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Le nombre b-chromatique de quelques classes de graphes généralisant les arbres

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Résumé: Une coloration des sommets de G s'appelle une b-coloration si chaque classe de couleur contient au moins un sommet qui a un voisin dans toutes les autres classes de couleur. Le nombre b-chromatique $\chi_b(G)$ de G est le plus grand entier k pour lequel G a une b-coloration avec k couleurs. Ces notions ont été introduites par Irving et Manlove en 1999. Elles permettent d'évaluer les performances de certains algorithmes de coloration.

Irving et Manlove ont montré que le calcul du nombre b-chromatique d'un graphe est un problème NP-difficile et qu'il peut être résolu en temps polynomial pour les arbres. Une question qui se pose naturellement est donc d'enquêter sur les graphes qui ont une structure proche des arbres: cactus, graphes triangulés, graphes série-parallèles, "block" graphes, etc.

Dans cette thèse, nous généralisons le résultat d'Irving et Manlove pour les cactus dont le "m-degré" est au moins 7 et pour les graphes planaires extérieurs dont la maille est au moins 8. (Le m-degré $m(G)$ est le plus grand entier d tel que G a au moins d sommets de degré au moins $d-1$.) Nous démontrons un résultat semblable pour le produit cartésien d'un arbre par une chaîne, un cycle ou une étoile. Pour ce qui concerne les graphes dont les blocs sont des cliques, nous montrons que le problème avec un nombre de couleurs fixé peut être résolu en temps polynomial et nous présentons des cas où le problème de décision peut être résolu. Toutefois, nous avons constaté que la différence $m(G) - \chi_b(G)$ peut être arbitrairement grande pour les graphes blocs, ce qui montre qu'avoir une structure arborescence n'est pas suffisant pour que le graphe satisfasse $\chi_b(G) \geq m(G) - 1$.

MOTS CLÉS: nombre b-chromatique, m-degré, arbres, cactus, graphe planaire exterieure, graph bloc, produit cartésien.

The b-chromatic number of some tree-like graphs

Abstract: A vertex colouring of a graph G is called a b-colouring if each colour class contains at least one vertex that has a neighbour in all other colour classes. The b-chromatic number $\chi_b(G)$ of G is the largest integer k for which G has a b-colouring with k colours. These concepts have been introduced by Irving and Manlove in 1999. They allow the analisys of the performance of some algorithms for colouring.

Irving and Manlove showed that finding the b-chromatic number is NP-hard for general graphs, while it can be found in polynomial time for trees. A question that naturally arises is to investigate the graphs that have a "tree structure", for instance: cactus, chordal graphs, series-parallel graphs, block graphs, etc.

In this thesis, we generalize the result of Irving and Manlove for cacti with "m-degree" at least 7 and for outerplanar graphs with girth at least 8. (The m-degree $m(G)$ is the largest integer d such that G has at least d vertices of degree at least $d-1$.) We prove a similar result for the cartesian product of a tree by a path, a cycle or a star. Regarding graphs whose blocks are cliques, we show that the fixed-parameter problem can be solved in polynomial time and we present cases where the decision problem can be solved. However, we found that the difference $m(G) - \chi_b(G)$ can be arbitrarily large for block graphs, which shows that the tree structure is not sufficient for having $\chi_b(G) \geq m(G) - 1$.

KEYWORDS: b-chromatic number, m-degree, tree, cactus, outerplanar graph, block graph, cartesian product.

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Résumé

Les problèmes de coloration de graphes sont parmi les sujets les plus étudiés dans la théorie des graphes. En fait, il existe une grande variété de problèmes de coloration (nous renvoyons le lecteur à [24] pour un panorama sur le sujet), les plus traditionnels d'entre eux étant le problème de colorier proprement un graphe avec le nombre minimum de couleurs. Relatif au problème cité on a le paramètre $\chi(G)$ appelé *nombre chromatique de* G. Il est connu que trouver $\chi(G)$ est NP-difficile, pour un graphe G générale. Ainsi, on peut essayer d'appliquer des heuristiques pour trouver une bonne coloration de G.

Une heuristique bien connue est l'heuristique gloutonne, où les sommets sont parcourus dans un ordre quelconque, $v_1, \cdots, v_n$, et, à la i-ème itération, le sommet v_i est coloré avec la couleur minimum qui ne figure pas sur les sommets de $N(v_i) \cap \{v_1, \cdots, v_{i-1}\}$. Évidemment, comme trouver le nombre chromatique d'un graphe est NP-difficile, cette heuristique ne produit pas toujours une coloration optimale: par exemple, si $P = \langle v_1, v_2, v_3, v_4 \rangle$ est une chaîne induite et nous colorons P en utilisant l'heuristique gloutonne dans l'ordre v_1, v_4, v_2, v_3, on obtient une coloration avec 3 couleurs, cependant que le nombre chromatic de P est 2. Ceci a motivé Christen et Selkow d'introduire la définition de *nombre de Grundy* [8], qui est intuitivement le pire nombre de couleurs d'une coloration qui peut être produite par l'heuristique gloutonne. Plus formellement, le nombre de Grundy est le nombre maximum de couleurs pour lequel il existe une coloration propre où chaque sommet de couleur i est adjacent à au moins un sommet de couleur j, pour chaque couleur $j \leq i - 1$.

Une autre approche pourrait consister à essayer de diminuer le nombre de couleurs utilisées par une coloration existante. Une façon d'y parvenir consiste à fusionner deux couleurs qui n'ont pas d'arêtes entre elles. Comme pour l'heuristique gloutonne, on ne peut pas espérer obtenir une coloration optimale avec cette heuristique, c'est à dire, étant donné une coloration Ψ de G, il n'est pas toujours possible de diminuer le nombre de couleurs utilisées par Ψ avec l'heuristique décrite. Ainsi, Harary et Hedetniemi ont introduit la notion de *nombre achromatique* [16], qui mesure le pire des colorations existantes qui ne peuvent pas être améliorées par la fusion de deux couleurs. Plus formellement, le nombre achromatique est le nombre maximum de couleurs pour lequel il existe une coloration propre telle que n'importe quelles deux

couleurs ont une arête entre elles.

Une autre idée pour essayer d'améliorer une coloration existante peut être, au lieu de recolorer tous les sommets dans une certaine classe de couleur avec la même couleur, de simplement essayer de recolorer chaque sommet séparement. Autrement dit, si nous avons une coloration propre de G et il existe une classe de couleur c telle que tous les sommets de la couleur c est non adjacent à au moins une autre couleur, nous pouvons alors séparément changer la couleur de chaque sommet de c, obtenant une coloration propre qui utilise moins de couleurs. Relatif à cette heuristique on a le paramètre appelé *nombre b-chromatique de G*, introduit par Irving et Manlove en [20], qui fait l'objet de cette thèse: une *b-coloration de G* est une coloration propre de G tel que chaque classe de couleur contient au moins un sommet qui est adjacent a chaque autre couleur. Le *nombre b-chromatique de G* est le plus grand entier $\chi_b(G)$ pour lequel il existe une b-coloration de G avec $\chi_b(G)$ couleurs.

Observez qu'une b-coloration, ainsi qu'une coloration obtenue avec l'heuristique gloutonne, ne peut pas être améliorée par la fusion de deux couleurs. En conséquence, le nombre achromatique est une borne supérieure pour le nombre de Grundy et le nombre b-chromatique. Toutefois, le nombre de Grundy n'a aucun rapport avec le nombre b-chromatique. Par exemple, si G est l'union de $n+1$ étoiles $K_{1,n}$, alors $\chi_b(G) = n+1$, cependant que le nombre de Grundy de G est 2.

Dans leur article séminal, Irving et Manlove ont prouvé que le problème de trouver le nombre b-chromatique d'un graphe est NP-difficile. Naturellement, une coloration propre de G avec $\chi(G)$ couleurs est une b-coloration de G, car ça ne peut pas être améliorée; donc, $\chi(G) \leq \chi_b(G)$. Pour trouver une borne supérieure, notez que si G a une b-coloration avec k couleurs, alors G a au moins k sommets de degré au moins $k-1$. Donc, si $m(G)$ est le plus grand entier m tel que G a au moins m sommets de degré au moins $m-1$, on sait que G ne peut pas avoir une b-coloration avec plus de $m(G)$ couleurs, i.e., $\chi_b(G) \leq m(G)$. Nous appelons ce paramètre le *m-degré de G* et nous disons qu'un sommet ayant un degré au moins $m(G)-1$ est *dense*. Cette borne supérieure a été donnée par Irving et Manlove en [20], où ils ont montré que la différence entre $\chi_b(G)$ et $m(G)$ peut être arbitrairement grande pour un graphe en général. En outre, Kratochvíl, Tuza et Voigt [28] ont prouvé que décider si $\chi_b(G)$ est égal à $m(G)$ est NP-complet, même si G est un graphe biparti ayant exactement $m(G)$ sommets denses, chacun de degré $m(G)-1$. Toutefois, la différence $m(G) - \chi_b(G)$ est au plus un pour les arbres [20], ce qui nous a fait nous demander quel genre de graphes ont la même propriété. Une approche naturelle est d'étudier les graphes en forme d'arbre, c'est à dire des graphes qui ont une structure arborescente, comme, par exemple, des cactus, des graphes bloc, des k-arbres, etc. Dans cette thèse, nous étudions le nombre b-chromatique des cactus, des graphes outerplanar, des graphes bloc et des produits cartésiens des arbres par certaines autres classes de graphes. Heureusement, nous avons pu trouver des réponses positives pour les cactus et les outerplanar. Toutefois, nous avons également constaté que la différence entre $\chi_b(G)$ et $m(G)$ peut être arbitrairement grande pour certains graphes "en forme d'arbre", comme nous le verrons dans le Chapitre 5. Ainsi, la structure arborescente n'est pas suffisant pour avoir $\chi_b(G) \geq m(G) - 1$. Notez que, trivialement, ce n'est pas non plus nécessaire, puisque tout graphe complet K_n a nombre b-nombre chromatique n.

Bien que l'approche de cette thèse soit purement théorique, on mentionne que la b-coloration peut être utilisée la classification des données [13] et dans la reconnaissance automatique des documents [15].

Dans le Chapitre 2, nous présentons les notations et donnons les définitions nécessaires. Nous présentons aussi l'algorithme de Irving et Manlove pour trouver le nombre b-chromatique d'un arbre. L'idée générale de cet algorithme, ainsi que certains de ses lemmes de base, sera important dans certaines de nos épreuves, spécialement dans le chapitre 6. Nous discutons aussi plus profondément l'état de l'art.

Dans le Chapitre 3, nous montrons que la différence entre $\chi_b(G)$ et $m(G)$ est au plus un pour les cactus dont le m-degré est au moins 7. Notre preuve fournit aussi un algorithme polynomial pour trouver une b-coloration optimale de ces graphes.

Dans le Chapitre 4, nous montrons le même résultat pour les graphes planaires extérieurs dont la maille est au moins 8. L'algorithme présenté permet également de trouver une b-coloration optimale du graphe en temps polynomial.

Dans le Chapitre 5, nous analysons le problème restreint au graphes blocs. Nous construisons un graphe bloc pour lequel la différence entre $\chi_b(G)$ et $m(G)$ est arbitrairement grande. Puis, nous prouvons que le problème avec un nombre de couleurs fixé peut être résolu en temps polynomial et nous présentons certains cas où le problème de décision peut aussi être résolu polynomialement.

Dans le Chapitre 6, nous montrons que $\chi_b(H) \geq m(H) - 1$, lorsque H est le produit cartésien d'un arbre par une chaîne, ou d'un arbre par un cycle, ou d'un arbre par une étoile. La preuve donne également un algorithme optimal pour trouver une b-coloration optimale de H.

Dans le Chapitre 7, nous analysons les résultats de cette thèse.

Dans l'Annexe A, nous montrons un résultat sur les cactus minimaux qui ne peuvent pas être colorés avec $m(G)$ couleurs. Ce résultat peut ultérieument aider a généraliser notre résultat sur les cactus pour les autres valeurs de $m(G)$.

Dans l'Annexe B, nous donnons les résumés en français de chacun des chapitres de cette thèse.

Contents

List of Symbols

Chapter 1

Introduction

The graph colouring problems are amongst the most researched topics in graph theory. In fact, there is a great variety of colouring problems (we direct the reader to [24] for a survey on the subject), the most traditional amongst them being the problem of properly colouring a graph with the minimum number of colours, which is called *chromatic number of* G and represented by $\chi(G)$. It is known that finding $\chi(G)$ is NP-hard, for a general graph G. Thus, one may try to apply heuristics to find a good colouring of G.

A well known heuristic is the greedy heuristic, where the vertices are iterated in some order, $v_1, \cdots, v_n$, and, at the i-th iteration, vertex v_i is coloured with the minimum colour that does not appear in $N(v_i) \cap \{v_1, \cdots, v_{i-1}\}$. Trivially, as finding the chromatic number of a graph is NP-hard, this heuristic does not always produce an optimal colouring: for example, if $P = \langle v_1, v_2, v_3, v_4 \rangle$ is an induced path and we colour P using the greedy heuristic in the order v_1, v_4, v_2, v_3 we obtain a colouring with 3 colours. This motivated Christen and Selkow to introduce the definition of *Grundy number* in [8], which intuitively is the worst colouring that can be produced by the greedy heuristic. More formally, the Grundy number is the maximum number of colours for which there exists a proper colouring where each vertex coloured with colour i is adjacent to at least one vertex coloured with colour j, for every colour $j \leq i - 1$.

Another approach could be to try to decrease the number of colours used by an existing colouring. One way of doing this is to merge two colours that have no edge between them. As with the greedy heuristic, one cannot expect to obtain an optimal colouring with this heuristic, i.e., given a colouring Ψ

of G, it is not always possible to decrease the number of colours used in Ψ with the described heuristic. Thus, Harary and Hedetniemi introduced the notion of *achromatic number* in [16], which measures the worst existing colouring that cannot be improved by merging two colours. More formally, the achromatic number is the maximum number of colours for which there exists a proper colouring where any two colours “see” each other (have an edge between them).

Another idea is to try to improve an existing colouring by, instead of trying to recolour all the vertices in a given colour class with the same colour, just trying to recolour each vertex separately. That is, if we have a proper colouring of G and there exists a colour c such that every vertex coloured with c is non-adjacent to at least one other colour, we can then separately change the colour of each vertex coloured with colour c, obtaining a proper colouring that uses less colours than before. Related to this heuristic is the parameter called *b-chromatic number*, introduced by Irving and Manlove in [20], which is the subject of this thesis: a *b-colouring* of G is a proper colouring of G such that every colour class contains at least one vertex that sees every other colour (called a *b-vertex*); the b-chromatic number $\chi_b(G)$ of G is the maximum integer for which there exists a b-colouring of G with $\chi_b(G)$ colours. Given a b-colouring ψ of G, a subset of $V(G)$ containing exactly one b-vertex of each colour is called a *basis of* ψ.

Observe that a b-colouring, as well as a colouring obtained with the greedy heuristic, cannot be improved by merging two colours; thus the achromatic number is an upper bound for both the Grundy number and the b-chromatic number. However, the Grundy number has no relation with the b-chromatic number; observe Figure 1.1 (the Grundy number is denoted by $\Gamma(G)$ - this figure appears in [20]).

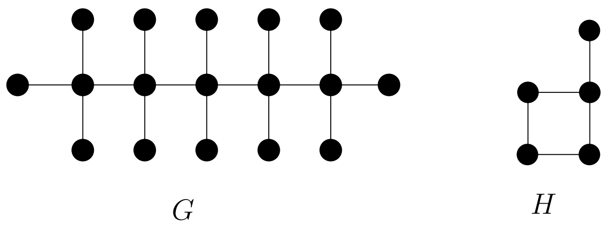


Figure 1.1: $\Gamma(G) = 4$ while $\chi_b(G) = 5$, and $\Gamma(H) = 3$ while $\chi_b(H) = 2$.

Naturally, a proper colouring of G with $\chi(G)$ colours is a b-colouring of

G, since it cannot be improved; so, $\chi(G) \leq \chi_b(G)$. For an upper bound, note that if G has a b-colouring with k colours, then G has at least k vertices with degree at least $k-1$ (the b-vertices). So, if $m(G)$ is the largest integer such that G has at least $m(G)$ vertices with degree at least $m(G)-1$, we know that G cannot have a b-colouring with more than $m(G)$ colours, i.e., $\chi_b(G) \leq m(G)$; we call this parameter the *m-degree of* G and we say that a vertex having degree at least $m(G)-1$ is *dense*. This upper bound is given by Irving and Manlove in [20], where they showed that the difference between $\chi_b(G)$ and $m(G)$ can be arbitrarily large. As an example, consider the complete bipartite graph $K_{n,n}$ with parts A, B. Observe that $m(K_{n,n}) = n+1$ and suppose that $\chi_b(K_{n,n}) > 2$. Then, there must exist at least two b-vertices of different colour classes in a part of $K_{n,n}$, say $u, v \in A$ are b-vertices of colour classes c, c', respectively, $c \neq c'$. We get a contradiction as u cannot have a neighbour in the colour class c' since all the neighbours of u are also adjacent to v (hence, cannot be coloured with the same colour as v). So, we have $m(K_{n,n}) = n+1$ and $\chi_b(K_{n,n}) = 2$.

In their seminal paper, Irving and Manlove prove that the problem of finding the b-chromatic number of a graph is NP-hard. Also, deciding if $\chi_b(G)$ equals $m(G)$ is NP-complete, even if G is either a bipartite graph [28] or a distance-hereditary chordal graph [17] having exactly $m(G)$ dense vertices, each with degree $m(G)-1$. However, Irving and Manlove show that if T is a tree then the difference $m(T) - \chi_b(T)$ is at most one [20]. More precisely, they characterize the trees with $\chi_b(T) < m(T)$, calling them "pivoted trees", and prove that a pivoted tree T can be b-coloured with $m(T)-1$ colours. Furthermore, they show that if T is a non-pivoted tree, then T has a special set of dense vertices, called "good set", that can play the role of the basis of a b-colouring of T with $m(T)$ colours. Their proof yields a polynomial algorithm that finds an optimal b-colouring of a tree.

Irving and Manlove's result for trees made us wonder what kind of graphs have the same property. A natural way of course is to investigate "tree-like" graphs, i.e., graphs that have a tree structure, as, for example, cacti, block graphs, k-trees, etc. In this thesis, we investigate the b-chromatic number of cacti, outerplanar graphs, block graphs and the cartesian product of trees and some other graph classes. Fortunately, we were able to find positive answers for cacti and outerplanar graphs. However, we also found that the difference between $\chi_b(G)$ and $m(G)$ can be arbitrarily large for some "tree-like" graphs, as we will see in Chapter 5; thus, the tree structure is not sufficient for having $\chi_b(G) \geq m(G)-1$. Note that, trivially, it is also not necessary, as any complete graph K_n has b-chromatic number n.

Although the focus of this thesis is purely theoretical, we mention that the b-colouring can be used for clustering data sets [13] and automatic recognition of documents [15].

In Chapter 2, we give the necessary definitions and notation and discuss the state of art. We also discuss about the existence of pivots in the graph classes studied in this thesis, present some existing results on the b-chromatic number of graphs with no $K_{2,3}$ as subgraph and state a conjecture about the b-chromatic number of graphs with no $K_{2,3}$ as subgraph. We then present the algorithm of Irving and Manlove for finding the b-chromatic number of a tree. The general idea of this algorithm, as well as some of its basic lemmas, will be important in some of our proofs, specially in Chapter 6. We also remark that Irving and Manlove's algorithm actually works on any graph with girth at least 11 (this is presented in the form of Corollary 2.18). Finally, we prove that if G has girth at least 8 and has no good set, then $\chi_b(G) \geq m(G) - 1$.

In Chapter 3, we generalize the result on trees by Irving and Manlove for the cacti with m-degree at least 7. We also give an algorithm that finds an optimum b-colouring of such a cactus. In fact, we characterize the cacti that do not have a good set and show some graphs that, although having a good set, cannot be b-coloured with $m(G)$ colours (we call them anomalous). Then, we prove that if G does not have a good set or is anomalous, then $\chi_b(G) = m(G) - 1$. And finally we prove that if G has a good set and $m(G) \geq 7$ (thus G is not anomalous), then $\chi_b(G) = m(G)$. We conjecture that if G has a good set and G is not anomalous, then $\chi_b(G) = m(G)$. It remains to prove this for $m(G) \leq 6$. Observe that, if this is true, then $\chi_b(G) \geq m(G) - 1$, for all cactus G. In Appendix A, we prove that if G is a minimal counter-example for this conjecture, then $|D(G)| = m(G)$, $d(v) = m(G) - 1$, for all $v \in D(G)$, $G \subseteq D(G) \cup N(D(G))$ and, for all $(u, v) \in E(G)$, at least one between u and v is a dense vertex.

In Chapter 4, we show that if G is an outerplanar graph with girth at least 8, then $\chi_b(G) \geq m(G) - 1$ and we also give a polynomial-time algorithm to find an optimal b-colouring of G. Note that every cactus is also an outerplanar graph; thus, this result generalizes the result presented in Chapter 3, but only to cacti with girth at least 8. The complexity of the proof presented in Chapter 3 shows us that it might require a much higher effort to generalize the result presented in Chapter 4 for general outerplanar graphs. Furthermore, as pointed out in Section 5.1, this result cannot be generalized for series-parallel graphs, which is a superclass of outerplanar graphs.

In Chapter 5, we show an example of a block graph G for which the difference $m(G) - \chi_b(G)$ is unbounded. Our construction can alternatively yield a claw-free block graph or a series-parallel graph. Then, we prove that the fixed parameter decision problem is polynomially solvable. Also, given a subset W of cardinality k such that $d(u) \geq k-1$, for all $k \in W$, we prove that the difficulty in obtaining a b-colouring with basis W lies on the existence of a special type of vertex, called side vertex. Finally, we show a special case where we can decide if $\chi_b(G) \geq k$, k given as the input and G being a claw-free block graph (i.e., the line graph of a tree).

Let $T \square G$ denote the cartesian product of graphs T and G. In Chapter 6, we prove that if T is a tree and G is a path of length greater than 4, or a cycle of length greater than 3 or a star, then $\chi_b(G \square T) \geq m(G \square T) - 1$. We also give polynomial-time algorithms to find optimal b-colourings of those graphs.

In Chapter 7, we present the cases left open in this thesis and discuss our perspectives.

Chapter 2

Definitions, notation and basic results

Consider G to be an undirected simple graph and denote by $V(G), E(G)$ the sets of vertices and edges of G, respectively (or simply V and E, if there is no ambiguity). We denote the neighbourhood of $u \in V$ by $N(u)$, the set $N(u) \cup \{u\}$ by $N[u]$ and the value $|N(u)|$ by $d(u)$; also, if $X \subseteq V$, then $N^X(u)$ represents the set $N(u) \cap X$. The minimum degree of a vertex of G is denoted by $\delta(G)$, while the maximum degree is denoted by $\Delta(G)$. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; and, given $X \subseteq V(G)$, the *subgraph of G induced by X* is the graph $G[X] = (X, E_X)$, where $(u, v) \in E_X$ if and only if $u, v \in X$ and $(u, v) \in E(G)$. Let $v_1, v_q \in V(G)$, $v_1 \neq v_q$; a *path between v_1 and v_q in G* (also called a v_1, v_q-path) is a sequence of vertices $\langle v_1, \cdots, v_q\rangle$ where $(v_i, v_{i+1}) \in E(G)$, for $i = 1, \cdots, q-1$, and $v_i \neq v_j$, for all $1 \leq i \neq j \leq q$; this path is a *cycle* of G if $(v_1, v_q) \in E(G)$. The *length* of a path or cycle equals the number of edges in the path or cycle. The *girth* of a graph G is the length of a shortest cycle in G. A graph is *connected* if there exists a path between every pair of vertices of G; otherwise, it is *disconnected*. A graph with no cycles is called a *forest* and a connected forest is called a *tree*. A *connected component of a graph G* is a maximal connected subgraph of G. A path or cycle is called *induced* if there are no other edges apart from the ones that define the path or cycle; if it is not the case, these additional edges are called *chords*. The induced path with k vertices is denoted by P_k , while the induced cycle with k vertices is denoted by C_k. The *distance between u and v* is the length of a minimum induced path with extremities u and v; we denote this value by $dist(u, v)$ and if there is no such path (u and v are in different connected components), we write $dist(u, v) = \infty$.

A graph G is *complete* if it has an edge between all pair of vertices. A subset $C \subseteq V(G)$ is called a *clique of* G if $G[C]$ is complete. The complete graph with n vertices is denoted by K_n. A subset $S \subseteq V(G)$ is a *stable set* if there is no edge between any two vertices of S. A *bipartite graph* is a graph G with vertex set $U \cup V$ where U and V are stable sets. A bipartite graph $G = (U \cup V, E)$ is said to be *complete bipartite graph* if it has an edge (u, v) for all pair $u \in U, v \in V$; if $p = |U|$ and $q = |V|$, we denote G by $K_{p,q}$. A *star* is the complete bipartite graph $K_{1,q}$, $q \geq 2$.

Let $X \subseteq V(G)$; we denote by $G - X$ the graph $G[V \setminus X]$ (if $X = \{x\}$, we write $G - x$). A *cut-vertex* is a vertex of G such that $G - x$ is not connected.

A *(proper) colouring of* G is an assignment of colours to the vertices of G such that no two adjacent vertices have the same colour. The *chromatic number of* G is the minimum integer $\chi(G)$ such that G has a proper colouring with $\chi(G)$ colours. In a proper colouring, the *colour class* of a colour c is the set of vertices of G coloured with colour c.

A vertex u in colour class C of a proper colouring ψ is said to be a *b-vertex* of C in ψ if u has at least one neighbour in each colour class other than C. A proper colouring ψ with k colours is a *b-colouring of* G if each colour class contains at least one b-vertex. The *b-chromatic number of* G is the largest integer $\chi_b(G)$ such that G has a b-colouring with $\chi_b(G)$ colours. Given a (partial) colouring ψ of G, we denote the colour of a vertex $u \in V$ in this colouring by $\psi(u)$; if u is not coloured, we write $\psi(u) = \emptyset$. Also, if $X \subseteq V$, we denote by $\psi(X)$ the set of colours $\{\psi(x) : x \in X\}$.

If ψ is a b-colouring of $G = (V, E)$ with k colours and $W \subseteq V$ contains exactly one b-vertex of each colour class of ψ, then W is said to be a *basis of* ψ; conversely, if W is the basis of some b-colouring of G, then we say that W is a $|W|$-basis. We recall that $\chi(G) \leq \chi_b(G) \leq m(G)$, where $m(G)$ is the *m-degree of* G; that is the largest integer such that G has at least $m(G)$ vertices with degree at least $m(G) - 1$. A vertex $u \in V$ is said to be a *dense vertex of* G if $d(u) \geq m(G) - 1$ and the set of dense vertices of G is denoted by $D(G)$. Also, more generally, if k is a positive integer, we say that u is *k-dense* if $d(u) \geq k - 1$ and denote by $D_k(G)$ the set of all k-dense vertices of G.

Let G be any graph. A *block* of G is a maximal 2-connected component of G (i.e., a maximal subgraph not containing a cut-vertex). In this thesis, we study the b-colouring of three graph classes that can be described by the types of their blocks: the *cacti*, where each block is either an edge or an induced cycle; the *outerplanar* graphs, where each block is a planar graph where all the vertices are in the external face; and the *block graphs*, where each block defines a clique.

Given two graphs F, G, the *cartesian product* of F and G, denoted by $F \square G$, is the graph $H = (V^*, E^*)$, where $uv \in V^*$, for every $u \in V(F)$ and $v \in V(G)$, and $(uv, xy) \in E^*$ if and only if either ($u = x$ and $(v, y) \in E(G)$), or ($v = y$ and $(u, x) \in E(F)$).

In Section 2.1, we give an overview of the existing results on b-colouring. In Section 2.2, we present the concept of pivot used to colour a tree and discuss the existence of pivots in other tree-like graphs. In Section 2.3, we present some existing results on the b-chromatic number of graphs with no $K_{2,3}$ as subgraph and state a conjecture on those graphs. In Section 2.4, we explain how to extend a certain partial colouring of G to a b-colouring of G. In Section 2.5, we present the algorithm from [20] for trees and remark that it also work for graphs with girth at least 11. Finally, in Section 2.6, we prove that if G is a graph with girth at least 8, then a good set of G can be found in polynomial time, if one exists; otherwise (i.e. G does not have a good set), we prove that $\chi_b(G) = m(G) - 1$.

2.1 State of Art

The concepts of b-colouring and b-chromatic number were introduced by Irving and Manlove in [20], where they also proved that finding the b-chromatic number of a graph is NP-hard. Also, Kratochvíl, Tuza and Voigt [28] show that deciding if $\chi_b(G) = m(G)$ is NP-complete, even if G is a bipartite graph with exactly $m(G)$ dense vertices, each with degree $m(G) - 1$. A similar result is proven by Havet, Linhares and Sampaio [17] for distance-hereditary chordal graphs. Concerning the approximation variant of the problem, Corteel, Valencia-Pabon and Vera [9] proved that there is no constant $\epsilon > 0$ for which the b-chromatic number can be approximated within a factor of $(120/113) - \epsilon$ in polynomial time, unless P=NP.

The b-chromatic number restricted to some graph classes has then been studied. Exact values were done for power graphs of complete caterpillars [10], power graphs of paths [12], power graphs of complete k-ary trees [11], Kneser graphs $K(n, k)$ for some values of n and k [22], hypercubes [26] and cubic graphs [25]. In [3], Bonomo et al. give a polynomial-time algorithm to find the b-chromatic number of cographs and P_4-sparse graphs. Recently, Campos et al. [6] generalized this result for $(q, q-4)$ graphs [1], for fixed q. Lower and/or upper bounds for the b-chromatic number can be found for power graphs of cycles [12], cartesian product of complete graphs [7, 23], vertex deleted subgraphs [32], d-regular graphs [4, 2, 33], $K_{1,s}$-free graphs and bipartite graphs [27]. Also, Kouider and Zaker give upper bounds for the b-chromatic number of general graphs depending on the clique number and clique partition number [27].

The b-chromatic number of cartesian products has been investigated for complete graphs [7, 23] and stars, paths and cycles [14, 26]. In [26], Kouider and Mahéo prove that $\chi_b(G \square H) \geq \chi_b(G) + \chi_b(H) - 1$ when both G and H have optimal b-colourings for which the basis are stable sets. Also, the b-chromatic number of the strong product, lexicographic product and direct product of two graphs is considered in [21].

Given a graph G, it is known that there does not necessarily exist a b-colouring of G with k colours for all value $k \in \{\chi(G), \cdots, \chi_b(G)\}$. For example, the cube can be b-coloured with 2 and 4 colours, but not with 3 colours. In his thesis, Faik [14] introduced the concept of b-continuous graphs: a graph G is b-continuous if G can be b-coloured with k colours for all $k \in \{\chi(G), \cdots, \chi_b(G)\}$. He proves that deciding if a graph G is b-continuous is NP-complete, even if G is a bipartite graph and both its chromatic and b-chromatic numbers are known. He also investigates the b-continuity of some graph classes. In particular, he proves that chordal graphs are b-continuous. Other graph classes known to be b-continuous are the Kneser graphs $K(n, 2)$ with $n \geq 17$ [22] and the P_4-sparse graphs [3].

Hoáng and Kouider [18] introduced and studied the b-perfect graphs (a graph G is b-perfect if $\chi_b(H) = \chi(H)$ for all induced subgraph H of G) and recently Hoáng, Maffray and Mechebbek [19] characterized all the b-perfect graphs by forbidden induced subgraphs.

The b-colouring and b-chromatic notions have also been used in data clustering [13] and in the automatic recognition of documents [15].

2.2 Pivots

Irving and Manlove proved that $\chi_b(T) \geq m(T) - 1$, where T is a tree [20]. Actually, they show that $\chi_b(T) = m(T) - 1$ if and only if there exists a special vertex that they called pivot. In this section, we give their definition of pivot (here we use the term encircled vertex), present some properties that will be used later in the text and discuss the existence of pivots in the graph classes studied in this thesis. We generalize the definitions and some results in [20] to apply for any positive integer k, not only for $m(G)$. This will be useful in some of our proofs, especially in Chapters 5 and 6.

Let $W \subseteq D_k(G)$ with cardinality k and let $u \in V(G) \setminus W$. If either $v \in W$ is a neighbour of u or there exists some $w \in N^W(u) \cap N^W(v)$ such that $d(w) = k - 1$, then we say that v *is reachable from* u *within* W and that w, if it exists, is a (u, v)*-bridge in* W. We say that W *encircles vertex* $u \in V \setminus W$ if every $v \in W$ is reachable from u.

Proposition 2.1. *Let $W \subseteq D_k(G)$ with cardinality k and suppose W encircles a vertex $u \in V \setminus W$ of degree less than k. Then, $|N^W(u)| \geq 2$ and $|W \setminus N(u)| \geq 1$.*

Proof: Since $d(u) < k = |W|$, there is a vertex v in $W \setminus N(u)$. As v must be reachable from u within W, there exists a vertex $w \in N^W(v) \cap N(u)^W$ with degree $k - 1$. If w is the only neighbour of u in W, then, since W encircles u, all vertices of $W \setminus w$ must be adjacent to w; but then $d(w) \geq k$, a contradiction. So u has at least two neighbours in W. □

Proposition 2.2. *Let $T = (V, E)$ be a tree and $W \subseteq V$ be a subset with cardinality at least 2. Then, there exists at most one vertex $u \in V \setminus W$ such that $|N^W(u)| \geq 2$ and $W \subseteq N(u) \cup N(N^W(u))$.* □

The propositions above gives us that W encircles at most one vertex if the graph being treated is a tree. We say that W is a *good set* if W does not encircle any vertex and every $v \in V \setminus W$ with degree at least $|W|$ is adjacent to some $w \in W$ with degree $|W| - 1$. If G does not have any good set of cardinality $m(G)$, then G is called a *pivoted*. In [20], Irving and Manlove proved the following:

Lemma 2.3. *Let T be a tree. Then T is pivoted if and only $|D(T)| = m(T)$ and $D(T)$ encircles a vertex $u \in V(T) \setminus D(T)$.*

Lemma 2.4. *Let G be a graph, and let $W \subseteq D_k(G)$ be a set of cardinality k that encircles some vertex $u \in V \setminus W$ of degree less than k. Then W is not a k-basis of G.*

Proof: Suppose on the contrary that W is the basis of a b-colouring of G with k colours. Let $W = \{v_1, \ldots, v_k\}$ and assume that v_i has colour i, for each $i = 1, \ldots, k$. If a vertex $v_i \in W$ is adjacent to u, then clearly u is not coloured i. On the other hand if v_i is not adjacent to u, then, since u is encircled by W, there is a vertex $w \in W$ with $d(w) = k-1$ that is a neighbour of u and v_i. Since w itself must be a b-vertex, and its degree is exactly $k-1$, its neighbours must all have distinct colours, so u cannot have colour i. In summary, u cannot have any colour, a contradiction. □

Theorem 2.5. *Let T be a tree. If T has a good set, then $\chi_b(T) = m(T)$; otherwise, $\chi_b(T) = m(T) - 1$.*

In Chapters 4 and 6, we prove that the theorem above also holds for outerplanar graphs with girth at least 8 and the cartesian product of trees by paths, cycles or stars. However, if G is outerplanar and we allow cycles of length less than 8, even if the blocks are either edges or induced cycles (hence, G is a cacti), the existence of a good set is not sufficient for having $\chi_b(G) = m(G)$. As an example, observe Figure 2.1. Note that $m(G) = |D(G)| = 5$ and that $D(G)$ does not encircle any vertex (the neighbours of v in $D(G)$ are not reachable from neither x nor y as $d(v) = 5 > m(G) - 1$). If we colour the dense vertices with the represented colours, vertex v cannot be a b-vertex as none of its uncoloured neighbours (vertices x and y) can be coloured with 1.

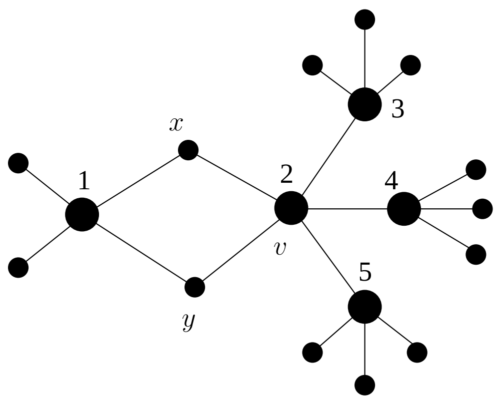


Figure 2.1: Example of a cacti G that has a good set of cardinality $m(G)$ but cannot be b-coloured with $m(G)$ colours.

Nevertheless, we characterize all cacti with m-degree at least 7 that cannot be b-coloured with $m(G)$ colours and show that they have at most two "pivots" and those are symmetric. We already knew that the analogue of Theorem 2.5 for block graphs is not valid. However, the known construction of block graphs with large difference $m(G) - \chi_b(G)$, mentioned to me by Frédéric Maffray, had the property that the "pivots" were all within the same clique. We then asked ourselves if this was always the case and arrived to the conclusion that it is not. The construction presented in Chapter 5 gives us block graphs with large difference $m(G) - \chi_b(G)$ and with a number of disjoint cliques containing "pivots" as large as that difference. Nevertheless, in Chapter 5 we solve the decision problem for some special cases and these cases also show the existence of a somewhat local structure (that can be seen as pivots) that makes it difficult to b-colour the graph with the desired number of colours. This suggests that these pivots may exist for the general case and could be characterized. However, even if we can characterize the existence of these pivots in the graph, it may be hard to recognize them. We note that deciding if $\chi_b(G) = m(G)$ is NP-complete for chordal distance-hereditary graphs (which is a super class of block graphs), even if the graph G has exactly $m(G)$ dense vertices, each with degree $m(G) - 1$ [17].

2.3 Graphs with no $K_{2,3}$ as subgraph

As mentioned in the previous section, in Chapter 5 we will construct a block graph G for which the difference $m(G) - \chi_b(G)$ is arbitrarily large. This construction requires G to have a big complete bipartite subgraph. Other known constructions also require this. For example, the complete bipartite graph $G = K_{p,q}$ itself is such that $m(G) = min\{p, q\} + 1$ and $\chi_b(G) = 2$; also, Faik [14] gave a construction of an interval graph G where $\chi_b(G) \leq m(G) - p$ and G has a subgraph $K_{2p-1,3p-4}$, $p \geq 2$. When we first observed this, we conjectured that graphs with no $K_{2,3}$ as subgraph would have $\chi_b(G) \geq m(G) - 1$ (observe that trees, cacti and outerplanar graphs are contained in this class). However, it does not hold for $C_3 \square C_3$ as shown in the following proposition (this is shown as a remark in [23]).

Proposition 2.6. $\chi_b(C_3 \square C_3) = 3$.

Proof: Denote $C_3 \square C_3$ by H and the vertex on the i-th row and j-th column by $v_{i,j}$. Suppose that $\chi_b(H) \geq m(H) - 1 = 4$. We can suppose that $v_{1,1}$ and

$v_{1,2}$ are b-vertices (there must be two in the same row) and are coloured 1 and 2, respectively. If $\chi_b(H) = 5$, as $d(v_{1,1}) = 4$, we can assume that $v_{1,3}, v_{2,1}, v_{3,1}$ are coloured 3, 4 and 5, respectively (observe Figure 2.2.(a)). But then, as $v_{1,2}$ is also a b-vertex, we must have $v_{2,2}, v_{3,2}$ coloured 5 and 4, respectively. We get a contradiction as there is no b-vertex of colour 4: $v_{2,1}$ and $v_{3,2}$ cannot be b-vertices (colour 5 is repeated in their neighbourhood) and neither $v_{2,3}$ nor $v_{3,3}$ can be coloured with 4. Now, consider $\chi_b(H) = 4$. Assume, without loss of generality, that $v_{1,3}, v_{2,1}$ are coloured 3 and 4, respectively (observe Figure 2.2.(b)). Then, $v_{3,2}$ must be coloured 4 and, as no other vertex can be coloured 4, then $v_{1,3}$ cannot be a b-vertex. If $v_{2,2}$ is a b-vertex of colour 3, then $v_{2,3}$ is coloured 1 and, as neither $v_{3,1}$ nor $v_{3,3}$ can be coloured 1, we get $v_{3,2}$ cannot be a b-vertex. Hence, $v_{2,1}$ must be a b-vertex of colour 4 and, consequently, $v_{3,1}$ is coloured 2. We then have situation in Figure 2.2.(b), where there is no colour with which we can colour $v_{3,3}$, a contradiction. In the case where $v_{3,1}$ is a b-vertex of colour 3, one can verify that we get an analogous situation. Finally, as $\chi_b(H) \geq \omega(H) = 3$, the result follows. □

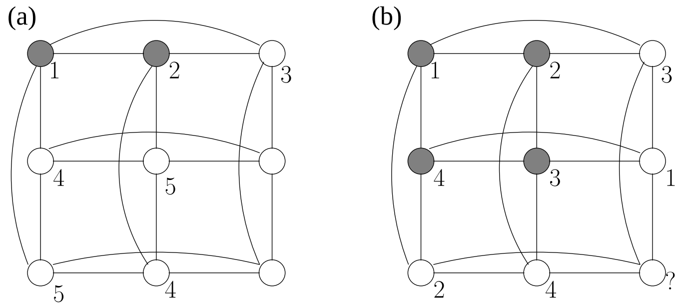


Figure 2.2: Cartesian product $H = C_3 \,\square\, C_3$ is such that $\chi_b(H) < m(H) - 1$.

Thus, $\chi_b(C_3 \,\square\, C_3) < m(C_3 \,\square\, C_3) - 1$ and the conjecture does not hold. Nevertheless, the conjecture seems to work in a great number of cases, such as for trees, the ones presented in this thesis and the ones cited below.

Theorem 2.7 (Faik[14]). *Let $H = P_{n_1} \,\square\, P_{n_2} \,\square \cdots \square\, P_{n_k}$, $n_i \geq 6$, for every $i = 1, \cdots, k$. Then $\chi_b(H) = m(H)$. The same is valid for the cartesian product of k cycles of length at least 6.*

Let $H = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_k}$ as in the theorem above. It is actually proven that $\chi_b(H) = 2k+1$. However, as $\Delta(H) = 2k$ and $2k+1 = \chi_b(G) \leq m(H) \leq$

$\Delta + 1$, we have that $m(H) = 2k + 1$. One can also verify that the values in the theorem below are at least $m(G) - 1$, where G is the corresponding graph.

Theorem 2.8 (Kouider, Mahéo[26]). *The following are valid:*

- $\chi_b(K_{1,n} \square K_{1,n}) = n + 2$, *if* $n \geq 2$;
- $\chi_b(K_{1,n} \square P_k) = min\{k, n+3\}$, $n \geq 3$, $k \geq 4$, *except when* $k = n + 3$ *or* $k = n + 4$, *when* $\chi_b(K_{1,n} \square P_k) = n + 2$;
- $\chi_b(Q_1) = \chi_b(Q_2) = 2$ *and* $\chi_b(Q_n) = n + 1$, *for* $n \geq 3$, *where* Q_n *is the hypercube with dimension* n;
- $\chi_b(C_n \square C_{n'}) = \chi_b(C_n \square P_k) = \chi_b(P_k \square P_{k'}) = 5$, $n, n' \geq 6$ *and* $k, k' \geq 7$.

Theorem 2.9 (Kouider, Mahéo[26]). *Let G and H be graphs such that G has a $\chi_b(G)$-basis W_G and H has $\chi_b(H)$-basis W_H where W_G and W_H are stable sets. Then $\chi_b(G \square H) \geq \chi_b(G) + \chi_b(H) - 1$ and $G \square H$ has a $\chi_b(G \square H)$-basis that is a stable set.*

The following corollary extends the result of Theorem 2.7 to the cartesian product of some paths of size smaller than 6.

Corollary 2.10. *Let $H = P_p \square P_q \square P_k$. If $k \geq 7$ and either $p \geq q \geq 5$ or $p = 4$ and $q \geq 7$ or $p = 3$ and $q \geq 11$, then $\chi_b(H) = m(H)$.*

Proof: Let $P_k = \{a_1, a_2, \cdots, a_k\}$, $k \geq 7$. Trivially, $\chi_b(P_k) = m(P_k) = 3$ and if we give the colours $1, 2, 3, 1, 2, 3, \cdots$ to $a_1, a_2, a_3, \cdots, a_k$ in this order, we obtain a b-colouring with 3 colours with basis $\{v_2, v_4, v_6\}$, which is also a stable set. Now, consider $H' = P_p \square P_q$ and denote the vertex on the i-th row and j-th column of H' by $v_{i,j}$. As $\Delta(H') = 4$, we have $m(H') = 5$. If $p, q \geq 5$, observe that the colouring presented in Figure 2.3 can be easily extended to a b-colouring of H' with 5 colours (every remaining uncoloured vertex has degree at most 4 and we already have the b-vertices needed). Also, the stable $\{v_{2,2}, v_{2,4}, v_{3,3}, v_{4,2}, v_{4,4}\}$ is a basis of this b-colouring. Analogously, if $p = 3$ and $q \geq 11$, the precolouring presented in Figure 2.5 can be extended to a b-colouring with basis $\{v_{2,2}, v_{2,4}, v_{2,6}, v_{2,8}, v_{2,10}\}$, and if $p = 4$ and $q \geq 7$, the precolouring in Figure 2.4 can be extended to a b-colouring with basis $\{v_{2,2}, v_{2,4}, v_{2,6}, v_{3,3}, v_{3,5}\}$. Finally, note that $m(H) = 7$ (as $\Delta(H) = 6$ and

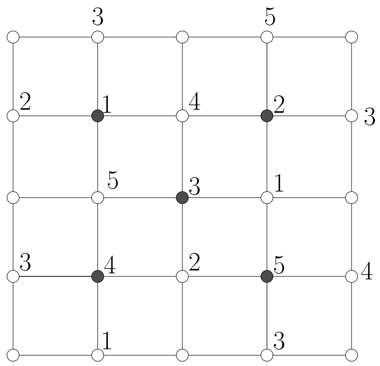


Figure 2.3: Partial colouring of $P_5 \square P_5$ with $m(P_5 \square P_5) = 5$ colours; the grey vertices are b-vertices.

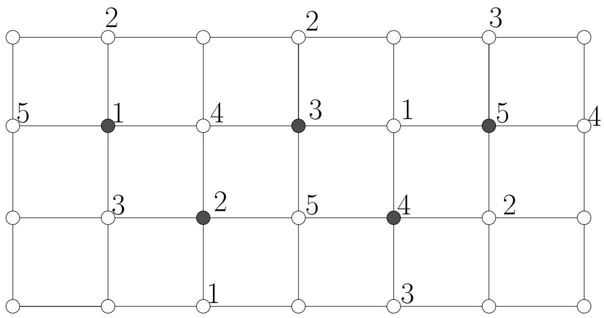


Figure 2.4: Partial colouring of $P_4 \square P_7$ with $m(P_4 \square P_7) = 5$ colours; the grey vertices are b-vertices.

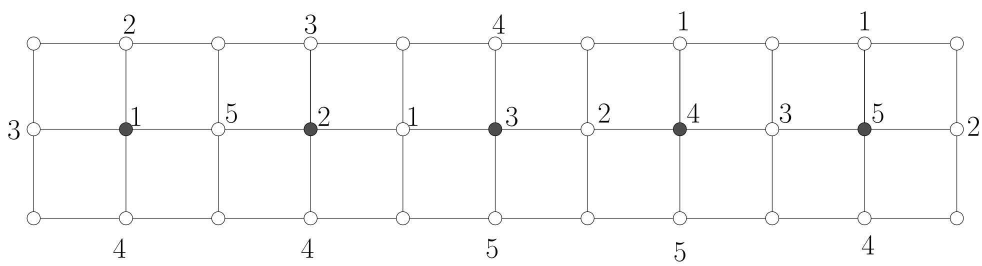


Figure 2.5: Partial colouring of $P_3 \square P_{11}$ with $m(P_3 \square P_{11}) = 5$ colours; the grey vertices are b-vertices.

there are more than 6 vertices with degree 6) and, by Theorem 2.9, $\chi_b(H) = 7$. □

In [33], Sahili and Kouider pose the question: *Is it true that every d-regular graph with girth at least 5 satisfies* $\chi_b(G) = d + 1$? Observe that this question has the same general form as our conjecture: a d-regular graph G with girth at least 5 has no $K_{2,3}$ as subgraph and $m(G) = d + 1$. Unfortunately, the Petersen graph gives the answer *no* to the question. This counter-example was shown by Blidia, Maffray and Zemir in [2], where they also conjecture that:

Conjecture 2.11 ([2])**.** *If G is a d-regular graph with girth at least 5 and G is not the Petersen graph, then $\chi_b(G) = d + 1$.*

Some partial positive answers to this conjecture can be found in [4], [2] and [33]. All these results on graphs with no $K_{2,3}$'s as subgraphs, as well as the results presentend in this thesis, indicate that there may be still some hope for our initial guess. We then make the following conjecture. (We remark that the Petersen graph is not a counter-example to our conjecture, as it can be b-coloured with 3 colours and has m-degree 4.)

Conjecture 2.12. *If G is a graph that does not have a $K_{2,3}$ as subgraph, not necessarily induced, and $G \neq C_3 \,\square\, C_3$, then $\chi_b(G) \geq m(G) - 1$.*

Observe that if G and H have no $K_{2,3}$ subgraphs (not necessarily induced), then so does $G \,\square\, H$. This motivated us to investigate the cartesian product of trees by other graphs with no $K_{2,3}$ subgraphs. We found that if T is a tree and G is either the path P_k, $k \geq 5$, or the cycle C_q, $q \geq 4$, or the star $K_{1,r}$, $r \geq 2$, then $\chi_b(T \,\square\, G) \geq m(T \,\square\, G) - 1$. The proofs of these results are the subject of Chapter 6.

Finally, we mention, and disproove, a conjecture proposed by Havet, Linhares and Sampaio in [17]. A graph G is a *tight graph* if $|D(G)| = m(G)$ and $d(v) = m(G) - 1$, for all $v \in D(G)$.

Conjecture 2.13. *Let G be a tight graph such that:*

- *For every edge $(u, v) \in E(G)$, one of its endpoints is dense, and the other is non-dense, and*
- $|N(u) \cap N(v)| \leq 1$, *for all pair of vertices $u, v \in D(G)$, $u \neq v$.*

Then, $\chi_b(G) = m(G)$.

