by

$$\phi(x,z) = \frac{G}{\pi}\left\{-\frac{\Phi_5}{2}\ln(x^2+a^2z^2)-\frac{\Omega_5}{4}\ln[x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4]\right.$$

$$\left.+\frac{\Psi_5}{2}\tan^{-1}\left[\frac{2bcz^2}{x^2+(b^2-c^2)z^2}\right]\right\} \quad (4.15)$$

$$D_z(x,z) = \frac{G}{\pi}\left[\Phi_6\frac{az}{x^2+a^2z^2}+\frac{(\Omega_6 b-\Psi_6 c)x^2z+(\Omega_6 b+\Psi_6 c)(b^2+c^2)z^3}{x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4}\right]. \quad (4.16)$$

The expressions for the variables σ_{xx} and D_x can be easily obtained using (2.8) and (2.9), a result that is omitted.

From a practical point of view, knowledge of the electric field behavior is usually more important than knowledge of the electric displacement. The component E_z is obtained using (2.7) and (4.15) which gives

$$E_z(x,z) = \frac{G}{\pi}\left\{\Phi_5\frac{a^2z}{x^2+a^2z^2}+\frac{[\Omega_5(b^2-c^2)-2\Psi_5 bc]x^2z+\Omega_5(b^2+c^2)^2z^3}{x^4+2(b^2-c^2)x^2z^2+(b^2+c^2)^2z^4}\right\}. \quad (4.17)$$

Equations (4.11)–(4.17) deserve a few comments. First, the electrical counterparts of the mechanical variables $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\mathbf{u}$ are $\mathbf{D}$, $\mathbf{E}$ and ϕ, respectively. Thus, it is not surprising that, apart from their coefficients, w and ϕ, on one hand, and σ_{zz} and D_z on the other, have exactly the same mathematical representation.

Second, while the results show that mathematically the electroelastic variables are independent of the nature of the load, physically this is not true. For example, the normal stress induced by a point force is different from the stress induced by a point charge from both a qualitative and quantitative point of view. This is so because the nature of the load G is also implicitly involved in the coefficients Φ_k, Ω_k and Ψ_k. To emphasize this point, examples showing the behavior of some of the variables under both kinds of loads are given in the next section.

Third, the expressions for the mechanical variables show some similarities with their counterparts of elasticity. For example, the stresses are given in terms of rational functions of polynomials, although with a more complicated structure. The same can be said of the displacement components which, as in the elastic case (LOVE, 1927), become unbounded for large values of the depth coordinate. In this case, this phenomenon is also displayed by the electric potential.

The mathematical difficulties associated with the solution to piezoelectric boundary-value problems makes almost inevitable the following question: can coupling effects be neglected? If the answer is affirmative, the solution to the electroelastic problem will follow steps very similar to those encountered in the theory of uncoupled thermoelasticity. In the remainder of this section we shall show that the results provided by the coupled and uncoupled theories are quite different at least in their mathematical structure. In the next section we reinforce the differences by a numerical example. To illustrate our point we consider the behavior of the normal stress σ_{zz} when a compressive point force is applied to the piezoelectric half-plane. The result using the coupled theory is given by (4.13) with $G = P$. If coupling effects are to be neglected, we must simply set $e_{31} = e_{33} = e_{15} = 0$ in (3.1) leading to a formulation based on a

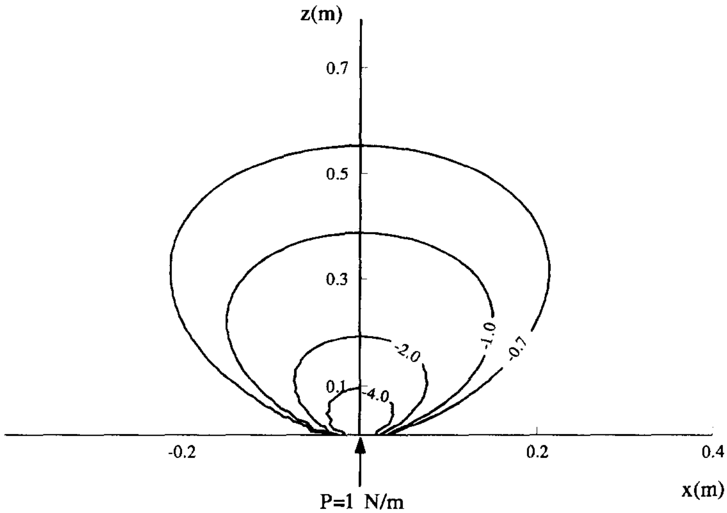

FIG. 1. Contours of σ_{zz} (N m^{-2}) induced by a point force.

