Technical Note

Modeling the Liquid Volume Flux in Bubbly Jets Using a Simple Integral Approach

Iran E. Lima Neto[1]

Abstract: This study presents a simple model to predict the liquid volume flux induced by round bubbly jets. The model is based on the classical integral equations for bubble plumes, but also accounts for the momentum at the source by including new conditions for the initial liquid jet velocity and radius. Moreover, an exponential functional relationship is used to relate the entrainment coefficient to a densimetric Froude number. The model predicts well the experimental results available in the literature for bubbly jets. Model simulations plotted together with experimental data for both bubbly jets and bubble plumes also reveals a clear jet/plume transition, resulting in entrainment rates in the bubbly jet zone larger than those in the bubble plume zone. Finally, potential applications of the model in aeration/mixing systems are presented. **DOI: 10.1061/(ASCE)HY.1943-7900.0000499.**

CE Database subject headings: Bubbles; Jets (fluid); Plumes; Two phase flow; Entrainment; Hydraulic models.

Author keywords: Bubbles; Jets; Plumes; Two phase flow; Entrainment; Integral equations.

Introduction

Bubble plumes and bubbly jets have a variety of applications in the fields of environmental, chemical, and mechanical engineering. Although bubble plumes are produced by the injection of gases in liquids (Cederwall and Ditmars 1970; Fannelop and Sjoen 1980; Milgram 1983; Wüest et al. 1992; Brevik and Kristiansen 2002; Lima Neto et al. 2008a, c), bubbly jets are produced by the injection of gas-liquid mixtures (Fast and Lorenzen 1976; Sun and Faeth 1986; Iguchi et al. 1997, Lima Neto et al. 2007, 2008b, d), as shown schematically in Fig. 1. Bubbly jets present some advantages over bubble plumes such as production of small bubbles without the need for porous diffusers, which are relatively expensive and susceptible to clogging, and higher energy efficiency for aeration and mixing purposes (see Lima Neto et al. 2007, 2008d).

Turbulence models have been proposed to assess the liquid flow structure in bubbly jets with gas volume fractions at the nozzle lower than 10% (Sun and Faeth 1986). On the other hand, for higher gas volume fractions (up to approximately 70%), only empirical relationships have been obtained (Iguchi et al. 1997; Lima Neto et al. 2008b, d). In the present study, a simple integral approach is proposed based on the classical theory for axisymmetric bubble plumes (see Socolofsky et al. 2002) to estimate the liquid volume flux induced by round bubbly jets with gas volume fractions ranging from very low to high. This information is important to analyze the mixing patterns in aerated tanks and water bodies.

Model Formulation

The model described in the present study is based on the similarity assumptions and integral techniques for single-phase axisymmetric jets and plumes (Morton et al. 1956; Rajaratnam 1976), but also incorporates the effects of bubble slip velocity and bubble expansion (Cederwall and Ditmars 1970) and the momentum amplification factor (Milgram 1983), which are widely used for modeling the liquid volume flux induced by bubble plumes (see Socolofsky et al. 2002). Hence, assuming similar Gaussian distributions of axial liquid velocity and density defect at different distances from the nozzle exit (see Fig. 1), the following governing equations can be obtained for the liquid volume conservation [Eq. (1)] and momentum conservation [Eq. (2)], respectively:

$$\frac{d(u_m b^2)}{dz} = 2\alpha u_m b \tag{1}$$

$$\frac{d(u_m^2 b^2)}{dz} = \frac{2gQ_{g,a}H_a}{\gamma\pi(H-z)(\frac{u_m}{1+\lambda^2}+u_s)} \tag{2}$$

in which u_m = the centerline liquid velocity; b = a measure of the liquid jet radius where the velocity $u = e^{-1} = 37\%$ of the centerline value [i.e., $u(z,r) = u_m(z)e^{-r^2/b^2}$]; z = the axial distance from the nozzle exit; r = the radial distance from the jet centerline; α = the entrainment coefficient; γ = the momentum amplification factor; λ = the spreading ratio of the bubble core radius relative to the liquid jet radius; u_s = the bubble slip velocity; $Q_{g,a}$ = the gas volume flow rate at atmospheric pressure; H_a = the atmospheric pressure head,; and H = the static pressure head at the nozzle. Note that the effect of bubble dissolution is neglected in the preceding analysis. The reader may refer to Wüest et al. (1992) for studies on large-scale bubble plumes where the effect of bubble dissolution on the flow structure is important.