We construct a class of graphs that violate the above conjecture. Let m be any positive integer greater than 3. We construct G_m as follows: let $W = \{w_1, \cdots, w_m\}$, $X = \{x_2, \cdots, x_m\}$, $Y = \{y_3, \cdots, y_m\}$ and $S^3_{m-4}, \cdots, S^m_{m-4}$ be a collection of $m-2$ disjoint stable sets of size $m-4$. G_m has vertex set equal to $\bigcup_{i=3}^m S^i_{m-4} \cup W \cup X \cup Y \cup \{x\}$ and edge set such that:

$$N(w_i) = S^i_{m-4} \cup \{x, x_i, y_i\}, \forall i \in [3, m]$$

$$N(w_1) = \{x_2, \cdots, x_m\}$$

$$N(w_2) = \{x_2, y_3, \cdots, y_m\}$$

We have that $d(x) = m-2$, $d(x_i) = 2$, for $i = 2, \cdots, m$, $d(y_i) = 2$, for $i = 3, \cdots, m$, $d(y) = 1$, for all $y \in \bigcup_{i=3}^m S^i_{m-4}$, and $d(w_i) = m-1$, for $i = 1, \cdots, m$. Thus, G_m is a tight graph with $m(G_m) = m$. Also, it is easy to verify that W and $V(G_m) \setminus W$ are stable sets and that $|N(w_i) \cap N(w_j)| = 1$, for all pair of vertices $w_i, w_j \in W$, $i \neq j$. So, G_m satisfies the constraints of Conjecture 2.13. Suppose that the conjecture holds and let ψ be an optimal b-colouring of G_m. As G_m is tight, we know that W is a basis of ψ and that there is no b-vertex in $V(G_m) \setminus W$ (i.e., W is the only basis of ψ). Assume, without loss of generality, that $\psi(w_i) = i$, for all $i \in \{1, \cdots, m\}$. As ψ is proper and $W \setminus \{w_1, w_2\} \subseteq N(x)$, we have that $\psi(x) \in \{1, 2\}$; without loss of generality, suppose that $\psi(x) = 1$. As $d(w_i) = m-1$, for all $i \in \{3, \cdots, m\}$, we have that x is the only neighbour of w_i coloured with 1. Thus, as $\psi(x_2) \neq 1$ and $\psi(y_i) \neq 1$, for all $i \in \{3, \cdots, m\}$, we have that w_2 has no neighbour coloured with colour 1, a contradiction. We remark that $\chi_b(G_m) = m-1$. It suffices to b-colour $G_m \setminus N[w_1]$ with $m-1$ colours, obtaining ψ, then give colour $\psi(w_2)$ to w_1, $\psi(w_3)$ to x_2 and $\psi(y_i)$ to x_i, for $i \in \{3, \cdots, m\}$. Observe the example in Figure 2.6, where $m = 5$ (the stable sets S^i_{m-4}'s are not represented as they can be coloured separatedly so that vertices w_3, w_4 and w_5 are b-vertices). So, G_m does not violate Conjecture 2.12.

2.4 Extending a precolouring

Let $W \subseteq V$. We say that a path P between $u, v \in W$ is a *link of* W if P has length at most three and every internal vertex of P is not in W. If $x \notin W$ is

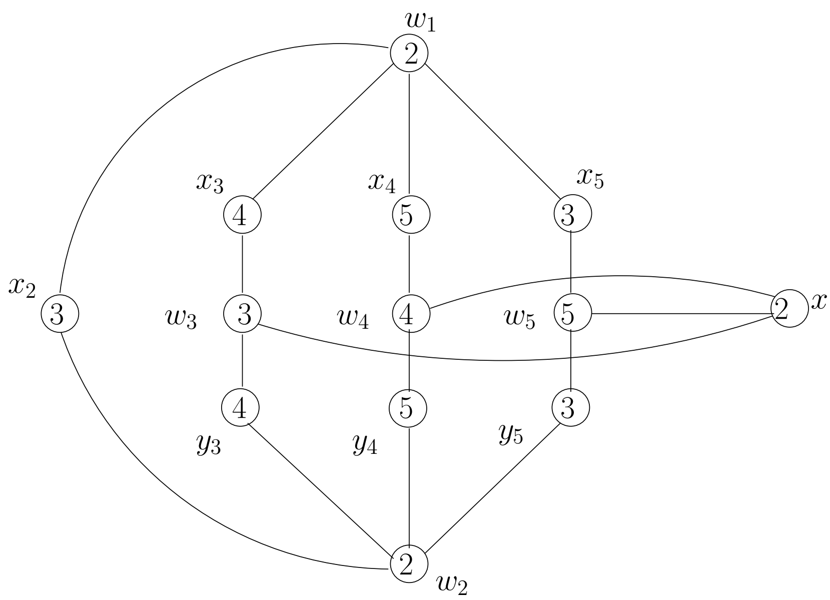


Figure 2.6: Precolouring G_5 with 4 colours.

in a link, we say that x is a *link vertex of* W (this definition is the same as inner vertex given in [20]) and if there exists a link between $u, v \in W$, we say that u and v are *linked*. We denote the set of link vertices of W by L_W and the set of extremities of links passing through $x \in L_W$ by $L_W(x)$. If there is no ambiguity, we use only L and $L(x)$. Note that $|L(x)| \geq 2$ and observe also that if $u \in N^W(x)$ and $w \in L(x) \setminus \{u\}$, then there is a link between u and w containing x (and hence $N^W(x) \subseteq L_W(x)$).

Let $G = (V, E)$ be any graph and $W \subseteq D_k(G)$ of cardinality k. Let ψ be a proper partial colouring of G with k colours where every vertex of W has a different colour in $\{1, \cdots, k\}$ and consider $w \in W$; we say that a colour c is *repeated* in $N(w)$ if there is more than one vertex in $N(w)$ coloured with c; the *redundancy of* ψ *in* $N(w)$ is the value $r_\psi(w) = |\{u \in N(w) : \psi(u) \neq \emptyset\}| - |\psi(N(w))|$ (number of coloured neighbours of w minus number of different colours appearing in the neighbourhood of w); and the *missing colours of* w *in* ψ are colours in the set $M_\psi(w) = \{1, \cdots, k\} \setminus \psi(N[w])$ (sometimes we may use only $r(w)$ and $M(w)$, if there is no ambiguity). A proper partial colouring ψ of G is called an *unsaturated precolouring of* G *with candidate set* W if:

- Each vertex of W is coloured with a different colour; and
- $r(w) \leq d(w) + 1 - |W|$, for all $w \in W$.

The following lemma will be of use in the construction of an unsaturated precolouring in the next chapters.

Lemma 2.14. *Let ψ be an unsaturated precolouring with candidate set W. Then, u has at least $M_\psi(u)$ uncoloured neighbours, for all $u \in W$.*

Proof: Suppose that ψ is such a precolouring and consider an ordering of W, $v_1, \cdots, v_k$, where v_i is coloured with colour i. For each $v_i \in W$, let U_i be the set of uncoloured neighbours of v_i. We want to prove that $|U_i| \geq |M(v_i)|$, for all $v_i \in W$. So, consider any $v_i \in W$ and denote by s the value $|\psi(N(v_i))|$. As ψ is unsaturated, we have: $r(v_i) \leq d(v_i) + 1 - k$; so, (I) $k - 1 \leq d(v_i) - r(v_i)$. In addition: (II)$|M(v_i)| = k - s - 1$ (number of missing colours equals number of colours, minus number of colours present in $N(v_i)$, minus colour i) and (III) $|U_i| = d(v_i) - (s + r(v_i))$ (number of uncoloured neighbours equals number of neighbours minus number of coloured neighbours). So: $|U_i| =^{(III)} d(v_i) - r(v_i) - s \geq^{(I)} k - 1 - s =^{(II)} |M(v_i)|$. □

Note that if all the vertices are coloured in an unsaturated b-precolouring with candidate set W, then this is also a b-colouring of G with basis W. This is true because there are no uncoloured vertices in the neighbourhood of a vertex $u \in W$ and, by Lemma 2.14, $M_\psi(u) = \emptyset$, for all $u \in W$.

Now, let ψ be an unsaturated precolouring with candidate set W. We show how to obtain a b-colouring of G from ψ, when W and ψ satisfy some constraints.

Lemma 2.15. *Let W be such that $|W| = k$ and every vertex not in W with degree at least k is a link vertex of W or is adjacent to a vertex in W with degree $k-1$. Also, let ψ be an unsaturated precolouring of G with candidate set W such that all link vertices of W are coloured. Then we can extend ψ to a b-colouring of G with basis W.*

Proof: Start by "uncolouring" every vertex which is not in $W \cup N(W)$. Define an ordering $v_1, \cdots, v_k$ of W such that $\psi(v_i) = i$ and let U_i be the set of uncoloured neighbours of v_i, $i = 1, \cdots, k$. Now, let $u \in U_i$. As all link vertices are coloured, the only vertex in W adjacent to u is v_i. In addition, note that if $u' \in V \setminus W$ is in a link between $v_j, v_l \in W$, then u' is adjacent to at least one between v_j and v_l. Suppose, without loss of generality, that it is adjacent to v_j. So, if $u \in N(u')$, then $j = i$, otherwise u is in the link $\langle v_i, u, u', v_j \rangle$, a contradiction as u is uncoloured. Thus, the only coloured neighbours of u is v_i and possibly some $u' \in N(v_i)$; then, u can be coloured with a missing colour in $N(v_i)$. So, we colour $|M(v_i)|$ arbitrary uncoloured neighbours of v_i, for every $v_i \in W$ (we know that $|U_i| \geq |M(v_i)|$, by Lemma 2.14). After taking this step for every v_i, we have that each v_i is already a b-vertex. Obviously, every vertex in W with degree $k-1$ has all of its neighbours already coloured. Thus, if u is still uncoloured, then $d(u) \leq k-1$ and there exists some colour that does not appear in the neighbourhood of u, with which we can colour u. So, we can colour the uncoloured vertices recursively until we obtain a proper colouring with k colours where each v_i is a b-vertex. This proves the lemma. □

Note that if W is a good set, then the constraint over W in the lemma above is satisfied.

2.5 Tree Strategy

In this section, we explain how to colour a tree using the algorithm presented in [20]. Actually, we present the algorithm in a more general form in order to be able to use it as a procedure in some of our proofs. As a result, we can see that it actually works for any graph with large girth. We also point out some properties that can be ensured by the algorithm.

Let $T = (V, E)$ be a forest and consider $W \subseteq D_k(G)$ of cardinality k. Number the vertices of W, $v_1, \cdots, v_k$, and colour v_i with colour i, $i = 1, \cdots, k$; let ψ be the obtained partial colouring. First, we want to colour the link neighbours of v_i, for all $v_i \in W$ such that $|N^L(v_i)| \geq 2$. So, consider v_i and let $N^L(v_i) = \{x_1, \cdots, x_s\}$, $s \geq 2$. Suppose, without loss of generality, that $x_1, \cdots, x_p$ are uncoloured, while $x_{p+1}, \cdots, x_s$ are already coloured, $1 \leq p \leq s$. If $p \geq 2$, let $v_{i_j} \in L(x_j) \setminus \{v_i\}$, for $1 \leq j \leq p$; permute the colours $\langle i_1, \cdots, i_p \rangle$ on the vertices $x_1, \cdots, x_p$ in such a way that, at the end, $\psi(x_j) \neq i_j$, for $j = 1, \cdots, p$. Otherwise (i.e., $p = 1$), give colour j to x_1, for any $v_j \in L(x_2) \setminus \{v_i\}$. Note that the following property holds:

Link Property (LP): if $x \in L$ is coloured with i, then there exists

$v_j \in N^W(x)$ such that v_i and v_j are linked.

Note that the only constraints imposed over T and W are that T is acyclic and W has cardinality k. It is easy to see that if T has girth greater than 10, the procedure above can still be applied. At the end, we obtain a precolouring satisfying (LP) where $r(v_i) = 0$, for all $v_i \in W$, and $N^L(v_i)$ is coloured, for all $v_i \in W$ such that $|N^L(v_i)| \geq 2$. Now, let $x \in L$ be still uncoloured; we know that $N^L(v_i) = \{x\}$, for all $v_i \in N^W(x)$. If x has a link neighbour y, then let $v_i \in N^W(y)$; trivially, $\psi(y) \neq i$ and, as we will see, we can suppose that (1) no other link neighbour of x is coloured with colour i. Also, as T is acyclic and v_j has no other link neighbours, for all $v_j \in N^W(x)$, we have that colour i does not appear in $N(v_j)$. Thus, we give colour i to x; note that assumption (1) can be made if we colour every uncoloured link vertex having some link neighbour in this way. Also, note that (LP) still holds and that, again, if T has girth greater than 10, the procedure still produces an unsaturated precolouring of T with k colours.

Finally, we want to colour any vertex $x \in L$ such that x has no neighbour in L and is the only link neighbour of v_i, for every $v_i \in N^W(x)$. We need to make an assumption about the set W; we suppose that it does not encircle any vertex. Thus, we know that there must exist a vertex $v_i \in W$ not reachable from x and we can colour x with i. Since $N^L(x) = \emptyset$ and $v_i \notin N(x)$, the colouring is still proper; also, if there exists $v_j \in N(x) \cap N(v_i) \cap W$, as v_i is not reached by x, we must have $d(v_j) > k-1$ and $1 = r(v_j) \leq d(v_j) - k + 1$. So, the following lemma holds:

Lemma 2.16. *Let $T = (V, E)$ be a forest and $W \subseteq D_k(G)$ be such that $|W| = k$ and W does not encircle any vertex. Then, there exists an unsaturated precolouring with candidate set W such that L_W is coloured. In addition, if $\psi(x) = i$, for $x \in L$, then either (LP) holds, or $N^L(x) = \emptyset$ and $N^L(v) = \{x\}$, for all $v \in N^W(x)$.*

The following remark will be used in Chapter 6.

Remark 2.17. *In Lemma 2.16, if $W \not\subseteq N(u) \cup N(N^W(u))$, for all $u \in V \setminus W$, we can also ensure that no colour is repeated in $N(v)$, for all $v \in W$.*

Observe that, although the obtained precolouring is an unsaturated precolouring of T with candidate set W, it cannot always be extended by Lemma 2.15 as there may exist vertices with degree greater than or equal to $|W|$ that are not link vertices and are not adjacent to any vertex of W with degree $|W| - 1$. However, we know the lemma can be applied when W is a good set. In the next section, we prove that if G has girth at least 8, then a good set of G can be found in polynomial time, if one exists; otherwise, we prove that $\chi_b(G) = m(G) - 1$. As a consequence, we have the following.

Corollary 2.18. *Let G be a graph with girth at least 11. If G does not have a good set, then $\chi_b(G) = m(G) - 1$; otherwise, $\chi_b(G) = m(G)$. Furthermore, an optimal b-colouring of G can be found in polynomial time.*

2.6 Graphs with no good set and large girth

The main result of this section is the following.

Theorem 2.19. *Let G be a graph with girth at least 8. Suppose that G has no good set. Then $\chi_b(G) = m(G) - 1$.*

It is easy to check in polynomial time if a given set W encircles any vertex x. On the other hand, $D(G)$ may in general contain exponentially many subsets of size $m(G)$. In the following lemma, we show that we can determine whether a good set exists by testing only a few subsets of $D(G)$.

Lemma 2.20. *Let G be a graph with girth at least 8. Then G does not have a good set if and only if $|D(G)| = m(G)$ and $D(G)$ encircles a vertex of $V \setminus D(G)$. Moreover, a good set of G (if any exists) can be found in polynomial time.*

Proof: Here is a polynomial time algorithm that determines a good set in G, if any exists.

First suppose that $|D(G)| = m(G)$. Then the only subset W of $D(G)$ of size $m(G)$ is $D(G)$ itself. If $D(G)$ encircles a vertex, then $D(G)$ is not a good set, and the algorithm returns the answer that G has no good set. Else, the algorithm returns the good set $D(G)$.

Now suppose that $|D(G)| > m(G)$. Let W be a subset of $D(G)$ of size $m(G)$ that contains all vertices with degree at least $m(G)$ (we know that there are at most $m(G)$ such vertices, by the definition of $m(G)$). If W does not encircle any vertex, then it is a good set and the algorithm returns it. Else, let u be any vertex that is encircled by W. Consider the sets $N_1 = N(u) \cap W$ and $N_2 = W \setminus N_1$. By Proposition 2.1, we have $|N_1| \geq 2$, $|N_2| \geq 1$, and every vertex of N_2 has a neighbour in N_1. Since G contains no C_3, C_4 or C_5, the sets N_1 and N_2 are stable sets and every vertex of N_2 has only one neighbour in N_1. Pick any $v_2 \in N_2$ and let v_1 be the unique neighbour of v_2 in N_1; also, let $v^3 \in N_1 \setminus \{v_1\}$ (recall that $|N_1| \geq 2$). Since u is encircled by W, we know that $d(v_1) = m(G) - 1$. Pick any $w \in D(G) \setminus W$ and consider the subset $W' = (W \setminus \{v_1\}) \cup \{w\}$. If W' does not encircle any vertex, then W' is a good set, and the algorithm returns it. Else, let u' be a vertex of $V \setminus W'$ that is encircled by W'.

Suppose that $w \neq u$. Note that u does not reach v_2 within W', since any path between u and v_2 different from $\langle u, v_1, v_2 \rangle$ must have length at least 6; hence, W' does not encircle u. Also, v_1 does not reach v_3, because N_1 is a stable set and every vertex in N_2 has exactly one neighbour in N_1; hence, W' also does not encircle v_1. Thus $u' \neq u, v_1$. Since v_1 is the only neighbour of v_2 in W, and $v_1 \notin W'$, it follows that either u' is adjacent to v_2 or $w \in N(u') \cap N(v_2)$. Also, since u' must reach v_3, there exists a path of length at most two between u' and v_3. In any case, we obtain a cycle of length at most 7, a contradiction.

Therefore we must have $w = u$. If there exists any $x \in N_2 \setminus N(v_1)$, then W' does not encircle any vertex (because any vertex that reaches v_2 and x within W' would lie in a cycle of length less than 8 in G). So, suppose that $W' \subseteq N(v_1) \cup N(u) \cup \{u\}$. Then the set $W'' = (W \setminus \{v_2\}) \cup \{u\}$ also does not encircle any vertex and, since $v_2 \in N(v_1)$ and $d(v_1) = m(G) - 1$, W'' is a good set, and the algorithm returns this set. □

Proof of Theorem 2.19. By Lemma 2.20, we know that $D(G) = m(G)$ and $D(G)$ encircles some vertex $u \in V \setminus D(G)$. By Lemma 2.4, G does not have a b-colouring with $m(G)$ colours. Let us show that it has a b-colouring with $m(G) - 1$ colours. Let $p = m(G) - 1$. As in the proof of Lemma 2.20, define $N_1 = N(u) \cap W$ and $N_2 = W \setminus N_1$. We know that N_1 and N_2 are stable sets, that every vertex of N_2 has a unique neighbour in N_1, and that $N_2 \neq \emptyset$. Call $v_1, \ldots, v_p, w$ the vertices of $D(G)$, such that $v_1 \in N_2$ and $w \in N(v_1) \cap N_1$. Assign colour i to v_i for all $i = 1, \ldots, p$, colour 1 to u, and colour h to w, for some h such that $v_h \in W \setminus N(w)$ (such an h exists, because $d(w) = p$ and $u \in N(w)$). This partial colouring is proper and, for all $i = 1, \ldots, p$, vertex v_i does not have two coloured neighbours of the same colour. Let $S = N(D(G)) \setminus \{u\}$. Then S is a stable set, for otherwise G would contain a cycle of length at most 7. So we can, for each $i = 1, \ldots, p$, colour the uncoloured neighbours of v_i in such a way that all colours different from i appear in $N(v_i)$ (because all uncoloured neighbours are in the stable set L). Finally, the vertices that are still uncoloured have degree strictly less that p, so the colouring can be extended to them greedily. Thus we obtain a b-colouring of G with $m(G) - 1$ colours, where $\{v_1, \ldots, v_p\}$ is a basis. □

Chapter 3

Cacti

The results presented in this chapter were obtained in colaboration with Victor Campos, Cláudia Linhares Sales and Frédéric Maffray. They were presented in the *V Latin-American Algorithms, Graphs and Optimization Symposium* (LAGOS'09) [5].

In this chapter, we consider G to be a *cactus*, which is a graph that does not contain two cycles that share an edge. As an example, observe Figure 3.1. In this chapter, we prove that if G is a cactus and $m(G) \geq 7$, then the difference between $\chi_b(G)$ and $m(G)$ is at most one. The proof resembles the one for trees. First, we present a family of cacti that cannot be b-coloured with $m(G)$ colours. These cacti are characterized either by the existence of some "pivots" or by being isomorphic to some "anomalous" configurations. This is done in Section 3.1, where we also prove that the number of "pivots" in G is small. Then, in Section 3.2, we define a quasi-good set (which is basically a subset of the dense vertices that has no pivots, with some further properties) and show how to find such a set, if one exists. Next, in Section 3.3, we show how to obtain a b-colouring with $m(G) - 1$ colours of a graph in the family presented in Section 3.1, thus proving that the b-chromatic number of those graphs is actually equal to $m(G) - 1$. Finally, in Section 3.4 we show how to construct a b-colouring of G with $m(G)$ colours when G has a good set and is such that $m(G) \geq 7$. The following trivial lemma will be useful in some of the proofs.

Lemma 3.1. *Let G be a cactus and U and U' be two disjoint subsets of $V(G)$. If $G[U]$ and $G[U']$ are connected, then U has at most two neighbours in U'.*

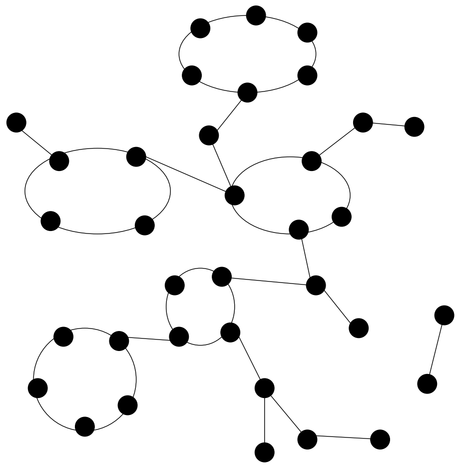


Figure 3.1: Example of a cactus.

3.1 Pivots and anomalous graphs

In this section, we present a family of cacti for which the difference between the b-chromatic number and the m-degree is at least 1. We already know, by Lemma 2.4, that if G is a cactus such that every subset of $D(G)$ with cardinality $m(G)$ encircles a vertex not in $D(G)$, then $\chi_b(G) < m(G)$. But also we define the following:

Let G be a cactus and W be a subset of $m(G)$ dense vertices of G. We say that W *encircles the pair* $x, y \in V$ if $x, y \notin W$, W does not encircle x or y and one of the following occurs:

E1. There are $W' \subset W$ and $u, v \in W'$ such that $|W'| = m(G)-1$, $\langle x, u, y, v\rangle$ is a cycle and:

(a) $d(u) = d(v) = m(G) - 1$, $N^{W'}(u) \neq \emptyset$, $N^{W'}(v) \neq \emptyset$ and every $w \in W' \setminus \{u, v\}$ is adjacent to u or v; or

(b) $d(u) = m(G) - 1$ and every $w \in W' \setminus \{u, v\}$ is adjacent to u; or

(c) $d(u) = m(G)$, $d(v) = m(G) - 1$, $N^{W'}(u) \neq \emptyset$, $N^{W'}(v) \neq \emptyset$ and every $w \in W \setminus \{u, v\}$ is adjacent to u or v; or

(d) $d(u) = m(G)$ and every $w \in W \setminus \{u, v\}$ is adjacent to u.

E2. There are $W' \subseteq W$ and $u, v, w \in W'$ such that $|W'| \geq m(G) - 1$, $\langle x, u, v, y, w \rangle$ is a cycle, $d(u) = d(v) = m(G) - 1$, every $w' \in W' \setminus \{u, v, w\}$ is adjacent to w, and either

(a) $W' \subset W$ and $d(w) = m(G) - 1$; or

(b) $W' = W$ and $d(w) = m(G)$.

Lemma 3.2. *Let G be a cactus and $W \subseteq D(G)$ be of cardinality $m(G)$. If W encircles a pair of vertices x, y, then W is not the basis of a b-colouring of G with $m(G)$ colours.*

Proof: Suppose on the contrary and let ψ be a b-colouring of G with W as basis. Consider that $\{1, \cdots, m(G)\}$ are the colours used in ψ and denote the vertex of W coloured with i by v_i, $i \in \{1, \cdots, m(G)\}$. Suppose first that E1 occurs and consider, without loss of generality, that $\langle v_1, x, v_2, y \rangle$ is a cycle in G. Suppose E1a or E1b occurs and let $W' = N^W[v_1] \cup N^W[v_2]$. If $\psi(x) = \psi(y)$, as at least one between v_1, v_2 has degree $m(G) - 1$, say v_1, we have that v_1 cannot be a b-vertex, a contradiction. So, consider that $\psi(x) = j$, for some $v_j \in W'$ (recall that $|W \setminus W'| = 1$). Obviously, $j \notin \{1, 2\}$ and, hence, v_j is adjacent to either v_1 or v_2, say v_1. Observe that, in this case, $d(v_1) = m(G) - 1$ and we get a contradiction as the colour j appears twice in $N(v_1)$. Now, consider that E1c or E1d occurs and let $\psi(x) = i$ and $\psi(y) = j$. We know that $i, j \notin \{1, 2\}$ and v_i is either adjacent to v_1 or to v_2, the same being valid for v_j. Suppose, without loss of generality, that $d(v_1) = m(G)$. If E1c occurs, as $d(v_2) = m(G) - 1$ (i.e., we cannot repeat colours in $N(v_2)$), we have $v_i, v_j \in N(v_1)$; this also trivially holds when E1d occurs. We then get a contradiction as $r(v_1) \geq 2 > d(v_1) - m(G) + 1$. Finally, consider that E2 occurs and suppose, without loss of generality, that $\langle v_1, x, v_2, v_3, y \rangle$ is a cycle in G. One can verify, by analogous arguments, that v_1 cannot be a b-vertex in ψ as there are too many colours repeated in $N(v_1)$. $\square$

So, we know that if G is such that every subset of $D(G)$ with cardinality $m(G)$ either encircles a vertex or a pair of vertices, then $\chi_b(G) < m(G)$. Unfortunately, these definitions are not sufficient to describe all the cacti with $\chi_b(G) < m(G)$. Observe, for example, the graph G of Figure 3.2. We have $m(G) = 4$, the big vertices represent the dense vertices; if we colour $D(G)$ with $\{1, 2, 3, 4\}$ from left to right, we get that both u and v must be coloured 1 in order to turn the dense vertices of the cycle into b-vertices. Thus, G cannot be b-coloured with $m(G) = 4$ colours. Now, consider G to be any graph in Figure 3.3 and let $H = G - \{(u,v)\}$ (remove only the edge (u,v)). We have $m(H) = m(G)$ and it is not hard to verify that any b-colouring of H with $m(H)$ colours is such that u and v have the same colour. Thus, G cannot be b-coloured with $m(G)$ colours. Actually, even if the black vertices in Figures 3.2 and 3.3 have degree bigger than $m(G)-1$, the graph still cannot be b-coloured with $m(G)$ colours. We then say that a cactus G is *anomalous* if there exists $H \subseteq G[D(G) \cup N(D(G))]$ isomorphic to the graph in Figure 3.2 or to some graph in Figure 3.3 such that $m(H) = m(G)$, $|D(G)| = m(G)$ and $d(v) = m(G) - 1$, for every grey vertex in the figures.

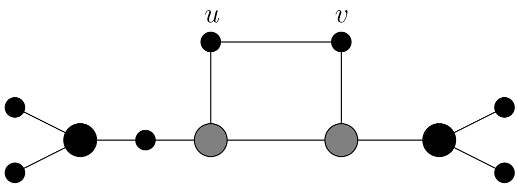


Figure 3.2: Anomalous graph with $m(G) = 4$.

We now analyse the possible number of encircled vertices and pairs of vertices of a subset W. The following proposition will be useful:

Proposition 3.3. *Let W be any set of $m(G)$ dense vertices and let $u \in V \backslash W$ be encircled by W, or be one of the vertices of a pair encircled by W. Then, there are at least two vertices in W adjacent to u.*

Lemma 3.4. *Let G be a cactus and W be a subset of $m(G)$ dense vertices of G. If W encircles two vertices, x and y, then one of the following occurs:*

F1. There are $u, v \in W$ such that $\langle x, u, y, v \rangle$ is a cycle, $d(u) = d(v) = m(G) - 1$ and every $w \in W \setminus \{u, v\}$ is adjacent to u or v; or

F2. There are $u, v, w \in W$ such that $\langle x, u, v, y, w \rangle$ is a cycle, $d(u) = d(v) = d(w) = m(G) - 1$ and every $w' \in W \setminus \{u, v, w\}$ is adjacent to w.

F3. $W = \{v_1, v_2, v_3, v_4\}$, $\langle x, v_1, v_2, y, v_3, v_4 \rangle$ is a cycle in G and $d(v_i) = 3$, $i = 1, \cdots, 4$.

Proof: Note that $|N^W(x)| \leq 2$, otherwise, by Lemma 3.1 applied to x and $y \cup N^W(y)$, some neighbour of x would not be reached by y. So, by Proposition

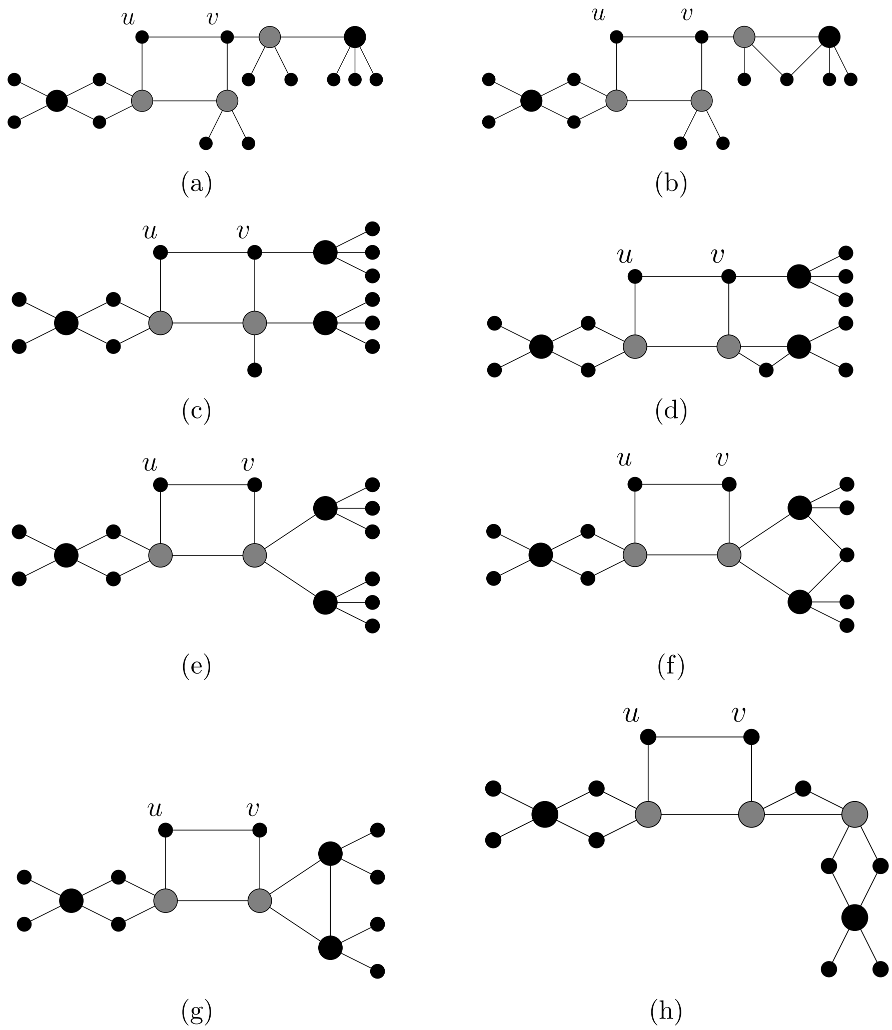


Figure 3.3: Anomalous graphs with $m(G) = 5$.

3.3, we have $|N^W(x)| = 2$. Analogously, we have $|N^W(y)| = 2$. First, suppose x and y have at least one common neighbour in W, say w, and let $w_x \in N^W(x) \setminus \{w\}$ and $w_y \in N^W(y) \setminus \{w\}$. Clearly, if $w_x = w_y$, we have situation F1, and if w_y is a (y, w_x)-bridge, then we have situation F2. So, suppose that $w_x \neq w_y$ and $(w_x, w_y) \notin E(G)$. Since $N^W(y) = \{w, w_y\}$, w must be the (y, w_x)-bridge; analogously, w is also the (x, w_y)-bridge. Observe Figure 3.4 and note that, in this case, w must also be a (x, w')-bridge, for all $w' \in W \setminus \{w_x\}$, otherwise, w' is not reached by y. However, in this case, $(W \setminus \{w\}) \cup \{x, y\} \subseteq N(w)$, i.e., $d(w) \geq m(G) + 1$, a contradiction. Now, suppose x, y have no common neighbours in W and let $N^W(x) = \{u_x, v_x\}$ and $N^W(y) = \{u_y, v_y\}$. Note that, if both u_x and v_x have the same bridge to y, say u_y, then x does not reach v_y, a contradiction. So, they have different bridges and must be bridges themselves, i.e., F3 occurs for x, y.

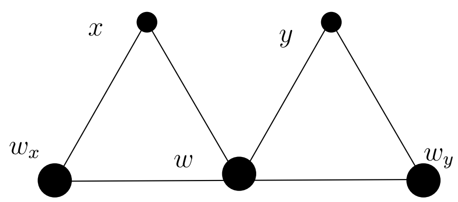


Figure 3.4: *Vertices x, y are encircled by W and have exactly one common neighbour in W.*

□

Given $W \subseteq D(G)$ and $x \in V(G) \setminus W$, we denote by W_x the set $W \cap (N(x) \cup N(N^W(x)))$. The following remarks are trivially valid.

Remark 3.5. *If x, y is an encircled pair, then $W_x = W_y$ and $|W \setminus W_x| \leq 1$.*

Lemma 3.6. *Let W be any set of $m(G)$ dense vertices, $m(G) \geq 4$. If W encircles at least one vertex or a pair of vertices, then either W encircles at most two vertices, or it encircles a pair, or it encircles two pairs and its structure is as represented in Figure 3.5.*

Proof: By Lemma 3.4, it is easy to see that W encircles at most two vertices. So, we analyse the cases where it encircles a pair of vertices.

Let x, y be an encircled pair. Note that, because of the existence of the cycle containing x,y and some vertices of W, there is no encircled vertex

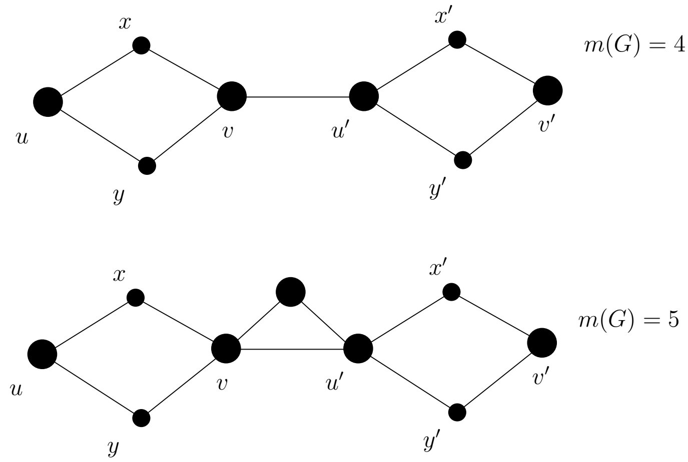


Figure 3.5: Cacti with two pairs encircled by W. The big vertices are in W and $d(v) = d(u') = m(G) - 1$.

$x' \notin \{x, y\}$. So, we analyse the existence of another encircled pair. Let x' be one of the vertices of such a pair, $x' \neq x, y$. First, suppose that E2 occurs for x, y and let $u, v, w \in W$ be such that $\langle x, w, y, v, u\rangle$ is a cycle in G. By Proposition 3.3 and Remark 3.5, x' must be adjacent to u or v. Suppose, without loss of generality, that $x' \in N(u)$ and let y' be the vertex encircled together with x'. As G is a cactus and x', y' must be within a cycle of G, we have $y' \notin \{x, y\}$. By E2 and Remark 3.5, there exists at most one vertex in $W \setminus \{u, v, w\}$ non-adjacent to w; so, we must have the situation in Figure 3.6. Certainly, $W = \{u, v, w, w'\}$, otherwise there would be at least two different vertices in $W \setminus Z$, for $Z = W_x$ or $Z = W_{x'}$, contradicting Remark 3.5. However, in this case, $d(u) = 4 = m(G)$, contradicting E2.

Now, suppose that E1 occurs for x, y and let $u, v \in W$ be such that $C = \langle x, u, y, v\rangle$ is a cycle in G. Let x', y' be an encircled pair different from x, y. By the paragraph above, we can suppose that E2 does not occur for x', y'. Furthermore, since two cycles may intersect in at most one vertex and by Remark 3.5 applied to the pairs x, y and x', y', we have $x' \notin \{x, y\}$ and $y' \notin \{x, y\}$. Let $C' = \langle x', u', y', v'\rangle$ be the cycle containing x', y', where $u', v' \in W$. Note that C and C' either intersect in one of the vertices u, v, u', v' or are connected through an edge between $\{u, v\}$ and $\{u', v'\}$. So, we can suppose that (1) $W \setminus (N[u'] \cup N[v']) = \{u\}$ and (2) $W \setminus (N[u] \cup N[v]) = \{v'\}$. Suppose C and C' intersect in vertex $v = u'$. Then, $W \setminus \{u, v, v'\} \subseteq N(v)$

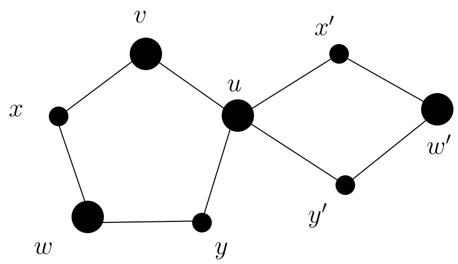


Figure 3.6: The pair x, y is encircled by W. Situation where E2 occurs for x and y and there exists another encircled pair $x', y,'$.

and $d(v) \geq m(G) + 1$, contradicting E1. Now, consider that the cycles are connected through the edge (v, u'). Note that every vertex w in $W^* = W \setminus \{u, v, u', v'\}$ must be adjacent to v and u'. So, $|W^*| \leq 1$ and the possible cases are the ones represented in Figure 3.5. Note that, by (1) and (2), we have that E1c and E1d do not occur; thus, $d(v) = d(u') = m(G) - 1$. $\square$

3.2 Quasi-Good Set

Consider a cactus G. In this section, we want to obtain a subset of the dense vertices of G that can play the role of a basis of a b-colouring of G with $m(G)$ colours. As we will see afterwards, we do not always know how to recognize such a subset. However, a good start is to pick a subset that does not encircle any vertex or pair of vertices. We say that $W \subseteq D(G)$ with cardinality $m(G)$ is a *quasi-good set* if (this definition is slightly different from the definition of "good set" in [20]):

- W does not encircle any vertex or pair of vertices; and
- Every $u \in V \setminus W$ with degree greater than $m(G) - 1$ is either adjacent to some vertex in W with degree $m(G) - 1$ or is an link vertex of W.

In Section 3.4, we will use this quasi-good set to obtain a b-colouring of G with $m(G)$ colours, when $m(G) \geq 7$. Trivially, if $|D(G)| = m(G)$, then: if $D(G)$ encircles a vertex or pair of vertices, then G does not have a quasi-good set; and if $D(G)$ does not encircle any vertex or pair of vertices, then $D(G)$ is a quasi-good set itself. So, it remains to analyse the existence of a quasi-good set in a cactus G that has more than $m(G)$ dense vertices. The main result of this section is the following:

Theorem 3.7. *Let G be a cactus with $|D(G)| > m(G)$, and let W be a subset of $m(G) + 1$ dense vertices of G containing all vertices with degree greater than $m(G) - 1$. Then, G does not have a quasi-good set if and only if $|D(G)| = m(G) + 1$ and either:*

(I) W induces a cycle of length 5 and $d(v) = 3$, for all $v \in W$, or W is as represented in Figure 3.7; or

(II) there exist vertices $u, v \in W$, with degree $m(G) - 1$, and $w \notin W$ such that $\langle u, v, w\rangle$ is a cycle and every vertex in W is adjacent to u or to v.

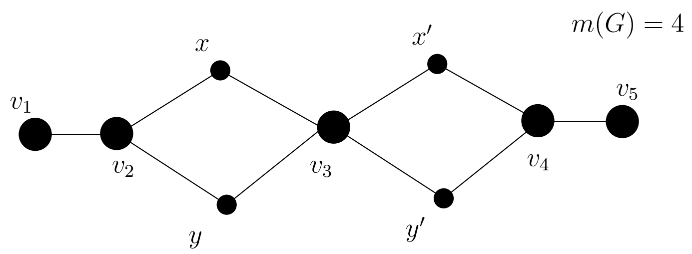


Figure 3.7: In this graph, $m(G) = 4$, W is represented by the bigger vertices and $d(v_2) = d(v_4) = 3$.

First, we prove the following part of the theorem.

Lemma 3.8. *Let G be a cactus with $|D(G)| = m(G)+1$. If (I) or (II) occurs for $D(G)$, then G has no quasi-good set.*

Proof: Let $D(G) = \{v_1, \cdots, v_{m(G)+1}\}$. We prove that $D(G) \setminus \{v_i\}$ is not a quasi-good set, for all $v_i \in D(G)$, thus proving the lemma. First, suppose that $D(G)$ induces a cycle of length 5 and $d(v_i) = m(G)-1$, for all $v_i \in D(G)$. Trivially, $W' = D(G)\setminus\{v_i\}$ encircles v_i (and, consequently, W' is not a quasi-good set) for all $v_i \in D(G)$. Now, suppose that $D(G)$ is as represented in Figure 3.7. Trivially, E1b occurs for $W' = D(G) \setminus \{v_i\}$ for all $i \in \{1, 2, 4, 5\}$ (hence, W' is not a quasi-good set). As for $W' = D(G) \setminus \{v_3\}$, observe that $v_3 \in D(G) \setminus W'$ has degree $m(G)$ but v_3 is neither adjacent to some $v_i \in W'$ of degree $m(G) - 1$ nor is within a link of W'. Thus, W' is not a quasi-good set and the lemma follows. Finally, suppose, without loss of generality, that $\langle w, v_1, v_2\rangle$ is a cycle in G such that $w \notin D(G)$, $d(v_1) = d(v_2) = m(G) - 1$ and $D(G) \subseteq N(v_1) \cup N(v_2)$ (i.e., (II) occurs for $D(G)$). Then, $D(G) \setminus \{v_i\}$ encircles w, for all $i \in \{3, \cdots, m(G) + 1\}$, $D(G) \setminus \{v_1\}$ encircles v_1 and $D(G) \setminus \{v_2\}$ encircles v_2. □

Now, we need to prove the other way of the equivalence, i.e., that if G does not have a quasi-good set, then $|D(G)| = m(G) + 1$ and (I) or (II) occurs. Actually, we prove that if one of the situations below occurs, then G has a quasi-good set (consider W to be defined as in the theorem):

1. $|D(G)| > m(G) + 1$ and (I) or (II) occurs for W; or
2. Neither (I) nor (II) occurs for W and some $W' \subseteq W$ with cardinality $m(G)$ encircles two vertices or a pair of vertices; or
3. Neither (I) nor (II) occurs for W and every $W' \subseteq W$ with cardinality $m(G)$ encircles at most one vertex and no pair of vertices.

Let W' be a set of $m(G)$ dense vertices containing all vertices with degree at least $m(G)$. If G does not have a quasi-good set, then W' encircles at least one vertex or a pair of vertices. Now, from 2 and 3, we get that (I) or (II) occurs and, from 1, we get that $|D(G)| = m(G) + 1$. The theorem, then, follows. Now, we present lemmas that cover each described situation.

Recall that $W_x = W \cap (N(x) \cup N(N^W(x)))$. If $|W \setminus W_x| \geq 2$, we know that x is not encircled by W and, by Remark 3.5, that x is not part of a pair encircled by W. Also, obviously, if x is encircled by W, then $W_x = W$.

Lemma 3.9. *Let $W \subseteq D(G)$ of cardinality $m(G) + 1$ containing all vertices with degree at least $m(G)$. If $|D(G)| > m(G) + 1$ and (I) or (II) occurs for W, then G has a quasi-good set.*

Proof: It is easy to see that, if the structure of W is as represented in Figure 3.7, then $(W \setminus \{v_2, v_4\}) \cup \{w\}$ is a quasi-good set, for any $w \in D(G) \setminus W$. Let us now prove that if W induces a cycle of length 5 or (II) occurs for W, then we can construct a quasi-good set.

Suppose that W induces a cycle of length 5 and $d(v) = 3 = m(G) - 1$, for all $v \in W$. Thus, $d(v) = 3$, for all $v \in D(G)$ and if we get a subset $W' \subseteq D(G)$ of cardinality $m(G)$ that does not encircle any vertex or pair of vertices, then W' is a quasi-good set. Let v be any dense vertex not in W

and let $u \in W$ be the vertex which separates v from the $W \setminus \{u\}$ (if v is in another connected component, just consider any $u \in W$). Remove u and one of the vertices adjacent to u in W and add v, obtaining W'. Note that situations E1 and E2 cannot occur, since three of the vertices in W' form an induced path embedded in a cycle and $u \notin W'$ separates v from $W' \setminus \{v\}$. Furthermore, any vertex $w \in V \setminus W'$ does not reach v or at least one vertex of $W' \setminus \{v\}$. Thus, W' is a quasi-good set.

Now, suppose that (II) occurs and let v_1, v_2 be such that $d(v_i) = m(G)-1$, $i = 1, 2$, $W \subseteq N(v_1) \cup N(v_2)$ and $\langle x, v_1, v_2 \rangle$ is a cycle, for $x \notin W$. Let W' be obtained from W by removing v_1, v_2 and adding any dense vertex in $D(G) \setminus W$. Since $d(v_1) = d(v_2) = m(G) - 1$ and W contains all vertices with degree greater than $m(G)-1$, we have that, if W' does not encircle any vertex, or pair of vertices, then W' is a quasi-good set. As $d(v_1) = m(G) - 1$ and $(W \setminus \{v_1\}) \cup \{x\}$ has cardinality $m(G)+1$, there must exist at least two vertices in $W \setminus N[v_1]$, say v_3, v_4; hence, v_3, v_4 are adjacent to v_2. The same is valid for v_2; so, let v_5, v_6 be vertices of $W \cap (N(v_1) \setminus \{v_2\})$. Note that $v_3, v_4 \in W'$ are separated from $v_5, v_6 \in W'$ by v_1, v_2, where $v_1, v_2, x \notin W'$. So, $|W \setminus W_{w'}| \geq 2$, for all $w' \in V \setminus W'$, and W' does not encircle any vertex or pair of vertices. □

Lemma 3.10. *Let $W \subseteq D(G)$ of cardinality $m(G)+1$ containing all vertices with degree at least $m(G)$. If neither (I) nor (II) occurs for W and some $W' \subseteq W$ with cardinality $m(G)$ encircles two vertices or at least one pair of vertices, then G has a quasi-good set.*

Proof: Let $W = \{v_1, \cdots, v_{m(G)+1}\}$ and denote by W^i the set $W \setminus \{v_i\}$. Suppose, without loss of generality, that W^1 encircles two vertices or at least one pair of vertices. First, consider the case where W^1 encircles more than one pair, i.e., the structure of G is as represented in Figure 3.5 (the big vertices represent W^1). Denote by S the set $W^1 \cup \{x, y, x', y'\}$. As $d(v) = d(u') = m(G) - 1$, we know that $N(v), N(u') \subseteq S$. If v_1 is in the cycle $C_l = \langle x, u, y, v \rangle$ or is separated from W^1 by a vertex in C_l or is in another connected component, replace v by v_1, obtaining W'. Trivially, $|W' \setminus W'_t| \geq 1$, for all $t \in V \setminus W'$; thus, W' does not encircle any vertex. Also, as W' intersects C_l in one vertex or one edge and v_1 cannot connect through v, one can verify that W' does not encircle any pair of vertices. Consequently, as $d(v) = m(G)-1$, W' is a quasi-good set. If v_1 is in $C_r = \langle u', x', v', y' \rangle$ or is separated from W^1 by a vertex in C_r, we have an analogous situation. Finally, if $m(G) = 5$ and v_1 is separated from W^1 by z, where $z \in N(v) \cap N(u')$, replace z by v_1, obtaining W'. Note that $|W' \setminus W'_t| \geq 2$, for all $t \in V \setminus W'$, and, as z is adjacent to v of degree $m(G)-1$, we have that W' is a quasi-good set.

Now, we may assume that W^1 encircles at most one pair of vertices. We analyse the possible cases, according to the definition of encircled pairs and to Lemma 3.4:

- E1 or F1 occurs for W^1: let x, y be two distinct vertices encircled by W^1 or a pair encircled by W^1 and suppose, without loss of generality, that v_2, v_3 are such that $C = \langle x, v_2, y, v_3 \rangle$ is a cycle in G. Let W' represent W^1_x, if E1a or E1b occurs, or W^1, otherwise. By E1 and F1, we can suppose that $N^{W'}(v_3) \neq \emptyset$ and, if E1a or E1c occurs, as both v_2 and v_3 have some neighbour in W', we can suppose that v_2 is the vertex with degree $m(G) - 1$.

  Now, without loss of generality, suppose that $v_4 \in N(v_3)$ and, if E1a or E1b occurs, let v_t be the vertex in $W^1 \setminus W'$. Trivially, W^3 cannot encircle any vertex different from v_3 itself and, as v_3 is within the path $\langle v_2, z, v_3, v_4 \rangle$, where $z \in \{x, y\}$, we have that if v_3 is not encircled by W^3 and W^3 does not encircle any pair of vertices, then W^3 is a quasi-good set. Observe that (1) if $v_1 \notin \{x, y\}$ or $N^W(v_2) \setminus \{x, y\} \neq \emptyset$, then v_3 is not encircled by W^3.

  First, assume $v_1 = x$. Suppose that E1a, F1 or E1c occurs for W^1 and let $v_i \in N^{W^1}(v_2)$. By (1), v_3 is not encircled by W^3. Suppose that x', y' is a pair encircled by W^3. By Remark 3.5, the pair x', y' is separated from C by v_2 and $N^W(v_3) \setminus \{v_1, v_4\} = \emptyset$. So, $(W^3 \cup \{y\}) \setminus \{v_2, v_4, v_t\} \subseteq N(v_2)$ and, as $d(v_2) = m(G) - 1$, at most one of x', y' is adjacent to v_2 and one can verify that x', y' cannot be encircled. Now, suppose that E1b or E1d occurs. Note that, because of the degree of v_3, either $N(v_3) = (W \setminus \{v_2, v_3, v_t\}) \cup \{y\}$ or $N(v_3) = (W \setminus \{v_2, v_3\}) \cup \{y\}$, i.e., y is the only neighbour of v_3 not in W. Also, as either both x and y are encircled by W^1 or none of x and y is encircled (as, in this case, x, y is an encircled pair), we have that $v_t \notin N(x) \cup N(y)$. Thus, one can verify that W^2 does not encircle any vertex or pair of vertices and, since v_2 is in a link between v_1 and v_3, we have that W^2 is a quasi-good set.