4×4 matrix **A**. That is, the problem reduces to one of anisotropic elasticity. We omit the details of these calculations, but we can show that in this case the stress is given by

$$\sigma^{\mathrm{E}}_{zz} = \frac{-P\delta_1 z^3}{\pi(x^4 + \delta_2 x^2 z^2 + \delta_3 z^4)} \tag{4.18}$$

where δ_1, δ_2 and δ_3 are functions of **C** only. Equation (4.18) can also be obtained from LEKHNITSKII (1981) after recasting his expressions into Cartesian coordinates. In passing we note that if the material is isotropic, $\delta_1 = 1$, $\delta_2 = 2$, $\delta_3 = 1$, whence the previous equation reduces to the well-known Flamant solution, namely

$$\sigma^{\mathrm{E}}_{zz} = \frac{-Pz^3}{\pi(x^2 + z^2)^2}. \tag{4.19}$$

Comparison of (4.13) with (4.18) reveals two additional terms in the electroelastic solution, one in z and the other in $x^2 z$. These terms are of the same order of magnitude as the term in z^3 since they involve the same coefficients. The differences, however, go beyond the mathematical form as we show in the next section through a numerical example.

5. NUMERICAL RESULTS

In this section we illustrate the qualitative and quantitative behavior of the electromechanical fields when the boundary of the half-plane is subjected to a point load. Toward this end we consider a piezoelectric ceramic known as PZT-4 (BERLINCOURT *et al.*, 1964), whose associated material coefficients are listed in Tables 1 and 2 for the cases of mechanical and electrical loads, respectively.

First we consider the case of a compressive line force of 1 N m^{-1} applied at the origin of coordinates. Contours of the induced normal stress are shown in Fig. 1. On physical grounds the results are rather predictable; the contours are similar to those found in anisotropic elasticity. The similarity, however, is only qualitative. To expose their differences we can calculate the magnitudes of the normal stress as evaluated using (4.13) and (4.18) along the line $x = 0$. For the material PZT-4, using Table 1 we find that the electroelastic theory predicts

$$\sigma_{zz}^{\mathrm{P}}(0, z) = -0.551\,\frac{P}{z}.$$

On the other hand, if piezoelectric effects are neglected, (4.18) yields

$$\sigma_{zz}^{\mathrm{E}}(0, z) = -0.7514\,\frac{P}{z}.$$

Thus the ratio of the elastic to piezoelectric solutions is $\sigma_{zz}^{\mathrm{E}}/\sigma_{zz}^{\mathrm{P}} = 1.26$. That is, the difference is by no means negligible and clearly suggests that the uncoupled approximation may lead to substantial errors.

The prediction of the behavior of the electric field merely on physical grounds is not a trivial matter, which in fact reinforces the importance of the expressions deduced in this article. As was mentioned in the introduction, the material not only can fail mechanically but also electrically. In the latter case the phenomenon is called dielectric breakdown and it takes place in the presence of very high electric fields. Because of these fields large numbers of electrons may suddenly be excited to energies within the conduction band, resulting in large currents that may deteriorate the material in an irreversible manner. Breakdown occurs when the electric field exceeds the material's dielectric strength. For a ceramic this strength is in the range 5×10^4–3×10^5 V m^{-1}. A limitation in the values of the applied or induced electric field must also be imposed in the case of poled ferroelectrics. If the poled material is exposed to very strong alternating electric fields, or direct fields opposing the direction of poling, depolarization may occur. In other words, the piezoelectric properties become less pronounced or vanish completely. The field strength necessary to cause serious depoling depends on the grade of the material, the duration of application and the temperature, but is typically in the range 5×10^5–10^6 V m^{-1}. Thus it is clear why information regarding the behavior of $\mathbf{E}$ is so important. The distribution of the E_z component is shown in Fig. 2. It is interesting to note the transition of the field from negative to positive values. The material lines for which $E_z = 0$ can be found from (4.14). Letting θ denote the angle that such lines form with the x axis we find that for a PZT-4 $\theta = 40.2°$. This angle strongly depends on the material properties. For $BaTiO_3$ and CdS, the electric field is zero along lines $\theta = 12.7°$ and $\theta = 27.7°$, respectively. Finally, from Figs 1 and 2 we can observe that in regions where the stress has a magnitude of approximately 1 MPa, the electric field can reach values of 10^4 V m^{-1}.