The starting conditions for the numerical integration of Eqs. (1) and (2) require the assumption of a source of buoyancy only, as in the case of bubble plumes (Cederwall and Ditmars 1970; Milgram 1983). To solve this problem while preserving the multiphase nature of the plume, Wüest et al. (1992) defined a densimetric Froude number of 1.6 at the source. This condition has been recently validated by Socolofsky et al. (2008) and Einarsrud and Brevik (2009). However, for bubbly jets, momentum is expected to dominate the flow and an alternative approach is required. Thus,

[1]Assistant Professor, Dept. of Hydraulic and Environmental Engineering, Federal Univ. of Ceará, Fortaleza, Brazil. E-mail: iran@deha.ufc.br

Note. This manuscript was submitted on January 12, 2011; approved on August 1, 2011; published online on August 3, 2011. Discussion period open until July 1, 2012; separate discussions must be submitted for individual papers. This technical note is part of the ***Journal of Hydraulic Engineering***, Vol. 138, No. 2, February 1, 2012. ©ASCE, ISSN 0733-9429/2012/2-210–215/$25.00.

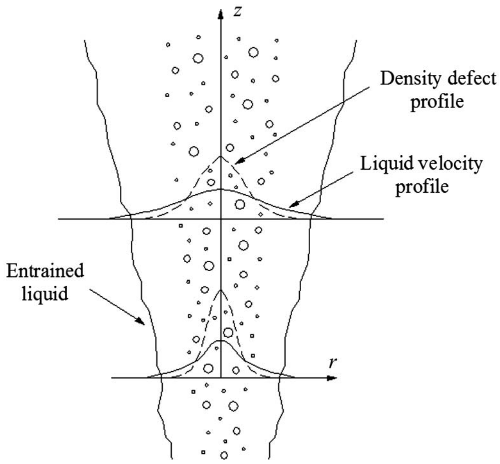

Fig. 1. Diagram of an axisymmetric bubble plume generated due to gas injection in water, which is similar to that of a bubbly jet generated due to a gas-liquid injection in water

rather than using the drift flux model, which considers a distribution parameter and a drift velocity using some empirical relationships (see Iguchi et al. 1997), or the concept of superficial water velocity, which neglects the effect of the gas phase on the liquid velocity at the nozzle exit (see Lima Neto et al. 2008b, d), the following equation was used in the present study for the initial liquid velocity:

$$u_{m,0} = \frac{Q_{l,0}}{(1-\varepsilon_0)(\pi d^2/4)} \tag{3}$$

in which d = the nozzle diameter; and ε_0 = the gas volume fraction at the nozzle, given by

$$\varepsilon_0 = \frac{Q_{g,0}}{Q_{g,0}+Q_{l,0}} \tag{4}$$

in which $Q_{g,0}$ and $Q_{l,0}$ = the gas and liquid volume flow rates, respectively.

Eq. (3) suggests that, for the same values of d and $Q_{l,0}$, the liquid velocity $u_{m,0}$ increases with $Q_{g,0}$ (or ε_0) because the cross-sectional area at the nozzle exit is assumed to decrease by a factor of $1-\varepsilon_0$.

Instead of using corrections for the virtual origin of the liquid jet, as is often done for bubble plumes (Cederwall and Ditmars 1970; Milgram 1983), the liquid jet radius at the nozzle exit was simply taken as equal to the nozzle diameter

$$b_0 = d \tag{5}$$

This assumption is supported by observations of the bubbly jets studied by Lima Neto et al. (2008b), in which extrapolation of the linear spreading of the liquid jet resulted in $b_0 \cong d$.

The momentum amplification factor (γ), spreading ratio of the bubble core to the water jet radius (λ), and bubble slip velocity (u_s) were considered constants, as is usually done in bubble plume studies (see Socolofsky et al. 2002). Nevertheless, the values for these parameters were obtained from investigations on bubbly jets. Thus, an average value of $\gamma = 2$ was obtained from the mean and turbulent components of the flow induced by the bubbly jets studied by Iguchi et al. (1997). Because the momentum amplification factor can be defined by $\gamma = (\bar{u}_i^2 + \overline{u_i'u_j'})/(\bar{u}_i^2)$, in which u_i' and u_j' represent the turbulent components of the flow, a value of $\gamma = 2$ implies that a fixed proportion of 50% of the total momentum flux is attributed to turbulent transport. This value lies within the region of $1 \le \gamma \le 3$ reported by Milgram (1983) for bubble plumes. An average value of $\lambda = 0.7$ was obtained from Lima Neto et al. (2008b), which is within the range of typical values of 0.4–1.0 reported by Milgram (1983) and Socolofsky et al. (2002) for bubble plumes. An average value of $u_s = 0.4$ m/s was also obtained from Lima Neto et al. (2008b). This value lies within the range of 0.2–0.6 m/s reported by Milgram (1983) and Lima Neto et al. (2008d) for bubble plumes and bubbly jets, respectively.