  Now, assume $v_1 \neq x, y$. By (1), W^3 does not encircle v_3. So, suppose that W^3 encircles the pair x', y'. By Remark 3.5, we know that $x', y' \neq x, y$ and v_1 is separated from C by same vertex as x', y'. Also, if v_1 is separated from C by v_3, then $N^W(v_2) = \emptyset$; otherwise, v_2 separates v_1 from C and $N^W(v_3) = \{v_4\}$. We analyse the following possibilities:

– v_3 separates v_1 from C: so, E1b or E1d occurs for W^1 and x, y. Note that, as $d(v_3) \leq m(G)$, (i) $N(v_3) \subseteq (W^1 \setminus \{v_2, v_3\}) \cup \{x, y\}$; also, (ii) $W \setminus N[v_3] \subseteq \{v_1, v_2, v_t\}$. Suppose, first, that $x' = v_3$. By (ii) and Proposition 3.3, we have that one of the situations in Figure 3.8 occurs. Note that W^4 does not encircle any vertex or pair of vertices and that v_4 is within a link, i.e., W^4 is a quasi-good set. Now, suppose that $x', y' \neq v_3$. By (i), we know that $x', y' \notin N(v_3)$. If $(N(x') \cup N(y')) \cap N(v_3) = \emptyset$, then we have the situation represented in Figure 3.9.(a) and one can easily verify that W^4 is a quasi-good set. Otherwise, suppose that $v_i \in N(v_3) \cap (N(x') \cup N(y'))$. As $x', y' \notin N(v_3)$, $(x', y') \notin E(G)$, E1 or E2 occurs for W^3, x', y' and by (ii), one can verify that the possible situations are illustrated in Figure 3.9.b,c,d and W^i is a quasi-good set.

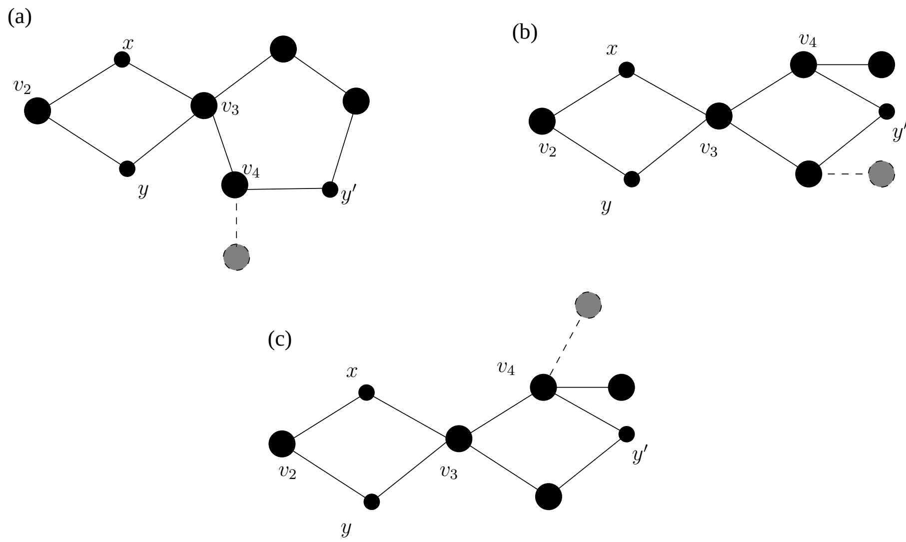


Figure 3.8: W^3 *encircles a pair when* v_3 *separates* v_1 *from* C. *The dotted edges and vertices may not exist.*

– v_2 separates v_1 from C: so, $N^W(v_3) = \{v_4\}$ and (i) $W^3_{x'} = W^3_{y'} =$

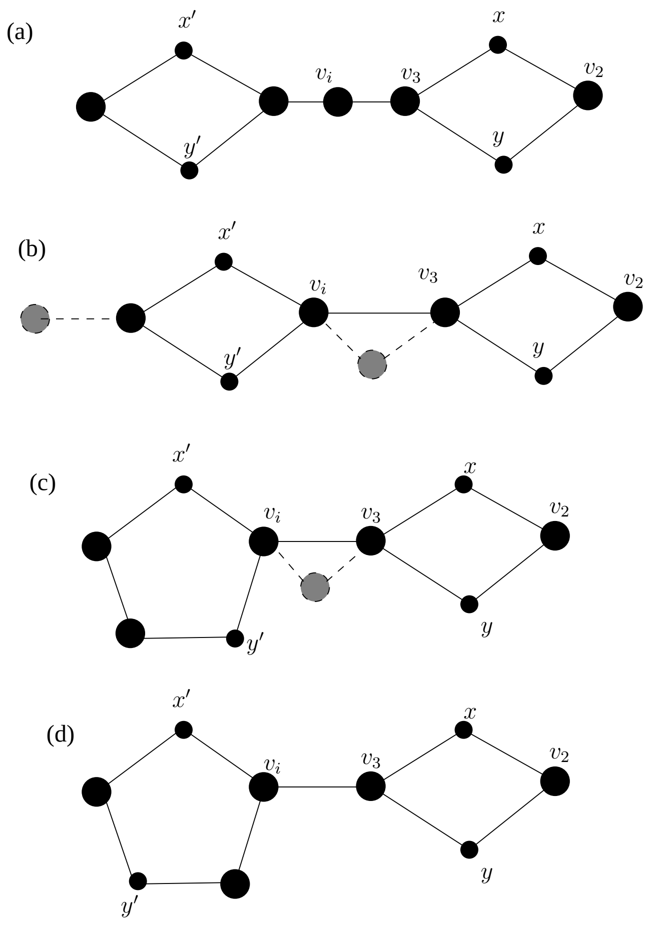


Figure 3.9: *W^3 encircles a pair when v_3 separates v_1 from C. The dotted edges and vertices may not exist, except for (b) where at least one of the dotted vertices must exist.*

$W^3 \setminus \{v_4\}$. By (i), we have that v_i is also separated from C by v_2, for all $v_i \in W^3 \setminus \{v_2, v_4\}$. Note that, if $N^{W^2}(v_2) \neq \emptyset$, then W^2 is a quasi-good set. So, suppose that $N^{W^2}(v_2) = \emptyset$. As $|W \setminus W^1_x| \leq 1$, we have that $x', y' \in N(v_2)$ and $m(G) = 4$. Also, note that E2 does not occur for x', y', by the existence of $v_4 \in W^3$ and the fact that $d(v_2) > m(G) - 1$. Thus, the structure of W is as in Figure 3.7, i.e., (I) occurs for W, a contradiction.

- E2 or F2 occurs for W^1: let x, y be two distinct vertices encircled by W^1 or a pair encircled by W^1. Suppose, without loss of generality, that $C = \langle x, v_2, v_3, y, v_4 \rangle$ is a cycle. By E2 and F2, we know that (i) $W^1 \setminus (N[v_4] \cup \{v_2, v_3\})$ has at most one vertex; if it is the case, let v_5 be such vertex. First, suppose that $N^{W^1}(v_4) \neq \emptyset$. It is easy to see that W^4 does not encircle any vertex and, if W^4 does not encircle any pair, as v_4 is within a link of W^4, we have that W^4 is a quasi-good set. So, suppose that W^4 encircles the pair x', y'. Note that x', y' are separated from C by either v_2 or v_3, say v_2. One can verify that $|W \setminus W^2_z| \geq 2$, for all $z \in V \setminus W^2$; thus, W^2 does not encircle any vertex or pair of vertices and, as $d(v_2) = m(G) - 1$ by E2, W^2 is a quasi-good set. Now, consider that $N^{W^1}(v_4) = \emptyset$. Consequently, as $m(G) \geq 4$, E2b must occur. Thus, $d(v_4) = m(G) - 1$ and if W^4 does not encircle any vertex or pair of vertices, then W^4 is a quasi-good set. Also, by (i), we have that $m(G) = 4$. First, suppose that W^4 encircles a vertex $z \in V \setminus W^4$. Note that if $z \in \{x, y\}$, then, as W^1 does not encircle z, we must have $d(v_1) = m(G) - 1$ and $v_1 \in N(v_5) \cap N(z)$. It is easy to verify that, in this case, either W^2 is a quasi-good set, if $z = x$, or W^3 is a quasi-good set, if $z = y$. Now, consider that $z \neq x, y$; then, z must be adjacent either to v_2 or v_3, say v_2. As $N(v_2) = \{x, y, z\}$ and $W^4 = W^4_z$ (i.e., z separates C from v_1 and v_5), it is easy to see that W^2 is a quasi-good set. Now, suppose that W^4 encircles a pair x', y'. As $d(v_2) = d(v_3) = 3$, neither v_2 nor v_3 are within the cycle containing x', y'; thus, the cycle containing x', y' contains v_1 and v_5. Also, by Remark 3.5, we have that $|W \setminus W^4_{x'}| \leq 1$. Observe that either v_2 or v_3 is in $W \setminus W^4_{x'}$; thus, either E1a or E1b occurs for W^4 and any between v_1, v_5 having a neighbour in W^4 must have degree $m(G) - 1$. So, $v_1 \neq x, y$ and one can verify that the only possible situation is the one illustrated in Figure 3.10 (up to symmetry), in which case W^3 is a quasi-good set.

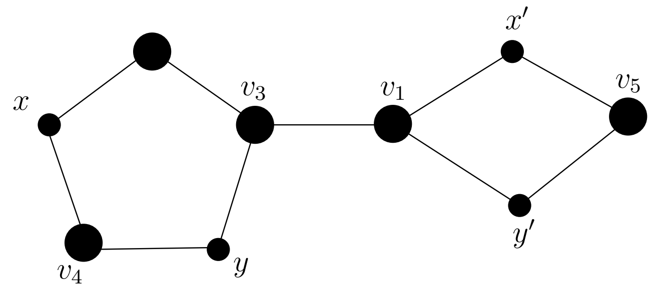


Figure 3.10: *E2 occurs for* W^1, x, y, W^4 *encircles* x', y' *and* $m(G) = 4$.

- F3 occurs for W^1: let $\langle x, v_2, v_3, y, v_4, v_5 \rangle$ be the cycle, where x and y are encircled by W^1. Observe that if $x = v_1$, then W^3 trivially does not encircle any vertex or pair of vertices and, as $d(v_3) = m(G) - 1$, we have that W^3 is a quasi-good set. If $v_1 = y$, we have an analogous situation. So, suppose that v_1 is not in the cycle. If v_1 is separated from the cycle by v_i, $i \in \{2, 3, 4, 5\}$, let $W' = W^i$; otherwise, if v_1 is separated from the cycle by x, let $W' = W^2$; finally, if v_1 is in another connected component of G or is separated from the cycle by y, let $W' = W^3$. Obviously, situations E1 and E2 cannot occur because of the disposition of three vertices of W' in the cycle of length six. Also, any vertex not in W' does not reach at least one vertex in W' and, since $d(v_j) = m(G) - 1$, for all $v_j \in W^1$, we have that W' is a quasi-good set.

□

Lemma 3.11. *Let* $W \subseteq D(G)$ *of cardinality* $m(G)+1$ *containing all vertices with degree at least* $m(G)$. *If neither (I) nor (II) occurs and every* $W' \subseteq W$ *with cardinality* $m(G)$ *encircles at most one vertex and no pair of vertices, then* G *has a quasi-good set.*

Proof: Consider the same notation as in the proof of the previous lemma. Suppose, without loss of generality, that $d(v_1) = m(G) - 1$ (note that there are at most $m(G)$ vertices with degree greater than $m(G) - 1$). Obviously, if W^1 does not encircle any vertex, then it is a quasi-good set. So, let u be encircled by W^1 and suppose, without loss of generality, that $N(u) \cap W^1 = \{v_2, \cdots, v_{p-1}\}$. By Proposition 2.1 and the fact that W^1 contains all vertices with degree greater than $m(G)-1$, we know that $p > 3$ and $p-1 < m(G)+1$.

We may also suppose that v_2 is a (u, v_p)-bridge; hence, $d(v_2) = m(G) - 1$. We analyse the following cases:

- $u = v_1$: since $v_p \in N(v_2)$ and $d(v_2) = m(G)-1$, if W^p does not encircle any vertex, then W^p is a quasi-good set. So, suppose that W^p encircles $v \in V \setminus W^p$. First, we analyse the case where v is adjacent to v_1. As $v_p \notin N(v_1)$, we have that $v \neq v_p$ and, consequently, $v \notin W$. By Proposition 2.1, there exists $v_i \in W^p$ adjacent to v, $i \neq 1$. We have that $N^{W^p}(v) = \{v_1, v_i\}$, otherwise v_1 does not reach some $v_j \in N^{W^1}(v)$. So, we know that every $v_j \in W^p \setminus \{v_1, v_i\}$ is either adjacent to v_1 or to v_i. Observe that, if $i = 2$, as $W \subseteq N(v_1) \cup N(v_2)$ and $d(v_1) = d(v_2) = m(G)-1$, we have that (II) occurs for W, a contradiction; so, consider $i \neq 2$. Since v_1 is encircled by W^1 and $v_i \in W^1$, then either $v^i \in N(v_1)$ or there exists a (v_1, v_i)-bridge, v_j. Observe Figure 3.11. In (a), if $N^{W^p}(v_i) \setminus \{v_1\} = \emptyset$, then $(W^p \setminus \{v_1\}) \cup \{v\} \subseteq N(v_1)$, a contradiction to the fact that $d(v_1) = m(G)-1$. So, let $v_k \in N^{W^p}(v_i) \setminus \{v_1\}$. Note that any vertex in $V \setminus W^i$ does not reach v_p or v_k and, since v_i must be the (v, v_k)-bridge (i.e., $d(v_i) = m(G)-1$), we have that W^i is a quasi-good set. In (b) and (c), there is no $v_k \in N^{W^p}(v_i) \setminus \{v_j\}$, otherwise v_1 would not reach v_k within W^1. So, $N(v_1) = (W^p \setminus \{v_1, v_i\}) \cup \{v\}$ and one can verify that if $z \in V \setminus W^i$ has two neighbours in W^i, then z does not reach v_j or some $v_k \in N^{W^i}(v_1) \setminus \{v_j\}$, i.e., z is not encircled by W^i. As v_i is adjacent to $v_j \in W^i$ with degree $m(G)-1$, we have that W^i is a quasi-good set.

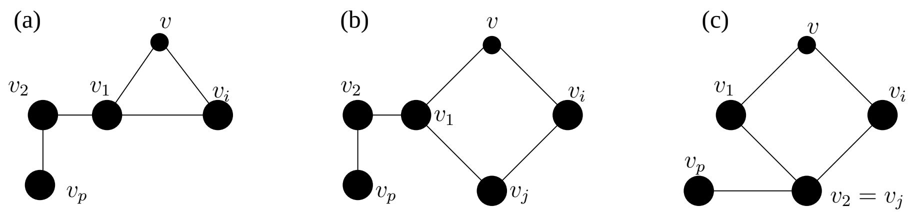


Figure 3.11: Situation of the proof of Lemma 3.11, where W^1 encircles v_1 and W^p encircles v, where $v \in N(v_1) \setminus N(v_2)$.

Now, suppose that v is not adjacent to v_1. Let v_i be a (v, v_1)-bridge (thus $d(v_i) = m(G) - 1$). By Proposition 2.1, there exists $v_j \in W^p$ adjacent to v, $j \neq 1, i$. Also, v_j is reachable from v_1 within W^1; so, $v_j \in N(v_1)$ or there exists a (v_1, v_j)-bridge, say v_k. If $k = i$, note that

$(W^p \setminus \{v_i\}) \cup \{v\} \subseteq N(v_i)$, a contradiction since $d(v_i) = m(G) - 1$. So, $k \neq i$ and $C = \langle v_1, v_k, v_j, v, v_i, v_1 \rangle$ is a cycle in G, $k = j$ or not. Trivially, $N^W(v) \setminus C = \emptyset$, since v_1 is encircled by W^1. Also, if there exists $v_l \in W^p \setminus C$, then W^l is a quasi-good set: since v_l must be reached by v within W^p and $N^{W^p}(v) \setminus C = \emptyset$, v_l must be adjacent to $v_h \in N^C(v)$ such that $d(v_h) = m(G) - 1$. So, $W = (C \setminus \{v\}) \cup \{v_p\}$. Since $|W| = m(G) + 1$ and $m(G) \geq 4$, we have that $k \neq j$. Consider, first, that $v \neq v_p$. Note that $k = 2$, otherwise W^1 encircles v and v_1. It is easy to verify that $W^i = N[v_2]$ (recall that $d(v_2) = m(G) - 1 = 3$) and, as v_i is within the link $\langle v_1, v_i, v, v_j \rangle$, W^i is a quasi-good set. Now, consider that $v = v_p$; thus, $i = 2$. As (I) does not occur for W and $d(v_1) = d(v_2) = d(v_k) = d(v_j) = 3$, we must have that $d(v_p) > m(G){-}1$. It is easy to see that W^2 is a quasi-good set.

- $u \neq v_1$: trivially, $u \notin W$. So, if $p > 4$, then v_2 is separated from at least one vertex $v_i \in W$ by u and, in this case, we can verify that W^2 does not encircle any vertex and, as $d(v_2) = m(G) - 1$, W^2 is a quasi-good set. Now, suppose $p = 4$. We claim that, if there is no quasi-good set, then W^2 encircles v_2 and W^3 encircles v_3 (Claim 3.12). Suppose $(v_2, v_3) \notin E$. So, there must exist a (v_2, v_3)-bridge, v_i. By Proposition 3.3, there exists $v_k \in N^{W^2}(v_2) \setminus \{v_i\}$, a contradiction to the fact that v_3 is encircled by W^3. So, $(v_2, v_3) \in E$. Since $u \neq v_1$ is encircled by W^1 and $N^{W^1}(u) = \{v_2, v_3\}$, we know that every vertex in $W^1 \setminus \{v_2, v_3\}$ is either adjacent to v_2 or to v_3. In addition, since v_1 must be reachable from v_2 within W^2 and from v_3 within W^3 and by the existence of the cycle $\langle u, v_2, v_3 \rangle$, we know that v_1 must be adjacent either to v_2 or to v_3. Finally, as $d(v_2) = m(G) - 1$ and by Proposition 2.1, there exists $v_i \in W^2 \setminus N(v_2)$. This vertex must be adjacent to v_3 and, as G is a cactus, v_3 is the only common neighbour of v_i and v_2, i.e., v_3 is a bridge and, hence, has degree $m(G){-}1$. So, (II) occurs for W, a contradiction. Now, we prove the claim.

  Claim 3.12. *Let u be encircled by W^1, $u \neq v_1$. If G has no quasi-good set, then W^2 encircles v_2 and W^3 encircles v_3.*

  Proof of the claim: By contradiction, suppose that G has no quasi-good set and W^2 encircles v, $v \neq v_2$. First, consider $v = u$. Since $N^{W^1}(u) = \{v_2, v_3\}$ and by Proposition 2.1, we know that v_1 must be adjacent to u. Note that, if $|N^W(v_2)| > 1$, then u does not reach some

$v_i \in W^2$, a contradiction. So, $N^W(v_2) = v_p$ and, as $m(G) \geq 4$ and $N(u) \cap W^1 = \{v_2, v_3\}$, there must exist $v_i \in N(v_3) \cap V_1$. Also, v_3 must be the (u, v_i)-bridge in W^1; so, $d(v_3) = m(G) - 1$. It is easy to verify that any vertex in $V \setminus W^3$ does not reach at least one between $\{v_1, v_2, v_p, v_i\}$ and, consequently, W^3 is a quasi-good set.

Now, suppose that $v \neq u$. If v_2 is reachable from v, we have that W^1 encircles two vertices, u and v, a contradiction. So, $v \notin N(v_2)$ and, if v_i is adjacent to both v and v_2, then $d(v_i) \geq m(G)$. We analyse the following cases:

(a) $v_p \in N(v)$: so, $d(v_p) \geq m(G)$ and, since v is encircled by W^2 and $v_3 \in W^2$, we have the cycle $C = \langle u, v_2, v_p, v, v_i, v_3 \rangle$, where v_i is either v_3 or has degree $m(G) - 1$ (and, thus, $v_i \neq v_p$). If $i \neq 3$, note that any $w \in V \setminus W^3$ does not reach at least one between $v_2, v_p, v_i \in W^3$, so W^3 does not encircle any vertex. Also, as v_3 is in the link $\langle v_2, u, v_3, v_i \rangle$, we have that W^3 is a quasi-good set, a contradiction. So, consider $i = 3$. As $m(G) \geq 4$, there must exist $v_i \in W^2 \setminus \{v_1, v_3, v_p\}$ and, since v_i is also in W^1, v_3 must be a (u, v_i)-bridge and a (v, v_i)-bridge (thus, $d(v_3) = m(G) - 1$). It is easy to see that any vertex $w \in V \setminus W^3$ does not reach at least one between v_2, v_p, v_i and W^3 is a quasi-good set, a contradiction.

(b) $v_p \notin N(v)$: since $v_p \in W^2$ and v is encircled by W^2, there must exist a (v, v_p)-bridge, v_i. First, we analyse the case where $v_i = v_3$ or is a (v, v_3)-bridge; thus, there exists a cycle $\langle u, v_3, v_i, v_p, v_2 \rangle$. By Proposition 2.1, there exists $v_k \in N(v) \setminus \{v_i\}$. Observe Figure 3.12. If $i \neq 3$, note that $k = 1$ and there is no $v_l \in W \setminus \{v_1, v_2, v_3, v_i, v_p\}$, otherwise v_l would not reach either u within W^1 or v within W^2. So, $m(G) = 4$ and, since $d(v_i) = m(G) - 1$, we have that $N(v_i) = \{v, v_3, v_p\}$. However, in this case, any $w \in V \setminus W^i$ does not reach at least one between v_1, v_2, v_3, v_p and, consequently, W^i is a quasi-good set, a contradiction. So, consider $i = 3$. Note that, there is no $v_l \in N^{W^1}(v_2) \setminus \{v_p\}$, otherwise v_l is distant from v within W^2. Since $N^{W^1}(u) = \{v_2, v_3\}$, we have that all vertices in $W^1 \setminus \{v_2, v_3\}$ must be adjacent to v_3. However, in this case, we have that $N(v_3)$ contains the set $(W^1 \setminus \{v_2, v_3\}) \cup \{u, v\}$ with cardinality $m(G)$, contradicting the fact that v_3 is a (v, v_p)-bridge.

Now, suppose that the (v, v_p)-bridge, v_i, is not in $N[v_3]$. Since v_2

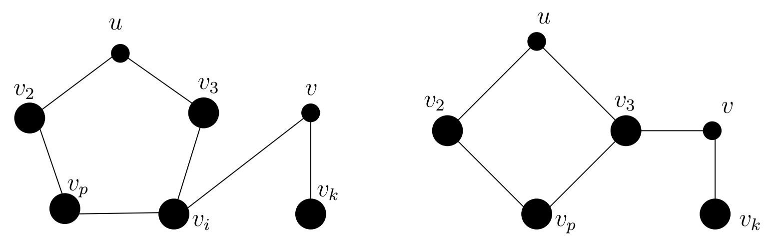


Figure 3.12: Situation of case (b) of the proof of Claim 3.12, where the (v, v_p)-bridge is within a cycle containing u, but not v.

is not reachable from v, we have that v_i cannot be adjacent to v_2 and, hence, v_i is not reachable from u within W^1, i.e., $i = 1$ (recall that $N^{W^1}(u) = \{v_2, v_3\}$). Also, as v_3 is reachable from v, we have that $\langle u, v_2, v_p, v_1, v, v_j, v_3 \rangle$ is a cycle in G, where $j = 3$ or v_j is a (v, v_3)-bridge. Observe Figure 3.13. Note that any vertex $w \in V \setminus W^p$ is distant from at least one between v_1, v_2, v_3, i.e., W^p does not encircle any vertex and, as v_p is adjacent to $v_2 \in W^p$ with degree $m(G) - 1$, we have that W^p is a quasi-good set, a contradiction.

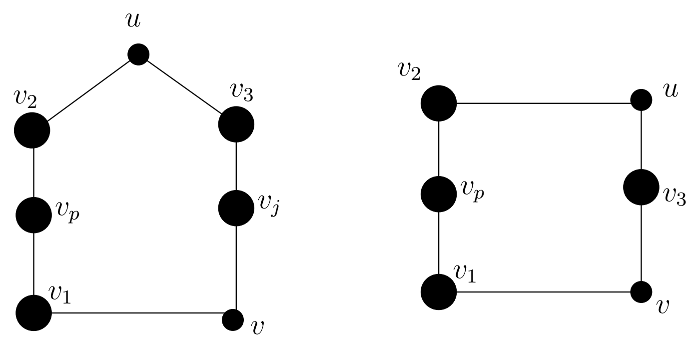


Figure 3.13: Situation of case (b) of the proof of Claim 3.12, where the (v, v_p)-bridge is within a cycle containing u and v.

This completes the proof that W^2 encircles v_2. Now, observe that if $d(v_3) = m(G) - 1$ and $N^{W^1}(v_3) \neq \emptyset$, then we can prove that v_3 is encircled by W^3 analogously. So, suppose otherwise. As $N^{W^1}(u) = \{v_2, v_3\}$, v_2 must be a (u, v_i)-bridge, for all $v_i \in W^1 \setminus \{v_2, v_3\}$. In fact,

$N(v_2) = (W^1 \setminus \{v_2, v_3\}) \cup \{u\}$, i.e., $v_3 \notin N(v_2)$. So, as v_2 is encircled by W^2, there exists $v_i \in N^{W^2}(v_2) \cap N(v_3)$ such that $d(v_i) = m(G) - 1$. Also, as $m(G) \geq 4$, there exists $v_j \in W^1 \setminus \{v_2, v_3, v_i\}$ and it is easy to see that W^3 does not encircle any vertex. Then, as v_3 is adjacent to $v_i \in W^3$ with degree $m(G) - 1$, we have that W^3 is a quasi-good set, a contradiction.

◇

□

3.3 b-Colouring cacti with $\chi_b(G) < m(G)$

In this section, we b-colour G with $m(G) - 1$ colours, where G is a given anomalous cactus or a cactus that has no quasi-good set. To do this, we choose a subset $W \subseteq D(G)$ with cardinality $m(G) - 1$ that satisfies the hypothesis of Lemma 2.15. Then, we construct an unsaturated precolouring ψ with candidate set W. Thus, by Lemma 2.15, we know that ψ can be extended to a b-colouring of G with $m(G) - 1$ colours. Observe first that, for anomalous cacti, the precolourings presented in Figure 3.14 are as desired. So, it remains to colour pivoted cacti. The following proposition will be useful in some of the upcoming subsections:

Proposition 3.13. *Let $W \subseteq D(G)$ with cardinality $m(G)$ and let $v \in W$ be such that $d(v) = m(G) - 1$. Then, $|W \setminus N[v]| = |N(v) \setminus W|$.*

Proof: Denote by q the value $|N(v) \setminus W|$. We have that $|N(v)| = |N(v) \setminus W| + |N^W(v)| \Rightarrow m(G) - 1 = q + |N^W(v)|$. So, $q = m(G) - |N^W(v)| - 1$. Also, we know that $|W \setminus N[v]| = |W| - |N^W(v)| - 1$ and the result follows.

□

By Lemma 3.6 and Theorem 3.7, we know that the situations where G is pivoted are: when G has exactly $m(G)$ dense vertices that encircles two pairs of vertices, or one pair of vertices, or one or two vertices; or when G has $m(G) + 1$ dense vertices and one of situations described in Theorem 3.7 occurs. We analyse each possible case in the following subsections.

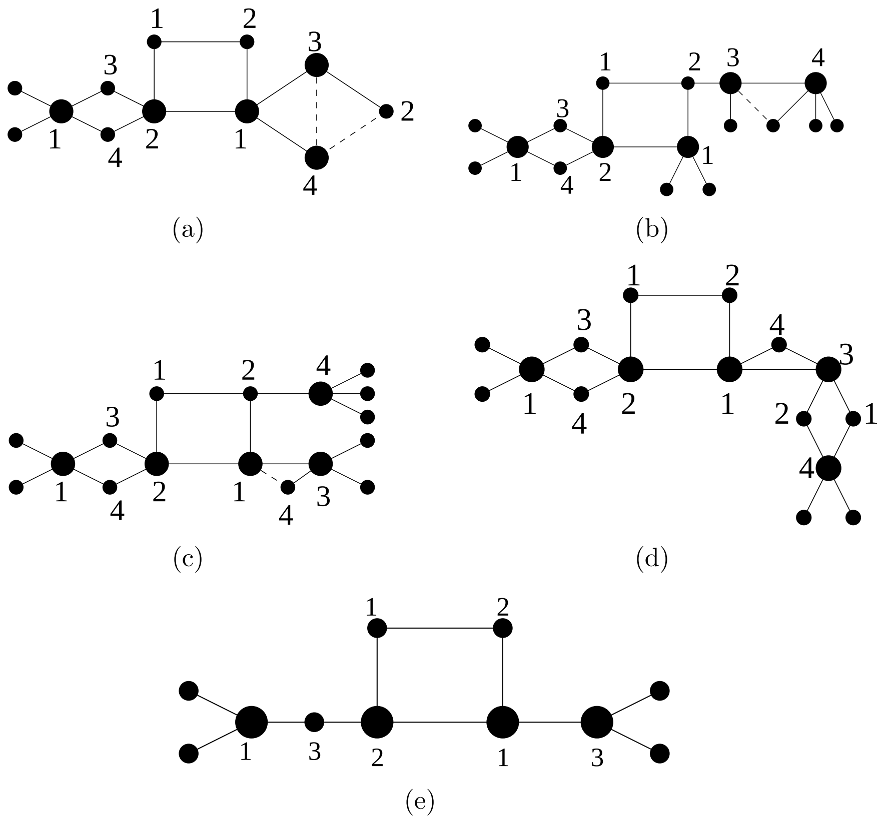


Figure 3.14: Unsaturated precolouring of anomalous cacti.

3.3.1 $|D(G)| = m(G)$ and $D(G)$ encircles two pairs of vertices

Suppose that $|D(G)| = m(G)$ and $D(G)$ encircles two pairs of vertices. By Lemma 3.6, G is as represented in Figure 3.5. As $|D(G)| = m(G)$, it is easy to verify that the precolouring ψ presented in Figure 3.15 is unsaturated with candidate set W, where W is formed by the grey vertices, and that ψ and W satisfy Lemma 2.15.

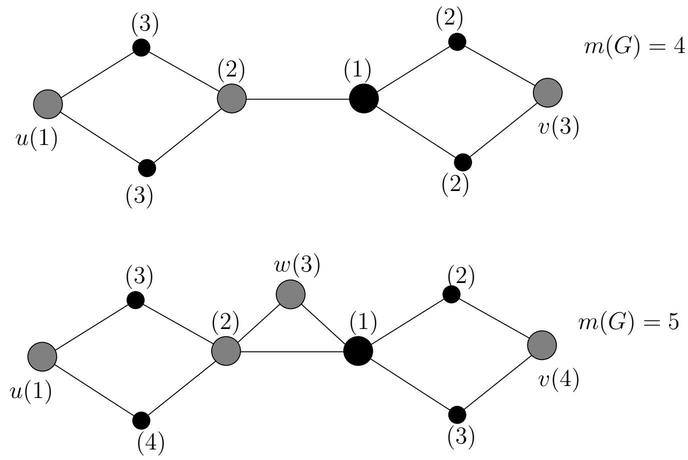


Figure 3.15: Partial colouring of a graph with structure as represented in Figure 3.5. The grey vertices can play the role of b-vertices.

3.3.2 $|D(G)| = m(G)$ and $D(G)$ encircles a pair of vertices

Suppose that $|D(G)| = m(G)$ and $D(G)$ encircles exactly one pair of vertices. In each possible situation where this happens, we choose a subset $W \subset D(G)$ of cardinality $m(G) - 1$ to be the basis of the b-colouring to be constructed. Recall the definition of redundancy of $v \in W$, denoted by $r(v)$, introduced in Section 2.4. As a dense vertex has degree at least $m(G) - 1$, if we ensure that $r(v)$ is at most one, for each $v \in W$, then the obtained precolouring is unsaturated. We need the following remark:

Remark 3.14. *Let $a, b \in W$. There are at most three links between a and b,* *not necessarily disjoint, and at most two neighbours of a and two neighbours of b lie in these paths (observe Figure 3.16).*

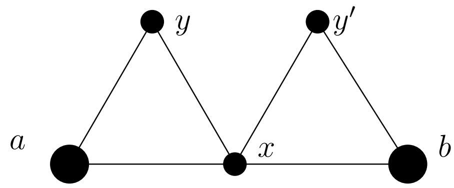


Figure 3.16: Situation when there are three links between $a, b \in W$: $\langle a, x, b\rangle$, $\langle a, y, x, b\rangle$ and $\langle a, x, y', b\rangle$.

Suppose that $D(G)$ encircles the pair, x, y and let $D(G) = \{v_1, \cdots, v_{m(G)}\}$. By definition, we have one of the following cases:

- E1 occurs: suppose, without loss of generality, that $\langle x, v_1, y, v_2\rangle$ is a cycle in G. As there is no encircled vertices, we can also suppose that $v_{m(G)}$ is not reachable from x and from y. Let $W' = D(G) \setminus \{v_{m(G)}\}$, if E1a or E1b occurs, or $W' = D(G)$, otherwise. We can suppose that $N^{W'}(v_1) \neq \emptyset$ and that, if E1a or E1c occurs, then $d(v_2) = m(G) - 1$.

  Now, let $W = D(G) \setminus \{v_1\}$ and suppose, without loss of generality, that $v_3 \in N(v_1)$. Thus, v_1 is within the link $\langle v_3, v_1, x, v_2\rangle$ and, as $d(z) < m(G) - 1$, for all $z \in V \setminus D(G)$, we have that W satisfies the constraint in Lemma 2.15. Assign colour i to v_i, for all $1 < i < m(G)$, colour 1 to $y, v_{m(G)}$, colour 3 to x and colour 2 to v_1. Since the edges (v_1, v_2), (x, v_3) and $(y, v_{m(G)})$ are not in the graph, we have a proper precolouring. Furthermore, note that there is no vertex in W, other than possibly v_2, simultaneously adjacent to more than one vertex coloured with colour i, for $i = 1, 2, 3$; so $r(v_j) = 0$, for $2 < j \leq m(G)$, and $r(v_2) \leq 1$. Now, we need to colour the remaining link vertices. Remark 3.15 follows directly from Remark 3.14 and the fact that x, y is a pair encircled by $D(G)$.

  Remark 3.15. *Vertex $v_{m(G)}$ has at most two link neighbours and, if $v_{m(G)} \in N(v_k)$, for some $v_k \in D(G)$, then $v_{m(G)}$ has at most one link neighbour different from v_1.*

Note that, by Remark 3.15, no matter which colour we assign to the link neighbours of $v_{m(G)}$, as long as the partial colouring remains proper, we will always have $r(v_{m(G)}) \leq 1$. So, from now on, we will only be concerned about the values $r(v_i)$, for $1 < i < m(G)$. We analyse the existence of the following types of links (they are coloured in the order presented below):

- Links with extremity in v_2: let $x_1, \cdots, x_q$ be all the link neighbours of v_2 and denote by v_{i_j} the other extremity of the link passing through x_j, $j = 1, \cdots, q$. First, suppose that $d(v_2) = m(G) - 1$. Thus, as $\{x_1, \cdots, x_q, x, y\} \subseteq N(v_2) \setminus D(G)$, by Proposition 3.13, we have that $|D(G) \setminus N[v_2]| \geq q + 2$. Consequently, there exist at least q vertices in $D(G) \setminus \{v_1, v_2, v_{m(G)}\}$ non-adjacent to v_2 (and, hence, adjacent to v_1) and we can give the colours of these vertices to $x_1, \cdots, x_q$. Now, suppose that $d(v_2) \geq m(G)$; then, we know that E1a and E1c do not occur and, consequently, $W' \setminus \{v_1, v_2\} \subseteq N(v_1)$. Also, observe that $q \leq 2$ and $i_j = m(G)$, for $j \in [1, q]$. So, we can colour x_1, x_q with colours from $M(v_2)$ (if there are no such colours, just repeat colour 3 in x_1, x_q). After this, give colour 2 to the uncoloured vertices on those links. Note that, at the end, $r(v_2) \leq 1$ and $r(v_{i_j}) \leq 1$, for $j = 1, \cdots, q$. Also, it is easy to verify that no two adjacent vertices are coloured in the last step with colour 2, as this would generate cycles intersecting in more than one vertex.
- Links between v_i and v_j, $i, j \neq 2, m(G)$: let $\langle v_i, x', y', v_j \rangle$ be such a link. If $x' \neq y'$, give colour j to x' and colour i to y' (if x' or y' is already coloured, do not change their colours); otherwise, give colour 3 to x'. Note that if some $v_k \in N(v_2)$ is the extremity of a link of this type, then there is no link between v_2 and v_k (i.e., v_2 is the only coloured neigbour of v_k until now). So, at the end, $r(v_k) \leq 1$, for $2 < k \leq m(G)$, and $r(v_2)$ does not increase.
- Links between v_i and $v_{m(G)}$, $i \neq 2$: let z, z' be all the uncoloured link neighbours of v_i within a link with extremity in $v_{m(G)}$ ($z = z'$ or not). Note that v_i has at most two coloured neighbours different from $v_{m(G)}$, namely v_j, $j = 1$ or $j = 2$, and some eventual x' in a path between v_i and some v_k, $k \neq 1, m(G)$. So, if $m(G) \geq 6$, then there exist at least two colours different from $1, i$ that do not

appear in the neighbourhood of v_i; then, colour z with c and z' with c'. Now, consider $m(G) \leq 5$. Suppose, first, that $v_i \in N(v_1)$ and that v_i has another coloured neighbour $x' \neq v_1, v_{m(G)}$. Since $v_1 \notin W$, we know that there exists a path of length at most three $\langle v_i, x', y', v_j \rangle$, where $v_j \in N(v_1)$. If $x' \neq v_j$, give colour 1 to x' and colour j to z and z'. Otherwise, suppose that $N(v_i) = \{v_1, v_j, z, z'\}$ and $z, z' \in N(v_{m(G)})$. It is easy to verify that, in this case, $D(G)$ encircles two pairs, z, z' and x, y, a contradiction. So, there must exist a neighbour of v_i non-adjacent to $v_{m(G)}$ that we can colour with 1; so, colour z, z' with j. Now, consider that v_1 is the only coloured neighbour of v_i. If $m(G) = 5$, then there exists a colour $k \neq 1, 2, i$ with which we can colour z and z'. Otherwise, if $m(G) = 4$ (thus, $i = 3$), suppose that $N(v_3) = \{x', y', v_1\}$ and $x', y' \in N(v_4)$. Note that, as $D(G)(v_2) = \emptyset$ and $v_4 \notin N(v_1)$, E1b must occur and $d(v_1) = 3$. However, in this case, $D(G)$ encircles two pairs, x', y' and x, y, a contradiction. So, there must exist a neighbour x' of v_3 non-adjacent to v_4; colour x' with 1 and z, z' different from x' with 2 (one can verify that $z, z' \notin N(v_1)$ as $|D(G)| = 4$). Finally, suppose that $v_i \in N(v_2)$. If $v_{m(G)} \in N(v_2)$, we colour the link between v_i and $v_{m(G)}$ as in the previous item; so, consider $v_{m(G)} \notin N(v_2)$. Note that $m(G) = 5$, $i = 4$, $N^{D(G)}(v_2) = \{v_4\}$ and $N^{D(G)}(v_1) = \{v_3\}$. Colour z, z' with 3 and, if there exists a link $\langle v_2, x', y', v_4 \rangle$, change the colour of y' to 1. If, at the end, $v_{m(G)}$ has a link neighbour y' still uncoloured, then give colour i to y', where v_i is the other extremity of the link passing by y'.

- E2 occurs: suppose, without loss of generality, that $\langle x, v_1, v_2, y, v_{m(G)} \rangle$ is a cycle in G and that, if E2a occurs, then $v_{m(G)-1} \in D(G) \setminus N(v_{m(G)})$. Let $W = D(G) \setminus \{v_{m(G)}\}$. Colour v_i with i, for all $i \in [1, m(G) - 1]$, x, y with $m(G) - 1$ and $v_{m(G)}$ with 1. Note that, as neither x nor y is encircled by $D(G)$, $v_{m(G)-1} \notin N(z)$, for all $z \in \{x, y, v_1, v_2\}$; thus, the precolouring is proper and $r(v_i) = 0$, for all $i \in [1, m(G) - 1]$. Now, we need to colour the link vertices of W. First, consider a link $\langle v_i, \cdots, v_j \rangle$, where $v_i, v_j \in N(v_{m(G)})$: if $x' \neq y'$, give colour i to y' and colour j to x' (if x' or y' is already coloured, do not change their colours); otherwise, give colour 2 to x'. Now, if there is still some uncoloured link vertex, note that such a vertex lies within a link with extremity in $v_{m(G)-1}$ and $v_{m(G)-1} \notin N(v_{m(G)})$ (hence, E2a occurs). So, let z, z' be all the uncoloured link neighbours of v_i, $i \in [1, m(G)-2]$. Note that $r(v_i) = 0$; thus, if there exists $c \in M(v_i) \setminus \{m(G)-1\}$, then we can colour z and z' with c. We prove that this colour exists. As v_1 and v_2 have exactly 2 coloured neighbours and $m(G) \geq 4$, we know that $M(v_j) \setminus \{m(G)-1\} \neq \emptyset$, for $j = 1$ and $j = 2$. So, consider $v_i \in N(v_{m(G)})$. By an analogous argument, we can suppose that v_i has more than two coloured neighbours. It is easy to verify that, in this case, $v_{m(G)-1} \in N(v_i)$ and there exists a path of length at most 3, $\langle v_i, x', y', v_j \rangle$, for some $j \in [3, \cdots, m(G)-2]$; thus, $m(G) \geq 6$ and, trivially, $\{1, \cdots, m(G)-2\} \setminus \psi(N[v_i]) \neq \emptyset$, i.e., colour c exists. We can apply an analogous argument to colour any uncoloured link neighbour of $v_{m(G)-1}$ at the end. We then obtain an unsaturated precolouring ψ with candidate set W that colours all link vertices of W. Also, observe that $v_{m(G)}$ is the only vertex not in W that may have degree larger than $m(G)-1$; if this is the case, we have that E2b occurs and $v_{m(G)}$ is within the link $\langle v_1, x, v_{m(G)}, v_{m(G)-1} \rangle$. So, ψ and W satisfy Lemma 2.15.

3.3.3 $|D(G)| = m(G)$ and $D(G)$ encircles a vertex u

Let $D(G) = \{v_1, \cdots, v_{m(G)}\}$ and suppose that $D(G)$ encircles $u \in V \setminus D(G)$. By Lemma 2.1, we can suppose, without loss of generality, that $v_1 \notin N(u)$ and that $v = v_{m(G)}$ is a (v_1, u)-bridge (i.e., $v \in N(v_1) \cap N(u)$ and $d(v) = m(G)-1$); also, there must exist $v_q \in N(u) \setminus \{v\}$ (if there exists v_q that is within a cycle with v, then choose this vertex).

Let $W = D(G) \setminus \{v\}$; clearly, W satisfies Lemma 2.15, as v is within the link $\langle v_1, v, u, v_q \rangle$. So, now we construct an unsaturated precolouring with candidate set W that colours all link vertices of W. Apply colour i to v_i, $1 \leq i < m(G)$, colour 1 to u and colour q to v. Note that the only situations where we repeat colours in the neighbourhood of some vertex of W are: if there exists $v_k \in N(u) \cap N(v_1)$, in which case, by the choice of v_q, we have $k = q$ and $r(v_1) = r(v_q) = 1$; or if there exists $v_k \in N(v) \cap N(v_q)$, in which case we can suppose that $k = 1$ (thus, again, $r(v_1) = r(v_q) = 1$). So, we can suppose that

(*) $r(v_i) = 0$, for all $v_i \in W \setminus \{v_1, v_q\}$, $r(v_1) = r(v_q) \leq 1$ and, if $r(v_1) = r(v_q) = 1$, then $\langle u, v, v_1, v_q \rangle$ is a cycle in G.

Now, we want to colour the remaining link vertices. Consider $v_i \in N^W(u)$ and let $S \subseteq N^W(v_i)$ be the neighbours of v_i having a link with v_i. Let

$\{x_1, \cdots, x_q\} \subseteq N(v_i)$ be such that $x_j \in N(u) \cap N(v_i)$ or x_j is within a link between v_i and some $v_{i_j} \in S$. Note that, if $v_j \in S$, then $v_j \notin N(u)$, the link between v_i and v_j is the only one with extremity in v_j and $N^W(v_j) = \{v_i\}$. Also, if $x_j \in N(u) \cap N(v_i)$, then v_i is the only neighbour of x_j in W. Now, if $q = 1$ and $x_1 \in N(u)$, then we colour x_1 with any colour in $M(v_i)$ (if $M(v_i) = \emptyset$, colour x_1 with any colour different from $1, i$). Otherwise, we have $S \neq \emptyset$ and, consequently, $d(v_i) = m(G) - 1$ (v_i must be a (u, v_j)-bridge, for all $v_j \in S$). By Proposition 3.13, since $\{x_1, \cdots, x_q, u\} \subseteq N(v_i) \setminus W$, there exists at least q vertices in $W \setminus \{v_1\}$ non-adjacent to v_i whose colours we can use to colour $x_1, \cdots, x_q$. By what was said before, one can verify that this colouring does not increase $r(v_j)$, for all $v_j \in W$, and (*) still holds.

Now, we colour the remaining link vertices. Note that if $r(v_i) = 0$ and v_i has only one uncoloured link neighbour x, then we can just colour x with any colour in $[1, \cdots, m(G)] \setminus \psi(N(x))$ (as $x \notin D(G)$, we have that $d(x) < m(G) - 1$ and such a colour must exist). We prove that this always occurs, i.e., that, for all $v_i \in W$, v_i has at most one uncoloured link neighbour, say x, and if x exists, then $r(v_i) = 0$. So, suppose that there exists a link $P = \langle v_i, x, y, v_j \rangle$ where x is uncoloured. If $(v_i, v_j) \in E(G)$, then we know that $x \neq y$ and $v_j \in N(u)$, as otherwise x would have been coloured in the previous paragraph. As pointed out before, this is the only link with extremity in v_i; also, trivially, there is no cycle $\langle u, v_q, v_1, v \rangle$ containing v_i and, by (*), we know that $r(v_i) = 0$. Now, suppose that $(v_i, v_j) \notin E(G)$. As u is encircled by $D(G)$, it is easy to see that there exists a cycle C containing the link P and either u or some $v_k \in N^{D(G)}(u)$. In either case, P is the only link of this type with extremity in v_i, i.e., x is the only uncoloured link neighbour of v_i. Also, as $(v_i, v_j) \notin E$ and by (*) and the existence of the cycle C, it easy to see that $r(v_i) = 0$.

3.3.4 $|D(G)| = m(G) + 1$ and G has no quasi-good set

Now, we colour the graphs that, although having more than $m(G)$ dense vertices, do not have a quasi-good set. By Theorem 3.7, we know that $|D(G)| = m(G) + 1$ and one of the following situations occurs:

- The structure of $D(G)$ is as represented in Figure 3.7: we know that $m(G) = 4$, $d(v_1) = d(v_2) = 3$ and the link vertices of $D(G) \setminus \{v_1, v_2\}$ are $\{v_1, v_2, x, y, x', y'\}$. Observe Figure 3.17. One can easily verify that

ψ and W satisfy Lemma 2.15, for ψ being the precolouring presented and W being the set of grey vertices.

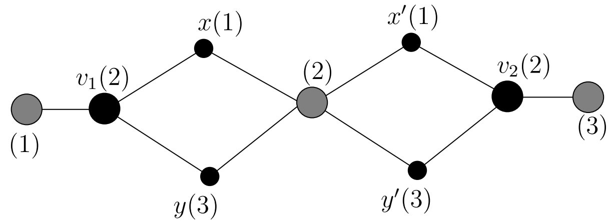


Figure 3.17: Partial colouring of a graph whose structure is as in Figure 3.7. The grey vertices can be the basis of a b-colouring of G with $m(G) - 1$ colours.

- $D(G)$ induces a cycle of length 5 and $d(u) = 3$, for all $u \in D(G)$: let $\langle v_1, v_2, v_3, v_4, v_5 \rangle$ be the cycle induced by $D(G)$. Colour v_i with i, for $i \in [1, 2, 3]$, v_4 with 1 and v_5 with 3; let ψ be the obtained precolouring. One can easily verify that $\{v_1, v_2, v_3\}$ and ψ satisfy Lemma 2.15.

- There are vertices u, v with degree $m(G) - 1$ and a non-dense vertex, x, such that $\langle u, v, x \rangle$ is a cycle in G and $D(G) \subseteq N(u) \cup N(v)$: let $W = D(G) \setminus \{u, v\} = \{v_1, \cdots, v_{m(G)-1}\}$ and colour each $v_i \in W$ with colour i. Since $d(u) = d(v) = m(G) - 1$ and $x \in N(u) \cap N(v)$, there exist $v_i \in N^W(u)$ and $v_j \in N^W(v)$. Colour u with j and v with i. Now, suppose that there is an uncoloured link $\langle v_a, x, y, v_b \rangle$. If $x \neq y$, then colour x with b and y with a. If $x = y$, then: if $v_a, v_b \in N(u)$, colour x with colour j; otherwise, colour x with colour i. Note that each $v_k \in N^W(u)$ has at most two link neighbours, namely u and some $v_j \in N^W(u) \setminus \{v_i\}$. The same is analogously valid for the vertices in $N^W(v)$. Then, clearly, $r(v_i) \leq 1$, for all $v_i \in W$. Thus, we obtain an unsaturated precolouring ψ with candidate set W that colours all link vertices of W and where every vertex $z \in V(G) \setminus W$ with degree at least $m(G) - 1$ is a link vertex of W (these vertices are exactly u and v). Thus, Lemma 2.15 can be applied to extend ψ to a b-colouring of G with $m(G) - 1$ colours.

3.4 b-Colouring a cactus that has a quasi-good set

In this sectio, we consider G to be a cactus that has a quasi-good set and with $m(G) \geq 7$. The main result of this section is the following.

Theorem 3.16. *Let G be a cactus with $m(G) \geq 7$ and W be a quasi-good set of G. Then, there exists a b-colouring of G with basis W.*

Let G be a cactus with $m(G) \geq 7$. Given a quasi-good set W of G, we will construct an unsaturated precolouring of G with candidate set W that colours $W \cup N(W)$. Then, by Lemma 2.15, we know that this partial colouring can be extended to the entire graph.

Let G' denote the induced subgraph $G[W \cup N[W]]$ and let H be a connected component of G'. A subset $R \subseteq V(H)$ is a *tight set of H* if $H[R]$ is connected and, for every $u \in W \cap R$, either $N(u) \setminus W \subseteq R$ or there is a cycle C in $H[R]$ that contains u and the only neighbours of u in R are its neighbours in C. R is called a *basic tight set of H* if it either induces a cycle or equals $(N(u) \setminus W) \cup \{u\}$, for some $u \in W \cap V(H)$. We denote the basic tight set $(N(u) \setminus W) \cup \{u\}$ by $[u]$.