The next example concerns the case of a concentrated charge. Knowledge of its effects has also practical relevance. A piezoelectric material is used to convert mechanical energy into electrical energy and vice versa. In the first case one must record or measure the induced voltage, while in the second case one must supply an external field. The application or recording of a voltage is done by attaching electrodes to the surface of the piezoelectric. Mathematically, this can be modeled by a distribution of

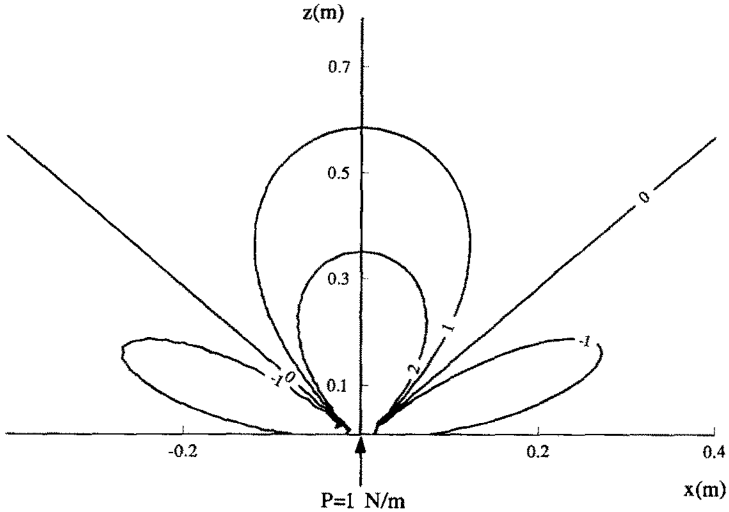

FIG. 2. Contours of E_z ($\times 10^{-2}$ V m^{-1}) induced by a point force.

charge. The study of the effects of an electrical load, whether it is distributed or concentrated, can be important to optimize electrode's shape and also to impose limits on the intensity of the applied voltages to avoid mechanical and electrical failure. As shown in Fig. 3 a positive line charge of 1 C m^{-1} applied at the origin of coordinates generates contours of normal stress σ_{zz} that change from compressive to tensile, a phenomenon that would be reversed by changing the sign of the charge. It can also be observed that the magnitudes of the normal stress are by no means negligible. Furthermore, the material lines for which $\sigma_{zz} = 0$ are found from (4.13). For the three materials given in Table 2 it is found that $\theta = 0°$, $\theta = 56.4°$ and $\theta = 64.2°$, respectively.

In this case it is also interesting to study the behavior of the vertical displacement

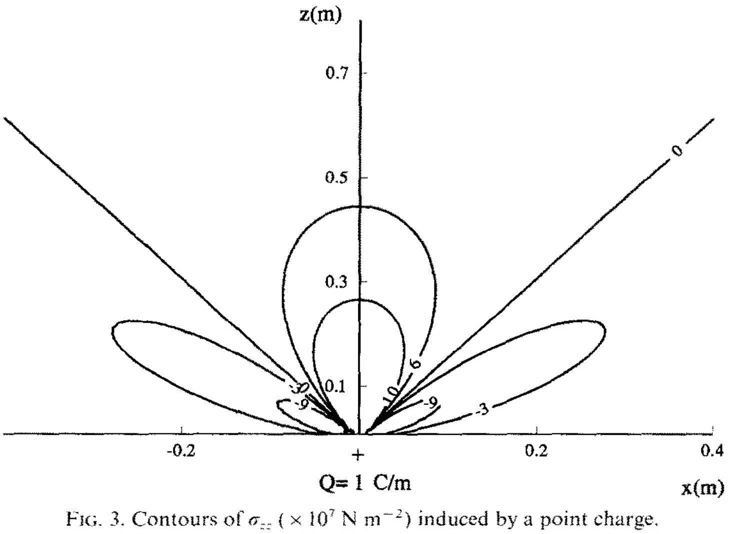

FIG. 3. Contours of σ_{zz} ($\times 10^7$ N m^{-2}) induced by a point charge.