Finally, using the Buckingham pi theorem assuming that the forces due to momentum and buoyancy dominate the flow structure, the entrainment coefficient can be described by the following dimensionless relationship:

$$\alpha = f(\mathsf{F}_0) \tag{6}$$

in which F_0 = a densimetric Froude number defined in this study as

$$\mathrm{Fr}_0 = \frac{u_{m,0}}{\sqrt{dg(\rho_l-\rho_g)/\rho_l}} \tag{7}$$

in which ρ_l and ρ_g = the liquid and gas density, respectively.

Note that Eq. (7) differs from the densimetric Froude number proposed by Wüest et al. (1992) for bubble plumes.

Therefore, Eq. (6) is the only function to be tested in order to adjust the model to experimental data. Hence, provided the form of Eq. (6) is known, Eqs. (1) and (2) can be solved numerically using a Runge-Kutta fourth-order method to yield values of u_m and b along the bubbly jet.

The next section shows a comparison of the model described in this study with the experimental data of Sun and Faeth (1986), Iguchi et al. (1997), and Lima Neto et al. (2008b). A sensitivity analysis was also conducted to investigate the importance of each of the empirical parameters (γ, λ, and u_s) in the model results. The present model is based on the assumption of constant values of γ, λ, and u_s within the bubbly jet. This condition is expected to be attained for bubbly jets with nearly monodisperse bubble diameters, which occurs for nozzle Reynolds numbers larger than 8.000 (Lima Neto et al. 2008b, d). In fact, such condition generated bubbles with relatively uniform diameters of 1–4 mm in all the aforementioned experimental studies. The next section also shows a comparison of model simulations with experimental data from Lima Neto et al. (2008a) for bubble plumes under shallow water conditions (similar to those considered in the aforementioned bubbly jet experiments) and from Fannelop and Sjoen (1980) and Milgram and Van Houten (1982) for bubble plumes under deeper water conditions. Table 1 shows a summary of the experimental conditions for these studies on bubbly jets and bubble plumes.

Results and Discussion

The best fit of the model to the experimental data of Sun and Faeth (1986), Iguchi et al. (1997), and Lima Neto et al. (2008b) provided values for the entrainment coefficient α varying from 0.060 to 0.095, which are higher than that for single-phase jets (0.054) but within the typical range of values, 0.04–0.12, reported by Milgram (1983) and Socolofsky et al. (2002) for bubble plumes. For all the tests, the densimetric Froude number was considerably higher than 1.0 (i.e., F_0 ranged from 3.2 to 46.9; see Table 1), suggesting that momentum, rather than buoyancy, dominated the flow structure. Fig. 2 shows the fitted values of α plotted as a function of Fr_0. The following functional relationship was obtained by adjusting an exponential curve to these values:

$$\alpha = 0.04 + 0.06\exp(-1/\mathsf{F}_0)^{4.6} \tag{8}$$

Fig. 2 shows that Eq. (8) fitted well to the data, with a correlation coefficient $C = 0.93$. This suggests that F_0 is an appropriate parameter to describe the mean flow generated by bubbly jets.

Table 1. Experimental Conditions for Bubbly Jets and Bubble Plumes

Authors	Flow type	Static pressure head (m)	d (mm)	Gas volume flow rate (l/ min)	Liquid volume flow rate (l/ min)	Gas volume fraction (%)	Froude number
Sun and Faeth (1986)	Bubbly jet	0.91	5.08	0.05–0.2	2	2–10	7.4–8
Iguchi et al. (1997)	Bubbly jet	0.4	5	0.6–2.4	2.5–5	11–49	12–28.3
Lima Neto et al. (2008b)	Bubbly jet	0.76	4–13.5	0.4–5	2–7	5–71	3.2–46.9
Milgram and Van Houten (1982)	Bubble plume	3.66	16	9.4–103.5	0	100	1.6[a]
Fannelop and Sjoen (1980)	Bubble plume	10	100	152.5–610	0	100	1.6[a]
Lima Neto et al. (2008a)	Bubble plume	0.76	3	2–3	0	100	1.6[a]

[a]Froude number at the source, as suggested by Wüest et al. (1992).

According to Eq. (8), $\alpha \to 0.04$ as $\mathsf{F}_0 \to 0$ and $\alpha \to 0.10$ as $\mathsf{F}_0 \to \infty$. This equation is similar to that proposed by Seol et al. (2007) for bubble plumes. Nevertheless, instead of using the densimetric Froude number F_0, they used a dimensionless parameter given by $u_s/(B/z)^{1/3}$, in which B is the kinematic buoyancy flux of the plume. Therefore, in their case, the entrainment coefficient varies with the axial distance from the source, z.