The general idea is to colour each vertex of W with a different colour and, then, colour each connected component H of G' separately, using a sequence of tight sets of H, $R_1, \cdots, R_k$, where R_1 is basic, $R_i \subset R_{i+1}$, $i = 1, \cdots, k-1$, and $R_k = V(H)$. So, we start by colouring R_1 and, at step i, we extend the precolouring that colours R_{i-1} to a precolouring that colours R_i, $i = 2, \cdots, k$.

In the next subsection, we show how to obtain this sequence. We also ensure some other properties for the tight sets of the sequence that will be important for colouring G'. Then, in Subsection 3.4.2 we show how to colour a basic tight set of H and how to extend the precolouring that colours R_{i-1} to a precolouring that colours R_i. We will also need the following definitions.

Let H be a connected component of G'. Given a tight set R of H, we say that X is an *R-flap* if X is the set of vertices of a connected component of $H \setminus R$. Also, if $u \in (R \cup N(R)) \cap W$ is such that $N(u) \setminus (W \cup R) \neq \emptyset$, we say that u is an *intermediate vertex of R*.

3.4.1 Tight sets

Before we explain how to obtain the desired sequence of tight sets, we show how to obtain a basic tight set having a convenient property. To better understand the necessity of this property, consider a connected component H of G' and a sequence of tight sets $R_1, \cdots, R_k$ of H as mentioned before. Suppose that we have an unsaturated precolouring ψ with candidate set W that colours exactly $R_i \cup W$ in H, $i \in [1, k-1]$. In order to extend ψ to colour $R_{i+1} \setminus (R_i \cup W)$, we would like to ensure that there are sufficiently many vertices of W "distant" from any R_i-flap X, so that we can use the colours of these "distant" vertices to colour $(X \cap (R_{i+1} \setminus R_i)) \setminus W$. More formally, we want to ensure the following for all tight set R in the sequence:

(Half Property) $|X \cap W| \leq \frac{1}{2}|W|$, for every R-flap X.

From now on, we write "R satisfies (HP)" when a tight set R satisfies the Half Property. Observe that if R is a tight set of H that satisfies (HP) and R' is a tight set of H containing R, then R' also satisfies (HP). Thus, we need only to ensure that the first tight set of the sequence satisfies (HP). We prove the existence of a basic tight set that satisfies (HP) in the following lemma, where we also ensure another property that will be useful later.

Lemma 3.17. *Let H be a connected component of G'. There exists a basic tight set R that satisfies (HP). Furthermore, if $H[R]$ is a cycle, then there is no $u \in W$ such that $R \subseteq (N(u) \setminus W) \cup \{u\}$.*

Proof: We first prove that there exists a basic tight set of H that satisfies (HP). Observe that, as $V(G') = N(W) \cup W$, H must have at least one basic tight set. So, let R be a basic tight set of H and $\{X_1, \ldots, X_k\}$ be the set of R-flaps with indices such that $|X_1 \cap W| \geq \cdots \geq |X_k \cap W|$. If $|X_1 \cap W| \leq \frac{1}{2}m(G)$, we are done; so, suppose otherwise. Observe that $|(V(H) \setminus X_1) \cap W| \leq \frac{1}{2}m(G)$. In the procedure described in the next two paragraphs, we obtain a basic tight set R' of H such that, for every R'-flap X, either (I) $X \subseteq V(H) \setminus X_1$, in which case $|X \cap W| \leq \frac{1}{2}m(G)$; or (II) $X \subset X_1$. Thus, if R' still has a flap X containing more than $\frac{1}{2}m(G)$ vertices of W, then $|X| < |X_1|$. So, as the graph is finite, we can run the procedure until we find the desired tight set.

Suppose, first, that $R = [u]$, for some $u \in V(H) \cap W$, and let $N = |N(X_1) \cap [u]|$. We know that $|N| \leq 2$ and N separates R from X_1. Suppose that there exists a cycle C intersecting R and X_1 and let $R' = V(C)$; trivially, R' is a basic tight set of H. As G is a cactus, one can verify that C contains N. Consequently, either (I) or (II) holds, for every R'-flap. Now, suppose that there is no such cycle. Then, we know that $N(X_1) \cap [u] = \{x\}$ and

$N(x) \cap X_1 = \{x'\}$. Also, x' separates X_1 from R and, trivially, if $x' \in W$, then every $[x']$-flap satisfies (I) or (II). So, suppose that $x' \notin W$. Observe that, in this case, $x \neq u$ as, otherwise, $x' \in [u]$. Also, as $V(H) \subseteq N(W) \cup W$, there must exist $v \in N^W(x')$, and every $[v]$-flap satisfies (I) or (II).

Now, assume that R induces a cycle in H and let $N(X_1) \cap R = \{x\}$. Suppose that $x \in W$ and consider $R' = [x]$. Let X be any R'-flap. Observe that, as x separates R from X_1, we have that $X \subseteq V(H) \setminus X_1$ or $X \subseteq X_1$, i.e., X violates (I) and (II) only if $X = X_1$, in which case $N^{X_1}(x) \subseteq W$. So, suppose that this is the case, i.e., that $N^{X_1}(x) \subseteq W$. If $N^{X_1}(x) = \{u\}$, let $R' = [u]$. Otherwise, we know that there exists a cycle C containing x and $N^{X_1}(x)$; thus, let $R' = V(C)$. It is easy to see that X satisfies (I) or (II), for all R'-flap X. Now, suppose that $x \notin W$. One can easily verify that an argument analogous to the one in the previous paragraph can be applied by analysing if x is within a cycle in $H[X_1 \cup \{x\}]$ or not.

Now, let R be a basic tight set of H satisfying (HP). If $H[R]$ is a cycle and $R \subseteq (N(u) \setminus W) \cup \{u\}$, for some $u \in W$, then $R \subseteq [u]$ and, consequently, $[u]$ is a basic tight set that satisfies the lemma. $\square$

Now, we want to construct a desired sequence from a basic tight set R satisfying the lemma above. So, we set R_1 to R and, while the current set R_i is not equal to $V(H)$, we obtain R_{i+1} from R_i by adding either $(N(u) \setminus W) \cup \{u\}$, for some intermediate vertex u of R_i, or $(N(R_i) \cap X) \setminus W$, for some R_i-flap X, in the case R_i has no intermediate vertex. The following two lemmas prove that this procedure works.

Lemma 3.18. *Let H be a connected component of G' and R be a tight set of H that satisfies (HP). Then $w \in R$, for all $w \in W$ such that $N(w) \setminus W \subseteq R$.*

Proof: Let $w \in W \setminus R$. Observe that if w is not in the same connected component as R and $N(w) \setminus W \subseteq R$, then $N(w) \setminus W = \emptyset$ and, as $d(w) = m(G) - 1$, we have $W = N[w]$, a contradiction since $V(H) \cap W \neq \emptyset$ (recall that $V(H) \subseteq V(G') \subseteq W \cup N(W)$). So, let X be the R-flap containing w and denote by S the set $N(w) \cap R$. By Lemma 3.1, we know that $|S| \leq 2$, and, by (HP) and the fact that $m(G) \geq 7$, there must exist at least 4 vertices in $W \setminus X$. Observe that, as $d(w) \geq m(G) - 1$, for each vertex in $W \setminus \{w\}$

non-adjacent to w, there must exist at least one vertex in $N(w) \setminus W$, i.e.,

$$\begin{aligned} |N(w) \setminus W| &\geq |W \setminus N[w]| \\ &\geq |W \setminus (S \cup X)| \\ &= |(W \setminus X) \setminus S| \\ &= |W \setminus X| - |S \cap W| \end{aligned}$$

Also, as $|W \setminus X| \geq 4$ and $N(w) \setminus W = (N(w) \setminus (S \cup W)) \cup (S \setminus W)$, we have:

$$|N(w) \setminus (W \cup S)| \geq 4 - (|S \cap W| + |S \setminus W|) = 4 - |S| \geq 2$$

Thus, $N(w) \setminus (W \cup R) \neq \emptyset$. □

Lemma 3.19. *Let H be a connected component of G' and R be a non-empty tight set of H that satisfies (HP), $R \neq V(H)$. Then, either $R' = R \cup [u]$ is a tight set, for some intermediate vertex u of R, or R has no intermediate vertex and $R' = R \cup N^X(R)$ is a tight set, for any R-flap X. Furthermore, $R \subset R'$.*

Proof: First, note that if there exists $u \in R \cap W$ such that u is an intermediate vertex of R, then $R' = R \cup [u]$ is tight as it is connected and $R' \cap W = R \cap W$. So, suppose that every intermediate vertex of R is not in R. Let u be any intermediate vertex of R and let $R' = R \cup [u]$. Suppose that R' is not tight. As R' is connected, there must exist $w \in W \cap R'$ such that $N(w) \setminus W \nsubseteq R'$. Obviously, $w \neq u$; hence, $w \in R$ and we have a contradiction, as in this case, w is also an intermediate vertex of R. So, if R has any intermediate vertex, then there exists an intermediate vertex u of R such that $R \cup [u]$ is tight. Observe that, by the definition of intermediate vertex, we have $R \subset R \cup [u]$.

Now, consider a tight set R of H that has no intermediate vertex, $R \neq V(H)$, and let X be any R-flap. Also, let $S = N^X(R)$ and $R' = R \cup S$. Obviously, $R \subset R'$ and R' is still connected. Additionally, by Lemma 3.18 and the fact that R has no intermediate vertex, we know that $S \cap W = \emptyset$ and, consequently, $N^S(u) = \emptyset$, for all $u \in W \cap R$. Thus, $W \cap R = W \cap R'$ and $N^{R'}(u) = N^R(u)$, for all $u \in R \cap W$. So, R' is also tight. □

We can then obtain a sequence of tight sets of H as desired with the additional property that every set on the sequence satisfies (HP). Now, consider a tight set R on the sequence and let X be an R-flap. As $m(G) \geq 7$, we know that there are at least four vertices in $W \setminus X$, say w_1, w_2, w_3, w_4.

As said before, we would like to use the colours of these vertices to colour $N(R) \cap X$. However, observe that if $w \in R \cap W$ separates X from R, it may occur that $\{w_1, \cdots, w_4\} \subseteq N(w) \setminus X$ and, thus, we cannot use the colours of those vertices to colour any $x \in N(w) \cap X$. To solve this problem, we introduce the following definitions.

Let R be a tight set of H. If $w \in W$ is an intermediate vertex of R, let $J(w, R)$ be the union of R-flaps that intersect $N(w)$. We say that R has a *4-gap* if $|W \setminus (J(w, R) \cup \{w\})| \geq 4$, for all intermediate vertex w of R. Observe that, if R, R' are tight sets such that $R \subseteq R'$, then the R'-flaps are contained in the R-flaps; so, one can easily see that if R has a 4-gap, then R' also has a 4-gap. Thus, it is sufficient to ensure that the initial basic tight set has a 4-gap. Unfortunately, if the first tight set of the sequence is a cycle, then it does not necessarilly have a 4-gap. So, if C is the vertex set of a cycle of H, we define $[C]$ as being the set $C \cup \bigcup_{w \in C \cap W}(N(w) \setminus W)$ and we prove, in the following lemma, that $[C]$ is tight and has a 4-gap.

Lemma 3.20. *Let H be a connected component of G' and R be a basic tight set of H satisfying (HP). If $R = [w]$, for some $w \in W$, then R has a 4-gap. Otherwise, $[R]$ is tight and has a 4-gap.*

Proof: If $R = [u]$, for some $u \in W$, let Q denote R; otherwise, let Q denote $[R]$. Trivially, Q is connected and, as $N(w) \setminus W \subseteq Q$, for all $w \in Q \cap W$, we have that Q is tight. Now, we want to prove that Q has a 4-gap. Let $w \in W$ be an intermediate vertex of Q (if there is no such vertex, Q has a 4-gap by definition). We know that $w \notin Q$, as $N(w) \setminus (W \cup R) \neq \emptyset$. So, let X be the Q-flap containing w. As $w \in X$, trivially, $J(w, Q) \cup \{w\} = X$. Also, as Q satisfies (HP) and $m(G) \geq 7$, we have $|W \setminus (J(w, Q) \cup \{w\})| = |W \setminus X| \geq 4$. □

3.4.2 Colouring Phase

Let G be a cactus with $m(G) \geq 7$, $W \subseteq D(G)$ be a quasi-good set of G and $G' = G[W \cup N(W)]$. We say that a precolouring ψ of G is *nice for W* (or simply nice, if there is no ambiguity) if it is an unsaturated precolouring with candidate set W that colours only vertices of G' and is such that, for every connected component H of G', the coloured vertices in H are exactly $(W \cap V(H)) \cup R$, where R is either empty or is a tight set of H that satisfies (HP). For simplicity, as W is the candidate set of ψ and must be coloured, we say only that ψ colours R. Also, from now on, we consider that the vertices of W are coloured with colours from the range $[1, m(G)]$ and we denote the vertex of W coloured with i by w_i.

So, given a nice precolouring ψ of G' and a connected component H such that ψ colours only vertices of W in H, we will pick a basic tight set R as explained in Lemma 3.17, extend ψ to colour R, obtaining a nice precolouring ψ^+, then we extend ψ^+ to colour: $N(u) \setminus W$, for some intermediate vertex u of R; or, if R has no intermediate vertex and $R \neq V(H)$, we extend ψ^+ to colour $N^X(R)$, for some R-flap X. We first show how to extend ψ to colour a basic tight set R of H.

Lemma 3.21. *Let G be a cactus with $m(G) \geq 7$, $W \subseteq D(G)$ be a quasi-good set of G, H be a connected component of $G' = G[W \cup N(W)]$, ψ be a nice precolouring that colours only vertices of H that are in W and $R = [w]$, for some $w \in H \cap W$. Then, there exists a nice precolouring that extends ψ and colours R.*

Proof: Let $X_1, \cdots, X_q$ be the vertex sets of the non-trivial connected components of $H - w$ containing at least one vertex of $[w]$ (i.e., $|X_i| \geq 2$ and $X_i \cap (N(w) \setminus W) \neq \emptyset$). Observe that if $x \in N(w) \setminus (W \cup \bigcup_{i=1,\cdots,q} X_i)$, then $\{x\}$ is a connected component of $H - w$. So, after colouring $N(w) \cap X_i$, for all $i \in [1, q]$, we can give any colour from $M(w)$ to x, if there exists such a colour; otherwise, i.e., if $M(w) = \emptyset$, we can colour x with any colour different from $\psi(w)$. An analogous argument can be made in the case where $X_i \cap W = \{u\}$ and $N(w) \cap X_i = \{u, x\}$, $x \neq u$: since w is the only coloured neighbour of u, we have $M(w) \subseteq M(u)$; thus, after colouring $X_j \cap N(w)$, for all $j \in [1, q]$, $j \neq i$, we can give any colour from $M(w)$ to x (again, if $M(w) = \emptyset$, colour x with any colour different from $\psi(w), \psi(u)$ - exists as $m(G) \geq 7$). So, suppose that $(X_i \cap W) \setminus N(w) \neq \emptyset$, for all $i \in [1, q]$. By Lemma 3.1, we know that $|X_i \cap [w]| \leq 2$, for all $i \in [1, q]$. So, consider, without loss of generality, that there exists an index $p \in [0, q]$ such that $|X_i \cap [w]| = 2$, for all $i \in [1, p]$, and $|X_i \cap [w]| = 1$, for all $i \in [p + 1, q]$. For each $i \in [1, q]$, denote the vertices in $X_i \cap [w]$ by x_i, y_i (if $i > p$, consider $x_i = y_i$). Also, denote the set $\{x_1, y_1, \cdots, x_q, y_q\}$ by Z. We want to construct a function $f : Z \to M(w)$ in such a way that the vertex of W coloured with $f(z_i)$ is in X_i, for all $z_i \in Z$. Then, we will use this function to colour $[w]$. So, consider the cases:

- $i > p$: let $u \in (W \cap X_i) \setminus N(w)$ and set $f(x_i)$ to $\psi(u)$;

- $i \leq p$: if there exist $u_1, u_2 \in W \cap X_i$ such that u_1 is reachable from x_i and not from y_i and u_2 is reachable from y_i and not from x_i, then set $f(x_i)$ to $\psi(u_1)$ and $f(y_i)$ to $\psi(u_2)$. Otherwise, suppose, without loss of generality, that every vertex of $W \cap X_i$ reachable from y_i is also reachable from x_i. Let $u \in W \cap X_i$ reachable from both x_i and y_i, if there exists one, or let u be any vertex in $W \cap X_i$, otherwise. Set $f(x_i)$ to $\psi(u)$ and $f(y_i)$ to *null*.

Now, let $J = \{z \in Z : f(z) \neq null\}$. Note that $x_i \in J$, for all $i \in [1, q]$, and that $f(z) \neq f(z')$, for all $z, z' \in J$. Furthermore, let $z, z' \in J$ and $f(z) = c$; we know that w_c is not adjacent to w and is not reachable from z'. Thus, if $|J| \geq 2$, we can permute the colours defined by f on the vertices of J in such a way that $\psi(z) \neq f(z)$, for all $z \in J$, and obtain an unsaturated extension of ψ that colours J. So, suppose that $|J| = 1$ (hence, $q = 1$). If $p = 0$, as x_1 is not encircled by W, there must exist $u \in W$ not reachable from x_1 and we can color x_1 with $\psi(u)$. So, consider $p = 1$. If there exists $u \in X_1 \cap W$ not reachable from y_i (recall the construction of f), then colour y_i with $\psi(u)$ and x_i with $\psi(u')$, for any $u' \in W$ not reachable from x_i (exists, as x_i is not encircled by W). So, suppose that every $u \in X_i \cap W$ is reachable from both x_i and y_i. We know that there exists a cycle C containing x_1, y_1, w and at least one $u \in W \cap X_i$. Trivially, any $V(C)$-flap X separated from C by w is also a connected component of $H - w$ and, as $q = 1$, we know that $|W \cap X| \leq 1$. Furthermore, let A be the subset of $[w]$-flaps containing any neighbour of x_1 or y_1. Note that any $V(C)$-flap separated from C by other vertex than w is contained in some $[w]$-flap in A. One can then verify that if $[w]$ satisfies (HP) or has a 4-gap, then $V(C)$ also does; thus, as $V(C) \nsubseteq [v]$, for all $v \in W$, we can consider the basic tight set $V(C)$ instead of $[w]$.

Denote by ψ^+ the extension of ψ obtained in the previous paragraph. Now, let $S = Z \setminus J$ (subset of uncoloured y_i's) and consider $y_i \in S$ (note that if $q = 1$, then $S = \emptyset$). Suppose that there exists $u \in X_i \cap W$ not reachable from y_i. If $f(x_i) \neq \psi^+(u)$, then colour y_i with $\psi^+(u)$ and remove it from S. Otherwise, by the construction of f, we have that $W \cap X_i = \{u\}$. Thus, $\psi^+(N(y_i)) \subseteq \psi^+(\{x_i, w\})$ and we can colour y_i either with a colour from $M_{\psi^+}(w)$, if one exists, or with any colour not in $\psi^+(N(y_i))$. So, we denote the subset $X_i \cap W$ by F_i and consider that every vertex in F_i is reachable from both x_i and y_i, for all $y_i \in S$. Also, note that $N(y_i) \cap [w] = \{w\}$, for all $y_i \in S$; thus, during the colouring of $[w]$, every coloured neighbour of y_i is in W, for all $y_i \in S$. Consequently, if $|S| > |M_{\psi^+}(w)|$ and we are able to colour

$S' \subset S$ with cardinality $|M_{\psi^+}(w)|$ each with a different colour from $|M_{\psi^+}|$, then we can colour y_i with any colour not in $N(y_i)$, for all $y_i \in S \setminus S'$ (such a colour exists as $q \geq 2$). So, from now on we consider that $|M(w)| \geq |S|$. Trivially, $F_i \cap F_j = \emptyset$, for every pair $y_i, y_j \in S$. Let $u \in W$ be such that $\psi^+(u) \in M(w)$. We know that if some y_i cannot be coloured with $\psi^+(u)$, then $u \in F_i$ and, consequently, y_j can be coloured with $\psi^+(u)$, for every $y_j \in S \setminus \{y_i\}$. So, if $|S| \geq 2$ and $\psi^+(F_i) \cap M(w) \neq \emptyset$, for every $y_i \in S$, then we can colour S with colours from $M(w)$. Now, suppose otherwise and consider, without loss of generality, that $y_1 \in S$ and $M(w) \subseteq \psi^+(F_1)$. As y_1 is not encircled by W, there must exist a vertex $u \in W$ not reachable from y_1 (and, consequently, not in X_1). As $u \notin F_1$ and $M(w) \subseteq \psi^+(F_1)$, we must have that $\psi^+(u) \notin M(w)$. So, let $z \in N(w)$ be such that $\psi^+(z) = \psi^+(u)$. If $z = u$, then $d(w) \geq m(G)$ (as u is not reachable from y_1) and we can repeat the colour $\psi^+(z)$ in y_1; and if $z \neq x_1$, then colour z with c, for any $c \in M(w)$, and y_1 with $\psi^+(u)$. So, suppose that $z = x_1$. Recall the definition of W_t and note that $W_x = W_y$. Thus, as W does not encircle the pair (x_1, y_1), one can verify that either $d(w) \geq m(G)$, in which case we can colour y_1 with $\psi^+(u)$, or there exists $u' \in W \setminus (W_x \cup \{u\})$. Thus, as $\psi^+(x_1) \neq \psi^+(u')$, we can apply the same argument as before to colour y_1. After this, just colour the remaining uncoloured vertices in $[w]$ with the colours missing in $N(w)$. $\square$

Lemma 3.22. *Let G be a cactus with $m(G) \geq 7$, $W \subseteq D(G)$ be a quasi-good set of G, H be a connected component of $G' = G[W \cup N(W)]$, $R \subseteq H$ be basic tight set satisfying Lemma 3.17 such that $H[R]$ is a cycle and ψ be a nice precolouring that colours only vertices of W in H. Then, we can extend ψ to colour either $[R]$, if $|R \cap W| = 1$, or R, otherwise.*

Proof: For each $x \in R \setminus W$, denote by $N^*(x)$ the subset $N^W(x) \setminus R$. Note that if $N^W(x) \cap R = \emptyset$, for some $x \in R \setminus W$, then $N^*(x) \neq \emptyset$. If $R \cap W = \emptyset$, note that it is easy to use the colours in $\psi(\bigcup_{x \in R} N^*(x))$ to colour R. If $R \cap W = \{w\}$, note that we can colour $[w]$ using Lemma 3.21. Then, let ψ^+ be the obtained precolouring and $R = \{w, x_1, \cdots, x_q\}$. Also, let $\psi^+(w) = c$, $\psi^+(x_1) = c_1$ and $\psi^+(x_q) = c_q$. Colour x_i with c, for each i even in $[2, q-1]$. Then, for each i odd in $[3, q-1]$, if w_{c_1} is not reachable form x_i, then colour x_i with c_1; otherwise, colour x_i with $\psi(w')$, for any $w' \in N^*(x_{i-1})$. At the end, if q is even and $\psi(x_{q-1}) = c_1 = c_q$, then change the colour of x_q to $\psi(w')$, for any $w' \in N^*(x_{q-1})$; otherwise, if q is even and $\psi(x_{q-1}) = c_q \neq c_1$ (in which case, we know that w_{c_1} is separated from R by x_{q-1} and $w_{c_q} \in N^*(x_{q-2})$), then change the colour of x_1 to c_q and of x_q to c_1. So, from now on, we suppose that $|W \cap R| \geq 2$.

First, consider that there exists at least one maximal subpath $P \subseteq H[R]$ such that $P \cap W = \emptyset$ and P has length greater than one. So, let $P_1, \cdots, P_q$ be all such subpaths and let x_i, y_i be the extremities of P_i, for every $i \in [1, q]$. We know that $N^*(x) \neq \emptyset$, for all $x \in P_i \setminus \{x_i, y_i\}$, $i \in [1, q]$. We first colour $S = R \setminus \bigcup_{i=1}^{q}(P_i \setminus \{x_i, y_i\})$. So, let P be a connected component of $H[S]$. Trivially, P is a path; so, let z, z' be the extremities of P. First, we colour $N^P(w)$, for all $w \in P \cap W$ non adjacent to z or z'. Let w be such a vertex and let $N^P(w) = \{t_1, t_2\}$. If $t_i \notin W$, let $w_i \in (W \cap P) \setminus \{w\}$ closest to t_i, $i = 1, 2$. We know w_i exists as $t_i \neq z, z'$; also, we know that t_i is within a link between w and w_i, $i = 1, 2$. If both t_1 and t_2 are not in W, then colour t_1 with $\psi(t_2)$, if t_1 is not coloured yet, and t_2 with $\psi(t_1)$, if t_2 is not coloured yet. Otherwise, suppose that $t_1 \in W$, t_1 is not coloured and $t_2 \notin W$ (if both are in W or t_1 is coloured, there is nothing to do). If (t_2, w_2) is not an edge, than colour t_2 with $\psi(w_2)$. Otherwise, let $t' \in N^P(w_2) \setminus \{t_2\}$. If $t' \in W$, as t_2 is not encircled by W, there must exist $w' \in W$ not reachable from t_2, in which case we colour t_2 with $\psi(w')$. Otherwise, we postpone the colouring of t_2 for the iteration of w_2. Now, consider $w \in W \cap P$ adjacent to z or z'. Let $N^P(w) = \{t_1, t_2\}$, where $t_2 \in \{z, z'\}$, without loss of generality. We know that t_2 is an extremity of some P_i, $i \in [1, q]$. Let $x \in N(t_2) \cap P_i$ and w' be any vertex in $N^*(x)$. If $t_1 \in W$, then colour t_2 with $\psi(w')$; so, suppose otherwise. If $|P \cap W| \geq 2$, then define w_1 related to t_1 as before. If $w_1 \notin N(t_1)$, then colour t_1 with $\psi(w_1)$ and t_2 with $\psi(w')$; otherwise, colour t_1 with $\psi(w')$, if t_1 is not coloured yet, and t_2 with $\psi(w_1)$. Now, suppose that $P \cap W = \{w\}$; then, t_1 is also the extremity of some P_j, $j \in [1, q]$, as $t_1 \in \{z, z'\}$. Note that, as $|W \cap R| \geq 2$, we have $i \neq j$. Thus, colour t_2 with $\psi(w')$ and t_1 with $\psi(w'')$, for any $w'' \in N^*(x')$, where $x' \in N(t_1) \cap P_j$. Now, we colour $P_i \setminus \{x_i, y_i\}$, for all $i \in [1, q]$. Let ψ^+ be the precolouring obtained above and consider $P_i = \langle x_i = v_1, v_2, \cdots, v_q = y_i \rangle$, $i \in [1, q]$. Also, let $u_1 \in N^R(v_1) \setminus P_i$ and $u_2 \in N^R(v_p) \setminus P_i$; as $|W \cap R| \geq 2$, we know that $u_1 \neq u_2$, and as ψ^+ is a proper precolouring, we know that $\psi^+(v_1) \neq \psi^+(u_1)$ and $\psi^+(v_p) \neq \psi^+(u_2)$. If $\psi^+(v_1) \neq \psi^+(u_2)$ and $\psi^+(v_q) \neq \psi^+(u_1)$, then we can easily colour $v_2, \cdots, v_{p-1}$ by alternating the colours $\psi^+(u_1), \psi^+(u_2)$ in P_i; so, suppose otherwise. One can verify that, in this case, $R \cap W = \{u_1, u_2\}$ and $N(u_1) \cap N(u_2) \neq \emptyset$, i.e., R forms the cycle $\langle u_2, x, u_1, v_1, \cdots, v_p \rangle$. So, let $w \in N^*(x_2)$. If there exists $w' \in W \setminus \{w\}$ not reachable from x, then colour x with $\psi^+(w')$, v_1 and v_p with $\psi^+(w)$ and, then, we can again alternate the colours $\psi^+(u_1), \psi^+(u_2)$ in $P_i \setminus \{v_1, v_p\}$. Otherwise, we have $p = 3$ and, as $m(G) \geq 7$, there must exist $w' \in W$ separated from R by x (and, obviously, $w' \neq u_1, u_2, w$). Thus, we colour x with $\psi^+(w)$, v_1 with $\psi^+(u_2)$, v_3 with $\psi^+(u_1)$ and v_2 with $\psi^+(w')$.

Now, suppose that $H[R]$ is a cycle such that every maximal subpath of R that does not intersect W has length at most one. Let $R \cap W = \{u_1, \cdots, u_q\}$. We write R as $\langle u_1, x_1, y_1, \cdots, u_q, x_q, y_q \rangle$ and assume that $x_i = u_i$ when $(u_i, u_{i+1}) \in E(G)$ and that $y_i = x_i$ when the path between u_i and u_{i+1} has length at most two. We analyse the following cases (recall that $q \geq 2$):

- $q \geq 5$: for $i = 1, \cdots, q$, if $x_i \neq u_i$ then give colour $\psi(u_{(i+3) \mod q})$ to x_i. After this, for each uncoloured y_i, let $j = (i+1) \mod q$. Then, choose any colour in $\psi(W \cap R) \setminus \{\psi(x_i), \psi(u_j), \psi(x)\}$, where x is the neighbour of u_j in R different from y_i, i.e., x is either x_j or $u_{(j+1) \mod q}$. See Figure 3.18 for a better understanding. Note that, as $x_1, \cdots, x_q$ are coloured first, if $x_i = y_i$, for some i, the colouring is still proper.

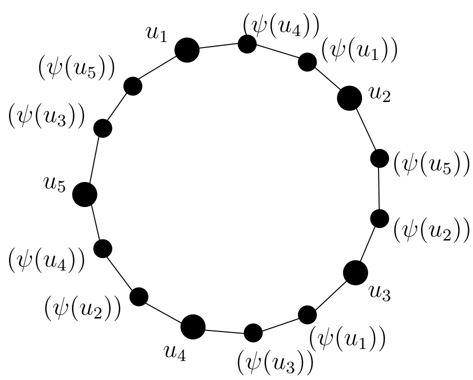


Figure 3.18: Representantion of a nice precolouring ψ that colours a basic tight set R when $H[R]$ is a cycle and $|R \cap W| = 5$. Some of the dense vertices may not exist, i.e., the paths between vertices of W in the cycle may have length 1,2 or 3.

- $q = 4$: all the possible cycles are represented in Figure 3.19, as well as a precolouring of $R' \subseteq R$. The only situations where there is some uncoloured vertex in R are in (a), (b) or (c). If (a) or (c) occurs, as x_1 is not encircled by W, there must exist a vertex $w \in W$ not reachable from x_3; then, just colour x_3 with $\psi(w)$. So, suppose that (b) occurs. If there exists $w \in (N^W(u_1) \cup N^W(x_1)) \setminus R$, then give colour $\psi(w)$ to x_3 and, as x_1 is not encircled by W, there must exist $w' \in W$ not reachable from x_1; then, give colour $\psi(w')$ to x_1. Otherwise, we can suppose that $(N(u_i) \cap W) \setminus R = \emptyset$, $i = 1, \cdots, 4$, and $(N(x_i) \cap W) \setminus R = \emptyset$, $i = 1, 2$. As $m(G) \geq 7$, we can choose any two colours $c, c' \in \{1, \cdots, m(G)\} \setminus \{\psi(u_1), \psi(u_2), \psi(u_3), \psi(u_4)\}$ to give to x_1 and x_3.

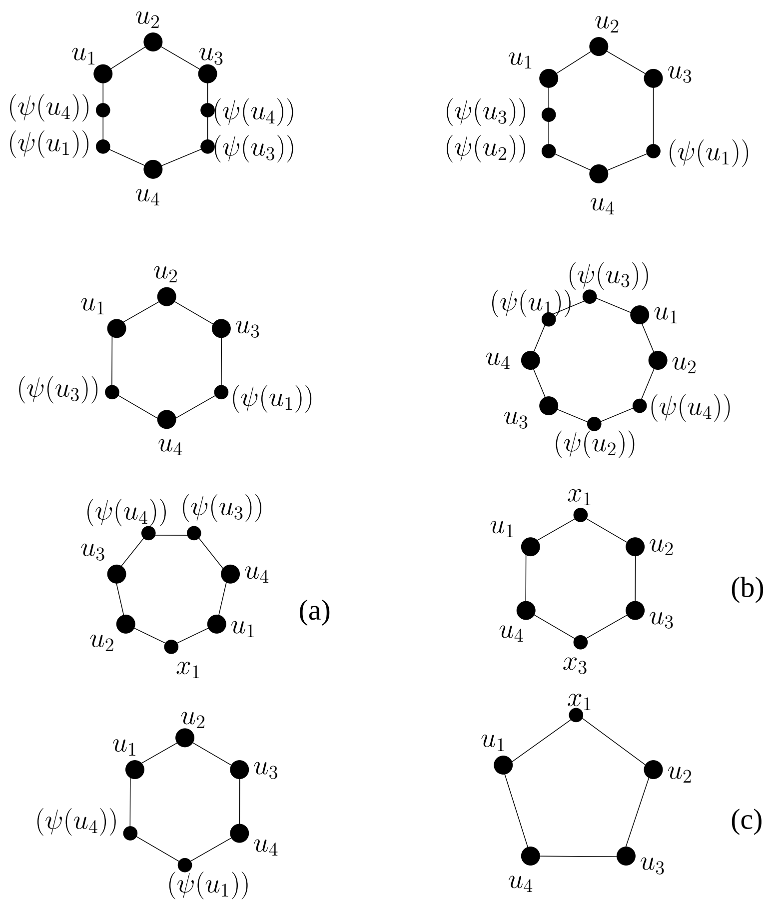


Figure 3.19: Cases where $|R \cap W| = 4$ and there is at least two edges in $G[R \cap W]$.

- $q = 3$: all the possible cycles are represented in Figure 3.20, as well as a precolouring of $R' \subseteq R$. The only situations where there is some uncoloured vertex in R are in (a), (b), (c) and (d). If (d) occurs, as x_3 is not encircled by W, then there must exist $w \in W$ not reachable form x_3 and we can just give colour $\psi(w)$ to x_3. Now, suppose that (a) occurs. If there exists $w \in (N^W(x_2) \cup N^W(u_2)) \setminus R$, then give colour $\psi(w)$ to y_2 and colour $\psi(u_1)$ to x_2. Otherwise, suppose that $(N^W(y_2) \cup N^W(u_3)) \setminus R$ is also empty (or we have an analogous situation) and give colour $\psi(u_1)$ to y_2 and any colour from $M(u_2)$ to x_2 (such a colour must exist as $m(G) \geq 7$ and $\psi(N[u_2] \cup N(x_2)) = \{\psi(u_1), \psi(u_2), \psi(u_3)\}$). Now, suppose that (b) occurs. If there exists any $w \in W \setminus \{u_2\}$ not reachable from x_3, then give colour $\psi(w)$ to x_3. Otherwise, as x_3 is not encircled by W, we must have that u_2 is the only vertex in W not reachable from x_3; consequently, we have that $d(u_1) \geq m(G)$ and we can give colour $\psi(u_1)$ to y_2 and colour $\psi(u_2)$ to x_3. Finally, consider that (c) occurs. Observe that if we can colour x_3 with $\psi(w)$, for some w reachable from x_1 not through u_1, then, as x_1 is not encircled by W, there must exist w' not reachable from x_1 and we can colour x_1 with $\psi(w')$ (by the choice of w, we know that $w \neq w'$). So, we can suppose that $d(u_3) = m(G) - 1$ (otherwise, x_3 can be coloured with $\psi(u_2)$) and $(N^W(x_1) \cup N^W(u_2)) \setminus R = \emptyset$. Analogously, we can suppose that $d(u_2) = m(G) - 1$ and $(N^W(x_3) \cup N^W(u_3)) \setminus R = \emptyset$. Thus, if there exist $w, w' \in W \setminus (N[u_1] \cup \{u_2, u_3\})$, $w \neq w'$, then we can colour x_1 with $\psi(w)$ and x_3 with $\psi(w')$. Otherwise, as E2 does not occur, we have that either $W \setminus (N[u_1] \cup \{u_2, u_3\}) = \{w\}$ and $d(u_1) \geq m(G)$, in which case we colour x_1 and x_3 with $\psi(w)$, or $W \setminus \{u_2, u_3\} \subseteq N[u_1]$ and $d(u_1) \geq m(G) + 1$, in which case we colour x_1 and x_3 with any colour $c \notin \psi(\{u_1, u_2, u_3\})$.

- $q = 2$: recall that $N^R(u_1) = \{x_1, y_2\}$ and $N^R(u_2) = \{y_1, x_2\}$. First, suppose that at least one of the paths between u_1 and u_2 in R has length two, say $x_1 \neq y_1$, and that $u_1 \notin N(u_2)$. Then, give colour $\psi(u_2)$ to x_1 and $\psi(u_1)$ to y_1. Assume that $u_2 \neq x_2 \neq y_2$. If there exists $w \in N(x_2) \setminus R$, then give colour $\psi(w)$ to y_2 and y_1 and colour $\psi(u_1)$ to x_2. Otherwise, if there exists $w \in N^W(u_2) \setminus R$, then give colour $\psi(w)$ to y_2 and then colour x_2 with any colour in $M(u_2)$ (if there is none, just repeat colour $\psi(u_1)$ in x_2). Finally, suppose that $N^W(y) \setminus R = \emptyset$, for all $y \in R$; then, we can pick two colours different from $\psi(u_1), \psi(u_2)$ to give to x_2 and y_2. Now, assume that $x_2 = y_2$. As x_2 is not encircled by W, there must exist $w \in W$ not reachable

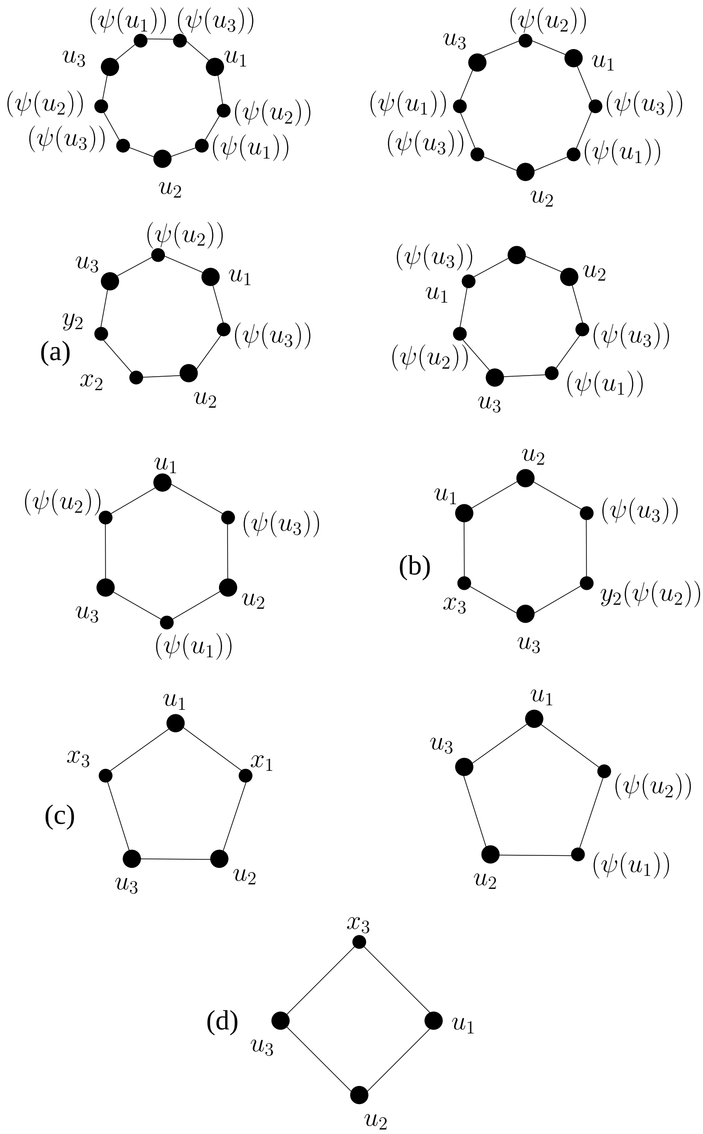


Figure 3.20: Cases where $|R \cap W| = 3$.

from x_2; so, give colour $\psi(w)$ to x_2. Observe that we can use analogous arguments to colour x_2, y_2 in the case where $u_1 = x_1$. So, consider the case where $u_1 \neq x_1 = y_1$ and $u_2 \neq x_2 = y_2$. We can again suppose that $N^W(x_i) \setminus R = \emptyset$, $i = 1, 2$. If there exists $w, w' \in W \setminus N[u_1] \cup N[u_2]$, $w \neq w'$, then we can colour x_1 with $\psi(w)$ and x_2 with $\psi(w')$; so, suppose otherwise. If $W \setminus (N[u_1] \cup N[u_2]) = \{w\}$, then, as neither E1.a nor E1.b occurs, at least one of u_1, u_2, say u_1, is such that $N^W(u_1) \neq \emptyset$ and $d(u_1) \geq m(G)$. Thus, we can colour x_1 with $\psi(w)$ and x_2 with $\psi(w')$, for any $w' \in N^W(u_1)$. Now, suppose that $W \subseteq N[u_1] \cup N[u_2]$. As x_i is not encircled, for $i = 1, 2$, then at least of u_1, u_2, say u_1, is such that $N^W(u_1) \neq \emptyset$ and $d(u_1) \geq m(G)$. If $d(u_1) = m(G)$, then, as E1.d does not occur, we have $N^w(u_2) \neq \emptyset$; consequently, as E1.c does not occur, then $d(u_2) \geq m(G)$ and we can colour x_1 and x_2 with one colour from $\psi(N^W(u_1))$ and one colour from $\psi(N^W(u_2))$. Now, assume that $d(u_1) > m(G)$. If $N^W(u_1)$ has more than one vertex, then we can colour x_1 and x_2 using colours from $\psi(N^W(u_1))$. Otherwise, as E1.b does not occur, we must have that $d(u_2) \geq m(G)$ and, again, we can use one colour from $\psi(N^W(u_1))$ and one colour from $\psi(N^W(u_2))$ to colour x_1 and x_2.

□

Now, we know how to extend a nice precolouring that colours only vertices of W in H to a precolouring that colours R, where R is a basic tight set of H. However, recall that we also want to ensure a 4-gap for the initial coloured tight set. Thus, if the basic tight set R of H constructed with Lemma 3.17 induces a cycle, we want to colour $[R]$. The lemma below constitutes one of the two possible cases of the "induction step" (see Lemma 3.19) and can sometimes be applied to colour $[R]$, depending on the cardinality of $R \cap W$. The colouring of $[R]$ is presented afterwards. We need the following further definition.

Let H be a connected component of G', R be a tight set of H and ψ be a nice precolouring that colours R. Let $x \in V(H) \setminus R$; we say that colour i is *forbidden for x in ψ* if w_i is reachable from x or there exists $w \in N^W(x) \cup \{x\}$ such that w has a neighbour in $R \setminus W$ coloured with i; we denote the set of colours forbidden for x in ψ by $F_\psi(x)$ (we ommit ψ, if there is no ambiguity). Recall that $J(u, R)$ is the union of all R-flaps that intersects $N(u)$, where u is an intermediate vertex of R.

Lemma 3.23. *Let H be a connected component of G', ψ be a nice precolouring of G' that colours $R \subseteq V(H)$ and u be an intermediate vertex of R such that $R \cup [u]$ is tight. If there exists $w_{c_1}, w_{c_2} \in W \setminus (J(u,R) \cup \{u\})$ such that $c_1, c_2 \in M_\psi(u)$ and $c_1, c_2 \notin F_\psi(x)$, for all $x \in N(u) \setminus (W \cup R)$, then there is a nice precolouring of G that extends ψ and colours $R \cup [u]$.*

Proof: Let U be the set of uncoloured neighbours of u (trivially, $U = N(u) \setminus (W \cup R)$) and Q be the bipartite graph $(U \cup M(u), E')$, where $(x, c) \in E'$ if and only if $c \notin F(x)$. By assumption, we have $(x, c_i) \in E'$, for all $x \in U$, $i = 1, 2$. Now, we prove that there exists a matching of Q that covers $M(u)$ and, then, show how to use this matching to extend ψ to a nice precolouring that colours $R \cup [u]$. But, first, we make some observations about the edges of Q.

Let $c \in M(u)$ and consider an R-flap X and $x \in X \cap U$. By definition, we know that if $(x, c) \notin E(Q)$, then either w_c is reachable from x or there exists $w \in N^W(x) \cup \{x\}$ such that w has a neighbour in $R \setminus W$ coloured with c. One can verify that, if the latter occurs, then $w \neq u$, as $c \in M(u)$, and $u \in X$. Also, by Lemma 3.1, x is the only vertex in U for which this occurs. Now, if the former occurs, then at most one other vertex $y \in X \cap U$ also reaches w_c and, as in this case u separates w_c from z, for every $z \in N(u) \setminus \{x, y\}$, we have that x, y are the only two vertices for which this occurs. Consequently, c has at most three non-neighbours in Q and, if it is the case, then c is unique, i.e., (I) $|U \setminus N^Q(c')| \leq 2$, for all $c' \in M(u) \setminus \{c\}$. Now, suppose that $N^Q(c) = \emptyset$. Trivially, $c \neq c_1, c_2$ and, thus, we have $|M(u)| \geq 3$. Also, by (I) and Lemma 2.14, we know that $|M(u)| \leq 3$. So, let $U = \{x, y, z\}$ and suppose, without loss of generality, that w_c is reachable from x and y and there exists $w \in (N(z) \cap W) \cup \{z\}$ such that w has a neighbour in $R \setminus W$ coloured with c. By Lemma 3.1, we know that u has at most one neighbour in R, say z'. Consequently, $u \notin R$ and u, x, y, z are all in the same R-flap, X. Also, note that any colour $c' \in \psi(W \setminus (X \cup \{w_{\psi(z')}\}))$ is missing in $N(u)$ and, as $|W \setminus X| \geq 4$ (ψ is nice and, hence, R satisfies (HP)), there must exist at least one colour $c' \in \psi(W \setminus X) \setminus \{\psi(z'), c_1, c_2\}$. Obviously, $c' \neq c$ as $w_c \in X$; but then we get a contradiction as $|M(u)| = 3$. Thus, $N^Q(c) \neq \emptyset$, for all $c \in M(u)$.

Now, suppose, by contradiction, that there is no matching in Q that covers $M(u)$. By Hall's Theorem (we direct the reader to [30]), we know that there exists a subset $C \subseteq M(u)$ such that $|C| > |N^Q(C)|$. So, let C be such a subset. As $N^Q(c) \neq \emptyset$, for all $c \in M(u)$, we know that $|C| > 1$ and, as there exists at most one colour $c \in M(u)$ such that $|U \setminus N(c)| > 2$, we have that $|N^Q(C)| \geq |U| - 2$. However, as $U \subseteq N(c_i)$, $i = 1, 2$, we have that $c_1, c_2 \notin C$, i.e., $|C| \leq M(u) - 2$. But then, $|U| - 2 \leq |N^Q(C)| < |C| \leq |M(u)| - 2$, contradicting Lemma 2.14.

Now, let $\mathcal{M}$ be a matching that covers $M(u)$ in Q. We want to colour x with c, for all $(x, c) \in \mathcal{M}$. Let $(x, c) \in \mathcal{M}$ and $w \in N(x) \cap W$, $w \neq u$. By the construction of Q, we know that no neighbour of w in $R \setminus W$ is coloured with c; thus, if c appears in the neighbourhood of w, then $w_c \in N(w)$ and, as w_c is not reachable from x, $d(w) > m(G) - 1$. If x is the only common neighbour of u and w, then after extending ψ using $\mathcal{M}$, $r(w) = 1 \leq d(w) - m(G) + 1$. So, suppose that there exists another $y \in N(w) \cap N(u)$. Trivially, if y is paired in $\mathcal{M}$ with a colour that does not appear in $N(w)$, we have the same result as before; so, suppose that $(y, c') \in \mathcal{M}$, for some $w_{c'} \in N(w) \cap W$. As u separates R from $w_c, w_{c'}$, we know that $c, c' \neq c_1$. So, let $(z, c_1) \in \mathcal{M}$. Again, as w_c is separated from z by u, we know that if $(z, c) \notin E(Q)$, then there exists $v \in N^W(z) \cup \{z\}$ such that v has a neighbour v' in R coloured with c. Obviously, $v' \notin W$ and, as $c \in M_\psi(u)$, we have $v \neq u$. One can see that, in this case, $(z, c') \in E(Q)$ and $(\mathcal{M} \setminus \{(z, c_1), (y, c')\}) \cup \{(z, c'), (y, c_1)\}$ is also a matching that covers $M(u)$. So, the precolouring ψ^+ obtained from ψ by colouring x with c, for every $(x, c) \in \mathcal{M}$, is unsaturated. Finally, observe that, for all $w \in W \setminus \{u\}$ such that $N(w) \cap U \neq \emptyset$, as $c_1, c_2 \in M_\psi(w)$ and $|N(u) \cap N(w)| \leq 2$, if there exists $x \in U$ still uncoloured, then we can colour x with c_1 or c_2 without increasing $r(w)$, for all $w \in W$, i.e., there exists an unsaturated precolouring that extends ψ^+ and colours $R \cup [u]$. As $R \cup [u]$ is tight and R satisfies (HP) (hence, $R \cup [u]$ also does), this precolouring is also nice. □

Now, we show how to colour $[R]$, when $H[R]$ is a cycle.

Lemma 3.24. *Let H be a connected component of G' and ψ be a nice precolouring that colours R, where R is a basic tight set satisfying Lemma 3.17 such that $H[R]$ is a cycle. Then, we can extend ψ to colour $[R]$.*

Proof: Let $W \cap R = \{u_1, \cdots, u_q\}$ and denote $J(u_i, R)$ by J_i, $i = 1, \cdots, q$. As observed in the proof of Lemma 3.19, we know that if $u_i \in R \cap W$ is an intermediate vertex of R, then $R \cup [u_i]$ is a tight set. Thus, if for some $i \in [1, q]$, there exists $w_{c_1}, w_{c_2} \in W \setminus (J_i \cup \{u_i\})$ such that $c_1, c_2 \in M(u_i)$ (and, hence, $c_1, c_2 \notin F(x)$, for all $x \in N(u_i) \setminus (W \cup R)$, as u_i separates x from w_{c_1}, w_{c_2}), then we can apply Lemma 3.23 to colour $N(u_i) \setminus (W \cup R)$.