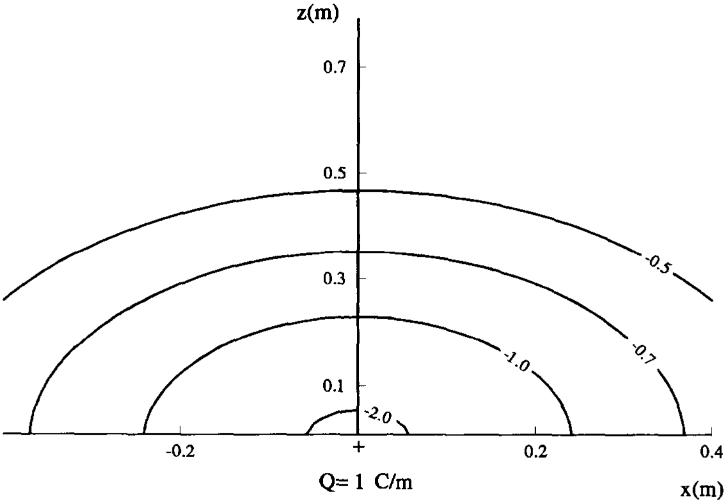

FIG. 4. Contours of w ($\times 10^{-2}$ m) induced by a point charge.

in the vicinity of the load, which is shown in Fig. 4. Notice that a positive charge gives rise to negative displacements, that is, the material moves towards the boundary. Naturally the behavior of the displacement is very much constrained to the neighborhood of the load, since as shown by (4.12) it is unbounded for large values of z. We also note that these curves are very similar to those representing equipotential lines. Finally, contours of electric field E_z are shown in Fig. 5. Their resemblance with the contours of stress due to a concentrated force is not surprising in view of the similarities of (4.13) and (4.17). Moreover, similar curves are found in the electrostatic theory of rigid dielectrics.

To conclude this section we want to point out that the behavior of other variables could also be represented using the fundamental results (4.11)–(4.17). Of particular

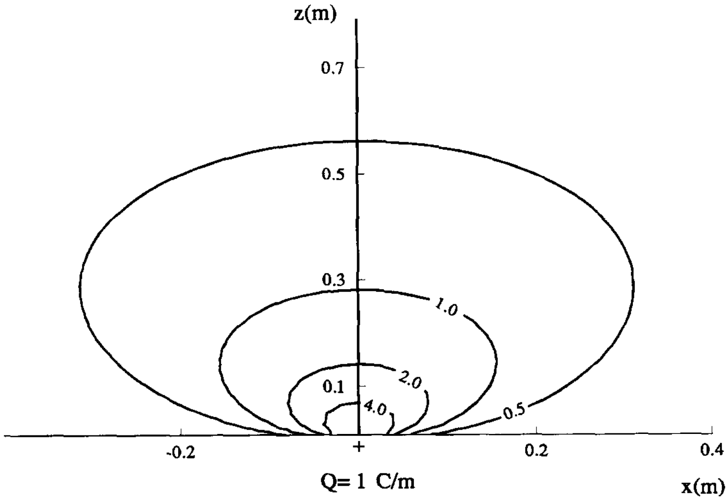

FIG. 5. Contours of E_z ($\times 10^{8}$ V m^{-1}) induced by a point charge.

importance would be finding the principal stresses σ_1 and σ_2 and contours along which the combination of these achieve maximum values. We omit these results for the sake of brevity. It can be verified, however, that in the case of $G = P$ the principal stresses are distributed in a manner similar to that found in the purely elastic case. For the case $G = Q$, σ_1 resembles E_z of Fig. 5, while σ_2 has the shape of the negative lobes of σ_{zz} given in Fig. 3.

6. CLOSURE

In this article we have provided exact expressions for the electroelastic variables induced by concentrated loads using a state space approach. The methodology is equally useful to solve in an exact or quasi-exact manner problems involving more complicated loading conditions. Moreover, we have shown that high values of stress and electric field arise in the neighborhood of point forces and point charges. These values being sufficiently high to produce mechanical and electrical failure, even under the assumption of a defect free material.

In closing we note that due to the complexities involved in the coupled electroelastic problem, it is virtually impossible to know *a priori* the behavior of the various physical variables when subjected to mechanical or electrical loads. Therefore, while our results appear to be physically reasonable and while they approach the behavior observed in the limiting cases of anisotropic elasticity and rigid dielectrics, only experimental verification could provide full validity to the present developments.

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APPENDIX

In this Appendix we present the values and units of the various constants appearing in the main body of the paper for three different materials: Barium Titanate ($BaTiO_3$), Lead Zirconate Titanate (PZT-4), and Cadmium Sulfur (CdS). These constants are in turn separated into Tables 1 and 2 according to the nature of the applied load. The experimentally found values of the elastic, piezoelectric and dielectric constants for these three materials can be found in the article of BERLINCOURT *et al.* (1964).