Fig. 3(a) demonstrates that model predictions fitted well ($C = 0.98$) with the normalized centerline velocity ($u_m/u_{m,0}$) of the dilute bubbly jets tested by Sun and Faeth (1986) (see Table 1). The results not only suggest that the liquid velocity at the nozzle exit is well represented by Eq. (3), but also that the combination of Eqs. (1)–(3), (5), and (8) describes well the velocity decay, especially for the range of normalized distance from the source (z/d) between 10 and 60. Fig. 3(b) shows the fit of the model to the values of $u_m/u_{m,0}$ for the bubbly jets investigated by Iguchi et al. (1997), which had higher gas volume fractions than those studied by Sun and Faeth (1986). The results show that the model predicts well ($C = 0.99$) the velocity decay for the range of z/d between 20 and 60. Measurements for $z/d < 20$ (where the gas volume fraction was higher) are not shown in Fig. 3(b) because of their inaccuracy, as reported by the authors. Fig. 4 also confirms a good fit of the model ($C \geq 0.98$) to the radial distributions of liquid velocity [given by $u(z,r) = u_m(z)e^{-r^2/b^2}$] for the bubbly jets studied by Lima Neto et al. (2008b), which included different nozzle diameters. The model also fitted well with the other data sets not shown here, resulting in correlation coefficients higher than 0.97 and standard deviations between model predictions and experiments of up to 17%. This gives credence to the considerations employed in the formulations.

The initial liquid velocities $u_{m,0}$ estimated from Eq. (3) ranged from 4.1 to 18.4 times larger than those obtained by using the methodology of Wüest et al. (1992) for bubble plumes, i.e., a densimetric Froude number of 1.6 at the source. Therefore, Eq. (3) is more suitable to predict this parameter in bubbly jets than their methodology. The initial velocities were also larger than those obtained using the drift flux model adopted by Iguchi et al. (1997) (from 1.02 to 2.13 times higher) and the superficial water velocity concept adopted by Lima Neto et al. (2008b, d) (from 1.06 to 3.5 times higher). Because of the lack of data near the nozzle exit for gas volume fractions higher than those of Sun and Faeth (1986) (i.e., $\varepsilon_0 < 10\%$, resulting in similar initial velocities for the three models), it cannot be affirmed that one model is superior to the other to estimate $u_{m,0}$. However, values of b_0 much larger than d had to be used to fit the experimental data applying the drift flux model or the superficial water velocity concept, especially for larger values of ε_0. This suggests that Eq. (3) is also more suitable to predict $u_{m,0}$ using the present study's integral approach, as in this case the simple relationship $b_0 = d$, verified experimentally by Lima Neto et al. (2008b), was applicable.

A sensitivity analysis was conducted to assess the relevance of each of the empirical parameters (γ, λ, and u_s) on the model results.

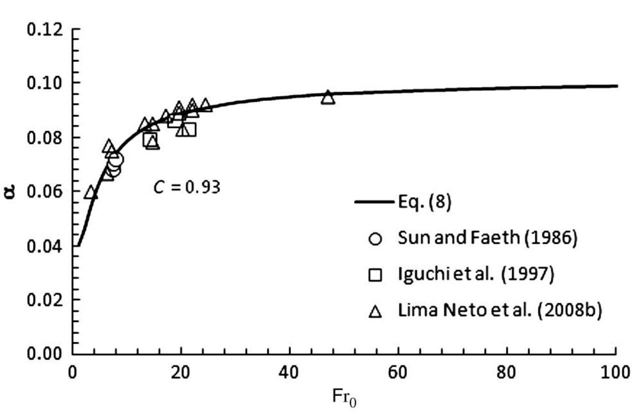

Fig. 2. Entrainment coefficient versus densimetric Froude number, indicating a fitted exponential curve