So, suppose that this is not the case, i.e., that (I) there exists at most one $w_c \in W \setminus (J_i \cup \{u_i\})$ such that $c \in M(u_i)$, for some $i \in [1, q]$. Consequently, we have $q \leq 5$. Note also that if $N(u_i) \cap R \subseteq W$, as the colours of $N(u_i) \setminus W$ have no influence over the colouring of R, we can colour $[u_i]$ separately using the strategy of the proof of Lemma 3.21. So, we also suppose that (II) $N^R(u_i) \setminus W \neq \emptyset$, for all $i \in [1, q]$. By Lemma 3.23, we also have $q \geq 2$. We then analyse the cases $q \in \{2, 3, 4\}$:

- $q = 4$: note that if $W \setminus (\bigcup_{i=1}^{4}(J_i \cap W) \cup R) \neq \emptyset$ or $J_i \cap W \neq \emptyset$ and $J_k \cap W \neq \emptyset$, for some $i, k \in [1, 4]$, $i \neq k$, then $|W \setminus (J_l \cup \{u_l\})| \geq 4$, for all $l \in [1, q]$, contradicting (I) as u_l has at most two vertices coloured with some colour in $\psi(W \setminus (J_l \cup \{u_l\}))$. So, suppose, without loss of generality, that $W \setminus R \subseteq J_1$. Then, after we colour $N(u_1)$, $N(u_i) \setminus W$ can be coloured independently with colours from $M(u_i)$, for every $i \in [2, 4]$. Let $N = N(u_1) \setminus (W \cup R)$ and denote by x_1, x_2 the neighbours of u_1 in R. If $\psi(x_i) \notin \psi(\{u_2, u_3, u_4\})$, for $i = 1$ or $i = 2$, then at least two colours of $\psi(\{u_2, u_3, u_4\})$ are in $M(u_1)$, contradicting (I). So, suppose, without loss of generality, that $\psi(x_1) = \psi(u_2)$ and $\psi(x_2) = \psi(u_4)$. By (II), we know that at least one of x_1, x_2 is not in W, say x_1. Let $y \in (N(x_1) \cap R) \setminus \{u_1\}$ and $z \in (N(y) \cap R) \setminus \{x_1\}$. Observe that $\psi(u_3) \in M(u_1)$ and suppose that there exists $c \in M(u_1) \setminus \{\psi(u_3)\}$ (trivially, $c \neq \psi(u_i)$, for all $i \in [1, 4]$). We show that it is possible to change the colour of x_1 to c, thus contradicting (I). Suppose otherwise; then either $\psi(y) = c$ (and, consequently, $y \notin W$) or $y \in W$ and $\psi(z) = c$. In both cases, if there is any other colour in $M(u_1) \setminus \{\psi(u_3), c\}$, then we can change the colour of x_1; so, suppose otherwise. If $\psi(y) = c$, then we know that $z \in W$ and we can change the colour of y to any colour in $\psi(W \cap R) \setminus \{\psi(z), \psi(t)\}$, where $t \in N(z) \cap R$, $t \neq y$, and then we can colour x_1 with c. Now, suppose that $y \in W$ and $\psi(z) = c$. Let $t \in N^R(z) \setminus \{y\}$ and $w \in N^R(t) \setminus \{z\}$. If $t \notin W$, change the colour of z to any colour in $\psi(W \cap R) \setminus \{\psi(t), \psi(y)\}$; otherwise, change its colour to any colour in $\psi(W \cap R) \setminus \{\psi(y), \psi(t), \psi(w)\}$. Then, colour x_1 with c. Finally, if $M(u_1) = \{\psi(u_3)\}$, then colour any $x \in N(u_1) \setminus (W \cup R)$ with $\psi(u_3)$ (x exists, by Lemma 2.14) and, then, colour the remaining uncoloured vertices in $N(u_1) \setminus (W \cup R)$ using the colours $\psi(u_2), \psi(u_4)$.

- $q = 3$: note that if $|W \setminus (J_i \cup R)| \geq 2$, for all $i \in [1, 3]$, then $|W \setminus (J_i \cup \{u_i\})| \geq 4$, contradicting (I). So, suppose, without loss of generality,

$|W \setminus (J_1 \cup R)| \leq 1$. As $m(G) \geq 7$, we have $|J_1 \cap W| \geq 3$ (hence, $|W \setminus (J_i \cup \{u_i\})| \geq 5$, $i = 2, 3$) and, consequently, we can apply Lemma 3.23 to colour $N(u_2) \cup N(u_3)$ after colouring $N(u_1)$. So, let $N = N(u_1) \setminus (W \cup R)$ and let $N(u_1) \cap R = \{z_1, z_2\}$. By (II), at least one of z_1, z_2 is not in W, say z_1. Also, by (I), we can suppose that at least one of z_1 and z_2 is coloured with a colour from $\psi(W \cap R)$ and that if $W \setminus (J_1 \cup R) \neq \emptyset$, then $\psi(z_1), \psi(z_2) \in \psi(W \cap R)$. If $W \setminus (J_1 \cup R) \neq \emptyset$, denote by u_4 such vertex.

First, we show how to change the colour of z_1 in ψ to some colour $c \in M(u_1) \setminus \psi(J_1 \cap W)$. We make this in such a way not to change the colour of z_2. This procedure will be useful later, when we extend ψ to colour N. Let $r_1 \in N^R(z_1) \setminus \{u_1\}$, $r_2 \in N^R(r_1) \setminus \{z_1\}$ and $r_3 \in N^R(r_2) \setminus \{r_1\}$ (as $z_1 \notin W$ and $|W \cap R| = 3$, we have $r_2 \neq u_1$). First, suppose that $r_1 \notin W$. If $\psi(r_1) \neq c$, then change the colour of z_1 to c. Otherwise, change the colour of r_1 to any colour in $\psi(\{u_1, u_2, u_3\}) \setminus \psi(\{r_2, r_3\})$ and, then, colour z_1 with c. Note that, as $r_2 \neq u_1$, we have $r_1 \neq z_2$ and the colour of z_2 is not changed. Now, suppose that $r_1 \in W$. If $\psi(r_2) \neq c$, colour z_1 with c. Otherwise, note that $r_2 \notin W$ and $r_3 \neq u_1$; let $r_4 \in N^R(r_3) \setminus \{r_2\}$. As $m(G) \geq 7$, there must exist a colour $c' \notin \{c, \psi(\{r_1, r_3, r_4, u_4\})$; then, we colour r_2 with c' and z_1 with c. Again, as $r_3 \neq u_1$, then $r_2 \neq z_2$ and the colour of z_2 is not changed.

We colour N similarly as we colour $[w]$ in the proof of Lemma 3.21. Let $X_1, \cdots, X_p$ be the non-trivial connected components of $H \setminus \{u_1\}$ containing at least one vertex of N and and assume the existence of an index $r \in [0, p]$ such that $|X_i \cap N| = 2$, for all $i \in [1, r]$, and $|X_i \cap N| = 1$, for all $i \in [r + 1, p]$, $r \geq 0$. As in the proof of Lemma 3.21, we can suppose that $(X_i \cap W) \setminus N(u_1) \neq \emptyset$, for $i = 1, \cdots, p$. Also, denote the vertices in $X_i \cap N$ by x_i, y_i (if $i > r$, consider $x_i = y_i$) and construct the function f as in the proof of Lemma 3.21. Let Q be the subset $\{z_i : f(z_i) \neq null\}$ (we know that $f(x_i) \neq$, for all $i \in [1, p]$). First, suppose that $|Q| \geq 2$. We permute the colours defined by f on the vertices of Q in such a way that $\psi(z) \neq f(z)$, for all $z \in Q$. By the construction of f, we know that $w_{f(z)} \in J_1 \setminus N(u_1)$, for all $z \in Q$. So, as at least one of z_1 and z_2 is coloured with a colour in $\psi(W \cap R)$, we know that at most one vertex $z \in Q$ is such that $f(z) \notin M(u_1)$, in which case one between z_1, z_2 is coloured with $f(z)$, say z_1. So, change the colour of the vertex of Q coloured with $f(z)$ to $c \in \psi(\{u_2, u_3\}) \setminus \psi(z_2)$. The obtained partial colouring is unsaturated. Now, let S be the subset of uncoloured y_i's. Suppose that, for some $y_i \in S$, there exists $u \in X_i \cap W$ not reachable from y_i. By the construction of y_i, we know that y_i does not reach any vertex of $X_i \cap W$. Thus, $N^W(y_i) = \{u_1\}$ and we can colour y_i with any colour in $M(u_1)$. So, suppose that y_i reaches every vertex of $X_i \cap W$, for all $y_i \in S$. Now, let $F_i = X_i \cap W$, for each $y_i \in S$. Trivially, $F_i \cap F_j = \emptyset$, for every pair $y_i, y_j \in S$. Let $c \in M(u_1)$. We know that if y_i cannot be coloured with c, then $w_c \in F_i$ and, consequently, y_j can be coloured with c, for any other $y_j \in S$, $j \neq i$. So, if $|S| \geq 2$ and $\psi(F_i) \cap M(u_1) \neq \emptyset$, for at least two vertices of S, then we can colour S with colours from $M(u_1)$. Now, suppose otherwise and consider, without loss of generality, that $y_1 \in S$ and $M(u_1) \subseteq \psi(F_1)$. As y_1 is not encircled, there must exist a vertex $u \in W$ not reachable from y_1 (and, consequently, not in X_1). As $u \notin F_1$ and $M(u_1) \subseteq \psi(F_1)$, we must have $\psi(u) \notin M(u_1)$. So, let $z \in N(u_1)$ be such that $\psi(z) = \psi(u)$. If $z = u$, then $d(w) \geq m(G)$ (as u is not reachable from y_1) and we can repeat the colour $\psi(z)$ in y_1; so, suppose otherwise. Let $v \in F_1$ be such that $\psi(v) \in M(u_1)$. If $z \neq x_1$, then colour z with $\psi(v)$ and y_1 with $\psi(u)$ (observe that here we may need to use the procedure explained in the previous paragraph in the case where $v \in \{z_1, z_2\}$). So, suppose that $\psi(x_1) = \psi(u)$. As x_1, y_1 is not an encircled pair and every vertex reachable from x_1 is also reachable from y_1 and vice-versa, there must exist another vertex $u' \in W \setminus \{u\}$ not reachable from y_1. Since $\psi(x_1) \neq \psi(u')$, we can colour y_1 using analogous arguments. At the end, we colour the remaining uncoloured vertices in S with the colours missing in $N(u_1)$. At the end, if there are still uncoloured vertices in $N(u_1)$, as $|N(w) \cap N(u_1)| \leq 2$, for all $w \in J_1$, we can use the colours $\psi(u_2), \psi(u_3)$ to colour these uncoloured neighbours.

Now, consider the case where $|Q| = 1$. We analyse the possible situations:

- $x_1 = y_1$: if $\psi(z_i) \notin \psi(\{u_2, u_3\})$, for $i = 1$ or $i = 2$, then we can easily colour x_1. So, suppose otherwise and assume, without loss of generality, that $\psi(z_1) = \psi(u_2)$. As X_1 is non-trivial and $W \cap X_1 \nsubseteq N(u_1)$, there must exist $w \in X_1$ such that $\psi(w) \in M(u_1)$. So, change the colour of z_1 to $\psi(w)$ as explained before and colour x_1 with $\psi(u_2)$.

- $x_1 \neq y_1$: let $c \in M(u_1)$. First, suppose that there exists $w \in W \setminus R$ such that w is not reachable from x_1 or from y_1. If w_c is not reachable from x_1, then we can colour x_1 with c and y_1 with $c' \in \psi(\{w, u_2, u_3\}) \setminus \psi(\{z_1, z_2\})$. So, we can suppose that w_c is reachable from x_1 and y_1. Then, change the colour of z_1 to c and colour x_1, y_1 with the colours in $\psi(\{w, u_2, u_3\}) \setminus \psi(z_2)$. Now, suppose that every vertex of $W \setminus R$ is reachable from x_1 or y_1 (consequently, $W \setminus R \subseteq J_1$). As X_1 is non-trivial and $x_1, y_1 \in X_1 \setminus W$, we know that there exists at least one vertex in $X_1 \cap W$. Thus, we can suppose that $\psi(z_1) = c$, where $w_c \in X_1 \cap W$ (we change the colour of z_1 as explained before, if necessary) and, consequently, $\psi(z_2) \in \psi(\{u_2, u_3\})$. As observed previously, we know that we can change the colour of z_1 without changing the colour of z_2 and the only assumption made for z_1 in the procedure to change its colour is that $z_1 \notin W$. Thus, if $z_2 \notin W$ and $|X_1 \cap W| > 1$, then we can analogously recolour z_2 to a colour from $\psi(X_1 \cap W) \setminus \psi(z_1)$ without changing the colour of z_1 and, then, colour x_1, y_1 with $\psi(u_2), \psi(u_3)$. Also, if $|X_1 \cap W| = 1$ then $d(u_1) \geq |(W \setminus \{u_1, \cdots, u_4\}) \cup \{x_1, y_1, z_1, z_2\}| \geq m(G)$, in which case we can colour both x_1 and y_1 with the colour in $\psi(\{u_2, u_3\}) \setminus \psi(z_2)$. So, suppose that $z_2 = u_2$ (without loss of generality) and $|X_1 \cap W| \geq 2$. In this case, as all vertices of $W \setminus R$ are reachable from x_1 or y_1, then either we can colour x_1 with $\psi(w)$, for some $w \in W \cap X_1$ not reachable from x_1, and then colour y_1 with $\psi(u_3)$; or we can colour y_1 with $\psi(w)$, for some $w \in W \cap X_1$ not reachable from y_1, and then colour x_1 with $\psi(u_3)$; or every vertex of $W \cap X_1$ is reachable from both x_1 and y_1. If the latter occurs, then $d(u_1) \geq m(G)$, as x_1, y_1 is not an encircled pair, and we can colour both x_1 and y_1 with $\psi(u_3)$.

Finally, if there are still uncoloured vertices in $N(u_1)$, say x, we know that the component X of $H \setminus \{u_1\}$ containing x either contains only x or is such that $X \cap W = \{w\}$ and $w \in N(x)$. So, $\psi(N(w)) = \{\psi(u_1)\}$, if w exists, and we can colour x with any colour in $M(u_1)$ or in $\{\psi(u_2), \psi(u_3)\}$, if $M(u_1) = \emptyset$.

- $q = 2$: we know that one of u_1, u_2 satisfies (I), say u_1. Thus, as at most two colours in $\psi(W \setminus (J_1 \cup R))$ appears in $N(u_1)$, we have (i)

$|W \setminus (J_1 \cup R)| \leq 2$. So, as $m(G) \geq 7$, we have $|J_1 \cap W| \geq 3$ (hence $|W \setminus (J_2 \cup \{u_2\})| \geq 4$) and we can apply Lemma 3.23 to colour $N(u_2)$ after colouring $R \cup N(u_1)$. Also, we know that if P is a maximal path in $H[R]$ not intersecting W of length greater than 1, then, as $H \subseteq G[W \cup N(W)]$, each internal vertex of P must have at least one neighbour in $W \setminus R$. Thus, by (i), at most two distinct vertices in R are internal vertices of such paths. As by (II) we also know that $N^R(u_1) \setminus W \neq \emptyset$, we have that all the possible structures of R are represented in Figure 3.21.

First, note that if (a) or (b) occurs, then we can colour $[u_1]$ as explained in the proof of Lemma 3.21 and, then, give colour $\psi(u_1)$ to x_2 in (a), and any two distinct colours in $\psi((J_1 \cup \{u_1\}) \cap W)) \setminus \psi(\{x_1, y_1\})$ to x_2, y_2 in (b) (as $|(J_1 \cup \{u_1\}) \cap W| \geq 4$, these colours exist). So, assume that (c)-(g) occurs. Note that, in each situation, $|W \setminus (J_1 \cup R)| \geq 1$ by the existence of paths of length greater than 1 that do not intersect W. If there exist $z_1, z_2 \in (J_1 \cap W) \setminus N(u_1)$, $z_1 \neq z_2$, observe the precolourings presented in Figure 3.21. As the precolourings do not use colours from $W \setminus (J_1 \cup R)$ in $N(u_1)$, we can use Lemma 3.23 to extend them to precolourings that colour $R \cup N(u_1)$. Now, suppose that $|(J_1 \cap W) \setminus N(u_1)| \leq 1$. Suppose that ψ^* is an extension of ψ that colours R such that: (ii) if $(J_1 \cap W) \setminus N(u_1) = \{w\}$, then $\psi^*(w) \in \psi^*(N^R(u_1))$. Note that, as ψ^* is unsaturated and w' is separated from u_1 by R, for all $w' \in W$ such that $\psi^*(w') \in M(u_1)$, we can colour the uncoloured vertices in $N(u_1) \cap J_1$ with colours in $M(u_1)$. So, suppose that the vertex w exists (otherwise the precolouring obtained in Lemma 3.22 satisfies our constraint). In the precolourings presented in Figure 3.21, consider z_1 to be w, z_2 to be u_2, replace the colour of $x' \in N^R(u_2)$ coloured with $\psi(z_2)$ by $\psi(u_1)$ and, in (f), colour the vertex t with $\psi(w)$ instead of $\psi(u_1)$. The obtained precolourings satisfy (ii).

□

Finally, we present a lemma to extend a precolouring that colours $R \subseteq V(H)$, where R has no intermediate vertex.

Lemma 3.25. *Let ψ be a nice precolouring of G and H be a connected component of G' where ψ colours $R \subseteq W$. If R has no intermediate vertex and $R \neq V(H)$, then there is a nice precolouring of G that extends ψ and colours $R' = R \cup N^X(R)$, for some R-flap X.*

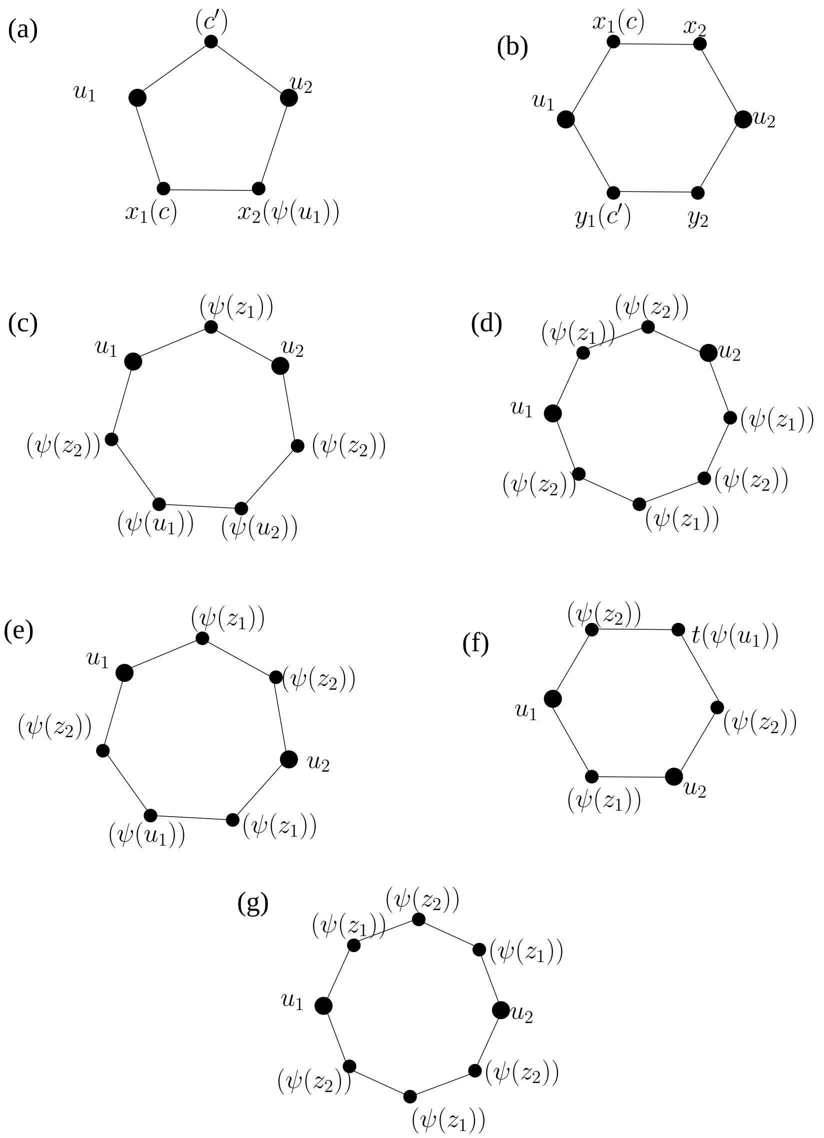


Figure 3.21: Cases where $|R \cap W| = 2$. The colours are represented in parenthesis.

Proof: Let X be any R-flap, $S_X = N^X(R)$ and $S_R = N^R(X)$. By Lemma 3.1, we know that $|S_X| \leq 2$ and $|S_R| \leq 2$. Thus, as ψ is nice, we have $|X \cap W| \leq \frac{1}{2}m(G)$ and, as $m(G) \geq 7$, there must exist at least two vertices w_1, w_2 in $W \setminus X$ such that $\psi(w_i) \notin \psi(S_R)$. So, we can give colours $\psi(w_1), \psi(w_2)$ to the uncoloured vertices in S_X, obtaining an unsaturated precolouring ψ^+. By Lemma 3.19, we know that $R' = R \cup N^X(R)$ is tight and, as R satisfies (HP), then R' also does; consequently, the ψ^+ is also nice. □

The presented lemmas imply Theorem 3.16 as the graph is finite and, with each application of Lemmas 3.23 and 3.25, we increase the size of the subset of coloured vertices of G', i.e., we eventually colour G' entirely.

Chapter 4

Outerplanar Graphs

The results in this chapter were presented in the 8th French Combinatorial Conference, at Orsay, 2010, and a complete version was submitted to the journal Discrete Mathematics [34].

In this chapter, we consider outerplanar graphs with girth at least 8. We recall that G is outerplanar if it has an embedding in the plane such that no two edges cross (i.e., the graph is planar) and all vertices lie on the same face. Our main result is the following:

Theorem 4.1. *Let G be an outerplanar graph with girth at least eight. Then $\chi_b(G)$ is equal to either $m(G)$ or $m(G)-1$. Moreover, we can determine the value of $\chi_b(G)$ (and a b-colouring with $\chi_b(G)$colours) in polynomial time.*

Let G be an outerplanar graph with girth at least 8. We already know, by Theorem 2.19, that if G does not have a good set, then $\chi_b(G) = m(G) - 1$. Also, we know by Lemma 2.20 that if G has a good set, then one can be found in polynomial time. It remains to prove that if G has a good set, then $\chi_b(G) = m(G)$. The proof is given in Section 4.1.

Given any graph G, the graph obtained from G by a *subdivision of an edge* $(u, v) \in E$ is the graph $(V \cup \{w\}, (E \setminus \{(u, v)\}) \cup \{(u, w), (v, w)\})$. A *subdivision of* G is a graph obtained from G by a sequence of edge subdivisions. Here is a classical characterization of outeplanar graphs.

Theorem 4.2. *Given a graph G, the following are equivalent:*
- G is outerplanar;
- G does not have a subgraph that is a subdivision of K_4 or of $K_{2,3}$;
- G is planar, and every block of G is either a vertex, an edge, or an hamiltonian cycle.

Here are a few properties of induced cycles in outerplanar graphs.

Theorem 4.3. *An outerplanar graph G with n vertices has $O(n)$ induced cycles. The intersection of any two induced cycles of G is either empty, or one vertex, or an edge. The edge intersection graph of induced cycles in G is a forest.*

One can produce the list $\mathcal{C}$ of induced cycles of an outerplanar graph G as follows. First produce the list of blocks of G; this can be done in time $O(n)$ [35]. Now let Q be the list of blocks of G that are cycles. Remove an element A of Q. If A is induced, put it in $\mathcal{C}$. Else, let e be a chord of A and let A', A'' be the two subcycles of A determined by e. Append A' and A'' to Q. Continue with the next element of Q. Stop when Q becomes empty.

The following proposition will be of use in our colouring procedure.

Proposition 4.4. *Let A be an induced cycle in an outerplanar graph G, and let x, y be two non-adjacent vertices of A. Then every induced path P between x and y is included in A.*

Proof: If at least one internal vertex of P is not in A, then $V(P) \cup V(A)$ contains a subdivision of $K_{2,3}$, a contradiction. □

4.1 Outerplanar graphs with a good set

In this section we prove the second part of the main theorem, namely:

Theorem 4.5. *Let G be an outerplanar graph with girth at least 8. Suppose that G has a good set. Then $\chi_b(G) = m(G)$.*

Let $W = \{v_1, \dots, v_{m(G)}\}$ be a good set of G and L_W be the set of link vertices of W. We want to construct an unsaturated precolouring with candidate set W that colours $W \cup L_W$; then, the theorem follows by Lemma 2.15. We start by assigning colour i to v_i ($i = 1, \dots, m(G)$); then we extend this precolouring to colour L in several phases. Before explaining these phases, we need to introduce some terminology and notation.

Let A be any cycle in G that contains vertices of W. We call *sector* of A any subpath P of A, of length at least one and using consecutive vertices of A, such that the extremities of P are in W and the interior vertices of P are in L. We say that a cycle is *special* if it contains either three or four vertices of W and every sector of A is a link. Note that in a special cycle A every vertex of $V(A) \cap L$ has a neighbour in $V(A) \cap W$. Moreover, every special cycle has length at most 12 and, consequently (since girth$(G) \geq 8$), is an induced cycle.

Let $\mathcal{S}$ be the collection of special cycles of G. One can obtain $\mathcal{S}$ easily by examining every member of the collection $\mathcal{C}$ of induced cycles of G. Let L_0 be the set of vertices of L that belong to special cycles.

Now, we colour the vertices of L. There will be four phases. In the first phase, we colour the vertices of L_0. In the second phase we colour the vertices of $N^{L\setminus L_0}(v_i)$ for every $v_i \in W$ that has at least two neighbours in $L \setminus L_0$. In the third phase we colour the uncoloured vertices of $L \setminus L_0$ that have a neighbour in L. In the fourth phase, we colour the remaining vertices of L. Throughout the colouring procedure, we shall ensure that the precolouring is proper and that no colour is repeated in $N(w)$ for all $w \in W$, except in the last phase, where we may repeat a colour in $N(w)$ if $d(w)$ allows it. Thus, the obtained precolouring is ideed an unsaturated precolouring. During Phases 1, 2 and 3, we shall ensure that the following property P holds.

(P) *If a vertex x gets colour j during Phases 1, 2 or 3, then either:*
P1 *there exists a path $\langle x, x', v_j\rangle$, or*
P2 *there exists a path $\langle x, u, x', v_j\rangle$ or $\langle x, u, x', x'', v_j\rangle$ for some $u \in W$.*

4.1.1 Phase 1

During this phase, we colour the vertices of L_0. We do this by considering each special cycle $A \in \mathcal{S}$ and colouring its uncoloured vertices. This is called the iteration of cycle A. The cycles of length at most 9 are iterated first. So let $A = \langle v_{i_1}, x_1, y_1, v_{i_2}, \cdots, v_{i_q}, x_q, y_q, v_{i_1}\rangle$ be any special cycle, where $A \cap W = \{v_{i_1}, \ldots, v_{i_q}\}$, and possibly $x_i = y_i$ for any $i = 1, \ldots, q$. By the definition of a special cycle, we have $q \in \{3, 4\}$. Some vertices of $L \cap V(A)$ may have been coloured during the iteration of another special cycle before A. The uncoloured vertices of A are coloured according to the following pattern. If $q = 3$ (so A has length 8 or 9), the pattern is shown in Figure 4.1 and we colour the vertices as represented there.

If $q = 4$ (so A has length between 8 and 12), the pattern is shown in Figure 4.2 and is formally defined as follows, where subscripts are understood modulo 4 and from the set $\{1, 2, 3, 4\}$.

- If x_j is uncoloured, then $\psi(x_j) \leftarrow i_{j-1}$, for $j = 1, \ldots, 4$;

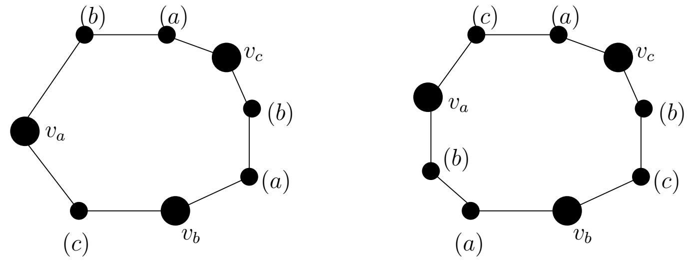


Figure 4.1: The attributed colours are in parenthesis.

- If y_j is uncoloured, then $\psi(y_j) \leftarrow i_{j+2}$, $j = 1, \ldots, 4$.

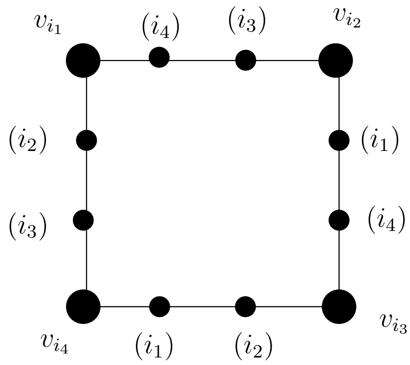


Figure 4.2: The attributed colours are in parenthesis.

It is easy to check that every vertex coloured during Phase 1 satisfies property P.

Lemma 4.6. *After Phase 1 is applied, in every special cycle A the precolouring is proper and no vertex $w \in V(A) \cap W$ has two neighbours of the same colour in A.*

Proof: Let A be a special cycle. Any colour that appears in A is either a colour given precedingly during the iteration of another special cycle or a colour j such that $v_j \in V(A) \cap W$. Consider any vertex $x \in V(A) \cap L$ that is already coloured before the iteration of A. So x was coloured during the iteration of another special cycle A'. By Theorem 4.3, the intersection of A and A' is either $\{x\}$ or $\{x, y\}$ for some neighbour y of x. In either case, the colour h of x is such that $v_h \in V(A') \setminus V(A)$, so x is the only vertex of A that receives colour h. If v_j is any vertex in $V(A) \cap W$, then clearly, by the definition of the colouring, colour j is not given to two adjacent vertices of A or to two neighbours of a vertex $w \in V(A) \cap W$. Thus the lemma holds. □

4.1.2 Phase 2

For every $w \in W$ and vertex $x \in L$ adjacent to w, let $D(w, x)$ be the set of colours j such that x lies on a link between w and v_j, i.e., on a link $\langle w, x, v_j \rangle$ or $\langle w, x, x', v_j \rangle$.

In this phase, for every vertex $w \in W$ that has at least two neighbours in $L \setminus L_0$ we color the uncoloured vertices of $N(w) \cap (L \setminus L_0)$. This is called the iteration of vertex w. Let $L_w = N(w) \cap (L \setminus L_0)$. For each $x \in L_w$, we pick a colour $f_x \in D(w, x)$ $(= \{j \mid \ x \text{ lies on a } w, v_j\text{-link}\})$. Since girth$(G) \geq 8$, we have $D(w, x) \cap D(w, y) = \emptyset$ for all $x, y \in N(w)$ with $x \neq y$. Consequently the set $\{f_x \mid x \in L_w\}$ has the same cardinality as L_w. Some vertices of L_w may have been coloured during this phase in the iteration of another member of W. We colour the uncoloured vertices of L_w with the colours from $\{f_x \mid x \in L_w\}$ in such a way that each vertex x receives a colour $\psi(x) \neq f_x$. This is possible because $|L_w| \geq 2$. Clearly, every vertex that is coloured during Phase 2 satisfies property P2.

4.1.3 Phase 3

Let R be the set of uncoloured vertices of $L \setminus L_0$ that have a neighbour in L. In this phase, we colour the vertices of R. So consider any $x \in R$. Pick any $x' \in N^L(x)$ and $v_j \in N^W(x')$. Assign colour j to x. Clearly, x satisfies property P1.

Lemma 4.7. *After Phases 1, 2 and 3 are applied, the precolouring is proper and does not repeat any colour in the neighbourhood of any vertex $w \in W$.*

Proof: Suppose on the contrary that there are two vertices x, y such that $\psi(x) = \psi(y) = j$ and either (i) x, y are adjacent or (ii) x, y are two neighbours of some vertex $w \in W$. Since x satisfies property P, there exists a path P_x equal to either $\langle x, x', v_j \rangle$ or $\langle x, u, x', v_j \rangle$ or $\langle x, u, x', x'', v_j \rangle$ for some $u \in W$ and $x', x'' \in L$. Likewise, there exists a path P_y equal to either $\langle y, y', v_j \rangle$ or $\langle y, v, y', v_j \rangle$ or $\langle y, v, y', y'', v_j \rangle$ for some $v \in W$ and $y', y'' \in L$. Since girth$(G) \geq 8$, each of P_x, P_y is induced. It may be that $P_x \setminus v_j$ and $P_y \setminus v_j$ are not disjoint, but anyhow it is a routine matter to check that in case (i), $V(P_x) \cup V(P_y)$ either contains a cycle of length at most 7 or induces a special cycle. In case (ii), as at least one between x and y is not coloured during Phase 2 on the iteration of w, again $V(P_x) \cup V(P_y) \cup \{w\}$ either contains a cycle of length at most 7 or induces a special cycle. In either case, this contradicts girth$(G) \geq 8$ or Lemma 4.6. □

4.1.4 Phase 4

Let R' be the set of vertices of L that are uncoloured after Phase 3. So $R' \subseteq L \setminus L_0$. Moreover, every vertex $x \in R'$ has no neighbour in L and is the only neighbour in $L \setminus L_0$ of each $w \in N^W(x)$. Thus, the following is valid:

P4a *If $\langle x, w, y \rangle$ is a path with $x \in R'$, $w \in W$, $y \in L$. Then $y \in L_0$.*

We divide this phase into three subphases. In the first two subphases, we deal with vertices $x \in R'$ such that there exists a path $\langle x, w, y \rangle$ with $w \in W$ and $y \in L_0$. To do this, for each vertex $y \in L_0$, let W_y be the set $\{w \in N^W(y) \mid N^{R'}(w) \neq \emptyset\}$, let R'_y be the set $\{x \in R' : N^{W_y}(x) \neq \emptyset\}$ and let us call *span* of y the value span$(y) = |W_y|$. For each vertex $y \in L_0$ we will colour every $x \in R'_y$. We call this the iteration of y. Vertices with span at least 2 are iterated in Subphase 4.1, and vertices with span equal to 1 are iterated in Subphase 4.2. In Subphase 4.3, we colour the remaining vertices of R'.

Subphase 4.1

Consider any vertex $y \in L_0$ with span$(y) \geq 2$. Let $q = |W_y| = $ span(y). Let $W_y = \{v_{i_1}, \ldots, v_{i_q}\}$ and $R'_y = \{x_1, \ldots, x_q\}$ where x_j is the neighbour of v_{i_j} in R' (recall that each vertex of W_y has only one neighbour in R', for otherwise its neighbours would have been coloured during Phase 2). For each x_j $(j = 1, \ldots, q)$, choose a colour $c_j \in D(v_{i_j}, x_j)$. As x_j has no neighbour in L, we know that $v_{c_j} \in N(x_j)$. It may happen that some of the x_j's are coloured during this phase in the iteration of some $y' \in L_0$ processed before y. So let us assume up to symmetry that $x_1, \ldots, x_r$ are the uncoloured vertices in $\{x_1, \ldots, x_q\}$, with $r \leq q$. If $r = 0$, just proceed to the next element of L_0. If $r = 1$, give colour c_2 to x_1. If $r > 1$, take any permutation $c'_1, \ldots, c'_r$ of $c_1, \ldots, c_r$ such that $c'_j \neq c_j$, and assign colour c'_j to x_j, $j = 1, \ldots, r$.

Lemma 4.8. *After Phase 4.1 is applied, the precolouring is proper and does not repeat any colour in the neighbourhood of any vertex $w \in W$.*

Proof: Suppose that the precolouring is not proper after the iteration of some vertex $y \in L_0$ in Subphase 4.1. With the notation above, this must be because some vertex $x_k \in R'_y$ receives colour c_j while x_k is already adjacent to a vertex z of colour c_j. We have $z \notin L$ because $N^L(x_k) = \emptyset$. So $z = v_{c_j}$. But then $\{y, v_{i_k}, x_k, v_{c_j}, x_j, v_{i_j}\}$ induces a cycle of length 6, a contradiction.

Now suppose that the precolouring repeats a colour in $N(w)$ for some $w \in W$. This must be because some vertex $x_k \in R'_y \cap N(w)$ receives a colour c_j while w already has a neighbour $z \neq x_k$ of colour c_j (possibly $w = v_{i_k}$). Since girth$(G) \geq 8$, we have $z \neq v_{c_j}$. So $z \in L$. By P4a, we have $z \in L_0$, so z was coloured during Phase 1. This means that some special cycle $A \in \mathcal{S}$ contains z and v_{c_j}. If $w = v_{i_k}$, then A and the path $\langle v_{c_j}, x_j, v_{i_j}, y, v_{i_k}, z\rangle$ between v_{c_j} and z, which is not contained in A because it contains $x_j, x_k \notin L_0$, contradict Proposition 4.4. If $w \neq v_{i_k}$, then A and the path $P = \langle z, w, x_k, v_{i_k}, y, v_{i_j}, x_j, v_{c_j}\rangle$ contradict Proposition 4.4. □

Subphase 4.2

During this second subphase, we may need to change the colour of vertices of L_0; as we make these modifications, we will ensure that:

P4b *If $y \in L_0$, then $\psi(y)$ can be changed only during the iteration of y in Subphase 4.2;*

P4c *If $\psi(y)$ is changed to j, then there exists a path $\langle y, w, x, v_j\rangle$ for some $w \in W$ and $x \in R'$.*

So consider any vertex $y \in L_0$ with span$(y) = 1$. Let $W_y = \{v_i\}$ and $R'_y = \{x\}$ where x is adjacent to v_i. Choose a colour $j \in D(v_i, x)$. As x has no neighbour in L, we know that $v_j \in N(x)$. Assume that x is still uncoloured. Let h be the colour of y. By P4b, h is the original colour of y. Let A be the special cycle such that y was coloured during Phase 1 in the iteration of A. So $v_h \in A$. Also, $h \neq j$, for otherwise $\langle y, v_i, x, v_j\rangle$ should be the path P_y of property P, with $P_y \subset V(A)$, and x would be coloured already. We assign colour h to x and recolour y with j. Clearly, y satisfies P4b and P4c.

Lemma 4.9. *After Phase 4.2 is applied, the precolouring is proper and does not repeat any colour in the neighbourhood of any vertex $w \in W$.*

Proof: Suppose that the precolouring is not proper after the iteration of some vertex $y \in L_0$ in Subphase 4.2. This must be because, with the notation above, either (i) x has a neighbour u of colour h, or (ii) y has a neighbour z of colour j.

First assume that (i) holds. We have $u \notin L$ because x has no neighbour in L. So $u = v_h$. By property P, there exists a path P_y of length at most four between y and v_h. Combining this path with $\langle y, v_i, x\rangle$, we obtain a cycle of length at most 7, a contradiction.

Now assume that (ii) holds. Clearly, $z \neq v_j$, so $z \in L$, and $(z, v_j) \notin E(G)$ because the colouring is proper. Also z did not have its colour changed to j before this iteration, for otherwise, by P4c, we would have a cycle of length 7. So j is its original colour. Moreover, $N(z) \cap N(v_j) = \emptyset$ (for otherwise, we have a cycle of length 6), and consequently z was not coloured during Phase 3. If z was coloured in Phase 2, we could find a special cycle containing x, a contradiction to $x \in R'$. Also $z \notin R'$, since z has a neighbour $y \in L$. So z was coloured during Phase 1 applied to a special cycle A' that contains v_j and z. But then A' and the path $\langle v_j, x, v_i, y, z\rangle$ between v_j and z, which contains $x \notin L_0$, contradict Proposition 4.4.

Now suppose that the precolouring repeats a colour in $N(w)$ for some $w \in W$ after the iteration of some vertex $y \in L_0$ in Subphase 4.2. We may assume that y is the first vertex for which there is such a repetition. The repetition occurs because either (iii) $w \in N^W(x)$ and w has a neighbour $z \neq x$ with $\psi(z) = h$, or (iv) $w \in N^W(y)$ and w has a neighbour $z \neq y$ with $\psi(z) = j$.

Suppose that (iii) holds. We know that $w \neq v_i$, for otherwise colour h was repeated already in $N(w)$ (on y and z) before this iteration, which contradicts the choice of y. If $z = v_h$, then A and the path $\langle y, v_i, x, w, z\rangle$ contradict Proposition 4.4. Therefore $z \neq v_h$. So $z \in L$. By P4a, $z \in L_0$, so z was coloured during Phase 1. By P4b, the colour of z has not changed (for otherwise, this would be during the iteration of z, and then x would be coloured already). So h is the original colour of z. By property P, there is a path P_z of length at most four between z and v_h. Then, by Proposition 4.4, the path $\langle y, v_i, x, w, z, P_z\rangle$ must join A on a neighbour of y, contradicting the fact that z is coloured in Phase 1 (there is no induced cycle containing z and v_j).

Suppose that (iv) holds. Since girth$(G) \geq 8$, we have $z \neq v_j$. So $z \in L$. Suppose that $w = v_i$. By P4a, $z \in L_0$, so z was coloured during Phase 1. The colour of z has not changed, for otherwise x would have been coloured already. So there exists a path P_z of length at most four between z and v_j, and by combining P_z and the path $\langle v_j, x, v_i, z\rangle$ we obtain a cycle of length at most 7, a contradiction. Therefore $w \neq v_i$. Now, either j is the original colour of z, or j was given to z during the iteration of z in Phase 4. In either case, by property P or P4c, there is a path P_z of length at most four between z and v_j, and by combining P_z and the path $\langle v_j, x, v_i, y, w, z\rangle$ we obtain either a cycle of length at most 7, or a special cycle that contains x, a contradiction. □

Subphase 4.3

Now, let x be a vertex of R' that is still uncoloured; so x and all vertices $v_i \in N^W(x)$ have no neighbour in L.

Suppose that every vertex of W is either in $N^W(x)$ or adjacent to a vertex of $N^W(x)$. Since W is a good set, x is not encircled by W, so there is a vertex $w \in N^W(x)$ with $d(w) \geq m(G)$ and a vertex $u \in N^W(w)$. In that case we give colour $\psi(u)$ to x. Note then that this situation cannot occur for another vertex $x' \neq x$, for otherwise there would be a cycle of length at most 6 containing x and x'. So w is the only vertex of W with a repeated colour in its neighbourhood, and that colour appears only twice in its neighbourhood.

In the opposite case, there exists a vertex $u \in W$ that is not in $N(x)$ and is not adjacent to any vertex of $N^W(x)$. So we can give colour $\psi(u)$ to x, and this colour will not appear twice in the neighbourhood of any vertex $w \in W$.

Chapter 5

Block Graphs

In this chapter, we consider the problem:

k,`b-Colouring`
-Input: graph G, positive integer k
-Question: does there exist a b-colouring of G with k colours?

We know that decinding if a bipartite graph G has b-chromatic number $m(G)$ is NP-complete [28]; consequently, the problem above is also NP-complete. In Section 5.2, we present a solution for this problem when k is fixed and in Section 5.3, we analyse a special case where we can decide, given a block graph G and a subset $W \subseteq D_k(G)$, if W can be the basis of a b-colouring of G with k colours. We remark that some of our results may lead the reader to ask if block graphs are b-continuous. The answer is yes, as all chordal graphs are b-continuous; the proof can be found in [14].

5.1 $m(G) - \chi_b(G)$ arbitrarily large

Let r be any positive integer; in this section, we construct a graph G such that $\chi_b(G) \leq m(G) - r$. Let k be an integer such that $k \geq r$. An (r, k)-*gadget* is the graph G' obtained from the complete graph C of size $2kr + 2r - k - 2$, as follows: add two vertices v^1, v^2 adjacent to all vertices of the clique and to each other, then add two stable sets of size k, S^1, S^2, S^1 adjacent to v^1 and S^2 to v^2, and, finally, for each vertex $u \in S^1 \cup S^2$, add a stable set S_u of size $2kr + 2r - 2$ adjacent to u. Observe Figure 5.1. We denote the starting clique of G' by $C(G')$, the two vertices adjacent to the clique by $v^1(G')$ and $v^2(G')$ and the stable set adjacent to $v^j(G')$ by $S^j(G')$, $j = 1, 2$.

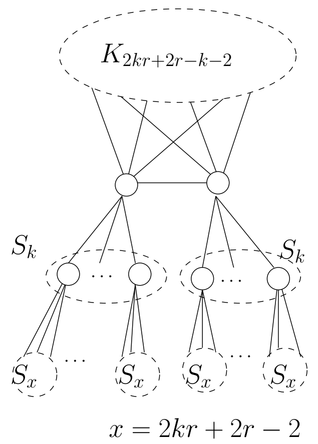


Figure 5.1: An (r, k)-gadget.

Our graph G is the disjoint union of r (r, k)-gadgets, $G'_1, \cdots, G'_r$. Note that, for all $i \in \{1, \cdots, r\}$, we have:

- A vertex $u \in C(G'_i)$ has degree equal to $2kr + 2r - k - 2 - 1 + 2 = 2kr + 2r - k - 1$;
- $d(v^j(G'_i)) = 2kr + 2r - k - 2 + 1 + k = 2k + 2r - 1$, for $j = 1, 2$;
- $d(u) = 2kr + 2r - 2 + 1 = 2kr + 2r - 1$, for all $u \in S^j(G'_i)$, $j = 1, 2$; and
- $d(u) = 1$, for all other vertex u.

So, there are $\sum_{i=1}^{r} |S^1(G'_i) \cup S^2(G'_i) \cup \{v^1(G'_i), v^2(G'_i)\}| = r(2k + 2) = 2rk + 2r$ vertices with degree $2kr + 2r - 1$ and all other vertex has degree at most $2kr + 2r - k - 1 < 2kr + 2r - 2$; hence, $m(G) = 2kr + 2r$. Suppose for a contradiction that $\chi_b(G) > m(G) - r$ and let ψ be an optimal b-colouring of G with basis V'. Trivially, the vertices of V' have degree at least $m(G) - r$. As, for all $i \in \{1, \cdots, r\}$ and all $u \in V(G'_i) \setminus D(G)$, we have that $d(u) \leq 2kr + 2r - k - 1 = m(G) - k - 1 \leq m(G) - r - 1$, then $u \notin V'$. So, $V' \subseteq D(G)$ and, as $|V'| > m(G) - r$, there must exist a gadget G'_i such that all the dense vertices of G'_i are in V'; denote $v^j(G'_i)$ by v_i^j, $j = 1, 2$, and $S^1(G'_i) \cup S^2(G'_i) \cup \{v_i^1, v_i^2\}$ by B_i. Observe that, as v_i^1 and v_i^2

are b-vertices and $N^{V'}(v_i^1) \cap N^{V'}(v_i^2) = \emptyset$, all the colours in $\psi(V' \setminus \{v_i^1, v_i^2\})$ must all appear in $C(G'_i)$. However, $|\psi(V' \setminus \{v_i^1, v_i^2\}| \geq m(G) - r - 1$, while $|C(G'_i)| = 2kr + 2r - k - 2 = m(G) - k - 2$; a contradiction, as $k \geq r$, i.e., $m(G) - (r+1) > m(G) - (k+2)$.

Remark that the constructed graph is a block graph and if we remove the edges of $C(G'_i)$, for all i, the argument is still valid and the obtained graph is series-parallel. Also, if instead of having a stable $S^j(G'_i)$, we add a clique $K^j(G'_i)$ of size k and, for each $u \in K^j(G'_i)$, we add a clique of size $x - k - 1$ adjacent to u, instead of a stable of size x, the argument again works and the obtained graph is the line graph of a tree (a claw-free block graph).

5.2 Deciding if $\chi_b(G) \geq k$, k fixed

In this section, we present a solution for the problem below restricted to block graphs.

`Fixed` k`,b-Colouring`
-Input: graph G
-Question: does there exist a b-colouring of G with k colours?

Let G be a block graph. As all chordal graphs are perfect, we know that $\chi(G) = \omega(G)$; hence, if $k < \omega(G)$, the answer is `no`. Also, if $k = \omega(G)$, as any optimal colouring is also a b-colouring, the answer to the problem is `yes`. Thus, from now on, we consider k to be greater than $\omega(G)$. In order to solve `Fixed` k`,b-colouring`, we will create a number of instances of the problem below. If for any of the instances the answer is yes, then the answer to the original problem is yes; otherwise, the answer is no.

`1-PreExtension`
-Input: a graph H and a precolouring ψ of H with k colours where each colour is used exactly once
-Question: can ψ be extended to a colouring of H with k colours?

An instance for `1-PreExtension` is created as follows:

- Let $W \subseteq D_k(G)$ of cardinality k;

- for each $u \in W$, let N'_u be any subset of $N(u)$ of cardinality $k-1$ and let $\mathcal{N}$ be the set $\{N'_u : u \in W\}$;
- Let $G(W, \mathcal{N}_W)$ be initially equal to G and, then, add to $G(W, \mathcal{N})$ all edges between the vertices of N, for all $N \in \mathcal{N}$;
- Finally, let ψ_W be a precolouring where each vertex of W has a different colour and all other vertices are not coloured.

Trivially, $(G(W, \mathcal{N}), \psi_W)$ obtained as explained before is an instance of `1-PreExtension`. Actually, the graph $G(W, \mathcal{N}))$ generalizes the idea of partial b-closure presented in [29]. Note that a colouring of $G(W, \mathcal{N})$ with k colours that extends ψ_W is also a b-colouring of G with k colours with basis W. In [31], it was proven that `1-PreExtension` is solvable in polynomial time for chordal graphs. Now, we prove that $G(W, \mathcal{N})$ is chordal.

Theorem 5.1 (Marx[31]). *`1-PreExtension` can be solved in polynomial time for chordal graphs.*

The following was obtained in coloraboration with Leonardo Sampaio and another version of this idea can be found in [17].

Lemma 5.2. *Let G be a block graph and $H = G(W, \mathcal{N})$ be obtained as explained before. Then H is chordal.*

Proof: By contradiction, suppose that H has an induced cycle $C = \{x_1, ..., x_q\}$, $q > 3$, where $(x_i, x_{(i+1) \mod q})$ are the edges of the cycle. Since this cycle is not in G, there must exist an edge $(x_i, x_{(i+1) \mod q})$ that is not in G. Suppose, without loss of generality, that $(x_i, x_{i+1}) \notin E(G)$, for some $1 \leq i < q$. Thus, there exists some $u \in W$ such that $x_i, x_{i+1} \in N'_u$. Since C is an induced cycle in H, we have that $x_j \notin N'_u$, for all $j \neq i, i+1$. So, the path between x_i, x_{i+1} in H defined by C also defines a path P between x_i and x_{i+1} in G that do not pass through u (it suffices to replace the edges of the cycle that are not in G by the corresponding common neighbours in G). But then, $P \cup \{u\}$ defines a cycle in G that is not a clique, since $(x_i, x_{i+1}) \notin E(G)$, a contradiction. □

Corollary 5.3. *Consider a block graph G with exactly $m(G)$ dense vertices each of degree $m(G) - 1$. It can be decided in polynomial time whether $\chi_b(G) = m(G)$ or not.*

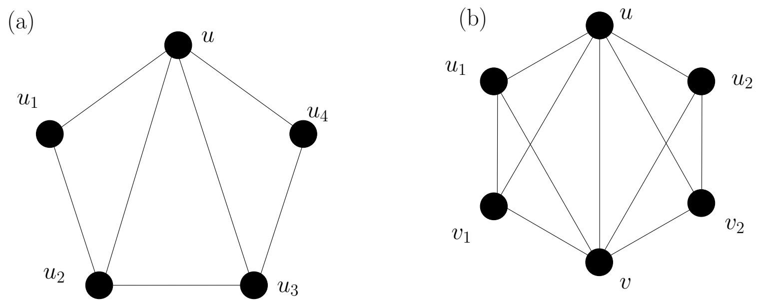


Figure 5.2: Denote by C the cycle formed in both examples. In (a), $u \in W$ and $N'_u \cap C = \{u_1, u_4\}$. In (b), $u, v \in W$, $N'_u \cap C = \{u_1, u_2\}$ and $N'_v \cap C = \{v_1, v_2\}$. In both cases, the graph $G(W, \mathcal{N})$ contains a C_4. The graph represented in (a) is a *gem*, while the graph in (b) is gem-free. Observe that both are also interval graphs.