TABLE 1. *Material coefficients when* G = P

Variable	$BaTiO_3$	PZT-4	CdS
a	0.9405	1.204	1.854
b	1.004	1.069	0.7006
c	0.2292	0.2004	0.1348
$k_1 \times 10^{-12}$ [m^2 N^{-1}]	−5.478	−7.524	−6.989
$k_2 \times 10^{-12}$ [m^2 N^{-1}]	14.15	17.73	34.48
$k_3 \times 10^{-2}$ [m^2 C^{-1}]	1.223	2.209	1.726
$\Phi_1 \times 10^{-12}$ [m^2 N^{-1}]	−1.606	−48.75	−20.10
$\Omega_1 \times 10^{-12}$ [m^2 N^{-1}]	−3.872	41.23	13.12
$\Psi_1 \times 10^{-12}$ [m^2 N^{-1}]	52.01	70.22	45.52
$\Phi_2 \times 10^{-12}$ [m^2 N^{-1}]	−2.624	−43.23	−7.284
$\Omega_2 \times 10^{-12}$ [m^2 N^{-1}]	16.77	60.96	41.76
$\Psi_2 \times 10^{-12}$ [m^2 N^{-1}]	45.69	48.00	66.12
Φ_3	0.02329	1.897	0.3470
Ω_3	−1.0233	−2.897	−1.347
Ψ_3	−4.386	−4.060	−2.218
Φ_4	0.02190	2.283	0.6435
Ω_4	−0.02190	−2.283	−0.6435
Ψ_4	−4.638	−4.921	−1.733
$\Phi_5 \times 10^{-2}$ [m^2 C^{-1}]	1.403	2.565	−12.75
$\Omega_5 \times 10^{-2}$ [m^2 C^{-1}]	0.1801	−0.3560	14.48
$\Psi_5 \times 10^{-2}$ [m^2 C^{-1}]	3.234	12.27	−110.7
$\Phi_6 \times 10^{-10}$ [C N^{-1}]	1.973	12.13	−0.1074
$\Omega_6 \times 10^{-10}$ [C N^{-1}]	−1.973	−12.13	0.1704
$\Psi_6 \times 10^{-10}$ [C N^{-1}]	−7.388	−5.913	−1.030

TABLE 2. *Material coefficients when $G = Q$*

Variable	$BaTiO_3$	PZT-4	CdS
a	0.9405	1.204	1.8584
b	1.004	1.069	0.7006
c	0.2292	0.2004	0.1348
$k_1 \times 10^{-2}$ [$m^2\,C^{-1}$]	−0.7089	−1.733	−0.2540
$k_2 \times 10^{-2}$ [$m^2\,C^{-1}$]	−1.223	−2.209	−1.726
$k_3 \times 10^{7}$ [$N\,m^2\,C^{-2}$]	7.334	8.831	1116
$\Phi_1 \times 10^{-2}$ [$m^2\,C^{-1}$]	−0.8304	−5.262	21.48
$\Omega_1 \times 10^{-2}$ [$m^2\,C^{-1}$]	0.1216	3.530	−21.74
$\Psi_1 \times 10^{-2}$ [$m^2\,C^{-1}$]	0.07846	−1.940	54.20
$\Phi_2 \times 10^{-2}$ [$m^2\,C^{-1}$]	−1.357	−4.667	7.782
$\Omega_2 \times 10^{-2}$ [$m^2\,c^{-1}$]	0.1336	2.458	−9.508
$\Psi_2 \times 10^{-2}$ [$m^2\,C^{-1}$]	0.01978	−2.947	95.96
$\Phi_3 \times 10^{8}$ [$N\,C^{-1}$]	1.204	20.48	−37.08
$\Omega_3 \times 10^{8}$ [$N\,C^{-1}$]	−1.204	−20.48	37.08
$\Psi_3 \times 10^{8}$ [$N\,C^{-1}$]	−0.3327	13.78	−317.7
$\Phi_4 \times 10^{8}$ [$N\,C^{-1}$]	11.33	24.65	−68.76
$\Omega_4 \times 10^{8}$ [$N\,C^{-1}$]	−11.33	−24.65	68.76
$\Psi_4 \times 10^{8}$ [$N\,C^{-1}$]	−0.6100	10.60	−217.3
$\Phi_5 \times 10^{7}$ [$N\,m^2\,C^{-2}$]	7.258	2.768	136.3
$\Omega_5 \times 10^{7}$ [$N\,m^2\,C^{-2}$]	0.07641	6.063	979.7
$\Psi_5 \times 10^{7}$ [$N\,m^2\,C^{-2}$]	0.04731	0.3468	−966.7
Φ_6	1.021	1.309	0.1148
Ω_6	0.02062	−0.3092	0.8852
Ψ_6	−0.004996	0.5914	−0.9204