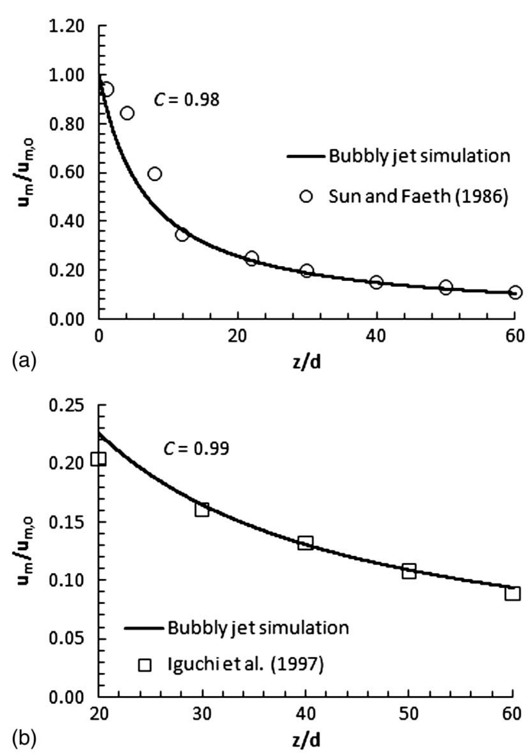

Fig. 3. Comparison of model predictions of normalized centerline velocity decay with experimental data of Sun and Faeth (1986) and Iguchi et al. (1997): (a) $d = 5.08$ mm, $H = 0.91$ m, $Q_{g,0} = 0.1$ l/ min, $Q_{l,0} = 2.0$ l/ min, $\varepsilon_0 = 5\%$, and $\mathsf{F}_0 = 7.6$; (b) $d = 5$ mm, $H = 0.4$ m, $Q_{g,0} = 2.4$ l/ min, $Q_{l,0} = 2.5$ l/ min, $\varepsilon_0 = 49\%$, and $\mathsf{F}_0 = 18.9$

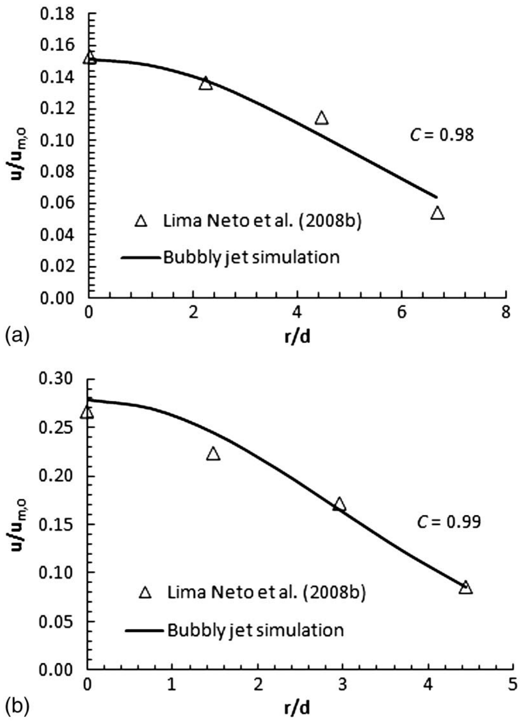

Fig. 4. Comparison of model predictions of normalized radial distributions of liquid velocity at $z = 0.43$ m with experimental data of Lima Neto et al. (2008b): (a) $d = 9$ mm, $Q_{g,0} = 2$ l/min, $Q_{l,0} = 5$ l/min, $\varepsilon = 29\%$, and $\mathsf{F} = 6.18$; (b) $d = 13.5$ mm, $Q_{g,0} = 3$ l/min, $Q_{l,0} = 7$ l/min, $\varepsilon = 30\%$, and $\mathsf{F} = 3.2$