Observe that this result cannot be extended to general chordal graphs, as shown in Figure 5.2.

Theorem 5.4. *Let G be a block graph and k be a positive integer, $k > \omega(G)$. There exists a b-colouring of G with k colours if and only if there exist subsets $W \subseteq D_k(G)$ of cardinality k and $N'_u \subseteq N(u)$ of cardinality $k-1$, for each $u \in W$, such that the answer to the problem `1-PreExtension(`$G(W, \{N'_u : u \in W\}), \psi_W$`)` is yes.*

Proof: $\Leftarrow$ Trivially, as $H = G(W, \{N'_u : u \in W\})$ is a supergraph of G, an extension ψ of ψ_W to H is also a proper colouring of G. Also, as $N'_u \cup \{u\}$ is a clique of size k, we have that u has a neighbour coloured with each other colour, i.e., u is a b-vertex of colour $\psi(u)$, for all $u \in W$.

$\Rightarrow$ Let ψ be a b-colouring of G with k colours and let W be a basis of ψ. Also, for each $u \in W$, let $N'_u \subseteq N(u)$ contain exactly one neighbour of u of each colour. Trivially, $|W| = k$ and $|N'_u| = k-1$, for all $u \in W$. Also, it is easy to see that ψ is a colouring of $G(W, \{N'_u : u \in W\})$ (we only add edges between vertices with different colours) and, as each vertex of W has a different colour, ψ is actually an extension of ψ_W with k colours, i.e., the answer to the problem `1-PreExtension(`$G(W, \{N'_u : u \in W\}), \psi_W$`)` is yes.

□

Corollary 5.5. *Let G be a block graph and k be a fixed positive integer. Then, `Fixed k,b-Colouring` can be solved in polynomial time.*

Proof: Note that there are $O(n^k)$ possible instances for the problem `1-PreExtension` and that, by Lemma 5.2 and Theorem 5.1, each instance can be solved in polynomial time. Thus, we enumerate these instances and solve `1-PreExtension` for each. If at some point we obtain the answer yes, then we know that there exists a b-colouring of G with k colours, i.e., the answer to `Fixed` k`,b-Colouring` is yes; otherwise, as all the possible instances are investigated, by Theorem 5.4 the answer is no. □

5.3 k,b-Colouring, k given

Consider a block graph G. In this section, we analyse the existence of a b-colouring with basis W, given a subset $W \subseteq D_k(G)$. The definition of blocking clique below can be thought of as an "encircled clique", although it does not cover the definition of encircled vertex.

Consider $W \subseteq D_k(G)$ with cardinality k. Let A be a block of G and $u \in W \cap A$. We define the *saturating index of* u *related to* A *in* W (and the *saturating index of* A *in* W) as being the values:

$$s^W(u, A) = min\{d(u) - k + 1, |N^W(u) \setminus A|\}$$

$$s^W(A) = \sum_{u \in A \cap W} s^W(u, A)$$

Furthermore, we define the set:

$$R^W(A) = (A \cap W) \cup N^W(A \cap W)$$

We omit the W label above when there is no ambiguity. We say that A is a *blocking clique of* W if:

$$|A \setminus W| > s^W(A) + |W \setminus R^W(A)|$$

As said before, the definition of blocking clique does not generalize encircled vertices. For example, if $x \in A \setminus W$ is encircled by W but $N^W(x) \nsubseteq A$, then A may not be a blocking clique. Thus, we say that W is *unblocked* if it does not have a blocking clique and also does not encircle any vertex. One can easily verify that if every subset of $D_k(G)$ with k vertices is blocked, then G cannot be b-coloured with k colours. So, we ask ourselves if the other way is also true, i.e., if there exists an unblocked subset of $D_k(G)$ with k vertices, can we b-colour G with k colours? Unfortunately, this is not the case as shown by the example in Figure 5.3.

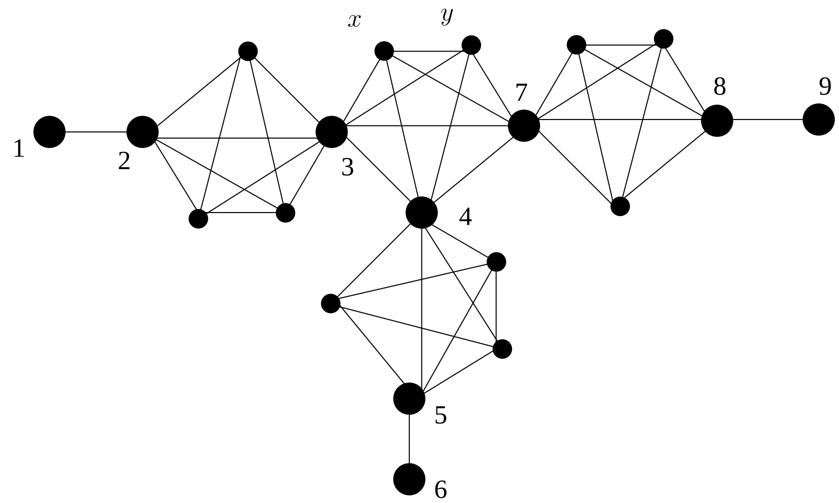


Figure 5.3: Unblocked set that is not a basis. The big vertices represent the dense vertices. We consider that $d(u) = m(G) - 1$, for all $m(G)$-dense vertex u. Note that there are three colours (1, 9 and 6) available for the vertices x and y, however, when colouring the other non-dense vertices, we need to use at least two of these colours.

If $u \in V \setminus W$ is a link vertex of W such that every link containing u is not induced, then we say that u is a *side vertex* . Observe that the link vertices presented in Figure 5.3 are side vertices; in that graph, each block has at least two side vertices. Another example for which being an unblocked subset of $D_k(G)$ is not sufficient is the following: let $W \subseteq D_k(G)$ with cardinality k, $x, y, z \in V \setminus W$ and $v_1, v_2, w_1, w_2 \in W$ be such that $(W \setminus \{v_1, v_2\}) \cup \{x\}$ is a block, $N^W(y) = \{v_1, w_1\}$, $N^W(z) = \{v_2, w_2\}$, $d(w_1) = d(w_2) = k - 1$ and $(v_1, w_1), (v_2, w_2) \notin E$. Observe Figure 5.4. In this example W is unblocked and, even though it has only one side vertex, one can see that it is still not a k-basis: if we give a different colour for each $w \in W$, then z must be coloured with $\psi(v_1)$ and y with $\psi(v_2)$; thus, as $d(w_1) = d(w_2) = k - 1$, then we cannot colour x without repeating a colour in $N(w_i)$, $i = 1, 2$. If W has the described structure, we say that W is a *nest*. Given an unblocked set $W \subseteq D_k(G)$, we can prove that if W has no side vertices, then $\chi_b(G) \geq k$.

Also, we prove that, if G is claw-free (and thus, is the line graph of a tree) and $W \subseteq D_k(G)$ is an unblocked set that is not a nest and is such that each block has at most one side vertex of W, then $\chi_b(G) \geq k$.

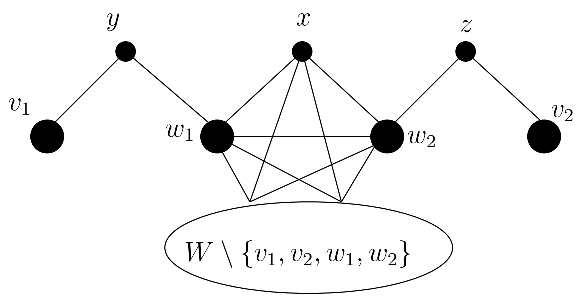


Figure 5.4: Representation of a nest.

Theorem 5.6. *Let G be a block graph and $W \subseteq D_k(G)$ be an unblocked set of cardinality k, $k \in \{\omega(G)+1, \cdots, m(G)\}$. If W has no side vertices, then there exists a b-colouring of G with k colours.*

Before proving this theorem, we need to prove the lemma below. We say that ψ is a *full b-precolouring of G with candidate set W* if ψ is proper, every vertex of W has a different colour and is a b-vertex in ψ. Trivially, if ψ^+ is a proper extension of ψ, then ψ^+ is also a full b-precolouring of G with candidate set W.

Lemma 5.7. *Let G be a block graph and ψ be a full b-precolouring of G with candidate set W, $|W| \in \{\omega(G)+1, \cdots, m(G)\}$. Then there exists a b-colouring of G with $|W|$ colours.*

Proof: Denote by k the value $|W|$; we want to obtain a b-colouring of G with k colours from this precolouring. For this, we iterate on the uncoloured vertices; whenever the picked vertex has degree lesser than k, we know that there exists a colour that does not appear in its neighbourhood; so we colour it with such a colour. However, if an uncoloured vertex u has degree greater than $k-1$, it may happen that all colours already occur in its neighbourhood; if this is the case, we treat it as described in the following.

First, consider the colours to be in $\{1, \cdots, k\}$ and let w_i be the vertex of W coloured with i, for all $i \in \{1, \cdots, k\}$. Consider $u \in V \setminus W$ to be such that $d(u) \geq k$, $\psi(u) = \emptyset$ and $\psi(N(u)) = \{1, \cdots, k\}$. Let $G_1, \cdots, G_q$ be the connected components of $G - u$. Denote by C_i the set $N^{G_i}(u)$ and by M_i the set $\{1, \cdots, k\} \setminus \psi(C_i)$. Also, if $y \in C_i$, denote by $G_{i,y}$ the connected component of $G - E(C_i \cup \{u\})$ containing y. Note that:

1. C_i is a clique of size at most $k-2$ (since $\omega(G) < k$), for all $i \in \{1, \cdots, q\}$;
2. $G - u$ is disconnected (otherwise, $N[u]$ is a clique of size at least $k+1$, a contradiction);
3. $|M_i| \geq 2$ (since $|C_i| < k - 1$), for all $i \in \{1, \cdots, q\}$.

Now, let $c \in \{1, \cdots, k\}$ and denote by A_c the set $\{i \in \{1, \cdots, q\} : c \in \psi(C_i)\}$. Also, for each $i \in A_c$, let $\alpha_i^c \in C_i$ be coloured with c (trivially, there is exactly one such vertex in C_i). We know that $|A_c| \geq 1$, for every $c \in \{1, \cdots, k\}$. Let $c \in \{1, \cdots, k\}$ and suppose that: (1) $w_c \notin G_{i,\alpha_i^c}$ and there exists $d_i \in M_i$ such that $w_{d_i} \notin G_{i,\alpha_i^c}$, for every $i \in A_c$. Note that we can switch the colours c and d_i in G_{i,α_i^c}, for every $i \in A_c$, and, then, colour u with c: after the switch, each C_i is still properly coloured, since $d_i \in M_i$, and if $w_j \in C_i$, for some $j \in \{1, \cdots, k\}$, we know that $j \neq c, d_i$ and, as we give colour c to u, w_j continues to be a b-vertex. Thus, the obtained extension is still a full b-precolouring of G with candidate set W. Now, if there exists $j \in A_c$ such that (1) occurs for every $i \in A_c \setminus \{j\}$ and: (2) $w_c \in G_{j,\alpha_j^c}$ and $w_d \in G_{j,\alpha_j^c}$, for some $d \in M_i$, then again we can do the switches as before and still obtain a full b-precolouring. So, suppose that, for every $c \in \{1, \cdots, k\}$, there exists $i \in A_c$ such that neither (1) nor (2) occurs for i and c, i.e., one of the following occurs, for every $c \in \{1, \cdots, k\}$:

(1') There exists $i \in A_c$ such that $w_c \notin G_{i,\alpha_i^c}$ and $\{w_d : d \in M_i\} \subseteq G_{i,\alpha_i^c}$; or

(2') There exists $i \in A_c$ such that $w_c \in G_{i,\alpha_i^c}$ and $\{w_d : d \in M_i\} \cap G_{i,\alpha_i^c} = \emptyset$.

Note that, as $|M_i| \geq 2$ and $\alpha_i^c \neq \alpha_i^d$, for $c \neq d$, then there are at most $\frac{k}{2}$ colours for which (1') occurs. Thus, there exists at least one colour $c \in \{1, \cdots, k\}$ for which (1') does not occur and, consequently, (2') occurs. As w_c is unique, there exists exactly one index $i \in A_c$ such that (2') is valid; so, for all $j \in A_c \setminus \{i\}$, we have that neither (1') nor (2') holds for j and c. Thus, we can do switches analogously as before for all $j \in A_c \setminus \{i\}$. As for C_i, we switch the colours c and d in G_{i,α_i^c}, for any $d \in M_i$, and then colour u with c. Note that, after this, as $d \notin G_{i,\alpha_i^c}$ and u is coloured with c, the only vertex that may cease to be a b-vertex is w_c itself. But in this case, note that the vertices in $N(u)$ that have their colour changed are the ones coloured with c, i.e., $\psi(N(u)) = \{1, \cdots, k\} \setminus \{c\}$ and u is a b-vertex of colour c. So, we have a full b-precolouring of G with candidate set $(W \setminus \{w_c\}) \cup \{u\}$. Observe that the changing in the candidate set does not interfere in a further extension of ψ, since, at each step, we can simply choose any basis of ψ. □

Now, we prove Theorem 5.6.
Proof: Let $W \subseteq D_k(G)$ with cardinality k, $k \in \{\omega(G)+1, \cdots, m(G)\}$, be an unblocked set with no side vertices. By Lemma 5.7, we know that it suffices to obtain a full b-precolouring of $G' = G[W \cup N(W)]$ with candidate set W. We use a similar idea as the one used for trees. Let $W = \{w_1, \cdots, w_k\}$ and colour w_i with colour i, for all $i \in \{1, \cdots, k\}$. Recall the definition of link vertices given in Section 2.4 and represent the set of link vertices of W by L. We first colour the blocks of G' containing at least two vertices of L. So, let A be such a block and let $L \cap A = \{x_1, \cdots, x_q\}$, $q \geq 2$. As x_j is not a side vertex, it must be within a link that has an extremity w_{i_j} separated from $A \setminus \{x_j\}$ by x_j, for all $j \in \{1, \cdots, q\}$. Thus, we can use the colours $\{i_1, \cdots, i_q\}$ to colour the uncoloured vertices in $\{x_1, \cdots, x_q\}$ in such a way that $\psi(x_j) \neq i_j$, $j \in \{1, \cdots, q\}$. Observe that if $(x, y) \in E(G)$, for $x, y \in L$, then both x and y are coloured. Now, let $y_1, \cdots, y_q \in N^L(u)$ still uncoloured, for some $u \in W$, and let $v_i \in L(y_i) \setminus N[u]$ (v_i exists, since y_i is not a side vertex). If $q > 2$, then permute the colours $\psi(v_1), \cdots, \psi(v_q)$ in $y_1, \cdots, y_q$ analogously. Otherwise, suppose that there exists $y \in N(u) \setminus W$ such that $\psi(y) \neq \emptyset$ and let $w \in L(y) \setminus N[u]$. Also, let A be the block containing y and u. Obviously, $y_1 \notin A$. Give colour $\psi(w)$ to y_1 and if there exists $y' \in A$ such that $\psi(y') = \psi(w)$, then give colour $\psi(v_1)$ to y'. Now, let $x \in L$ be still uncoloured. We know that $N^L(x) = \emptyset$ and $N^L(w_i) = \{x\}$, for all $w_i \in N^W(x)$. So, every block A containing x such that $A \cap W \neq \emptyset$ is actually contained in $W \cup \{x\}$. Thus, as x is not encircled by W, either there exists $w_i \in W \setminus (N(x) \cup N(N^W(x)))$ or there exists $w_i \in N^W(w_j)$ for some $w_j \in N^W(x)$ such that $d(w_j) > k - 1$. Then we can give colour i to x. At the end, as $k > \omega(G)$ and any uncoloured vertex in G' has exactly one neighbour in W, we can colour the remaining vertices in G' until each vertex of W is a b-vertex, thus obtaining a full b-precolouring of G with k colours. □

Corollary 5.8. *Let G be a block graph and let $k \in \{\omega(G)+1, \cdots, m(G)\}$. If* *there exists $W \subseteq D_k(G)$ with cardinality k such that W is a stable set, then there exists a b-colouring of G with k colours.*

Proof: Let $W \subseteq D_k(G)$ be a stable set with k vertices. Note that, if $W \nsubseteq N(x)$, for all $x \in V \setminus W$, then the hypothesis of Theorem 5.6 hold and the corollary follows. So, suppose the contrary and let x be such a vertex. Obviously, $x \in D_k(G)$. Then, let W' be obtained from W by removing any vertex and adding x. Colour each vertex of W' with a different colour. Trivially, x is already a b-vertex and any $y \in V \setminus W'$ is adjacent to at most one vertex in $W' \setminus \{x\}$. Thus, for each $w \in W' \setminus \{x\}$, we can colour $N(w)$ until w is a b-vertex without repeating any colour in $N(z)$, for all $z \in W' \setminus \{x\}$. The obtained precolouring is trivially a full b-precolouring with k colours.

□

Corollary 5.9. *Let G be a block graph and let $k \in \{\omega(G)+1, \cdots, m(G)\}$. If $|D_k(G)| > \Delta^2 + \Delta$, then G has a b-colouring with k colours.*

Proof: The proof is similar to the one given by Kratochvíl et al in [28] for d-regular graphs. By Corollary 5.8, we just need to find a stable set in $D_k(G)$. Take any vertex of $D_k(G)$, put it in W and remove it from G together with all its neighbours; trivially, the number of removed vertices is at most $1+\Delta$. Repeat the process $k-1$ times; at the end, the number of removed vertices is at most $(k-1)(\Delta+1) \leq (m-1)(\Delta+1) \leq \Delta(\Delta+1)$. So, there exists at least one more vertex in $D_k(G)$ to put in W and the corollary follows. □

Corollary 5.10. *Let G be a block graph and denote $m(G)$ by m. If $|D(G)| > m^2 + m$, then $\chi_b(G) = m$.*

Proof: Again, by Corollary 5.8, we need to find a stable set in $D(G)$. Let D^+ represent the set of dense vertices with degree at least m. We know that $|D^+| \leq m$; thus, there are more than m^2 dense vertices with degree $m-1$. So, if we iterate on $D(G) \setminus D^+$ removing at each time a vertex and its neighbourhood, as at each step we remove exactly m vertices, we know that we can do at least m iterations, i.e., the stable set can be obtained. □

Theorem 5.11. *Let G be a claw-free block graph and $W \subseteq D_k(G)$ be an unblocked set of cardinality k, $k \in \{\omega(G)+1, \cdots, m(G)\}$. If each block has at most one side vertex of W and W is not a nest, then there exists a b-colouring of G with k colours.*

Proof: We partition the set of link vertices, L, as follows:

S: contains all side vertices;

L_1: set of link vertices in $L \setminus S$ having a neighbour in $L \setminus S$ or a neighbour $w \in W$ such that $|N^{L\setminus S}(w)| \geq 2$; and

$L_2 = L \setminus (S \cup L_1)$.

Note that $|N^{L_2}(w)| \leq 1$, for all $w \in W$, $N^{L_2}(x) = \emptyset$, for all $x \in L_1 \cup L_2$, and if $N^{L_2}(w) \neq \emptyset$, $w \in W$, then $N^{L_1}(w) = \emptyset$. Also, for all $x \in L_2$, since $x \in L \setminus S$ and $N^{L\setminus S}(x) = \emptyset$, we have $|N^W(x)| \geq 2$. For each block A of G, define W_A^* as the set $\{w \in A \cap W : N^W(w) \setminus A \neq \emptyset\}$ and, for each $z \in A$, denote by A_z the block containing z different from A (if there is no such block, set $A_z = \emptyset$). Let $x \in S$. We know that $N^W(x) \cup N^L(x) \subseteq A$, for some block A of G, and, as each block has at most one side vertex, we know that $N^S(x) = \emptyset$. Denote the block containing x by $B(x)$. Observe that the following is valid:

(*) If there exists a colour $c \in \{1, \cdots, k\} \setminus \psi(B(x) \cup \bigcup_{w\in B(x)\cap W} N(w))$, then we can colour x with colour c.

We first colour each vertex of W with a different colour and then colour L_1 as explained in the proof of Theorem 5.6. Suppose that $x \in S$ is such that $N^W(x) = \{w\}$. Note that if we can obtain an unsaturated pre-colouring of $W \cup (L \setminus \{x\})$, at the end we can just colour x with any colour in $M(w)$ (if $y \in N(x) \setminus \{w\}$ is coloured, as $N^L(x) \subseteq N(w)$, we have that $\psi(y) \notin M(w)$). So, we can suppose that $|N^W(x)| \geq 2$, for all $x \in S$.

Now, we colour $N^S(A)$, for every block A such that $|W_A^*| \geq 2$. Let A be such a block. Note that if $x \in N^S(w) \setminus A$, for some $w \in A \cap W$, then, as $|N^W(x)| \geq 2$, we have $w \in W_A^*$. So, let $A_1, \cdots, A_q$ be all the the blocks such that $A_i \cap A \subseteq W$ and $A_i \cap S \neq \emptyset$. Denote by w_i the common vertex of A and A_i, by x_i the side vertex of A_i and let v_i be any vertex in $(A_i \cap W) \setminus \{w_i\}$. As G is claw-free and $|B \cap S| \leq 1$, for all block B of G, we know that $w_i \neq w_j$, for all $1 \leq i \neq j \leq q$. Suppose, without loss of generality, that $x_1, \cdots, x_p$ are all the uncoloured vertices in $\{x_1, \cdots, x_q\}$. If $p \geq 2$, give colour $\psi(v_1)$ to x_i, $i = 2, \cdots, p$, and colour $\psi(v_2)$ to x_1. If $p = 1$, let $w \in W_A^* \setminus \{w_1\}$ and $w' \in N^W(w) \setminus A$. If $q \geq 2$, colour x_1 with $\psi(v_2)$; otherwise, colour it with $\psi(w')$. Note that (I) at most two colours in $\psi(N^W(W_A^*))$ are used to colour $N^S(A)$ and that, for each $w \in W_A^*$, at most one colour in $N^W(w) \setminus A$ is used.

Now, let x be an uncoloured vertex of S and let $A = B(x)$. We know that $W_{A_w}^* = \{w\}$, for every $w \in W_A^*$ (otherwise, x would have been coloured). Thus, $\psi(x') \in \psi(R(A) \setminus A)$, for all $x' \in N^S(A)$. Note that if $N^L(w) = \{x\}$, for all $w \in A \cap W$, then, as x is not encircled, there must exist $w \in W$ not reachable from x and we can just colour x with $\psi(w)$. So, we analyse the cases:

- There exists $y \in A \cap L_2$: note that, in this case, $N^{L\setminus S}(w) = \{y\}$, for all $w \in A \cap W$; so, $\psi(N(w)) \subseteq \psi(R(A))$, for all $w \in A \cap W$. As $N^{L\setminus S}(y) = \emptyset$ and $y \notin S$, there must exist $w_1 \in N^W(y) \setminus A$. Also, as y is not encircled by W, there must exist $w_2 \in W$ not reachable from y. Suppose that there exists $w \in W_A^*$ such that $w_2 \in N(w)$. As w_2 is not reached by y, we have $d(w) > k - 1$. If there exists $x' \in N^S(A)$ such that $\psi(x') = \psi(w_2)$, let $w' = B(x') \cap A$ (observe that, by (I), x' must have been coloured during the iteration of A; thus, $w' \in W$). As G is claw-free, we have $d(w) = |A| + |A_w| - 2$ and $d(w') = |A| + |A_{w'}| - 2$; so, either there exists $w'' \in A_w \setminus \{w, w_2\}$, in which case we can suppose that $\psi(x') \neq \psi(w_2)$ (recall (I)), or $d(w')$ is also greater than $k - 1$. In any case, we can colour x with $\psi(w_1)$ and y with $\psi(w_2)$. Now, suppose that there exists $x' \in N^S(W_A^*)$ still uncoloured and let $w' = B(x') \cap A$; observe that $W_A^* = \{w'\}$. If any $u \in A_{w'} \cap W$ has a coloured neighbour $z \notin W$, we know that $z \notin A_{w'}$ and we can give colour $\psi(z)$ to x and $\psi(w_1)$ to x'. Otherwise, either there exists $w_3 \in W \setminus (R(A) \cup \{w_1, w_2\})$, in which case we colour x' with $\psi(w_3)$, or w' is already a b-vertex and we can colour x' with $\psi(w'')$, for any $w'' \in A \cap W$, $w'' \neq w'$.

- $A \cap L_2 = \emptyset$ and $N^{L_2}(A) \neq \emptyset$: let $N^{L_2}(A) = \{y_1, \cdots, y_q\}$ and let $w_i \in A \cap N(y_i)$, $i = 1, \cdots, q$. As $N^{L\setminus S}(y_i) = \emptyset$ and $N(x) \subseteq A$, we have that $w_i \in W$, $i = 1, \cdots, q$. For each y_i, let v_i be any vertex in $N^W(y_i) \setminus A_{w_i}$ (v_i exists, since $N^{L\setminus S}(y_i) = \emptyset$ and y_i is not a side vertex). Suppose that $q = 1$. If (II) $A_{w_1} \cap S = \{x'\}$ and $\psi(x') = \emptyset$, we have a case analogous to the previous one; so, suppose otherwise. As y_1 is not encircled, there must exist $z \in W$ distant from y_1; so, give colour $\psi(v_1)$ to x and colour $\psi(z)$ to y_1. If there exists $x' \in N^S(A)$ such that $\psi(x') = \emptyset$, we know that $w' \in B(x') \cap A$ is different from w_1, by (II). One can verify that it is possible to make an analogous argument as the one made at the end of the previous case. If $q > 2$, then give colour $\psi(v_1)$ to y_i, $i = 2, \cdots, p$, $\psi(v_2)$ to y_1 and $\psi(v_3)$ to x. If there exists $x' \in N^S(W_A^*)$, we can suppose that either $x' \notin N(y_i)$, for $i = 1, \cdots, q$, or $x' \in N(y_1)$; in any case, we can colour x' with $\psi(v_1)$. So, now, suppose that $q = 2$ and consider the following cases:
  - There exists $u_1 \in N^W(y_1) \setminus (A_{w_1} \cup \{v_1\})$: then give colour $\psi(v_1)$ to

y_2, $\psi(u_1)$ to x and $\psi(v_2)$ to y_1; if there exists $x' \in N^S(W_S^*)$, then give colour $\psi(v_1)$ to x', if $x' \notin N(y_2)$, or colour $\psi(v_2)$, otherwise. Now, suppose that $N^W(y_i) \subseteq A_{w_i} \cup \{v_i\}$, $i = 1, 2$.

– There exists $w' \in W_A^* \setminus \{w_1\}$: let $v' \in N^W(w') \setminus A$. Then, give colour $\psi(v')$ to y_1, $\psi(v_2)$ to x, $\psi(v_1)$ to y_2 and if, there exists $x' \in N^S(y_1)$ such that $\psi(x') = \psi(v')$, then give colour $\psi(v_1)$ to x'. After this, if there exists $x' \in N^S(W_A^*)$ such that $\psi(x') = \emptyset$, then either we can colour x' with $\psi(v_1)$, or $x' \in N(y_2)$. If the latter occurs, then either there exists a colour $c \neq \psi(v_1), \psi(v_2)$ with which we can colour x', or w_2 is already a b-vertex and we can just give colour $\psi(v_2)$ to x'.

– $W_A^* = \emptyset$: as W is not a nest, either there exists $w' \in W \setminus (A \cup \{v_1, v_2\})$, or $d(w_i) > k-1$, for $i = 1$ or $i = 2$. If the latter occurs, say $d(w_1) > k-1$, then give colour $\psi(v_2)$ to y_1 and x and colour $\psi(v_1)$ to y_2. Otherwise, as $N^L(w) = \{y_i\}$, for all $w \in N^W(y_i) \setminus A$, $N^L(w_i) = \{x, y_i\}$ and $\psi(x) = \emptyset$, $i = 1, 2$, we have that at least one of y_1, y_2, say y_1, is such that $\psi(w') \notin \psi(N(w))$, for all $w \in N^W(y_1)$. Thus, give colour $\psi(w')$ to y_1, $\psi(v_1)$ to y_2 and $\psi(v_2)$ to x.

- $A \cap L_2 = \emptyset$ and $N^{L_2}(A) = \emptyset$: first, suppose that there is no uncoloured side vertex in $N^S(W \cap A)$. Let $J = L_1 \cap A$. If $N^{L_1}(A) = \emptyset$, as W is unblocked, we have (observe that if $y \in A \setminus (W \cup J)$, then $y \in S$ and, consequently, $y = x$): $|A \setminus W| = |J + 1| \leq s(A) + |W \setminus R^W(A)|$ and, hence, $s(A) + |W \setminus (R^W(A) \cup J)| \geq 1$. So, there must exist a colour $c \in \{1, \cdots, k\} \setminus \psi(R_A \cup J)$, in which case we give colour c to x, or there exists $w \in W_A^*$ such that $d(w) > k-1$, in which case we give colour $\psi(w')$ to x, for any $w' \in (A_w \cap W) \setminus \{w\}$. Now, let $w_1, \cdots, w_q \in A \cap W$ be such that $J_i = A_{w_i} \cap L_1 \neq \emptyset$ (observe that w_i is not necessarily in W_A^*). Note that, for $i = 1, \cdots, q$, (II) if $y \in J_i$ and $w \in L(y) \setminus A_{w_i}$, then $\psi(w) \notin \psi(A_z \setminus \{z\})$, for all $z \in A$, $z \neq w_i$. So, if there exists $y_i \in J_i$, for some $i \in [1, q]$, and $w \in L(y_i) \setminus A_{w_i}$ such that $\psi(w) \notin \psi(A \cup A_{w_i})$, then we can colour x with $\psi(w)$. Suppose then that such a pair of vertices y_i, w do not exist.

  Let $R^* = \bigcup_{y \in J}(L(y) \setminus A)$. We analyse the existence of vertices in J_i coloured with some colour in $\psi(R^*)$. First, note that if $y_i \in J_i$ is coloured with $c \in \psi(R^*)$, then y_i is coloured during the iteration of vertex w_i; hence, $J_i = \{y_i\}$. So, suppose, without loss of generality, that $J_1, \cdots, J_{p-1}$ are all the sets formed by such vertices and denote by y_i the vertex in J_i, $i \in [1, p-1]$, $p \geq 1$. Also, suppose that y_1 is the first vertex to be coloured between $y_1, \cdots, y_{p-1}$. Let $v \in R^*$ such that $\psi(y_1) = \psi(v)$ and $y \in J$ such that $v \in L(y) \setminus A$. Also, let c' be any colour in $\psi(L(y_1) \setminus A_{w_1})$ and consider the moment before the iteration of w_1. If J is not coloured yet, we know that $J = \{y\}$ and, in this case, we colour y with c' and y_1 with $\psi(v)$. If J is already coloured and there exists $y' \in J$ such that $\psi(y') = \psi(v)$, then we change the colour of y' to c' and then colour y_1 with $\psi(v)$. Finally, if J is coloured and there is no vertex in A coloured with $\psi(v)$, then we just colour y_1 with $\psi(v)$. In any of those cases, at the end of the iteration of w_1, J is coloured and there is no vertex in J coloured with $\psi(v)$. So, we can suppose that in the subsequent iterations of w_i, $i \in [2, p-1]$, we always give colour $\psi(v)$ to y_i; i.e., we can suppose that: (III) there exists a colour $c \in \psi(R^*)$ such that $\psi(y_i) = c$, for all $i \in [1, p-1]$; and (IV) $\psi(z) \notin \psi(J)$, for all $z \in L(y_i) \setminus A_{w_i}$, for all $i \in [2, q]$. If $p > 2$, by (IV), there must exist $w \in L(y_2) \setminus A_{w_2}$ such that $\psi(w) \notin \psi(A \cup A_{w_2})$, a contradiction; thus, $p \leq 2$. We analyse the following cases:

- $q \geq 2$: we recall (II) and remark that the colours in $L(y) \setminus A_{w_2}$ do not appear in A, for all $y \in J_2$, as well as the colours in $L(y) \setminus A$ do not appear in A_{w_2}, for all $y \in J$. So, let $y_1 \in J_1$, $y_2 \in J_2$ and $w \in L(y_2) \setminus A_{w_2}$. Give colour $\psi(y_1)$ to x and colour $\psi(w)$ to y_1.
- $q = 1$: let $y_1 \in J_1$. First, note that, for all $w \in A_{w_1} \cap W$, $N^{L_2}(w) = \emptyset$ and if $w \neq w_1$, as $|W^*_{w_1}| = 1$, we have that $N^S(w) \subseteq A_{w_1}$. If there exists $w \in A_{w_1} \setminus \{w_1\}$ and $y \in N^{L \setminus S}(w) \setminus J$ (consequently, $y \in L_1$ and $\psi(y) \neq \emptyset$), then give colour $\psi(y)$ to x. Otherwise, note that $\psi(N(w)) \subseteq \psi(A_{w_1})$, for all $w \in (A_{w_1} \cap W) \setminus \{w_1\}$. Then, give colour $\psi(y_1)$ to x and, if w_1 is already a b-vertex, just colour y_1 with any colour in $\psi((A \cap W) \setminus \{w_1\})$; otherwise, there must exist a colour $c \notin \psi(A \cup A_{w_1})$, in which case, we can colour y_1 with c.

Now, suppose that there exists an uncoloured vertex $x' \in N^S(W \cap A)$ and let $w \in B(x') \cap A$. We know that $W^*_A = W^*_{A_w} = \{w\}$. If there exist $y \in N^{L_1}((A \cap W) \setminus \{w\})$ and $y' \in N^{L_1}((A_w \cap W) \setminus \{w\})$, then give colour $\psi(y')$ to x and $\psi(y)$ to x'. Now consider, without loss of generality, that $N^{L_1}((A_w \cap W) \setminus \{w\}) = \emptyset$. Observe that $N^L(w') \subseteq A_w$, for all $w' \in (A_w \cap W) \setminus \{w\}$. Thus, we can colour x as explained before and, after this, either there exists a colour $c \in \{1, \cdots, k\} \setminus \psi(A \cup A_w)$ that does not appear in $N(w)$ (and, consequently, in $N(w')$, for all $w' \in A_w \cap W$) with which we can colour x', or w is already a b-vertex and we can colour x' with $\psi(w')$, for any $w' \in (A \cap W) \setminus \{w\}$.

$\square$

Chapter 6

Cartesian Product of Trees by other Graphs

In this chapter, we investigate the cartesian product of trees and some other graph classes. Let $H = T \,\Box\, G$ be the cartesian product of a tree T by a graph G. Generally, we will number the vertices of graph G and represent the copy of T related to the i-th vertex of G by T^i. Also, we represent the copy of $u \in V(T)$ in T^i by u^i.

Now, let $W \subseteq V(H)$ be a set of $m(H)$ dense vertices. Let T^i be one of the copies of T in H; denote by V^i the set $V(T^i)$ and by X^i the set $X \cap V^i$, for any $X \subseteq V(H)$. Also, given $X \subseteq V(H)$, denote by X/T the set $\{x \in V(T) : x^i \in X\}$, and given $x^i \in V(H)$, denote by x the vertex of T corresponding to x^i. We say that $x^i \in V^i \setminus W^i$ is *locally encircled by* W^i if $|N(x^i) \cap W^i| \geq 2$ and $W^i \subseteq N(x^i) \cup N(N^W(x^i))$. Since T is a tree and $|N^{W^i}(x^i)| \geq 2$, for all locally encircled vertex x^i, the following proposition is trivially valid.

Proposition 6.1. *Each T^i has at most one locally encircled vertex.*

A link P such that $V(P) \subseteq V^i$, for some i, is said to be an *internal link* ; if P is a non-internal link, we say that P is a *cross link*. Figure 6.1 represents the different possible types of cross links, where the big vertices represent the vertices of W.

Actually, there may exist a cross link of the form $\langle u^i, u^l, v^l, v^j \rangle$, $i \neq l \neq j$, which is not represented in the figure. However, in this case, if $v^i \in W$, then v^l is within a cross link of Type 6, and otherwise, v^l is within a cross link of Type 5. The same argument can be applied to u^l; thus, all vertex that

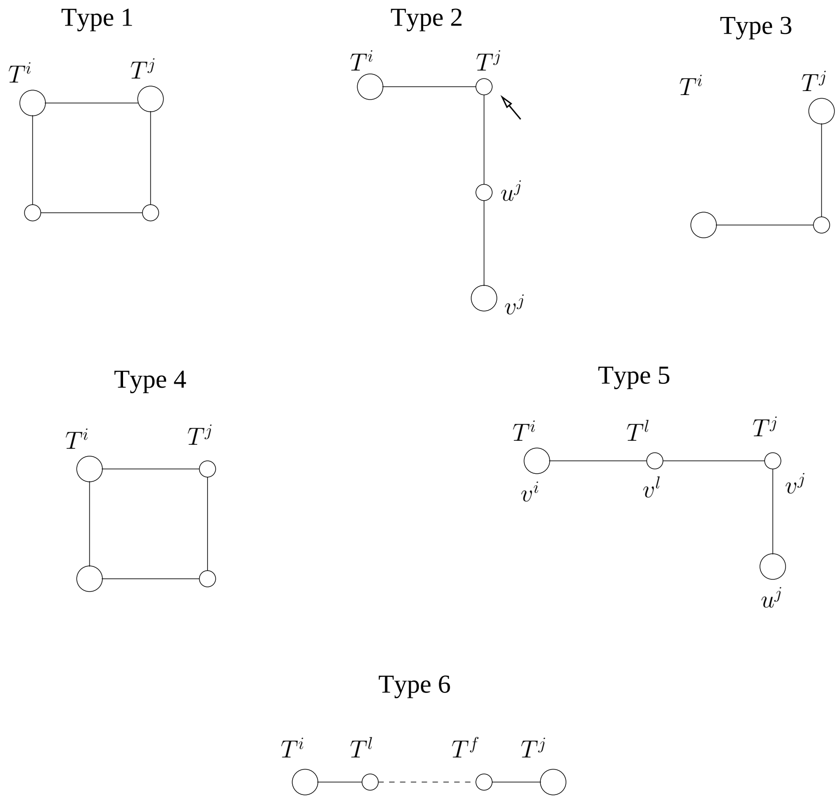


Figure 6.1: Different types of cross links. The big vertices are in W and the small ones are not in W. The dotted edge in Type 6 represents that either the edge exists or $l = f$.

lies within a cross link, also lies within a cross link of one of the presented types. Note that, in Type 5, if $u^l \in W$, then v^l is within a link of Type 3 and v^j is within a link of Type 1. Thus, we say that $\langle v^i, v^l, v^j, u^j \rangle$ is a cross link of W of Type 5, $i \neq l \neq j$, only if $u^l \notin W$. Later we will see that this actually ensures that the cartesian products being considered in this paper do not have cross links of Type 5.

Let $x^i \in V^i$ be a link vertex of W. If it is within an internal link of T^i, we say that x^i is an *internal link vertex of* W^i; and if x^i is not within any internal link, we say that x^i is a *cross link vertex*. Denote by L the set of link vertices of W, by L_I the set of internal link vertices of W, by L_C the set of cross link vertices of W and by R the set $V(H) \setminus (W \cup L_I)$.

In Sections 6.2, 6.3 and 6.4, we obtain an unsaturated precolouring of H with candidate set W, $|W| \in \{m(H) - 1, m(H)\}$, for the cartesian product H of a tree by a cycle, a path and a star, respectively. Both W and Ψ will be assured to satisfy Lemma 2.15, so we can extend it to a b-colouring of H with $|W|$ colours. In the next section, we analyse the general case, i.e., the cartesian product of a tree by a general graph.

6.1 General graph

In the following sections, whenever $H = T \,\square\, G$ has a good set W, where G is in some determined graph class, we will try to obtain an unsaturated precolouring with candidate set W where all the link vertices of W are coloured. In this section, we explain how to obtain an initial precolouring of H and, also, how to colour cross links of Type 1, requiring that some properties are valid.

Suppose that W is a good set of H. The first step to construct an initial unsaturated precolouring is to colour each vertex of W with a different colour in the range $[1, m(H)]$; we denote the vertex of W coloured with i by $(\gamma)_i$ and the obtained precolouring by Ψ. The next step is to extend Ψ by colouring the internal link vertices of T^i using the Tree Strategy (Section 2.5), whenever it is possible (i.e., whenever W^i does not locally encircle any vertex), for each copy T^i of T in H. The final step is to colour the cross link vertices in T^i that are within links of Type 2 but that do not have any neighbours in W^i (pointed vertex in Figure 6.1). Denote this set of vertices by Z. We colour Z in such a way to preserve the following property:

(*) If $z^i \in Z$ is coloured with colour c, then there exists $x^i \in N(z^i)$ such that $(\gamma)_c \in N^{W^i}(x^i)$.

We know that (II) $\Psi(V^j) \subseteq \Psi(W^j)$, for all j. Consider $z^i \in Z$. Note that if we colour z^i with a colour that satisfies (*), then $r(v^j)$ does not increase, for all $v^j \in W \setminus W^i$, since v^j has at most one neighbour in V^i, namely v^i, and by (II). Also, as $N^{W^i}(z^i) = \emptyset$, any proper extension of Ψ is also an unsaturated precolouring. So, we just want to find a vertex $(\gamma)_c \in W^i$ that satisfies (*) and such that c does not appear in $N(z^i)$. Let $X = \{x_1^i, \cdots, x_q^i\}$ be the neighbours of z^i in V^i such that $N^{W^i}(x_j^i) \neq \emptyset$, $j = 1, \cdots, q$. As z^i is within a link of Type 2 and $N^{W^i}(z^i) = \emptyset$, we have $q \geq 1$. Also: (I) if $y^j \in N(z^i)$ is coloured, then $y^j \in Z \cup L_I \cup W$. Now, consider $x_l^i \in X$ and $(\gamma)_c \in N^{W^i}(x_l^i)$. Observe that $X \cap Z = \emptyset$ and that if $y^j \in N^Z(z^i)$, then, by (*) and (II), $\Psi(y^j) \neq c$. So, as $(\gamma)_c \notin N(z^i)$ and by (I), if $c \notin \Psi(X)$, we can give colour c to z^i and obtain an unsaturated precolouring. So, suppose that there exists $x_j^i \in X$ such that $\Psi(x_j^i) = c$. By (I) and the fact that x_j^i is coloured and $x_j^i \notin Z \cup W$, we have that $x_j^i \in L_I^i$. By Lemma 2.16 and the fact that $z^i \notin L_I^i$ separates x_j^i from $(\gamma)_c$ in T^i, we know that $N^{L_I^i}(x_j^i) = \emptyset$ and that x_j^i is the only local link neighbour of $(\gamma)_d$, for all $(\gamma)_d \in N^{W^i}(x_j^i)$. Thus, $|N^{W^i}(x_j^i)| \geq 2$. This implies that $|\bigcup_{f=1}^q N^{W^i}(x_f^i)| > |X|$ and, consequently, there exists $(\gamma)_d \in N^{W^i}(x_f^i)$, for some $x_f^i \in X$, such that $d \notin \Psi(X)$. So, we extend Ψ by colouring y^i with d. Now, observe that the following is valid:

Lemma 6.2. *Let W be a good set and Ψ be a pre-colouring obtained as explained before. If $y^i \in L$ is coloured with colour c, then $(\gamma)_c \in W^i$ and either $y^i \in L_I^i$ and Lemma 2.16 holds, or $y^i \in Z$ and there exists $x^i \in N^{L^i}(y^i) \setminus W$ such that $(\gamma)_c \in N^{W^i}(x^i)$.*

In the following lemma, given an unsaturated precolouring, Ψ, we explain how to extend it to colour links of Type 1, requiring that some properties are met.

Lemma 6.3. *Consider an unsaturated precolouring Ψ with candidate set W. Let $v^i \in W$ and $x_1^i, \cdots, x_q^i \in N^{R^i}(v^i)$ be uncoloured neighbours of v^i in a path of Type 1 such that $N^W(x_j^i) = \{v^i\}$, for $j = 1, \cdots, q$. If there exists $j \neq i$ such that v^i, x_l^j are the only coloured neighbours of x_l^i, for $l = 1, \cdots, q$, and $|\Psi(\{x_1^j, \cdots, x_q^j\})| \geq 2$ then we can obtain an unsaturated precolouring that extends Ψ and colours $x_1^i, \cdots, x_q^i$.*

Proof: Colour $x_1^i, \cdots, x_q^i$ with the colours in $M(v^i)$ taking care not to give colour $\Psi(x_l^j)$ to x_l^i, $l \in [1, q]$. As $\Psi(\{x_1^j, \cdots, x_q^j\})$ has at least two colours (thus, $q \geq 2$), it is possible to do this even if $M(v^i) = 1$. Suppose that there are still some uncoloured vertices in $\{x_1^i, \cdots, x_q^i\}$, i.e., $q > M(v^i)$. If $|M(v^i)| > 1$, repeat the colour of some x_k^i in the remaining uncoloured vertices. Otherwise, if $|M(v^i)| = 1$, let x_k^i be the coloured vertex in $\{x_1^i, \cdots, x_q^i\}$; for each $l \in [1, q]$, $l \neq k$, if $\Psi(x_l^j) \neq \Psi(x_k^j)$, colour x_l^i with $\Psi(x_k^j)$, otherwise colour x_l^i with $\Psi(x_k^i)$. Note that as there are at least two colours in $\{x_1^j, \cdots, x_q^j\}$, at the end there will also be at least two colours in $\{x_1^i, \cdots, x_q^i\}$; thus, the lemma can be applied recursively. Trivially, the obtained precolouring is still unsaturated. □

6.2 Trees and Cycles

The main result of this section is the following:

Theorem 6.4. *Let T be a tree. Then, $\chi_b(T \,\square\, C_k) = m(T \,\square\, C_k)$, $k \geq 4$.*

First, we analyse the case where $T = P_2$ in the lemma below. In the proof, we will see that if $k \geq 4$, then $\chi_b(P_2 \,\square\, C_k) = m(P_2 \,\square\, C_k)$. Thus, in the remaining of this section, we consider T to be different from P_2. This lemma will be also useful in Section 6.3.

Lemma 6.5. *$\chi_b(P_2 \square P_k) \geq m(P_2 \square P_k) - 1$ and $\chi_b(P_2 \square C_{k'}) \geq m(P_2 \square C_{k'}) - 1$, $k \geq 2$, $k' \geq 3$.*

Proof: Trivially, $P_2 \,\square\, P_2$ equals a C_4 and can be b-coloured with $m(P_2 \,\square\, P_2) - 1 = 2$ colours. Now, let $\langle a_1, \cdots, a_k \rangle$ represent either the path P_k or cycle C_k, $k \geq 3$. Note that $m(P_2 \,\square\, P_3) = 3$ and, trivially, $\chi_b(P_2 \,\square\, P_3) \geq \omega(P_2 \,\square\, P_3) = 2 = m(P_2 \,\square\, P_3) - 1$. Analogously, we have that $m(P_2 \,\square\, C_3) = 4$ and $\chi_b(P_2 \,\square\, C_3) \geq \omega(P_2 \,\square\, C_3) = 3 = m(P_2 \,\square\, C_3) - 1$. So, we consider $k \geq 4$. Denote the product being treated by H and let $P_2 = \langle u, v \rangle$. Colour u^i with $(i-1) \mod 4$ and v^i with $(i+1) \mod 4$, for all $i \in \{1, \cdots, 4\}$ (observe Figure 6.2). This precolouring can be extended to the entire graph H by alternating the colours 1 and 3 in u^i, v^i, for $i \geq 5$. It is easy to verify that u^2, u^3, v^2, v^3 are b-vertices and, as $d(u^i), d(v^i) \leq 3$, for $i = 1, \cdots, k$, we know that $m(H) = 4$ and the lemma follows. □

Now, consider the cartesian product $H = T \,\square\, C_k$, $k \geq 4$. Let $C_k = \langle a_1, \cdots, a_k \rangle$. We know that:

$$d^H(u^i) = d^T(u) + 2, \; i = 1, \cdots, k$$

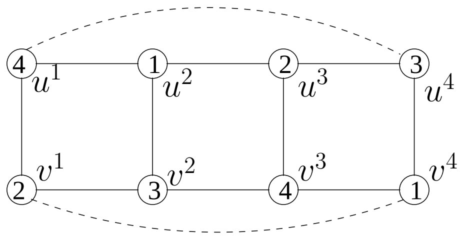


Figure 6.2: Precolouring of $P_2 \,\square\, P_k$ or $P_2 \,\square\, C_k$, $k \geq 4$.

So, we have at least $4m(T) \geq m(T)+2$ vertices in H with degree at least $m(T)+1$, i.e., $m(H) \geq m(T)+2$. Also, if $m(H) > m(T)+2$ and $u^i \in D(H)$, then:

$$d^H(u^i) \geq m(H) - 1 \Rightarrow d^T(u) + 2 > m(T) + 2 - 1 \Rightarrow d^T(u) > m(T) - 1$$

As there exist at most $m(T)$ such vertices in T, we have:

$$m(T) + 2 \leq m(H) \leq km(T)$$

Let $W \subseteq D(H)$ be a subset of $m(H)$ dense vertices of H containing every vertex of degree at least $m(H)$. Trivially, if $d^H(u^i) > m(H) - 1$, for some $i \in [1, k]$, then $\{u^1, \cdots, u^k\} \subseteq W$. Actually, we can suppose that:

(*) If $u^i \in W$, for some $i \in [1,k]$, then $\{u^1, \cdots, u^k\} \subseteq W$, except for at most one vertex $\alpha \in V(T)$ for which there exists an index i_α such that $\alpha^i \in W$, for all $i \in [1, i_\alpha]$, and $\alpha^i \notin W$, for all $i \in [i_\alpha + 1, k]$.

Lemma 6.6. *Let $W \subseteq D(H)$ with cardinality $m(H)$ containing all $u^i \in D(H)$ with $d(u^i) \geq m(H)$ and such that (*) holds. Then, W is a good set.*

Proof: Suppose that $x^i \in V^i \setminus W$ is encircled by W. First, note that if $W \subseteq \{u^1, \cdots, u^k\}$ and $u \neq x$, then $N^W(x^i) \subseteq \{u^i\}$, contradicting Proposition 2.1. Also, if there exists $u^i \in W$ such that $\{u^1, \cdots, u^k\} \subseteq W$, then x^i does not reach $u^{(i+2) \mod k}$, a contradiction. Thus, $W \subseteq \{x^1, \cdots, x^{i-1}\}$ (and, hence, $x = \alpha$). Actually, as $k \geq 4$, $m(H) \geq 4$ and $x^j \in N(x^i) \cup N(N^W(x^i))$, for all $x^j \in W$, we must have $k = 5$ and $m(H) = 4$. However, we know that T has at least one vertex with degree at least 2, say v, and, consequently, H has at least 5 vertices with degree at least 4, $v_1, \cdots, v_5$, and $m(H)$ should be at least 5, a contradiction. □

Now, given a good set W satisfying Lemma 6.6, we prove that we can obtain a b-colouring of H with W as basis. Actually, we show how to obtain an unsaturated precolouring that colours all link vertices of W; the theorem then follows as by Lemma 2.15 this precolouring can be extended to a b-colouring of H with $m(H)$ colours. But before we move on, we make some observations that will lead to an useful equation. Let $u^i \in W^i$. We know that:

$$N(u^i) = N^{W^i}(u^i) \cup N^{L_I^i}(u^i) \cup N^{R^i}(u^i) \cup \{u^{(i-1) \mod k}, u^{(i+1) \mod k}\}$$

Also, note that either $v^i \in W^i$ is a neighbour of u^i or there exists at most one vertex of L_I^i that is within an internal link between v^i and u^i, i.e.:

$$|N^{W^i}(u^i) \cup N^{L_I^i}(u^i)| \leq |W^i| - 1$$

Thus, we have:

$$d(u^i) = |N^{W^i}(u^i) \cup N^{L_I^i}(u^i)| + |N^{R^i}(u^i)| + 2$$

$$m(H) - 1 \leq |W^i| - 1 + |N^{R^i}(u^i)| + 2$$

$$|W \setminus W^i| \leq |N^{R^i}(u^i)| + 2 \tag{6.7}$$

In the following two lemmas, we analyse the easier cases.

Lemma 6.8. *Let $H = T \square C_k$, where T is a tree different from P_2 and $k \geq 4$, and $W \subseteq D(H)$ be a good set satisfying Lemma 6.6. If $W \subseteq \{w^1, \cdots, w^k\}$, for some $w \in V(T)$, then $\chi_b(H) = m(H)$.*

Proof: Denote $m(H)$ by m and let $W = \{w^1, \cdots, w^m\}$. First, colour w^i with i, for all $i \in \{1, \cdots, m\}$. Observe that, as $|W^i| \leq 1$, for all $i \in \{1, \cdots, k\}$, there is no internal link and the existing types of cross links are Type 1 and Type 6. Suppose first that $m < k$. If there exists a link of Type 6, then $w = \alpha$ and $m \in \{k-2, k-1\}$. If $m = k-2$, we have $k \geq 6$ (as $m \geq 4$) and we can colour w^{m+1} with $\Psi(w^1)$ and w^k with $\Psi(w^m)$. Now consider $m = k-1$. Since $T \neq P_2$, we have $\delta(T) \geq 2$ and, hence, $\delta(H) \geq 4$; thus, as $d^H(w^k) = m - 1 \geq 4$, we have that $m \geq 5$ and we can give colour $\Psi(w^3)$ to w^k. After this, note that we can recursively apply Lemma 6.3 to colour $N^L(w^i)$, starting from w^1 up to w^m ($|N^L(w^i)| \geq 2$, since $\delta(H) \geq 2$).

Now, consider that $m = k$ and let $N^T(w) = X = \{x_1, \cdots, x_q\}$. Trivially, $x_j^i \in L_C$, for all $i \in \{1, \cdots, k\}$, $j \in \{1, \cdots, q\}$. By Lemma 2.14 and the fact that w^i has exactly two coloured neighbours, namely $w^{(i-1) \mod k}$ and $w^{(i+1) \mod k}$, for all $w^i \in W$, we have $q \geq k-3$. For each $i \in \{1, \cdots, k\}$, colour x_j^i with $(i+j+1) \mod k$, if $j \in \{1, \cdots, k-3\}$, and with $(i-1) \mod k$, otherwise. Observe that $\Psi(\{x_1^i, \cdots, x_{k-3}^i\}) = \{1, \cdots, m(G)\} \setminus \{i, (i-1) \mod k, (i+1) \mod k\}$; thus, each w^i is a b-vertex. To see that the precolouring is also proper, note that colour i does not appear in X and: for each $j \in \{1, \cdots, q\}, i \in \{1, \cdots, k\}$, as the coloured neighbourhood of x_j^i is $\{w^i, x_j^{(i-1) \mod k}, x_j^{(i+1) \mod k}\}$ and $\Psi(N[x_j^i])$ is either $\{i, (i+j) \mod k, (i+j+1) \mod k, (i+j+2) \mod k\}$ or $\{i, (i-2) \mod k, (i-1) \mod k, i \mod k\}$ of cardinality 4, we know that the precolouring is proper. □

Lemma 6.9. *Let $H = T \square C_k$, where T is a tree different from P_2 and $k \geq 4$, and $W \subseteq D(H)$ be a good set satisfying Lemma 6.6. If $m(H) \leq 6$, then $\chi_b(H) = m(H)$.*

Proof: Let $u \in T$ with maximum degree; as $T \neq P_2$, we have $d^T(u) \geq 2$ and $d^H(u^i) \geq 4$, for all $i \in \{1, \cdots, k\}$. So, as $k \geq 4$ and by Lemma 6.8, we need to consider only the values 5 and 6 for $m(H)$. First, consider $m(H) = 5$. So, as $k \geq 4$ and by Lemma 6.8, we can suppose that $k = 4$. Also, as T is connected, there must exist $\alpha \in N(u)$ such that $d^T(\alpha) \geq 2$, i.e., we can suppose that $W = \{u^1, u^2, u^3, u^4, \alpha^1\}$, where $(u, \alpha) \in E(T)$. So, H contains the graph represented in Figure 6.3. Observe that the presented precolouring does not repeat colours in $N(z^i)$, for all $z^i \in W$, and all link vertices are coloured (unless $d^T(u) > 2$, in which case we can just repeat colours in $N(u^i)$, since u^i is already a b-vertex, $i = 1, \cdots, 4$). Thus, the presented precolouring can be extended to a b-colouring of H with 5 colours.

Now, consider $m(H) = 6$. By Lemma 6.8 and the uniqueness of α, we can suppose that $k \leq 5$. So, there exist $u, \alpha \in V(T)$ such that either $k = 4$ and $W = \{u^1, u^2, u^3, u^4, \alpha^1, \alpha^2\}$ or $k = 5$ and $W = \{u^1, u^2, u^3, u^4, u^5, \alpha^1\}$. Observe the precolourings presented in Figures 6.4 and 6.5. In Figure 6.4.(b), note that the remaining links of Type 1 between α^1 and α^2 can be coloured as in Figure 6.4.(a). Also, if H has more links of Type 1 than it is represented in the figures, these links can be easily coloured. Finally, if the distance between u and α in T is greater than 2, the precolourings presented in Figures 6.4.(c) and 6.5.(c) can be easily adapted to H.

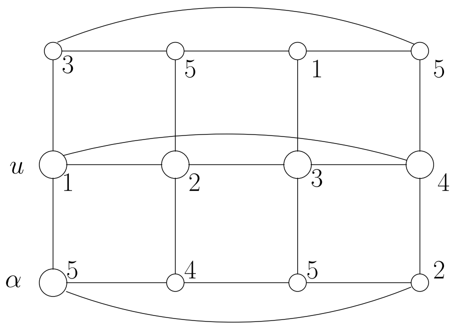


Figure 6.3: Precolouring of W and the link vertices when $m(H) = 5$.

□

Proof of Theorem 6.4: Let W be a good set satisfying Lemma 6.6. By Lemmas 6.5, 6.8 and 6.9, we can suppose that $T \neq P_2$, $|W^1| \geq 2$ and $m(H) \geq 7$. Let Ψ be an unsaturated precolouring of W obtained as explained in Section 6.1. We want to colour the vertices locally encircled by W and the remaining cross link vertices of W. Note that by the way we classify vertices to be within cross links of Type 2 and Type 5 and by the uniqueness of α, we have that the uncoloured cross link vertices lie in links of Type 1, Type 3 or Type 6. Also, note that only $\alpha^{i_\alpha+1}$ and α^k can lie within links of Type 2, 3 or 6.

We first colour links of Type 3 and Type 6. Let $i = i_\alpha$. First, suppose that there exists $v^i \in N^{W^i}(\alpha^i)$. Then give colour $\Psi(v^{(i-1) \bmod k})$ to α^{i+1} and, if $i+1 < k$, give colour $\Psi(v^{k-2})$ to α^k (note that, in this case, $k-2 \geq i$, i.e., $\Psi(\alpha^{i+1}) \neq \Psi(\alpha^k)$). Now, suppose that $N^{W^i}(\alpha^i) = \emptyset$. So, α^{i+1}, α^k are not within links of Type 3 and, if they are link vertices and are still uncoloured, then $i \geq k-2$ (i.e., $\langle \alpha^i, \alpha^{i+1}, \alpha^k, \alpha^1 \rangle$ is link of Type 6). Let $v^i \in W^i \setminus \{\alpha^i\}$ (exists as $|W^1| \geq 2$). Since $N^{W^{i+1}}(\alpha^{i+1}) = \emptyset$ and α^{i+1} is still uncoloured, we have $N^{L^i_I}(\alpha^i) = \emptyset$ (otherwise, α^{i+1} is within a link of Type 2 and should already be coloured). The same is valid for v^k. Thus, we can colour α^{i+1} with $\Psi(v^{i+1})$ and α^k with $\Psi(v^k)$, if $i+1 < k$. Note that: (I) if α^j is coloured with a colour not in $\Psi(W^j)$, then $j \in \{i_\alpha + 1, k\}$ and there exists $v^{i_\alpha} \in N^W(\alpha^{i_\alpha})$ such that $\Psi(\alpha^{i_\alpha+1}), \Psi(\alpha^k) \in \Psi(\{v^1, \cdots, v^k\})$.

Now, we proceed to colour the locally encircled vertices. By Proposition 6.1, we know that there exists at most one locally encircled vertex in T^i, for $i = 1, \cdots, k$. Also, by the uniqueness of α, it is easy to see that if x^i, y^j are locally encircled and $i \neq j$, then $x = y$. So, let $x^p, \cdots, x^q$ be all the

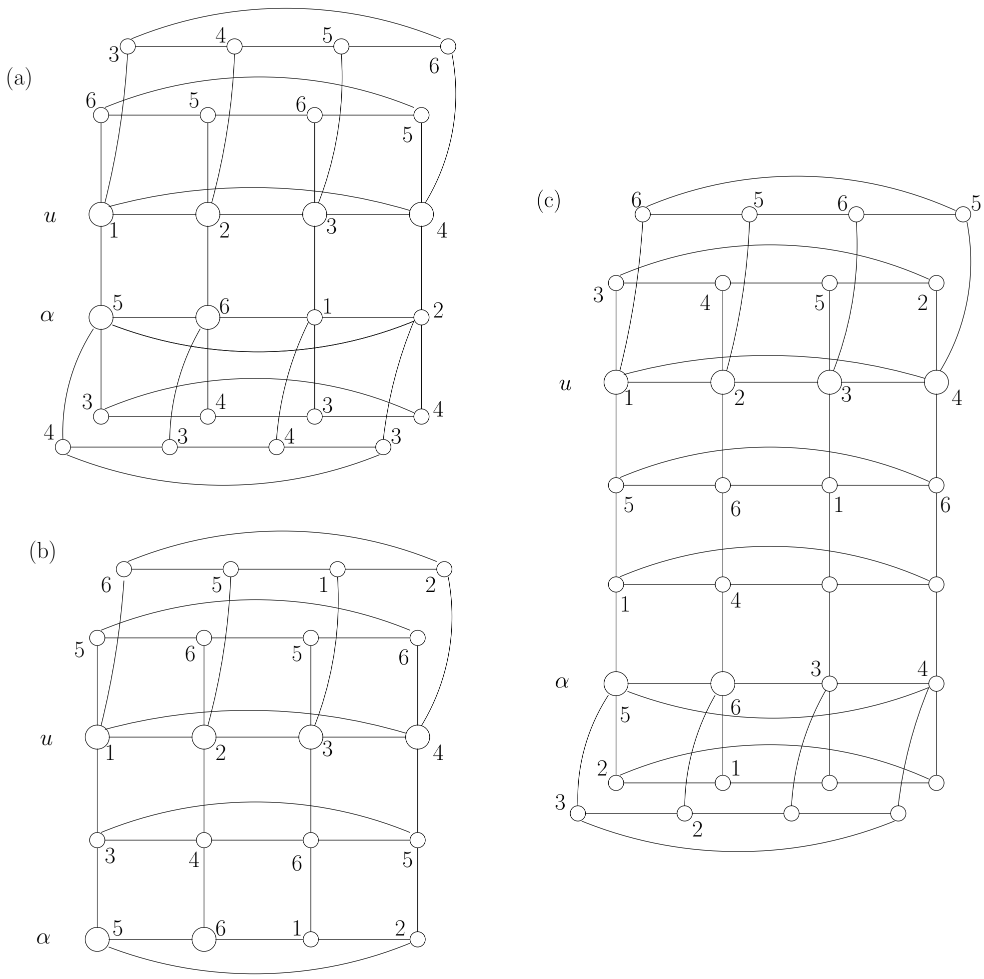


Figure 6.4: Precolouring of W and the link vertices when $m(H) = 6$ and $k = 4$.

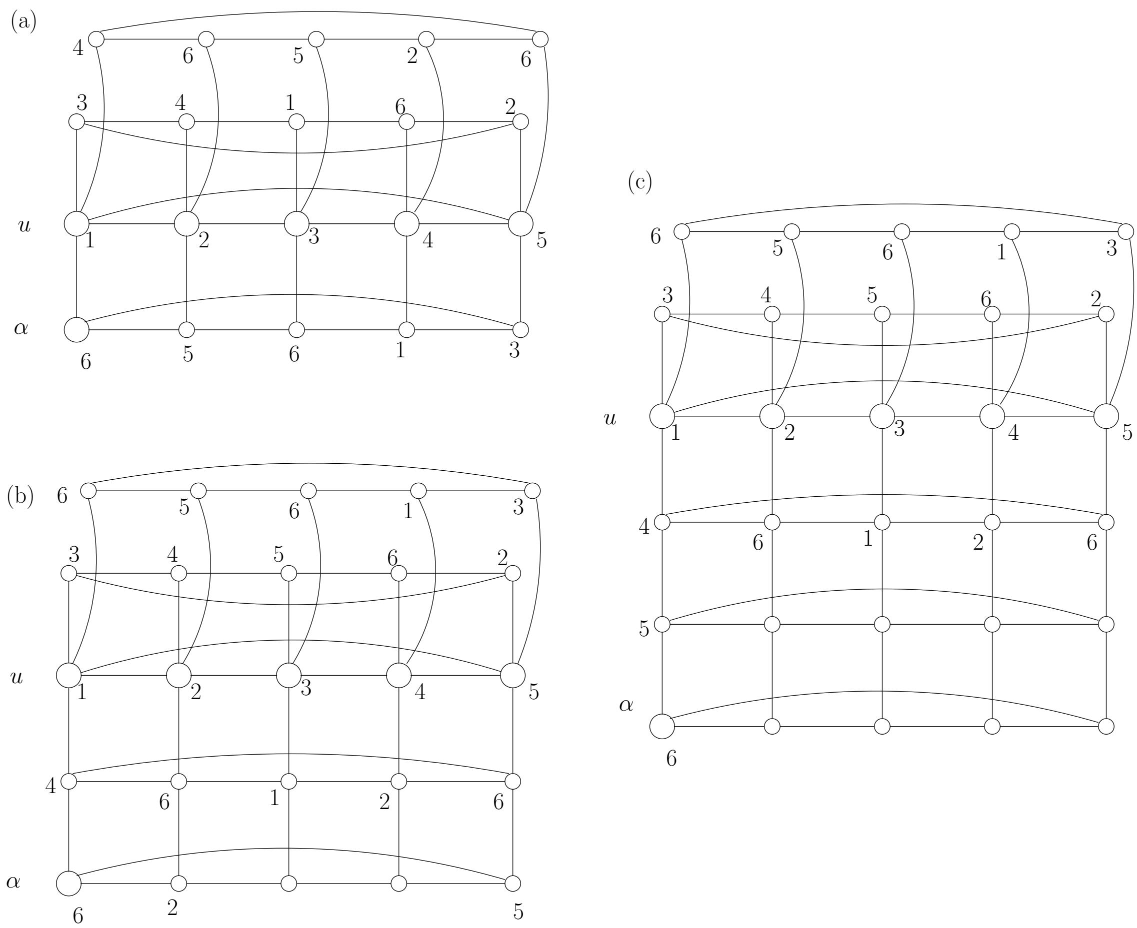


Figure 6.5: Precolouring of W and the link vertices when $m(H) = 6$ and $k = 5$.

locally encircled vertices (note that $p \in \{1, i_\alpha + 1\}$ and $q \in \{i_\alpha, k\}$). We know that, until now, $\Psi(V^i) \subseteq \Psi(W^i \cup \{\alpha^i\})$. By (I) and the fact that $|N^{W^p}(x^p)| \geq 2$ implies that there exists $u^p \in N^{W^p}(x^p) \setminus \{\alpha^p, v^p\}$. We colour x^j with $\Psi(u^{(j-2) \mod k})$, for $j = p, \cdots, q$.

Finally, we colour links of Type 1. We know that any internal link vertex is already coloured and that if y^i is within a cross link of Type 3 or 6, then $y^i \in \{\alpha^{i_\alpha+1}, \alpha^k\}$. Thus, if $y^i \in L$ is still uncoloured, then y^i has exactly one neighbour in W and such neighbour is in V^i. Also, if $v^i \in W^i$, then v^i has at most $|W^i| + 2$ coloured neighbours, namely $\{v^{(i-1) \mod k}, v^{(i+1) \mod k}\}$ and $(W \cup L_I \cup \{\alpha^{i_\alpha+1}, \alpha^k\}) \cap V^i$ of cardinality at most $|W^i|$. As $m(H) = k|W^k| + i_\alpha$ and $|W^i| \leq |W^k| + 1$, we have:

$$\begin{aligned} |M(v^i)| &\geq m(H) - 1 - (|W^i| + 2) \\ &\geq k|W^k| + i_\alpha - 1 - (|W^k| + 1 + 2) \\ &\geq (k-1)|W^k| + i_\alpha - 4 \end{aligned}$$

Note that if $|W^k| \geq 2$, then $|M(v^i)| \geq 2$. Otherwise, we have $|M(v_i)| \geq k + i_\alpha - 5 = m(H) - 5$ and, as $m(H) \geq 7$, we have $|M(v^i)| \geq 2$. Thus, we can assume $|M(v^i)| \geq 2$, for all $v^i \in W$. Denote by L_1 the set of uncoloured link vertices and let $v^i \in W$. Note that if $v \neq \alpha$ or $i_\alpha \geq 2$, then every $y^i \in N^{R^i}(v^i)$ is within a link of Type 1 or 3, namely $\langle v^i, y^i, y^j, v^j \rangle$ or $\langle v^i, y^i, y^j \rangle$ (in the case $y = \alpha$ and $i \in \{i_\alpha + 1, k\}$), where $j \in \{(i-1) \mod k, (i+1) \mod k\}$. So, if $N^{L_1}(v^i) \neq \emptyset$, then either $N^{L_1}(v^i) = N^{R^i}(v^i)$ or $N^{L_1}(v^i) = N^{R^i}(v^i) \setminus \{\alpha^i\}$ and $i \in \{i_\alpha + 1, k\}$. Also, note that $\Psi(u^{(i-1) \mod k}) \neq \emptyset$ and $\Psi(u^{(i+1) \mod k}) \neq \emptyset$, for all $u^i \in W$. Thus, $N^{L^1}(v^i)$ is the set of all the uncoloured neighbours of v^i and, by Lemma 2.14, we have $|N^{L^1}(v^i)| \geq |M(v^i)|$. Finally, note that v^i has at most one more coloured neighbour than v^j, for all $j \in \{i+1, k\}$, and if it is the case then either (a) $i \leq i_\alpha < j$ and some $y^i \in N(v^i)$ is within an internal link between v^i and α^i, or (b) $i \leq i_\alpha + 1 < j < k$ and $\alpha \in N(v)$. We choose a subset $X^1 \subseteq N^{L_1}(v^1)$ with cardinality $|M(v^1)|$ and construct a subset X^j, for all $j \in \{2, \cdots, k\}$, as follows:

- If (a) occurs:

$$X^j = \begin{cases} \{x^j : x \in X^1/T\}, & j \in \{2, \cdots, i_\alpha\} \\ \{x^j : x \in X^1/T\} \cup \{y^j\}, & j \in \{i_\alpha + 1, \cdots, k\} \end{cases}$$

- If (b) occurs:

$$X^j = \begin{cases} \{x^j : x \in X^1/T\}, & j \in \{2, \cdots, i_\alpha + 1, k\} \\ \{x^j : x \in X^1/T\} \cup \{\alpha^j\}, & j \in \{i_\alpha + 2, \cdots, k-1\} \end{cases}$$

We know that $X^i \subseteq N^{L_1}(v^i)$ and $|X^i| = |M(v^i)|$, for all $i \in \{1, \cdots, k\}$. We will colour X^i with colours from $M(v^i)$, for every $i \in \{1, \cdots, k\}$, starting from X^1 up to X^k. As each vertex of X^i will be coloured with a different colour from $M(v^i)$, $i \in \{1, \cdots, k\}$, the following holds throughout the procedure:

(i) $\Psi(y^i) \neq \Psi(t^i)$, for all $y^i, t^i \in X^i$, $y \neq t$.

Let $X^1/T = \{x_1, \cdots, x_q\}$, where $q = |M(v^1)|$ (recall that $q \geq 2$). Start by colouring X^1 with the colours from $M(v^1)$. Now, consider the i-th iteration, $i > 1$, and let $X^i = \{y^i_1, \cdots, y^i_l\}$. Suppose, without loss of generality, that $x_j = y_j$, for $j \in \{1, \cdots, q\}$, and that $(X^i/T) \setminus (X^1/T) = \{y_l\}$, if $l > q$. We first prove that $l \geq 4$. Suppose that $l = 3$. By Equation 6.7, we know that there exists at most three vertices in $W \setminus (W^k \cup \{v^{k-1}, v^1\})$. So, as $k \geq 4$ and $W \not\subseteq \{v^1, \cdots, v^k\}$, we have that $W^k = \{v^k\}$ and $m(H) = 6$, a contradiction. Now, suppose that $l = 2$. As $q \geq 2$, we have $l = q$ and $\alpha^k \notin N(y^k_2)$ (otherwise, we would have $l = q + 1$ as, in this case, y^1_2 is within the internal link $\langle v^1, y^1_2, \alpha^1$, i.e., $y^1_2 \notin X^1$). Also, by Equation 6.7, we know that there exists at most two vertices in $W \setminus (W^k \cup \{v^{k-1}, v^1\})$. So, as $k \geq 4$, we have $W^k = \{v^k\}$ and $m(H) = 5$, a contradiction.

Now, consider $l \geq 4$. Let $H' = (X^i, M(v^i))$ be a bipartite graph where $(y^i_j, c) \in E(H')$ if and only if $c \notin \Psi(N(y^i_j))$. Observe that a perfect matching of H' gives us an extension of Ψ that colours X^i with colours from $M(v^i)$. We now make some observations about the edges in H'. Note that each y^i_j has at most two coloured neighbours different from v^i, for all $j \in \{1, \cdots, l-1\}$, namely y^{i-1}_j and $y^{(i+1) \mod k}_j$, while y^i_l may have one more coloured neighbour, namely α^i when $i \in \{i_\alpha + 1, k\}$. So, y^i_j has at most two non-neighbours in H', for all $j \in \{1, \cdots, l-1\}$, and y^i_l at most three. Also, by (i) each colour $c \in M(v^i) \setminus \Psi(\alpha^i)$ has at most two non-neighbours in H', while $\Psi(\alpha^i)$ may have three non-neighbours (in the case $i \in \{i_\alpha + 1, k\}$ and $\Psi(\alpha^i) \in M(v^i)$). By Hall's Theorem, H' has a perfect matching if and only if $|N^{H'}(A)| \geq |A|$, for all $A \subseteq X^i$. As $l \geq 4$, we already know that $N^{H'}(y^i_j) \neq \emptyset$, for all $j \in \{1, \cdots, l\}$. If $|A| = 2$, as $l \geq 4$ and at least one vertex of A is different from y^i_k, say y^i_j, we have $|N^{H'}(A)| \geq |N^{H'}(y^i_j)| \geq 2$. If $|A| = 3$, as at most one colour has three non-neighbours in H' and $l \geq 4$, we have $|N^{H'}(A)| \geq 3$. Finally, if $|A| \geq 4$, as no colour has more than three non-neighbours in H' we have $N^{H'}(A) = M(v^i)$.

At the end, if any $x^i \in L_1$ is still uncoloured, we know that x^i has at most 4 neighbours in $H[W \cup N(W)]$, namely $x^{(i-1) \mod k}$, $x^{(i+1) \mod k}$, α^i and

$v^i \in N^{W^i}(x^i)$. So, as $m(H) \geq 7$, we can colour x^i with any colour not in its neighbourhood, for all x^i still uncoloured. $\square$

6.3 Trees and paths

The main result of this section is the following:

Theorem 6.10. *Let $H = T \square P_k$, $k \geq 5$. If H has a good set, then $\chi_b(H) = m(H)$; otherwise, $\chi_b(H) = m(H) - 1$.*

Consider the cartesian product $H = T \square P_k$, $k \geq 5$. Let $P_k = \langle a_1, \cdots, a_k \rangle$. Note that:

$$d^H(u^i) = d^T(u) + 2,\ i = 2, \cdots, k-1$$

$$d^H(u^i) = d^T(u) + 1,\ i = 1, k$$

As we have at least $3m(T) \geq m(T) + 2$ vertices in H with degree at least $m(T) + 1$, we have $m(H) \geq m(T) + 2$. Suppose that $m(H) > m(T) + 2$; if $u^i \in D(H)$, as $d^H(u^i) \leq d^T(u) + 2$, $i = 1, \cdots, k$, we have $d^H(u^i) \geq m(H) - 1 \Rightarrow d^T(u) + 2 > m(T) + 2 - 1 \Rightarrow d^T(u) > m(T) - 1$. As there exist at most $m(T)$ such vertices in T, we have:

$$m(T) + 2 \leq m(H) \leq km(T)$$

Let $W \subseteq D(H)$ be a subset of $m(H)$ dense vertices of H containing every vertex of degree at least $m(H)$. Trivially, if $d^H(u^i) > m(H) - 1$, then $\{u^2, \cdots, u^{k-1}\} \subseteq W$. Actually, we can make the following assumptions:

- A1 At most one vertex $\alpha \in V(T)$ is such that there exists an index $i_\alpha < k - 1$ for which $\alpha^i \in W$, for all $2 \leq i \leq i_\alpha$, and $\alpha^i \notin W$, for all $i_\alpha < i \leq k$;

- A2 $\{u^2, \cdots, u^{k-1}\} \subseteq W$, for all $u \in V(T) \setminus \{\alpha\}$ such that some $u^i \in W$;

- A3 If there exists $w^2 \in W^2$ with $d(w^2) = m(H) - 1$, then $u^1, u^k \in W$, for all $u^i \in W$ such that $d^H(u^i) > m(H) - 1$;

- A4 If α does not exist, then at most one vertex, denoted by β, is such that $\beta^1 \in W$ and $\beta^k \notin V^k$; all other vertex $v^i \in W$ is such that $v^1 \in W$ iff $v^k \in W$.

Observe that, if α exists, as $\alpha^2 \in W$ and $\alpha^{k-1} \notin W$, we have that $d^H(\alpha^i) = m(H) - 1$, for $i = 2, \cdots, k-1$.

Let $u^i \in W^i$. Note that $N(u^i) \subseteq N^{W^i}(u^i) \cup N^{L^i_I}(u^i) \cup N^{R^i}(u^i) \cup \{u^{i-1}, u^{i+1}\}$ and we can make an argument analogous to the one made for cycles, i.e., Equation 6.7 is also valid here. Now, we analyse the existence of a good set in H.

Lemma 6.11. *Suppose that $T \neq P_2$. Then $H = T \,\square\, P_k$, $k \geq 5$, does not have a good set if and only if $k = 5$, $|D(H)| = m(H)$, $d(v^i) = m(H) - 1$, for all $v^i \in D(H)$, and there exist $u^2, u^3, u^4 \in V(H) \setminus D(H)$ such that $D(H) \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$.*

Proof: First, note that if $k = 5$, $|D(H)| = m(H)$, $d(v^i) = m(H) - 1$, for all $v^i \in D(H)$, and there exist $u^2, u^3, u^4 \in V(H) \setminus D(H)$ such that $D(H) \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$, then $D(H)$ encircles u^3 and, as $|D(H)| = m(H)$, H does not have a good set. Now, it remains to prove the other way of the equivalence.

Let $W \subseteq D(H)$ be a subset of cardinality $m(H)$ that satisfies A1-A4. First, we prove that if W encircles some vertex $u^i \in V(H) \setminus W$, then $k = 5$, $i = 3$ and $W \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$. Note that u^i does not reach any vertex in W^{i+2}, since $u^{i+1} \notin W$. Also, by Proposition 2.1 and the uniqueness of α, there must exist $v^i \in W^i$ such that $\{v^2, \cdots, v^{k-1}\} \subseteq W$. Thus, $i \geq k-2$. Similarly, u^i cannot reach any vertex in $V^{i-2} \setminus \{u^{i-2}\}$. Consequently, as $v^{i-2} \neq u^{i-2}$, we have $i - 2 \leq 1$. Hence, as $k \geq 5$, we have $i = 3$ and $k = 5$. Now, suppose that there exists $w^j \in W \setminus N(u^j)$, for some $j \in \{1, \cdots, 5\}$. Trivially, if $j \neq 3$, then u^3 does not reach w^j. Furthermore, if $w^3 \in W$ we know that $w^2 \in W$ and u^3 does not reach w^2, a contradiction.

Now, let $W \subseteq D(H)$ be a subset of cardinality $m(H)$ that satisfies A1-A4 and contains all vertices of H with degree at least $m(H)$. Trivially, if W does not encircle any vertex, then it is a good set. So, suppose that W encircles $u^3 \in V(H) \setminus W$. Note that, as $u^4 \notin W$, w^3 is the only (u^3, w^4)-bridge, for all $w^4 \in W$, and, consequently, $d(w^3) = m(H) - 1$, for all $w^3 \in W^3$. Also, since $d(w^2) = d(w^3) = d(w^4)$, for all $w^3 \in V^3$, and $W \subseteq N(u^2) \cup N(u^3) \cup N(u^4) \subseteq W^3 \cup \{w^2, w^4 : w^3 \in W^3\} \cup \{u^2\}$, we have $d(w^i) = m(H) - 1$, for all $w^i \in W$. So, as W contains all vertices with degree at least $m(H)$, we have $d(w^i) = m(H) - 1$, for all $w^i \in D(H)$. Note that $D(H) \cap (V^1 \cup V^k) = \emptyset$. Now, let $v \in V(T)$ be such that $v^2, v^3, v^4 \in W$ (v exists by Proposition 2.1 and the uniqueness of α). If $u^3 \in D(H)$, note that either $(W \setminus \{v^3, v^4\}) \cup \{u^3, u^4\}$, if $u = \alpha$, or $(W \setminus \{v^2, v^3, v^4\}) \cup \{u^2, u^3, u^4\}$, otherwise, is a good set that satisfies Assumptions A1-A4. So, suppose that $u^3 \notin D(H)$ and let $w^j \in D(H) \setminus W$. We know that $j \neq 1, k$. So, if $w \notin N(u)$, we know that $(W \setminus \{v^2, v^3, v^4\}) \cup \{w^2, w^3, w^4\}$ is a good set. Otherwise, consider that $D(H) \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$. Let W' be either $(W \setminus \{w^3\}) \cup \{w^4\}$, if $w = \alpha$ (note that i_α must be equal to 3, since α^2 must be reached by u^3 and $u^2 \notin D(H)$), or $(W \setminus \{v^3\}) \cup \{w^3\}$, if $w \neq \alpha$. Note that in both cases W' is a good set, but Assumptions A1-A4 are not valid; these situations are stated in Lemma 6.12. □

The following lemma follows directly from the proof of the lemma above.

Lemma 6.12. *If H has a good set W then either (I) there exists a good set for which Assumptions A1-A4 are valid or (II) $k = 5$, there exist $u^2, u^3, u^4 \in V(H) \setminus D(H)$ such that $D(H) \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$ and $x^2, x^3, x^4 \subseteq W$, for all $x^i \in W$, except for at most two vertices, in which case one of the following occurs for W:*

- *there exist $w^2, w^4 \in W$ such that $w^3 \notin W$; or*
- *there exist $v^3, w^2, w^4 \in W$ such that $v^2, v^4, w^3 \notin W$.*

We first prove the following part of Theorem 6.10:

Lemma 6.13. *Let $H = T \,\square\, P_k$, $k \geq 5$. If H does not have a good set, then $\chi_b(H) = m(H) - 1$.*

Proof: By Lemma 2.4, we know that $\chi_b(H) < m(H)$. We choose a subset $W \subseteq D(H)$ with cardinality $m(H) - 1$ such that $v^i \in D(H) \setminus W$ is a link vertex of W (by Lemma 6.11, we know that $|D(H)| = m(H)$). Then, we construct an unsaturated precolouring Ψ with candidate set W that colours all link vertices of W. As a consequence, by Lemma 2.15, Ψ can be extended to a b-colouring of H with $m(H) - 1$ colours and the lemma follows. By Lemma 6.11, we know that $k = 5$, $|D(H)| = m(H)$ and there exist $u^2, u^3, u^4 \in V(H) \setminus D(H)$ such that $D(H) \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$. Note that $m(H) \geq 6$, since $m(H) = 3|D(H) \cap V^3|$ and $d^H(u^3) \geq 3$ (i.e., $m(H) \geq 5$, as $u^3 \notin D(H)$). So, let $v^3, w^3 \in D(H)$ and consider $W = D(H) \setminus \{w^3\}$. Trivially, w^3 is a link vertex of W. Colour each vertex of W with a different colour and give colour: $\Psi(v^2)$ to u^4; $\Psi(v^4)$ to u^2; $\Psi(w^2)$ to u^3; and $\Psi(v^3)$ to w^3. After this, note that the only uncoloured links are of Type 1 and, since $d^{T^i}(z^i) \geq 3$ and u^i is the only coloured neighbour of z^i in T^i, then z^i has at least two uncoloured neighbours in such a link, for all $z^i \in W \setminus \{w^2, w^4\}$ (observe that, as $w^3 \notin W$, then w^2, w^4 are not extremities of links of Type 1). So, for each $z^i \in W \setminus \{w^2, w^4\}$, we colour the link neighbours of z^2 with the colours in $M(z^2)$ and, then, by Lemma 6.3, we can colour the link neighbours of z^3 and z^4, in this order. □

Now, we colour H with $m(H)$ colours, in the case where H does not have a good set satisfying A1-A4.

Lemma 6.14. *Let $H = T \Box P_k$, $k \geq 5$ and W be a good set of H that satisfies the item (II) of Lemma 6.11. Then, there exists a b-colouring of H with basis W.*

Proof: Let Ψ be a precolouring where each vertex of W is coloured with a different colour. We know that $k = 5$, there exist $u^2, u^3, u^4 \in V(H) \setminus D(H)$ such that $D(H) \subseteq N(u^2) \cup N(u^3) \cup N(u^4)$ and $x^2, x^3, x^4 \subseteq W$, for all $x^i \in W$, except for at most two vertices, in which case one of the following occurs:

- there exist $w^2, w^4 \in W$ such that $w^3 \notin W$: as $d^H(u^3) \geq 3$ and $u^3 \notin D(H)$, we have that $m(H) \geq 5$. Thus, there exist $v^2, v^3, v^4 \in W$ (and consequently, $m(H) \geq 6$ as, in this case, $d(u^3) \geq 4$). Give colour $\Psi(v^2)$ to u^4; $\Psi(v^4)$ to u^2; $\Psi(w^2)$ to u^3 and $\Psi(v^3)$ to w^3. Note that no colour is repeated in $N(z^i)$, for all $z^i \in W$. Furthermore, as $m(H) \geq 6$ and z^i has only coloured neighbour in T^i, u^i, we have that z^i has at least two uncoloured neighbours in a link of Type 1; thus, by Lemma 6.3, Ψ can be extended to colour the links of Type 1.

- there exist $v^3, w^2, w^4 \in W$ such that $v^2, v^4, w^3 \notin W$: as $d^H(u^3) \geq 4$ and $u^3 \notin D(H)$, we have that $m(H) \geq 6$. Consequently, there exists $z^2 \in W^2 \setminus \{w^2\}$. Give colour: $\psi(z^2)$ to u^4 and v^2; $\Psi(w^2)$ to u^3; $\Psi(v^3)$ to w^3; and $\Psi(w^4)$ to u^2 and v^4. After this, since $m(H) \geq 6$, we have that each $x^j \in W \setminus \{v^3, w^2, w^4\}$ has at least two neigbours in a link of Type 1; thus, by Lemma 6.3, we can colour these links.

As W is a good set and Ψ colours all link vertices of W, by Lemma 2.15, Ψ can be extended to a b-colouring of H with basis W. □

The following lemma colours the easier case, when H has a good set satisfying A1-A4.

Lemma 6.15. *Let $H = T \,\square\, P_k$, $k \geq 5$, and W be a good set of H that satisfies A1-A4. If $W \subseteq \{u^1, \cdots, u^k\}$, for some $u \in V(T)$, or $m(H) \leq 5$, then $\chi_b(H) = m(H)$.*

Proof: First, note that if $W \subseteq \{w^1, \cdots, w^k\}$, then every link is of Type 1 and we can colour the link vertices of V^1 up to V^k using Lemma 6.3. So, suppose that $|\{w^1, \cdots, w^k\} \cap D(H)| < m(H)$, for every $w \in V(T)$. Let $u \in V(T)$ with maximum degree. If $T = P_2$, the lemma follows by Lemma 6.5. Thus, $d(u) \geq 2$ and, as $k \geq 5$ and $|\{u^1, \cdots, u^k\} \cap D(H)| < m(H)$, we have $m(H) = 5$. Also, as T is connected, there must exist $\alpha \in N(u)$ such that $d(\alpha) \geq 2$. Note that if $d(u) > 2$ or $k \geq 7$, then $|\{u^1, \cdots, u^k\} \cap D(H)| \geq 5$, a contradiction. Thus, $d^T(u) = d^T(\alpha) = 2$ and $k \leq 6$. Let W' be either $\{u^2, u^3, u^4, \alpha^2, \alpha^3\}$, if $k = 5$, or $\{u^2, u^3, u^4, u^5, \alpha^2\}$, if $k = 6$. As $|D(H)| > 5$, by Lemmas 6.11 and 6.12 we know that W' is a good set. Observe that the precolouring presented in Figure 6.6 or in Figure 6.7 is an unsaturated precolouring of H with candidate set W'.

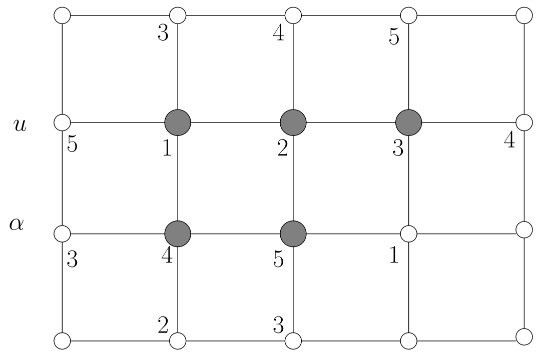


Figure 6.6: Precolouring of H, where $m(H) = 5$ and $k = 5$. The grey vertices represent W'.

□

Before we move on to prove Theorem 6.10, we make the following remarks.

Remark 6.16. *Let $v^1 \in V^1 \setminus W$. If v^1 is within a link of Type 3 or Type 4, then $v^2 \in W$ and $N^{W^2}(v^2) \neq \emptyset$.*

Remark 6.17. *If x^i, y^j are locally encircled vertices, $i \neq j$, $i, j \in [1, k]$, then either $x = y$ or $i \in \{1, k\}$ and $j \in [2, k-1]$.*

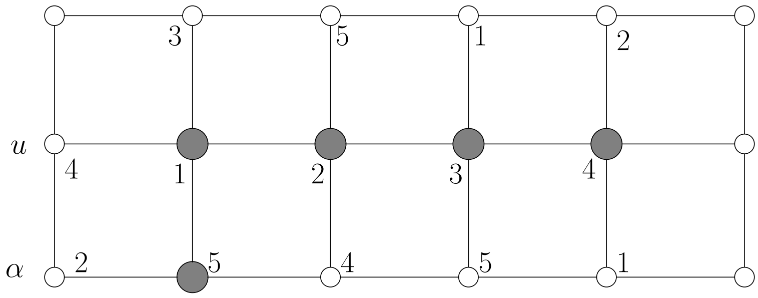


Figure 6.7: Precolouring of H, where $m(H) = 5$ and $k = 6$. The grey vertices represent W'.

Proof of Theorem 6.10: By Lemmas 6.11, 6.13, 6.14 and 6.15, we can suppose that H has a good set W that satisfies Assumptions A1-A4, $m(H) \geq 6$ and $|W^2| \geq 2$. Let Ψ be obtained as explained in Section 6.1 (thus, Ψ sastifies Lemma 6.2). Observe that, by Assumptions A1-A4, we can suppose that all cross link vertices still uncoloured are in links of Type 1, Type 2, Type 3 and Type 4.

We first colour the locally encircled vertices, starting by the ones in $V^2, \cdots, V^{k-1}$. Denote the set of encircled vertices by L_e and the set $L_e \setminus (V^1 \cup V^k)$ by L_e^*. By Lemma 6.1, we know that each T^i contains at most one encircled vertex and, by Remark 6.17, we know that all the vertices in L_e^* are copies of the same vertex of T. So, let $x \in V(T)$ be such that $L_e^* \subseteq \{x^2, \cdots, x^{k-1}\}$. Actually, observe that $L_e^* = \{x^p, \cdots, x^q\}$, where $p \in \{2, i_\alpha + 1\}$ and $q \in \{i_\alpha, k-1\}$. If there exists $w^p \in W^p \setminus (N(x^p) \cup \{\alpha^p\})$, then, for all $j \in \{p, \cdots, q\}$, colour x^j either with $\Psi(w^{j+1})$, if $j < k-1$, or with $\Psi(w^2)$, otherwise. Now, consider $W^p \setminus \{\alpha^p\} \subseteq N(x^p)$. Note that, as $|N^{W^p}(x^p)| \geq 2$, there exists $u^p \in W^p \setminus \{\alpha^p\}$ such that $N^{W^p}(u^p) = \emptyset$. By A2, we have $\{u^2, \cdots, u^{k-1}\} \subseteq W$. If $k \geq 6$, then colour x^j either with $\Psi(u^{j+2})$, if $j \leq k-3$, or with $\Psi(u^{j-2})$, otherwise, for all $j \in \{p, \cdots, q\}$. Now, consider $k = 5$; colour x^2 with $\Psi(u^4)$, if $x^2 \in L_e^*$, and x^4 with $\Psi(u^2)$, if $x^4 \in L_e^*$. Suppose that $x^3 \in L_e^*$ (otherwise, we are done). If $u^1 \in W$, colour x^3 with $\Psi(u^1)$; otherwise, if $u^k \in W$, colour x^3 with $\Psi(u^k)$; otherwise, if $d(u^3) > m(H) - 1$, colour x^3 with $\Psi(u^2)$. Now, consider $u^1, u^k \notin W$ and $d(u^3) = m(H) - 1$. As x^3 is not encircled by W, then either α exists and is not adjacent to x in T or there exists $w^3 \in W$ such that $d(w^3) > m(H) - 1$. If the former occurs, colour x^3 with $\Psi(\alpha^2)$; if the latter occurs, colour x^2 with $\Psi(w^2)$. Finally, we colour encircled vertices in $V^1 \cup V^k$. If $y^1 \in V^1 \setminus W^1$ is locally encircled, as

$|N^{W^1}(y^1)| \geq 2$ and by A4, there must exist $u^1 \in N^{W^1}(y^1)$ such that $u^k \in W$. Then, colour y^1 with $\Psi(u^3)$ and, if y^k is also locally encircled, colour y^k with $\Psi(u^{k-2})$. Observe that the following property holds:

(1) If $x^i \in L_e$ is coloured with $\Psi(w^j)$, for some $w^j \in W$, then $j \in \{i-1, i-2, i+1, i+2\}$. Furthermore, $w \neq \alpha$ and if $i \notin \{1,k\}$ and $w^i \in N(x^i)$, then $W^i \setminus \{\alpha^i\} \subseteq N(x^i)$ and $\alpha^i \notin N(w^i)$.

Now, we colour the cross link vertices that lie within cross links of Type 3 or Type 4, starting by the ones in V^1 and V^k different from α^1 (note that, as $\alpha^{k-1} \notin W$, α^k is not within such a link). Let $X_1^* = \{x^1 \in V^1 \setminus (W^1 \cup \{\alpha^1\}) : x^2 \in W^2 \textit{ and } N^{W^2}(x^2) \neq \emptyset\}$. Define X_k^* analogously. Let $x^k \in X_k^*$ and note that $x^1 \in X_1^* \cup W$. By Remark 6.16, we know that $N^{W^{k-1}}(x^{k-1}) \neq \emptyset$; thus, by (1), we know that $\Psi(x^2) \notin \Psi(N(x^{k-1}))$ and $\Psi(x^{k-1}) \notin \Psi(N(x^2))$. By (1) and Lemma 6.2, we also know that $\Psi(V^1) \cap \Psi(W^{k-1}) = \emptyset$ and $\Psi(V^k) \cap \Psi(W^2) = \emptyset$. So, give colour $\Psi(x^2)$ to x^k and, if $x^1 \notin W$, give colour $\Psi(x^{k-1})$ to x^1. Now, let $x^1 \in X_1^*$ such that $x^k \notin X_k^*$. As $\alpha^1 \notin X_1^*$, we know that $N^{W^2}(x^2) = \{\alpha^2\}$. Thus, by (1) and Lemma 6.2, we know that $\Psi(x^{k-1}) \notin \Psi(N(x^2))$; we can then colour x^1 with $\Psi(x^{k-1})$ as before. The following trivially holds.

(2) $\Psi(x^1) = \Psi(x^{k-1})$, for all $x^1 \in X_1^*$, and $\Psi(x^k) = \Psi(x^2)$, for all $x^k \in X_k^*$.

Now, consider that α^1 lies within a link of Type 3 or 4 and is still uncoloured. First, consider any $y^1 \in N(\alpha^1) \setminus W$. If $N^{W^1}(y^1) \neq \emptyset$, then either $N^{W^1}(\alpha^1) \neq \emptyset$ and α^1 is an internal link vertex, or $N^{W^1}(\alpha^1) = \emptyset$ and α^1 lies within a link of Type 2. We get a contradiction as in both cases α^1 should have already been coloured. Thus, (i) $N^{W^1}(y^1) = \emptyset$ and, consequently, if y^1 is coloured, then $y^1 \in X_1^*$, for all $y^1 \in N(\alpha^1) \setminus W$. By Remark 6.16, there exists $v^2 \in N^{W^2}(\alpha^2)$. First, consider $v^1 \notin W$. By (i), $N^{W^1}(v^1) = \emptyset$ and $\Psi(v^1) = \Psi(v^{k-1})$. Also, by (i), we know that $\Psi(v^3) \notin \Psi(N(\alpha^1))$. So, if $\Psi(v^3) \notin \Psi(N(\alpha^2))$, then we colour α^1 with $\Psi(v^3)$. So, suppose that $y^2 \in N(\alpha^2)$ is coloured with $\Psi(v^3)$. By Lemma 6.2, we know that y^2 is locally encircled. We then colour α^1 with $\Psi(u^2)$, for any $u^2 \in N^{W^2}(y^2) \setminus \{\alpha^2\}$. As y^2 is the only coloured neighbour of α^2 not in W and by (i), we know that $\Psi(u^2) \notin \Psi(N(\alpha^1) \cup N(\alpha^2))$. Now, consider $v^1 \in W$. We know that a colour not in $\Psi(W^1)$ appears in $N(v^1) \setminus \{v^2\}$ only if there exists $y^1 \in N^{T^1}(v^1)$ such that y^1 is locally encircled or $y^2 \in W$. If the former occurs, by (1), $\Psi(y^1) \notin \Psi(W^4)$, and if the latter occurs, by (2), $\Psi(y^1) = \Psi(y^{k-1})$. Also, by (i), we know that $\Psi(v^4) \notin \Psi(N(\alpha^1))$ and one can verify that if α^2 has a locally encircled neighbour y^2, by the existence of $v^2 \in W^2 \setminus (N(y^2) \cup \{\alpha^2\})$, we have $\Psi(y^2) \in \Psi(W^3)$. Thus, we can colour α^1 with $\Psi(v^4)$.

Now, let $i = i_\alpha$ and consider that α^{i+1} is also within a link of Type 3 and is still uncoloured. Note that, as α^{i+1} is not an internal link vertex and by the uniqueness of α, we have: (ii) $N^{W^i}(y^i) = \{\alpha^i\}$, for all $y^i \in N(\alpha^i)\setminus W$. So, α^i does not have encircled neighbours in T^i and, consequently, $\Psi(N(\alpha^i)) \subseteq \Psi(W^i \cup \{\alpha^{i-1}\})$. Also, the coloured neighbours of α^{i+1} are α^i and v^{i+1}; hence, we just need to colour α^{i+1} with a colour $c \in M(\alpha^i) \cap M(v^{i+1})$. Suppose, first, that α^i has a coloured neighbour $x^i \in V^i \setminus W^i$. By (ii), there exists a link of length three $\langle \alpha^i, x^i, y^i, w^i \rangle$. Note that there is no locally encircled vertex in V^{i+1}. Thus, $F = \Psi(N(\alpha^i) \cup N(v^{i+1})) \subseteq \Psi(W^i \cup W^{i+1} \cup \{\alpha^{i-1}, v^{i+2}\}) \setminus \{\alpha^i, v^{i+1}\})$, i.e., $|F| \leq |W^i| + |W^{i+1}|$. So, if $|W^{i+1}| \geq 3$, as $|W| \geq 2|W^{i+1}| + |W^i|$, we have at least 3 colour in $\{1, \cdots, m(G)\} \setminus F$. Then, as least one such colour is different from $\Psi(\alpha^i)$ and $\Psi(v^{i+1})$, we can properly colour α^{i+1}. Now, suppose that $W^{i+1} = \{w^{i+1}, v^{i+1}\}$. In this case, v^{i+1} has no coloured link neighbour and at least one colour $c \in \Psi(\{w^1, \cdots, w^k\} \cap W)$ does not appear in $N(\alpha^i)$. We can then colour α^{i+1} with c. Now, consider that α^i has no coloured neighbour in $V^i \setminus W^i$; thus, by (ii), α^i has exactly two coloured neighbours, namely α^{i-1} and v^i. If $W^{i+1} = \{v^{i+1}\}$, then v^{i+1} has exactly two coloured neighbours, namely v^i, v^{i+2}, and, as $m(H) \geq 6$, there exists a colour $c \notin \Psi(\{\alpha^{i-1}, \alpha^i, v^i, v^{i+1}, v^{i+2}\})$ with which we can colour α^{i+1}. Otherwise, let $u^{i+1} \in W^{i+1} \setminus \{v^{i+1}\}$ and denote $\Psi(\{u^2, \cdots, u^{k-1}\})$ by C. Note that if $\Psi(\alpha^{i-1}) \in C$, then $i = 2$, and if $\Psi(v^{i+2}) \in C$, then $i + 2 = k$ and, as $k \geq 5$, $i \geq 3$; consequently, at most one between α^{i-1} and v^{i+2} can be coloured with a colour from C. So, as v^{i+1} has at most one vertex in V^{i+1} coloured with some colour in C and $|C| \geq 3$, we have that $C \setminus \Psi(N(v^{i+1}) \cup \{\alpha^{i-2}\}) \neq \emptyset$ and we can colour α^{i+1} as desired.