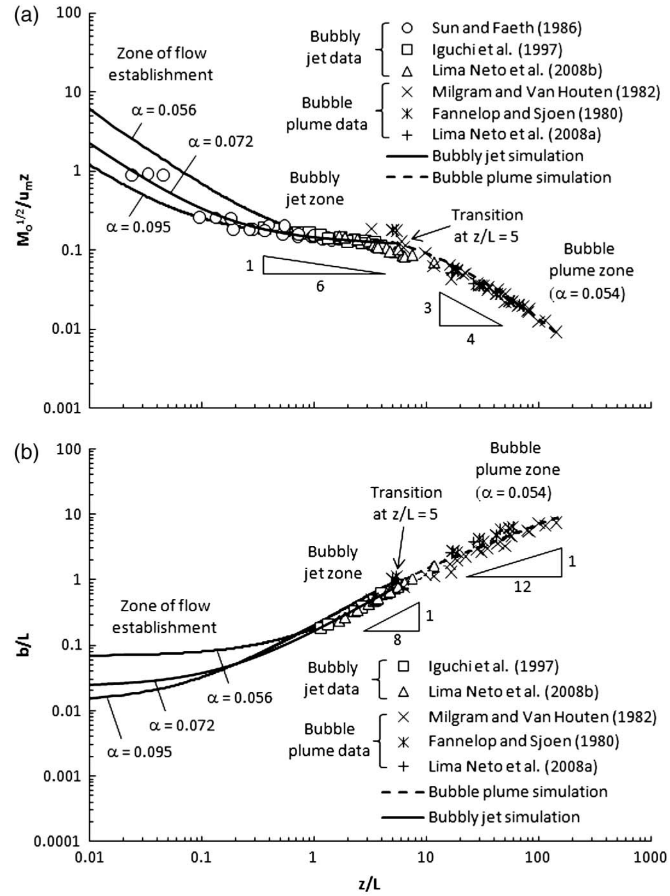

Fig. 5. Model simulations plotted together with experimental data in a dimensionless framework: (a) centerline liquid velocity inverse as a function of distance from the source; (b) liquid jet radius as a function of distance from the source; different regions of the flow, including their curve slopes and a transition from a bubbly jet zone to a bubble plume zone, are also indicated

The values of γ varied from 1 to 3 (as reported by Milgram 1983), λ varied from 0.4 to 1.0 (as reported by Milgram 1983 and Socolofsky et al. 2002), and u_s varied from 0.2 to 0.6 m/s (as reported by Milgram 1983 and Lima Neto et al. 2008b). The most sensitive parameter was the momentum amplification factor (γ). However, because the maximum variation of u_m and b from the model calculations using the standard values ($\gamma = 2$, $\lambda = 0.7$, and $u_s = 0.4$ m/s) was lower than 20%, it can be inferred that the model is not highly sensitive to the range of parameters and experimental conditions evaluated here.

Bubbly jets are expected to be controlled by the kinematic fluxes of momentum and buoyancy at the source, which can be expressed by $M_0 = Q_{l,0}u_{m,0}$ and $B_0 = Q_{g,0}g(\rho_l - \rho_g)/\rho_l$, respectively. Thus, a length scale L can be defined as $L = M_0^{3/4}/B_0^{1/2}$ (see Lima Neto et al. 2008d). Similarly to single-phase buoyant jets (see Papanicolaou and List 1988), L describes the relative importance of momentum and buoyancy fluxes. Figs. 5(a) and 5(b) show, respectively, the normalized centerline velocity inverse $M_0^{1/2}/(u_m z)$ and the normalized liquid jet radius b/L as a function of a normalized distance from the source, z/L, both obtained from experimental data for bubbly jets and bubble plumes (see Table 1). The figures indicate the behavior of the curves in three regions: zone of flow establishment, bubbly jet zone, and bubble plume zone. The zone of flow establishment was defined here as the axial distance from the nozzle exit up to $z = 0.1L$, which coincided with the classical limit of $z = 5d$ (see Rajaratnam 1976) when considering the experiments of Sun and Faeth (1986), the only set of data obtained from measurements taken within this region. The bubbly jet zone was defined as the region beyond $z = 0.1L$ where $M_0^{1/2}/(u_m z)$ decreased with z/L following a slope of $-1/6$, while b/L increased with z/L following a slope of $1/8$. A change in these slopes at approximately $z/L = 5$ clearly shows a transition from the momentum-dominated region (bubbly jet zone) to the buoyancy-dominated region (bubble plume zone). Therefore, the bubble plume zone was defined as the region beyond $z = 5L$ where $M_0^{1/2}/(u_m z)$ decreased with z/L following a slope of $-3/4$, while b/L increased with z/L following a slope of $1/12$. It is interesting to observe that similar transitions and slopes have been obtained for single-phase buoyant jets, as reported by Papanicolaou and List (1988). The preceding trends can be confirmed by the bubbly jet and bubble plume simulations depicted in Figs. 5(a) and 5(b). The bubbly jet simulations were performed for $z \leq 5L$ using the present model, whereas the bubble plume simulations were performed for $z > 5L$ by including the entrainment coefficient relationship of Seol et al. (2007) into the present model. In the bubbly jet case, three values for the entrainment coefficient $\alpha = 0.056$, 0.072, and 0.095 were used, which were within the range of α shown in Fig. 2. The intermediate value of 0.072 was selected because it provided the best fit of the model to the data at the zone of flow establishment. The results shown in Figs. 5(a) and 5(b) illustrate the sensitivity of the model with respect to α, specially for $z/L < 1$. In the bubble plume case, an average value of $\alpha = 0.054$ was used to extrapolate the intermediate condition for the bubbly jet simulation (i.e., $\alpha = 0.072$). This implies that closer to the source (i.e., in the bubbly jet zone) the flow is expected to have higher entrainment rates than away from the source (i.e., in the bubble plume zone). This result seems reasonable, as a decay of α with z has been verified experimentally by Seol et al. (2007) for bubble plumes.

Application

The integral approach proposed in the present study can be applied to estimate the liquid volume flux ($Q_l = \pi u_m b^2$) induced by round bubbly jets in aeration/mixing systems. Consider for example a nozzle with $d = 38$ mm discharging wastewater ($Q_{l,0} = 2.3$ l/s) from the bottom of a tank with water depth $H = 5$ m, which results in a single-phase jet with a volume flux at $z = 4.17$ m of $Q_l = 100$ l/s [assuming an initial surface jet thickness of $H/6 = 0.83$ m, as suggested by Lima Neto et al. (2008a, c), and using the integral model for single-phase jets, as described by Rajaratnam (1976)]. Hence, if a higher liquid volume flux is required to provide additional mixing, aeration, and/or prevent suspended solids deposition, an air line can be connected to the wastewater line to produce bubbly jets. This solution has the advantage of using the existing wastewater discharge system instead of installing new bubble plume diffusers (see Lima Neto et al. 2007). Therefore, the integral approach proposed in this study can be used to predict the increase in the normalized liquid volume flux ($Q_l/Q_{l,0}$) with the normalized distance from the source (z/d) for different gas volume fractions ε_0, as shown in Fig. 6. It can be seen that liquid volume fluxes at $z = 4.17$ m ranging from approximately 1.3 to 5.5 times higher than that for the single-phase jet can be obtained by varying ε_0 from 10% to 70% (i.e., keeping $Q_{l,0} = 2.3$ l/s and increasing $Q_{g,0}$ from 0.25 to 5.33 l/s). Fig. 6 also confirms that the larger the values of ε_0, the shorter the bubbly jet zone as compared to the bubble plume zone. A similar analysis can also be performed for the cases of aerated ponds, reservoirs, and other water bodies. However, caution should be taken when applying the present model for much deeper water conditions (H~50 m), because in such cases bubble dissolution may play an important role in the flow dynamics, as pointed out by Wüest et al. (1992) and Socolofsky et al. (2002). Additional limitations include gas volume fractions of up to approximately 70% and nozzle Reynolds numbers larger than 8.000, which were conditions imposed from the experiments used to validate the model (see Lima Neto et al. 2008b, d).

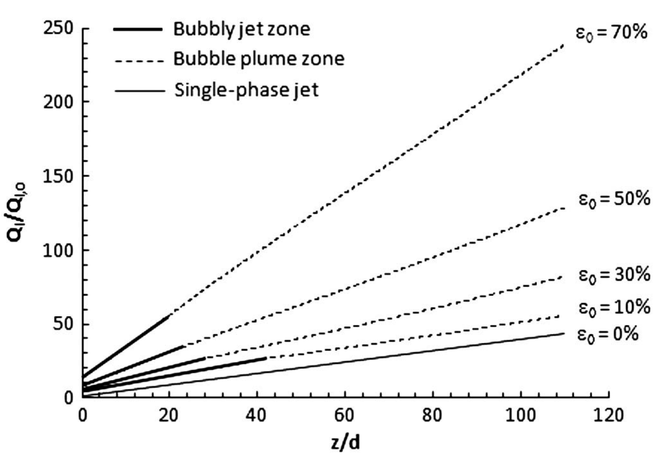

Fig. 6. Model simulations of normalized liquid volume flux as a function of normalized distance from the source for a single-phase jet ($\varepsilon_0 = 0\%$) and bubbly jets with different gas volume fractions ($\varepsilon_0 = 10$ to 70%); a liquid flow rate at the nozzle of $Q_{l,0} = 2.3$ l/s is considered for all the cases

Conclusions

In this study, a simple integral approach is proposed to estimate the liquid volume flux in axisymmetric bubbly jets. The model is similar to the classical theory for bubble plumes, but also accounts for the effect of momentum at the nozzle, including new conditions for the initial velocity and radius of the liquid jet. Using a simple exponential equation to relate the entrainment coefficient to a densimetric Froude number, a good fit of the model to the experimental data available for bubbly jets was obtained. Experimental data and model predictions also including a bubble plume regimen were plotted in a dimensionless framework using appropriate scales based on the kinematic fluxes of momentum and buoyancy at the source. This revealed three regions: zone of flow establishment, momentum-dominated zone (or bubbly jet zone), and buoyancy-dominated zone (or bubble plume zone), similar to single-phase buoyant jets. A clear transition between these regions was obtained, such that model simulations could be performed considering a bubbly jet zone and a bubble plume zone separately. Model results yielded larger entrainment rates in the bubbly jet zone than in the bubble plume zone, which is consistent with recent experimental results available in the literature. Finally, an application of the model for analysis of the circulation flow patterns in aerated tanks and water bodies is presented.

Acknowledgments

The author thanks the anonymous reviewers for their valuable comments and suggestions.

References

Brevik, I., and Kristiansen, Ø. (2002). "The flow in and around air-bubble plumes." *Int. J. Multiphase Flow*, 28(4), 617–634.

Cederwall, K., and Ditmars, J. D. (1970). "Analysis of air-bubble plumes." *Rep. No. KH-R-24*, W. M. Keck Laboratory of Hydraulics and Water Resources, Division of Engineering and Applied Science, CA Institute of Technology, Pasadena, CA.

Einarsrud, K. E., and Brevik, I. (2009). "Kinetic energy approach to dissolving axisymmetric multiphase plumes." *J. Hydraul. Eng.*, 135(12), 1041–1051.

Fannelop, T. K., and Sjoen, K. (1980). "Hydrodynamics of underwater blowouts." *AIAA 8th Aerospace Sciences Meeting*, Pasadena, CA, 80–0219.

Fast, A. W., and Lorenzen, M. W. (1976). "Synoptic survey of hypolimnetic aeration." *J. Environ. Eng. Div.*, 102(6), 1161–1173.

Iguchi, M., Okita, K., Nakatani, T., and Kasai, N. (1997). "Structure of turbulent round bubbling jet generated by premixed gas and liquid injection." *Int. J. Multiphase Flow*, 23(2), 249–262.

Lima Neto, I. E., Zhu, D. Z., and Rajaratnam, N. (2008a). "Air injection in water with different nozzles." *J. Environ. Eng.*, 134(4), 283–294.

Lima Neto, I. E., Zhu, D. Z., and Rajaratnam, N. (2008b). "Bubbly jets in stagnant water." *Int. J. Multiphase Flow*, 34(12), 1130–1141.

Lima Neto, I. E., Zhu, D. Z., and Rajaratnam, N. (2008c). "Effect of tank size and geometry on the flow induced by circular bubble plumes and water jets." *J. Hydraul. Eng.*, 134(6), 833–842.

Lima Neto, I. E., Zhu, D. Z., and Rajaratnam, N. (2008d). "Horizontal injection of gas-liquid mixtures in a water tank." *J. Hydraul. Eng.*, 134(12), 1722–1731.

Lima Neto, I. E., Zhu, D. Z., Rajaratnam, N., Yu, T., Spafford, M., and McEachern, P. (2007). "Dissolved oxygen downstream of an effluent outfall in an ice-covered river: Natural and artificial aeration." *J. Environ. Eng.*, 133(11), 1051–1060.

Milgram, H. J. (1983). "Mean flow in round bubble plumes." *J. Fluid Mech.*, 133, 345–376.

Milgram, H. J., and Van Houten, R. J. (1982). "Plumes from subsea well blowouts." *Proc., 3rd Int. Conf., Behavior of Offshore Structures (BOSS)*, Vol. I, MIT, Cambridge, MA, 659–684.

Morton, B. R., Taylor, G. I., and Turner, J. S. (1956). "Turbulent gravitational convection from maintained and instantaneous sources." *Proc. R. Soc. London, Ser. A*, 234(1196), 1–23.

Papanicolaou, P., and List, E. J. (1988). "Investigations of round vertical turbulent buoyant jets." *J. Fluid Mech.*, 195, 341–391.

Rajaratnam, N. (1976). *Turbulent jets*, Elsevier Scientific, Amsterdam, Netherlands.

Seol, D. G., Bhaumik, T., Bergmann, C., and Socolofsky, S. A. (2007). "Particle image velocimetry measurements of the mean flow characteristics in a bubble plume." *J. Eng. Mech.*, 133(6), 665–676.

Socolofsky, S. A., Bhaumik, T., and Seol, D. G. (2008). "Double-plume integral models for near-field mixing in multiphase plumes." *J. Hydraul. Eng.*, 134(6), 772–783.

Socolofsky, S. A., Crounse, B. C., and Adams, E. E. (2002). "Multiphase plumes in uniform, stratified, and flowing environments." *Environmental fluid mechanics theories and applications*, H. Shen, A. Cheng, K.-H. Wang, M. H. Teng and C. Liu, eds., ASCE, Reston, VA, 85–125.

Sun, T. Y., and Faeth, G. M. (1986). "Structure of turbulent bubbly jets—I. Methods and centerline properties." *Int. J. Multiphase Flow*, 12(1), 99–114.

Wüest, A., Brooks, N. H., and Imboden, D. M. (1992). "Bubble plume modelling for lake restoration." *Water Resour. Res.*, 28(12), 3235–3250.