Now, we colour the vertices in links of Type 2 that are not coloured by the application of Lemma 6.2, i.e., y^j within a link of Type 2 such that $N^{W^j}(y^j) \neq \emptyset$. Consider the link $\langle x^i, x^j, y^j, w^j \rangle$. Note that if $j \in \{1, k\}$, then $y^i \notin W$, otherwise y^j would be within a link of Type 3 and would have been coloured previously; otherwise (i.e. if $j \in \{2, \cdots, k-1\}$), by A1 and A2, we have $x = \alpha$, $i = i_\alpha$, $j = i + 1$ and, again, $y^i \notin W$. Hence, $N^W(y^j) = \{w^j\}$. We prove that $\Psi(x^i) \notin \Psi(N(w^j) \cup N(y^j))$; thus, if we colour y^j with $\Psi(x^i)$, the obtained precolouring is proper and still unsaturated. Trivially, $\Psi(y^i) \neq \Psi(x^i)$ as Ψ is a proper precolouring. Also, note that if $z^j \in N(y^j) \setminus W$, then, as $y^j \notin L_I$, we have $N^{W^j}(z^j) = \emptyset$. Consequently, if z^j is coloured, then $z^i \in W$, $\langle z^i, z^j, y^j, w^j \rangle$ is a link of Type 2 and, by Lemma 6.2, $\Psi(z^j) \in \Psi(W^j)$. Now, consider $z^l \in N(v^j) \setminus W$ coloured with a colour not in $\Psi(W^j)$. We know that either $l = j$ and z^l is locally encircled or $j = i + 1 = i_\alpha + 1$, $l = j + 1 = k$, $z = y$ and z^k is either locally encircled or z^k is within a link of Type 3 or 4. One can verify in the procedure that colours encircled vertices that if u^1 is a locally encircled vertex coloured with c, then $\psi_c \in N^{W^3}(u^3)$ and if u^k is locally encircled coloured with c', then $\psi_{c'} \in N^{W^{k-2}}(u^k)$. Thus, if $l \in \{1, k\}$ and z^l is locally encircled, we know that $\Psi(z^l) \neq \Psi(\alpha^i)$. Also, if z^j is locally encircled and $j = i_\alpha + 1$, then $x = \alpha$ and, by (1), $\Psi(z^j) \neq \Psi(\alpha^i)$. So, consider the case $j = i + 1 = i_\alpha + 1$, $l = j + 1 = k$ and y^k is within a link of Type 3 or 4. By (2), we know that $\Psi(y^k) = \Psi(y^2)$.

Finally, we colour links of Type 1. Denote by L_1 the set of uncoloured link vertices. We want to prove that we can apply Lemma 6.2 to colour $N^{L_1}(u^i)$, for all $u^i \in W$, starting from $V^1 \cap L_1$ up to $V^k \cap L_1$. Let $v^i \in W$ be such that $N^{L_1}(v^i) \neq \emptyset$. Observe that it suffices to prove that $|N^{L_1}(v^i)| \geq 2$. As L_1 contains only vertices that lie in links of Type 1, we have $\{v^{i-1}, v^{i+1}\} \cap W \neq \emptyset$. Thus, by A1, A2 and A4, we know that $v^2, v^3 \in W$. Recall that $R = V(H) \setminus (W \cup L_I)$ and note that if $x^2 \in N^{R^2}(v^2)$, then $z^i \notin W$, for all $z^2 \in N^{T^2}[x^2] \setminus \{v^2\}$, for all $i \in \{1, \cdots, k\}$, and, consequently, $x^2 \in N^{L_1}(v^2)$, i.e., $N^{L_1}(v^2) = N^{R^2}(v^2)$. Also, note that if $x^2 \in N^{L_1}(v^2)$, then, as $x^i \notin W$, for all $i \in \{1, \cdots, k\}$, we have $x^i \in N^{L_1}(v^i)$, for all $v^i \in W$. So, $|N^{L_1}(v^i)| \geq |N^{L_1}(v^2)|$ and it suffices to prove that $|N^{L_1}(v^2)| \geq 2$. It is easy to see that, as $m(H) \geq 6$, then $|W \setminus W^2| \geq 4$. Thus, by Equation 6.7, we have $|N^{R^2}(v^2)| \geq |W \setminus W^2| - 2 \geq 2$. □

6.4 Trees and stars

Let T be a tree and consider a star $K_{1,p} = (a_0, a_1, \cdots, a_p)$, where a_0 is the center of the star, $p \geq 2$. Let $H = T \,\square\, K_{1,p}$. We want to prove the following:

Theorem 6.18. *Let $H = T \,\square\, K_{1,p}$, $p \geq 2$. If H has a good set, then $\chi_b(H) = m(H)$; otherwise, $\chi_b(H) = m(H) - 1$.*

We also give an algorithm that finds an optimal b-colouring of H. First, we analyse the case where $m(H) = 3$. Observe that $m(H) \geq 3$, unless $T = \{v\}$, in which case $H = K_{1,p}$ and the theorem follows from the result from Irving and Manlove for trees [20]. We prove that T is an edge. Suppose otherwise; thus, it must have a vertex v with degree at least 2. Let $u \in N(v)$ and note that $d(v^i) \geq 3$, $i = 0, \cdots, p$, and $d(u^0) \geq 3$, i.e., there are at least 4 vertices with degree at least 3, namely v^0, v^1, v^2, u^0, and $m(H) \geq 4$, a contradiction. So, $T = K_2 = \{v_1, v_2\}$. Suppose that Ψ is a b-colouring of H with $m(H) = 3$ colours. Without loss of generality, suppose that $\Psi(v_1^0) = 1$

and $\Psi(v_2^0) = 2$. Observe that, in this case, if v_1^i is coloured with 3, for some $i \in [1,p]$, as $\{2,3\} \in \Psi(N(v_2^i))$, then v_2^i must be coloured with 1; thus, v_1^i is not a b-vertex of colour 3. Analogously, there is no v_2^i that is a b-vertex of colour 3, a contradiction. So, $\chi_b(H) < 3$ and Ψ where $\Psi(v_1^0) = \Psi(v_2^i) = 1$ and $\Psi(v_2^0) = \Psi(v_1^i) = 2$, $i = 1, \cdots, p$, is an optimal b-colouring of H.

From now on we suppose that $m(H) \geq 4$. We know that

$$d^H(u^0) = d^T(u) + p \tag{6.19}$$

$$d^H(u^i) = d^T(u) + 1,\ i = 1, \cdots, p \tag{6.20}$$

Also, as $d^H(u^0) = d^H(u^i) + p - 1$, for $i \in \{1, \cdots, p\}$, the following is valid:

$$if\ d^H(u^i) \geq m(H) - 1, i \geq 1,\ then\ d^H(u^0) \geq m(H) + p - 2 \geq m(H) \tag{6.21}$$

Trivially, as $p \geq 2$, H has at least $3m(T) \geq m(T)+1$ vertices with degree at least $m(T)$; thus, $m(H) \geq m(T) + 1$. Also, note that if $m(H) > m(T) + p$ and $u^i \in D(H)$, $i \in \{0, \cdots, p\}$, then: $d^T(u) + p \geq d^H(u^i) \geq m(H) - 1 \geq m(T) + p \Rightarrow d^T(u) \geq m(T)$. So, as T has at most $m(T)$ vertices with degree greater than $m(T) - 1$, we have that $m(H) \leq (p+1)m(T)$. Thus:

$$m(T) + 1 \leq m(H) \leq (p+1)m(T)$$

Now, we present some lemmas that will help us construct a convenient good set.

Lemma 6.22. *Let $W \subseteq D(H)$ be of cardinality $m(H)$ containing all vertices with degree greater than $m(H) - 1$. If $W \not\subseteq V^0$ and W encircles $u^i \in V^i \setminus W$, then $i \neq 0$ and $W \subseteq N^{V^0}(u^0) \cup N(u^i)$.*

Proof: Suppose that $W \not\subseteq V^0$ and that W encircles $u^0 \in V^0 \setminus W^0$. Let v^j be any vertex in $W \setminus V^0$. By Equation 6.21, we have that $u^j \notin W$ (otherwise, $d(u^0) \geq m(H)$ and u^0 should be in W) and $d(v^0) \geq m(H)$; thus, v^j is not reached by u^0, a contradiction. Now, consider that W encircles u^i, $i \in [1,p]$. Suppose that there exists $v^j \in W \setminus (V^0 \cup V^i)$. As $N(u^i) \subseteq V^i \cup \{u^0\}$, we must have $v = u$ and $u^0 \in W$, otherwise, u^j is not reached by u^i. But then $d(u^0) > m(H) - 1$ and $v^j = u^j$ is not reached by u^i, a contradiction. Finally, it is easy to see that if there exists $v^i \in W^i \setminus N(u^i)$ then $v^0 \in W$ is not reached by u^i, i.e., $W \subseteq N(u^i) \cup N^{V^0}(u^0)$. □

Lemma 6.23. *H does not have a good set if and only if $|D(H)| = m(H)$, $D(H) \subseteq V^0$ and $D(H)$ encircles a vertex.*

Proof: Let $W \subseteq D(H)$ of cardinality $m(H)$ containing all vertices with degree at least $m(H)$. Trivially, if W does not encircle any vertex, then we are done. So, suppose that W encircles $u^i \in V^i \setminus W$, $i \in [0,p]$. First, suppose that $W \subseteq V^0$. By Proposition 2.1, we have $u^i \in V^0$. If $D(H) \subseteq V^0$, the result follows from the proof of Irving and Manlove in [20]; so, suppose otherwise and let $v^j \in D(H) \cap V^j$, $j \in [1,p]$. By Equation 6.21, we know that $d(v^0) \geq m(H)$ and, consequently, $v^0 \in W$. Also, by Proposition 2.1, we know that as W encircles u^0, there must exist some vertex in W non-adjacent to u^0 and, hence, adjacent to some $w^0 \in N^W(u^0)$ with degree $m(H)-1$. So, $(W \setminus \{w^0\}) \cup \{v^j\}$ also contains every vertex with degree at least $m(H)$ and is not contained in V^0, i.e., we can suppose that $W \not\subseteq V^0$. Thus, by Lemma 6.22, we know that $u^i \notin V^0$ and $W \subseteq N^{V^0}(u^0) \cup N(u^i)$. Let $w^i \in W^i$; as $\{w^1, \cdots, w^p\} \setminus W \neq \emptyset$, $d(w^j) = d(w^l)$, for all $j, l \in [1,p]$, and W contains all vertices with degree at least $m(H)$, we know that $d(w^i) = m(H)-1$. If there exists $v^i \in W^i \setminus \{w^i\}$, let $W' = (W \setminus \{w^i\}) \cup \{v^j\}$, for any $j \neq 0, i$. By Lemma 6.22, we know that W' does not encircle any vertex and, as W' still contains all vertices with degree at least $m(H)$, W' is a good set. Otherwise (i.e., $W^i = \{w^i\}$), by Proposition 2.1 and the fact that $N(u^i) = N^{V^i}(u^i) \cup \{u^0\}$, we have that $u^0 \in W$. Also, as $m(H) \geq 4$, there must exist $y^0 \in W \setminus \{u^0, w^0, w^i\}$ and, since the only way y^0 is reached by u^i is through u^0, we have that $d(u^0) = m(H) - 1$ and $(W \setminus \{u^0\}) \cup \{w^j\}$ is a good set, for any $j \neq 0, i$. □

The following lemma describes the structure of the chosen good set:

Lemma 6.24. *If H is not pivoted, then there exists a good set $W \subseteq D(H)$ such that:*

G1. Either W contains all vertices with degree at least $m(H)$ or $D(H) \subseteq V^0$;

G2. $u^1, \cdots, u^p \in W$, for all $u^i \in W$, $i > 0$, except for at most one vertex, α, for which there exists an index $0 < i_\alpha < p$ such that $\{\alpha^1, \cdots, \alpha^{i_\alpha}\} \subseteq W$ and $\{\alpha^{i_\alpha+1}, \cdots, \alpha^p\} \cap W = \emptyset$; and

G3. If $i_\alpha = 1$, then $d(u^0) \geq m(H)$, for every $u^0 \in W^0$.

Proof: Let W be the a subset of $D(H)$ with cardinality $m(H)$ containing all vertices with degree greater than $m(H)-1$. If W is not a good set, observe that in the construction of a good set in Lemma 6.23, either $D(H) \subseteq V^0$ or we can suppose that $W \not\subseteq V^0$ and then construct a good set from W by removing a vertex with degree $m(H)-1$ and adding some $v^j \in D(H) \setminus V^0$.

Now, suppose that W is a good set. Trivially, as the vertices $v^1, \cdots, v^p$ have same degree, we can suppose that for each vertex $v^i \in W$, we have that $\{v^0, \cdots, v^{i-1}\} \subseteq W$. So, while there exist two distinct vertices $u, v \in V(T)$ such that $\{v^1, \cdots, v^i\} \subseteq W$ and $\{u^1, \cdots, u^j\} \subseteq W$, $i, j < p$, let $q = min\{i, p-j\}$; we remove q vertices from $v^i, \cdots, v^1$ and add from $u^{j+1}, \cdots, u^p$.

Finally, let W be a good set that satisfies G1 and G2 and suppose that $i_\alpha = 1$ and that there exists $u^0 \in D(H)$ such that $d(u^0) = m(H)-1$. By Equation 6.21, we know that $u \neq \alpha$ and $u^i \notin W$, for $i = 1, \cdots, p$. Thus, by Lemma 6.22, we know that $(W \setminus \{u^0\}) \cup \{\alpha^2\}$ is a good set satisfying G1, G2 and G3. □

Now, we proceed to the colouring of H. We first colour non-pivoted graphs. So, consider a good set W that satisfies G1, G2 and G3. We will construct an unsaturated precolouring with candidate set W where all link vertices of W are coloured; after this, by Lemma 2.15, we know we can extend this precolouring to a b-colouring of H with $m(H)$ colours. So, we partition L as follows:

- L_E - contains all the locally encircled vertices;
- $L_0 = L_I \setminus L_E$;
- $L_1 = \{x^i \in L_C : x^0 \notin W \ and \ \exists y^i \in N^{L_C}(x^i) \ s.t. \ y^0 \in W\}$;
- $L_2 = \{x^i \in L_C : x^0 \in W \ and \ N^{W^i}(x^i) \neq \emptyset\}$;
- $L_3 = \{x^i \in L_C \setminus L_2 : x^0 \in W\}$;
- $L_4 = L_C \setminus (L_1 \cup L_2 \cup L_3)$.

Figure 6.8 illustrates the structure of the links that define the partition above. Note that, by the uniqueness of α, there are no links of Type 5 or Type 6. Also, observe that L_1 contains vertices in links of Type 2, L_2 of Type 3, L_3 of Type 2 and Type 4 and L_4 of Type 1. Finally, observe that $L_i \cap V^0 = \emptyset$, for $i = 1, 2, 3$, and that if $\alpha^i \in L_C$, then $\alpha^i \in L_2 \cup L_3$ (since $\alpha^0 \in W$). The following remark trivially holds:

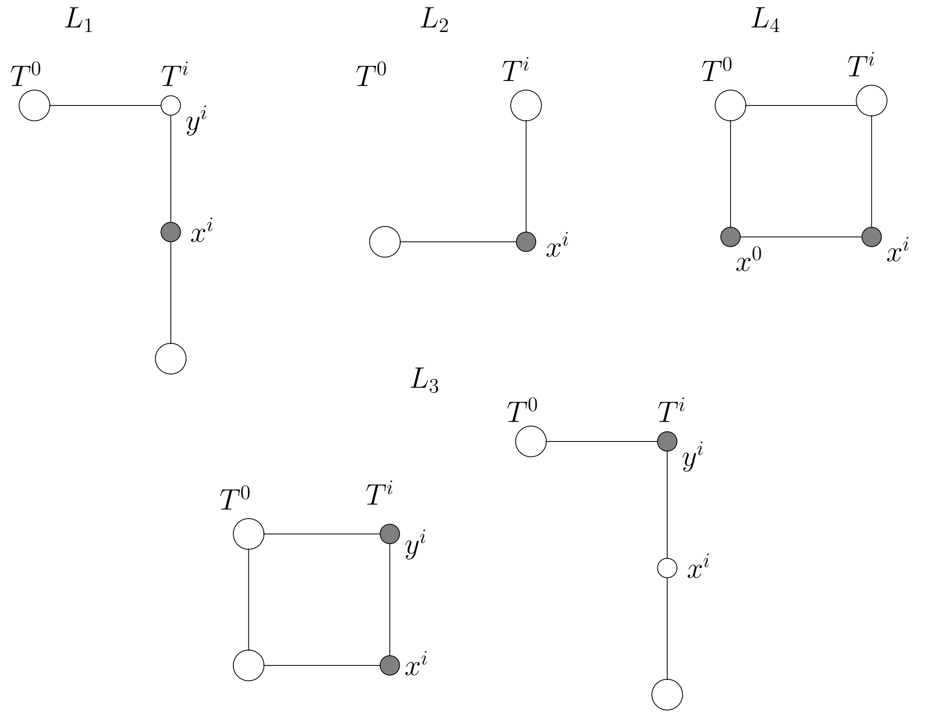


Figure 6.8: Structure of the sets L_1, L_2, L_3, L_4. The bigger vertices are in W and the grey vertices are within the corresponding subset.

Remark 6.25. *If $x^i \in L_I$ is locally encircled and x^{i+1} is not, $0 < i < p$, then $i = i_\alpha$, $\alpha^i \in N(x^i)$ and either $|N^{W^i}(x^i)| = 2$ or $N^{W^i}(\alpha^i) \neq \emptyset$. Conversely, if x^i is not locally encircled and x^{i+1} is, $0 < i < p$, then $i = i_\alpha$ and either $x = \alpha$ or $W^i \setminus (N(x^i) \cup N(N^W(x^i)) = \{\alpha^i\}$.*

By the above remark, if $x^i, \cdots, x^j$ are all the copies of x different from x^0 that are locally encircled, then either $i = 1, j = p$, or $i = 1, j = i_\alpha$, or $i = i_\alpha + 1, j = p$. The following lemma will be used to colour links of Type 4.

Lemma 6.26. *Suppose that Ψ is an unsaturated precolouring with candidate set W, $m(H) - 1 \leq |W| \leq m(H)$. Let J^0 be the set of vertices of a connected component of $T^0[W^0]$ and $J^{i_1}, \cdots, J^{i_r}$ be all the copies of J^0 such that $J^{i_f} \cap W = \emptyset$ and $N^L(J^{i_f}) = \emptyset$, for $f = 1, \cdots, r$. If $r \geq 2$, then, we can obtain an unsaturated precolouring of W from Ψ where J^{i_f} is coloured, for all $f \in \{1, \cdots, r\}$.*

Proof: First, note that, as $u^1, \cdots, u^{i-1} \in W$, for all $u^i \in W$, we have that $j \in \{i_1, \cdots, i_r\}$, for all $j \in [i_1, p]$, and, by the uniqueness of α, either $i_1 = 1$ or $i_1 = i_\alpha + 1$. So, denote by i the index i_1. Choose any root x^0 for J^0 and denote by q the value $|M(x^0)|$. Give the colours in $M(x^0)$ to the vertices $x^i, \cdots, x^{i+q-1}$ and let $x^{i+q}, \cdots, x^p$ remain uncoloured. Then, for each child t^0 of x^0, let C represent the set $\Psi(\{x^i, \cdots, x^p\}) \cap M(t^0)$ and let $\Psi^{-1}(C)$ be the set $\{x^k : \Psi(x^k) \in C\}$. If $|C| > 1$, do a permutation of these colours on the set $\{t^k : x^k \in \Psi^{-1}(C)\}$ in such a way that $\Psi(t^k) \neq \Psi(x^k)$. Otherwise, give the colour $c \in C$ to any t^k such that $\Psi(x^k) \neq c$ (remember that every coloured vertex in $\{x^i, \cdots, x^p\}$ has a different colour and that $r \geq 2$). After this, use the remaining colours of $M(t^0) \setminus C$ on the uncoloured vertices in $\{t^i, \cdots, t^p\}$, again leaving the "surplus" neighbours uncoloured. Continue this procedure from the children of x^0 in J^0 down to the leaves of J^0.

Now, suppose that $y^k \in J^k$ is still uncoloured, for some $k \in [i, p]$. If W is a good set, by G1, we know that $d(y^k) < m(H)$; hence, there exists some colour in $\{1, \cdots, m(H)\} \setminus \Psi(N(y^k))$ with which we can colour y^k. Otherwise, by Lemma 6.23, we know that $y^k \notin D(H)$; hence, $d(y^k) < m(H) - 1$ and there exists a colour in $\{1, \cdots, m(H) - 1\} \setminus \Psi(N(y^k))$ with which we can colour y^k. □

We first prove the following part of Theorem 6.18.

Theorem 6.27. *Let $H = T \,\square\, K_{1,p}$, $p \geq 2$. If H does not have a good set, then $\chi_b(H) = m(H) - 1$.*

Proof: By Lemma 2.4, we know that $\chi_b(H) < m(H)$. We want to construct a b-colouring of H with $m(H) - 1$ colours. By Lemma 6.23, we know that $|D(H)| = m(H)$, $D(H) \subseteq V^0$ and $D(H)$ encircles some vertex. Let x^0 be encircled by $D(H)$ and let $u^0 \in W \setminus N(x^0)$ (u^0 exists, by Proposition 2.1). Also, let $v^0 \in N^W(u^0) \cap N^W(x^0)$; we know that $d^H(v^0) = m(H) - 1$. Give colours $\{1, \cdots, m(H)-1\}$ to $W \setminus \{v^0\}$, colour $\Psi(u^0)$ to x^0 and $\Psi(w^0)$ to v^0, for any $w^0 \in N^W(x^0) \setminus \{v^0\}$ (by Proposition 2.1, w^0 exists). Note that the remaining uncoloured link vertices are in links of Type 4. By Lemma 6.26, it is possible to extend Ψ to colour these links. As v^0 is the only vertex in $V(H) \setminus W$ with degree at least $m(H) - 1$ and v^0 is also a link vertex of W, we can apply Lemma 2.15 to extend this precolouring to a b-colouring of H with $m(H) - 1$ colours. □

Now, consider H to be non-pivoted. The following lemma colours H when $W \subseteq W^0 \cup \{x^1, \cdots, x^p\}$, for some $x \in V(T)$.

Lemma 6.28. *Let W be a good set of H satisfying Lemma 6.24. If $W \subseteq W^0 \cup \{x^1, \cdots, x^p\}$, then $\chi_b(H) = m(H)$.*

Proof: Colour each vertex of W with a different colour and, then, colour the local internal vertices of W using the Tree Strategy (Section 2.5). If there exists a locally encircled vertex $y^0 \in V^0$ and $W \subseteq V^0$, as y^0 is not encircled by W, there must exist $v^0 \in d(w^0)$, for some $w^0 \in N(y^0)$ with degree at least $m(H)$; thus colour y^0 with $\Psi(v^0)$. Otherwise ($W \neq W^0$), colour y^0 with $\Psi(x^1)$. After this, if $W = W^0$, colour the remaining link vertices using Lemma 6.26. Otherwise, let $i = p$, if $x^p \in W$, or $i = i_\alpha$, otherwise. If there exists $y^j \in N(x^j)$, $j \in [1, i]$, such that $y^0 \notin W$ and $N^{W^0}(y^0) \nsubseteq \{x^0\}$, then, colour y^j with $\Psi(w^0)$, for any $w^0 \in N^{W^0}(y^0) \setminus \{x^0\}$, and v^j with $\Psi(x^j)$, for every $v^0 \in N^{W^0}(y^0) \setminus \{x^0\}$. Now, let J^0 be a connected component of $T^0[W^0]$. Note that if v^j is coloured, for some $v^0 \in W$ and some $j \in \{1, \cdots, p\}$, then $j \leq i$ and either $v^j = x^j$ or v^0 is separated from x^0 in T^0 by some $y^0 \in V^0 \setminus W$. Thus, if v^j and u^j are coloured, for some $v^0, u^0 \in W$, $u \neq v$, and some $j \in \{1, \cdots, p\}$, then v^0 and u^0 are in different connected components of $T^0[W^0]$, i.e., there is at most one vertex $y^0 \in V(J^0)$ such that y^i is coloured, for some $i > 0$. Thus, as $p \geq 2$, one can verify that we can apply the same argument of the proof of Lemma 6.26 to colour $\{z^1 : z^0 \in J^0\}, \cdots, \{z^p : z^0 \in J^0\}$ by choosing y^0 as the root of J^0. □

From now on, we consider that W is a good set of H satisfying G1, G2 and G3 and is such that $|W^1| \geq 2$ (hence, $m(H) \geq 5$). First, we present some lemmas, each of which is used to colour a subset of L.

Lemma 6.29. *Suppose that W is a good set of H satisfying G1, G2 and G3 and is such that $|W^1| \geq 2$. Then, there exists an unsaturated precolouring with candidate set W, Ψ, that colours $L_I \cup L_1$ such that the following hold:*
(c1) If $x^i \in L_E \cup L_1$ is coloured with c, $i \in [1,p]$, then there exists $v^i \in N(x^i)$ such that $(\gamma)_c = v^j$, for some $j \in \{0, \cdots, p\} \setminus \{i\}$. Furthermore, if $x^i \in L_1$, then $j = 0$ and $v^i \in L_3$; and
(r1) $r(v^i) = 0$, for all $v^i \in W$, except for at most one vertex, denoted by β^0, in which case we have: $r(\beta^0) = 1$ and there exists $y^0 \in N^{L_E}(\beta^0)$ such that $\Psi(y^0) = \Psi(\beta^j)$, for some $\beta^j \in W$, $j \neq 0$, and $W \subseteq N(y^0) \cup N(N^W(y^0))$.

Proof: We first colour each vertex of W with a different colour and then colour the internal link vertices of T^i that are not locally encircled using the Tree Strategy (Section 2.5), $i = 0, \cdots, p$. Observe that, by the definition of locally encircled vertex and Remark 2.17, we have $r(v^i) = 0$, for all $v^i \in W$. We colour L_e and L_1, in this order.

Let $x \in V(T)$ be such that $x^i \in L_e$, for some $i \in \{0, \cdots, p\}$. First, suppose that x^0 is locally encircled. If there exists $w^i \in W \setminus (N(x^0) \cup N(N^W(x^0))$, as x^0 is locally encircled, we know that $i \neq 0$; so, we colour x^0 with $\Psi(w^i)$. Otherwise, let w^i be any vertex in $W \setminus W^0$ (remember that $W \nsubseteq V^0$) and colour x^0 with $\Psi(w^i)$. Observe that (r1) holds and if $w \in N(x)$, then $d^H(w^0) > m(H) - 1$ and the repetition of $\Psi(w^i)$ in $N(w^0)$ is allowed. Now, suppose that x^i is locally encircled, for some $i \neq 0$. By the uniqueness of α and the fact that $|N^{W^i}(x^i)| \geq 2$, we know that there exists $v \in N(x)$ such that $\{v^0, \cdots, v^p\} \subseteq W$ and we can suppose that $\Psi(x^0) \notin \Psi(\{v^1, \cdots, v^p\})$. Let $L_e \cap \{x^1, \cdots, x^p\} = \{x^l, \cdots, x^q\}$. If $l - q > 0$, colour x^j with $\Psi(v^{j+1})$, for $j \in \{l, \cdots, q-1\}$, and x^q with $\Psi(v^l)$; otherwise, colour x^l with $\Psi(v^{(l+1) \bmod p})$. Note that (c1) holds.

Now, consider $x^i \in L_1$. As x^i is not an internal vertex and $x^0 \notin W$, we know that $N^W(x^i) = \{v^i\}$. Also, by the construction of L_1, there exists $y^i \in N^{L_C}(x^i)$ such that $y^0 \in W$. We colour x^i with $\Psi(y^0)$. Note that, as $x^i \notin L_I$, we have that $N^{W^i}(y^i) = \emptyset$ and, consequently, $y^i \in L_3$; thus, (c1) holds. Trivially, $\Psi(x^0) \neq \Psi(y^0)$, as $x^0 \in N(y^0)$, and also no two neighbours of y^i are adjacent, i.e., the precolouring is proper. In addition, if v^i has some neighbour z^i coloured with some colour $c \notin \Psi(W^i)$, then $z^i \in L_e \cup L_1$ and, by (c1), we have that $c \neq \Psi(y^0)$. Thus, $r(v^i)$ does not increase during this procedure, for all $v^i \in W$, and the obtained precolouring is unsaturated. □

Lemma 6.30. *Suppose that W is a good set of H satisfying G1, G2 and G3 and is such that $|W^1| \geq 2$. Let Ψ be an unsaturated precolouring satisfying Lemma 6.29. Then, there exists an unsaturated extension of Ψ that colours L_2. Also, the following hold:*
(r2) $r(v^i) = 0$, for all $v^i \in W \setminus \{\beta^0\}$, except for at most one vertex, denoted by ρ^0, and if ρ^0 exists, then $r(\rho^0) = 1$ and either $i_\alpha = 1$ and $\rho \in N(\alpha)$ or $i_\alpha = p-1$ and $\rho^0 = \alpha^0$.

Proof: First, observe that, as $N^{W^i}(x^i) \neq \emptyset$ and $x^i \notin L_I$, for all $x^i \in L_1 \cup L_2$, we have that $L_1 \cup L_2$ is stable and that $L_2 \cap N(L_I) = \emptyset$. So, given $v^i \in W^i$ and $x^i \in N^{L_2}(v^i)$, we just need to ensure that we colour x^i with a colour different from $\Psi(x^0)$ and $\Psi(v^i)$ and that we do not repeat too many colours in $N(v^i), N(x^0)$. Now, consider $v^i \in W$. We know that if $v^p \in W$ and $x^p \in N^{L_2}(v^p)$, then $x^0 \in W$ and, for all $i \in \{1, \cdots, p-1\}$, either $x^i \in N^{L_2}(v^i)$, or $x^i = \alpha^i$, or $x^i \in L_I$ (and, in this case, $i \leq i_\alpha$ and $N^{W^i}(x^i) = \{v^i, \alpha^i\}$). Also, if $v^p \notin W$, then $v = \alpha$ and $N^{L_2}(v^i)/T = N^{L^2}(v^j)/T$, for all $i, j \in \{1, \cdots, i_\alpha\}$. So, let $l = min\{i : N^{L_2}(v^i) \neq \emptyset\}$, $q = max\{i : v^i \in W\}$ and, for each $i \in \{l, \cdots, q\}$, let $X^i = N^{L_2}(v^i)$. By what was said before, we know that either $X^i/T = X^j/T$, for all $i, j \in \{l, \cdots, q\}$, or there exists $x \in V(T)$ such that $X^i/T \subseteq (X^j \cup \{x^j\})/T$, for all $i, j \in \{l, \cdots, q\}$. Let $X^q/T = \{x_1, \cdots, x_r\}$ and $X^0 = \{x_1^0, \cdots, x_r^0\}$. We want to use the colours in $\Psi(X^0) \cup \Psi(\{v^l, \cdots, v^q\})$ to colour $X^l, \cdots, X^q$. However, this is not always possible as these colours may already appear in $N(x_i^0)$, for some $i \in \{1, \cdots, r\}]$, or in $N(v^i)$, for some $i \in \{l, \cdots, q\}$. So, let $F = X^0 \cup \{v^l, \cdots, v^q\}$ and C_F be the set of colours $\Psi(N(F))$. We construct a bijective function $f : F \to \{1, \cdots, m(H)\}$ such that:

(*) $f(z) \notin C_F$, for all $z \in F$.

We know that if $y^i \in N^L(v^i)$ is already coloured, for some $i \in \{l, \cdots, q\}$, then either $\Psi(y^i) \in \Psi(W^i)$ or (c1) occurs. Thus, $\Psi(x_i^0) \notin \Psi(N(v^j))$, for all $i \in \{1, \cdots, r\}$, $j \in \{l, \cdots, q\}$. However, it may occur that for some $i \in \{1, \cdots, r\}$:

(i) $\Psi(x_i^0) \in \Psi(N(x_j^0))$, for some $j \in \{1, \cdots, r\}$.

Also, it may occur that for some $i \in \{l, \cdots, q\}$:

(ii) $\Psi(v^i) \in \Psi(N(v^j))$, for some $j \in \{l, \cdots, q\}$; and/or

(iii) $\Psi(v^i) \in \Psi(N(x_j^0))$, for some $j \in \{1, \cdots, r\}$.

Let $x_i^0 \in X^0$. If (i) does not occur, then set $f(x_i^0)$ to $\Psi(x_i^0)$. Otherwise, let $x_j^0 \in X^0$ such that $\Psi(x_i^0) \in \Psi(N(x_j^0))$. Note that, as $r \geq 2$, V^0 has no locally encircled vertex. By (c1), we know that $\Psi(x_j^k) \neq \Psi(x_i^0)$, for all $k \in \{1, \cdots, p\}$. Thus, let $y^0 \in N(x_j^0)$ such that $\Psi(y^0) = \Psi(x_i^0)$. By Lemma 2.16 and the fact that $v^0 \in W$ separates x_i^0 from x_j^0 in T^0, we know that there exists $z^0 \in N^{W^0}(y^0) \setminus \{x_j^0\}$. By (c1), we know that $\Psi(z^0)$ does not appear in $N(v^k)$, for all $k \in \{l, \cdots, q\}$. So, if $\Psi(z^0) \notin N(x_k^0)$, for all $k \in \{1, \cdots, r\}$, we set $f(x_i^0)$ to $\Psi(z^0)$. Otherwise, we can apply the same argument as before and, as H is finite, we eventually find some $w^0 \in W^0$ such that $\Psi(w^0) \notin C_F$; thus, set $f(x_i^0)$ to $\Psi(w^0)$. One can also verify that, if we apply this procedure to find $f(x_i^0)$ and $f(x_j^0)$, $i \neq j$, then the iterated sequence of vertices of $W^0 \setminus X^0$ will be different for x_i^0 and x_j^0, i.e., $f(x_i^0) \neq f(x_j^0)$.

Now, let $v^i \in W$. If neither (ii) nor (iii) occurs, set $f(v^i)$ to $\Psi(v^i)$. So, suppose first that (ii) occurs. Let $j \in \{l, \cdots, q\}$ and $y^j \in N(v^j)$ such that $\Psi(y^j) = \Psi(v^i)$. By (c1) and the facts that $\Psi(y^j) \notin \Psi(W^j)$ and $N^{L_2}(v^j)$ is still uncoloured, we have that $y^j \in L_e$. Let $\{y^a, \cdots, y^b\}$ be all the locally encircled copies of y different from y^0(recall Remark 6.25 for a better understanding). One can verify in the proof of Lemma 6.29 that there exists $u \in V(T)$ such that $\Psi(\{y^a, \cdots, y^b\}) \subseteq \Psi(\{u^1, \cdots, u^p\} \cap W)$. As $\Psi(y^j) = \Psi(v^i)$, we have $u = v$. Let $w \in V(T)$ such that $w^k \in N^{W^k}(y^k) \setminus \{v^k\}$, for all $k \in \{a, \cdots, b\}$ (w exists as $|N^{W^k}(y^k)| \geq 2$, for all $k \in \{a, \cdots, b\}$, and by the uniqueness of α). Consider any $h \in \{a, \cdots, b\}$; we prove that $\psi(w^h) \notin C_F$. Note that, for all $k = \{1, \cdots, r\}$, because of the distance between x_k^0 and w^0 in T^0, we know that x_k^0 has no locally encircled neighbour; hence, $\Psi(w^h) \notin \Psi(N(x_k^0))$. Now, consider $k \in \{l, \cdots, q\}$. Recall Remark 6.25 and note that all the locally encircled vertices in $V^1 \cup \cdots \cup V^p$ are copies of the same vertex of $V(T)$. So, if $k \notin \{a, \cdots, b\}$, we know that $N^{L_e}(v^k) = \emptyset$ and, as $\Psi(N^{L_I}(v^k) \cup N^{L_1}(v^k)) \subseteq \Psi(W^k \cup W^0)$, we have $\Psi(w^h) \notin \Psi(N(v^k))$. Otherwise ($k \in \{a, \cdots, b\}$), we know that $N^{L_I}(v^k) = \{y^k\}$; so, as $\Psi(y^k) \in \Psi(\{v^1, \cdots, v^p\} \cap W)$ and $\Psi(L_1) \subseteq \Psi(W^0)$, we have $\Psi(w^h) \notin \Psi(N(v^k))$. So, if (ii) occurs for v^i, set $f(v^i)$ to $\Psi(w^i)$. Note that $f(v^i) \in \Psi(W^i)$ and $f(x_j^0) \in \Psi(W^0)$, for all $j \in \{1, \cdots, r\}$, i.e., until now $f(z) \neq f(z')$, for all pair $z, z' \in F$. Now, consider that (iii) occurs for v^i, for some $i \in \{l, \cdots, q\}$. Let $y^a \in N(x_j^0)$ such that $\Psi(y^a) = \Psi(v^i)$. Note that if $a \neq 0$, then $y^a = x_j^a$ and $a \neq i$ (as the precolouring is proper). Thus, $\Psi(v^i) \in \Psi(N(v^a))$, i.e., (ii) also occurs and we set $f(v^i)$ as in the previous case. So, suppose that $a = 0$. As

$\Psi(y^0) \notin \Psi(W^0)$ and $V^0 \cap L_1 = \emptyset$, y^0 must be locally encircled and, hence, $r = 1$. Let $w^0 \in N^{W^0}(y^0) \setminus \{x_1^0\}$. We know that if x_1^k is coloured, for some $k \in \{1, \cdots, p\}$, then either $\Psi(x_1^k) \in \Psi(W^k)$ or $x_1^k \in L_e \cup L_1$ and, by (c1), $\Psi(x_1^k) \neq \Psi(w^0)$. Also, by Lemma 2.16 and (c1), $\Psi(w^0) \notin \Psi(N(v^k))$, for all $k \in \{l, \cdots, q\}$. Thus, $\Psi(w^0) \notin C_F$ and we set $f(v^i)$ to $\Psi(w^0)$. Note that, as $r = 1$, (i) cannot occur for x_1^0, i.e., $f(x_1^0) = \Psi(x_1^0) \neq \Psi(w^0)$. Also, this case may occur for at most one vertex $v^i \in \{v^l, \cdots, v^q\}$. Thus, as $f(v^j) \in \Psi(v^j)$, for all $j \in \{l, \cdots, q\}$, $j \neq i$, we have that $\Psi(z) \neq \Psi(z')$, for all pair $z, z' \in F$.

Now, we colour $X^l, \cdots, X^q$ using the function f. Observe Figure 6.9 for a better understanding of the following attribution of colours (in the figure, we suppose $l = 1$ and $q = p$). For simplicity, suppose $l = 1$. If $q > r$, then, for each $i \in \{1, \cdots, q\}$, give colours: $f(v^{i+j})$ to x_j^i, for every $j \in \{1, \cdots, q-i\}$; and $f(v^j)$ to x_{q-i+j}^i, for every $j \in \{1, \cdots, r-q+i\}$. If $r > q$, for each $i \in \{1, \cdots, r\}$, give colours: $f(x_{i+j}^0)$ to x_i^j, for every $j \in \{1, \cdots, r-i\}$; and $f(x_j^0)$ to x_i^{r-i+j}, for every $j \in \{1, \cdots, q-r+i\}$. Finally, if $r = q > 1$, colour $X^1, \cdots, X^q$ as in the first case, except that x_{r-1}^1 is coloured $f(x_r^0)$, x_r^1 is coloured $f(v^q)$ and $x_r^2, \cdots, x_r^q$ are coloured $f(x_1^0), \cdots, f(x_{r-1}^0)$, respectively. Note that, by the construction of f, $r(w^i)$ does not increase, for all $w^i \in W$.

Finally, we analyse the case where $r = l - q + 1 = 1$. We know that either $x = \alpha$ and $i_\alpha = p - 1$ or $v = \alpha$ and $i_\alpha = 1$. Let $col(w)$ denote the set $N(w) \cap (W \cup L_I \cup L_1)$ (set of coloured neighbours of w) and $W^* = \{w^0 \in W : \{w^1, \cdots, w^p\} \cap W = \emptyset\}$. We first consider the case $v = \alpha, r = i_\alpha = 1$. By G3, we know that $d^H(x_1^0) > m(H) - 1$; and by (r1) and the fact that $x_1^i \notin W$, for all $i \in \{1, \cdots, p\}$, we have that $r(x_1^0) = 0$. So, if there exists a colour $c \in M(\alpha^1) \setminus \{\Psi(x_1^0)\}$, we can colour x_1^1 with c. We prove that this colour exists. By the uniqueness of α, we know that $|W| \geq 3(|W^1| - 1) + |W^*| + 2$. Also, as each neighbour of α^1 in $W \cup L_I$ defines a different vertex of $W^1 \cup \{\alpha^0\}$ and each neighbour of α^1 in L_1 defines a different vertex of W^*, we have $|col(\alpha^1)| \leq |W^1| + |W^*|$. Thus, $|M(\alpha^1) \setminus \{\Psi(x_1^0)\}| \geq |W \setminus \{\alpha^1, x_1^0\}| - |col(\alpha^1)| \geq 3|W^1| - 3 + |W^*| + 2 - 2 - |W^1| - |W^*| = 2|W^1| - 3 \geq 1$, as $|W^1| \geq 2$. Now, consider the case $x_1 = \alpha$, $i_\alpha = p - 1$. By Equation 6.21, we know that $d(\alpha^0) \geq m(H)$, and by the existence of $v^1, \cdots, v^p \in W$ and (r1), we know that $r(\alpha^0) = 0$. Thus, if we colour α^p with any colour in $M(v^p) \setminus \Psi(\alpha^0)$, we obtain a proper unsaturated extension of Ψ. We prove that such a colour exists. We know that $|W| \geq 3|W^p| + |W^*| + i_\alpha + 1$ and, analogously as before, $|col(v^p)| \leq |W^p| + |W^*|$; thus, as $i_\alpha + 1 \geq 2$, we have that $|M(v^p) \setminus \{\Psi(\alpha^0)\}| \geq |W \setminus \{v^p, \alpha^0\}| - |col(v^p)| \geq 3|W^p| + |W^*| + 2 - 2 - |W^p| - |W^*| = 2|W^p| \geq 2$.

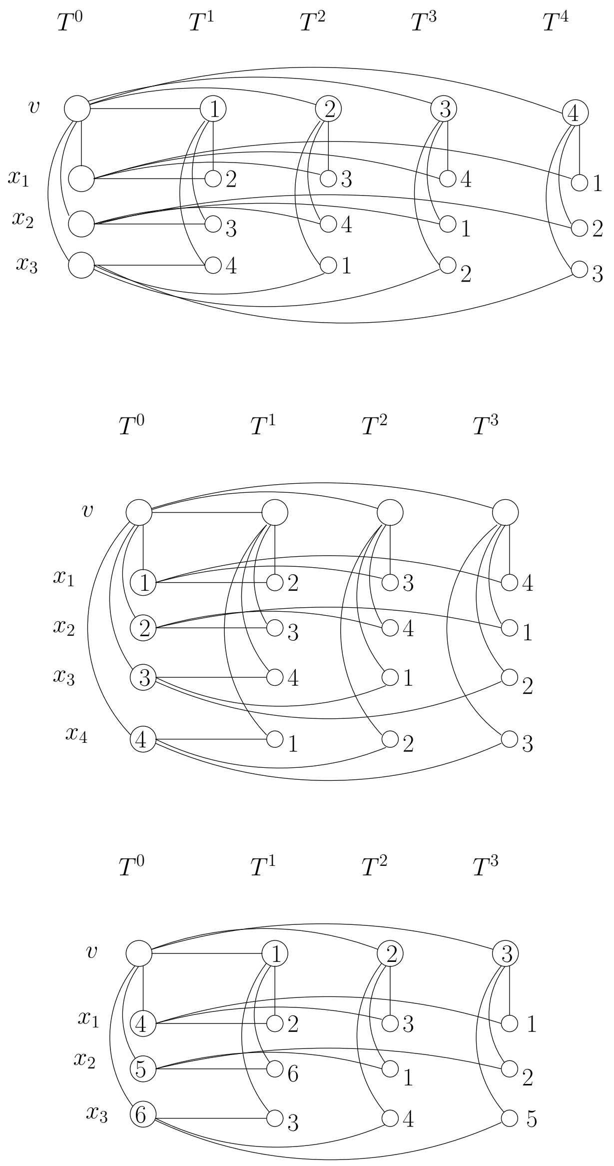


Figure 6.9: Example of the colouring of $x_1^i, \cdots, x_r^i \in N^{Z^i}(v^i)$, when $p > r$, $p < r$ and $p = r$.

Trivially, (r2) holds. □

Finally, we are able to finish the proof of Theorem 6.18.

Theorem 6.31. *Let $H = T \square K_{1,p}$, $p \geq 2$. If H has a good set, then $\chi_b(H) = m(H)$.*

Proof: Let W be a good set satisfying Lemma 6.24. Suppose that $|W^1| \geq 2$, otherwise the theorem follows from Lemma 6.28. Let Ψ be an unsaturated precolouring obtained using Lemmas 6.29 and 6.30. It remains to colour L_3 and L_4.

We start by colouring L_3. First, consider the case where there exists $y^0 \in V^0 \cap L_e$. Note that, as $W^i/T \subseteq W^0/T$, for all $i \in \{1, \cdots, p\}$, and $W^0 \subseteq N(y^0) \cup N(N^W(y^0))$, every connected component of $H[L_3]$ is either an isolated vertex or a star. Suppose that $v^i \in L_3$ is an isolated vertex in $H[L_3]$. As $N^{W^i}(v^i) = \emptyset$ (otherwise it should be in L_2), v^i must be adjacent to y^i and $N^{W^0}(v^0) = \emptyset$. Thus, every link containing $v^i \in L_3$ is of the form $\langle v^0, v^i, y^i, w^i \rangle$. So, let $i = max\{j : v^j \in L_3\}$ and let $w^i \in N^{W^i}(y^i)$. We know that $\{w^1, \cdots, w^i\} \subseteq W$ and that at most one colour in $\Psi(\{w^1, \cdots, w^i\})$, say $\Psi(w^j)$, appears in $N(v^0)$, in which case $\Psi(y^0) = \Psi(w^j)$. Also, $\Psi(y^k) \neq \Psi(w^k)$, for all $k \in \{1, \cdots, i\}$, as the precolouring is proper. Thus, for all $v^k \in \{v^1, \cdots, v^i\} \setminus (W \cup \{v^j\})$, colour v^k with $\Psi(w^k)$. As for v^j, note that as v^0 has at most p coloured neighbours and $|W| \geq |W^0| + p + 1 \geq p + 3$ (recall that $|W^1| \geq 2$), there must exist a colour $c \in M(v^0) \setminus \{\Psi(y^j)\}$ with which we can colour v^j. Now, consider a connected component $\{v^p, x_1^p, \cdots, x_q^p\}$ of $T^p[L_3]$ that is a star, where $v^p \in N(y^p)$ and v^p is the center of the star. Let $i = max\{j : N^{W^j}(y^j) \neq \emptyset\}$ (i.e., i is the maximum index such that y^i is coloured). Colour $\{v^1, \cdots, v^i\} \cap L_3$ as before and $\{v^{i+1}, \cdots, v^p\}$ with colours from $M(v^0)$. Then, consider $x_j \in \{x_1^p, \cdots, x_q^p\}/T$ and let $J = \{x_j^1, \cdots, x_j^p\} \cap L_3$. If $|J| \geq 2$, permute the colours from $M(x_j^0)$ in J in a way that $x_j^i \in J$ is not coloured with $\Psi(v^i)$. Now, consider $J = \{x_j^p\}$. As $|W^1| \geq 2$, we have $|W \setminus W^0| \geq p + 1$, and since x_j^0 has at most $p - 1$ neighbours coloured with some colour in $\Psi(W \setminus W^0)$, there must exist a colour $c \in M(x_j^0) \setminus \Psi(v^p)$ with which we can colour x_j^p.

Now, suppose that V^0 has no locally encircled vertex. For each $i \in \{1, \cdots, p\}$, denote by Q^i the set of vertices of the subgraph $T^i[L_3]$. We know that $N^W(x^i) = \{x^0\}$, for all $x^i \in L_3$. Also, we know that if $y^i \in L_I \cup L_1 \cup L_2$, then $N^{W^i}(y^i) \neq \emptyset$. Denote the set $L_I \cup L_1 \cup L_2$ by S and let Q^* be the set $\bigcup_{i=1}^p \{x^i \in Q^i : N^S(x^i) \neq \emptyset\}$. We first colour the vertices in Q^*. We want the following property to hold:

(c3) If $x^i \in Q^*$ is coloured with c, then $\psi_c \in N^{W^i}(y^i)$, for some $y^i \in N^S(x^i)$.

Let $x^i \in Q^*$, $y^i \in N^S(x^i)$ and $w^i \in N^{W^i}(y^i)$. We want to colour x^i with $\Psi(w^i)$. First, we prove that if x^j is coloured, for some $j \in \{1, \cdots, p\}$, $j \neq i$, then $\Psi(x^j) \in \Psi(W^j)$ (consequently, as $V^0 \cap L_e = \emptyset$, we have $\Psi(w^i) \in M(x^0)$). By contradiction, suppose otherwise. As $x^0 \in W$, we know that $x^j \notin L_1$. Also, every vertex of $(L_I \setminus L_e) \cap V^j$ is coloured with a colour from $\Psi(W^j)$, as well as the vertices of $L_3 \cap V^j$ coloured until now. Thus, x^j must be in $L_e \cup L_2$. But then, as $N^{W^i}(x^i) = \emptyset$, we must have that $j < i$ and $N^{W^j}(x^j) = \{\alpha^j\}$; consequently, as α is unique and $w^i \in W$, we have that the path $\langle \alpha^j, x^j, y^j, w^j \rangle$ is a link of W, contradicting the fact that $x^j \in L_e \cup L_2$. Now, we prove that $\Psi(w^i) \notin \Psi(N(x^i))$. Suppose that there exists $z^i \in N(x^i)$ such that $\Psi(z^i) = \Psi(w^i)$. By (c3), we know that $z^i \notin L_3$. Also, we know that if $z^i \in L_1 \cup L_2$, then $\Psi(z^i) \notin \Psi(W^i)$. Thus, $z^i \in L_I$ and, by Lemma 2.16, Remark 2.17 and the fact that $x^i \notin L_I \cup W^i$ separates z^i from w^i in T^i, we have that $|N^{W^i}(z^i)| \geq 2$. So, $|N^{W^i}(N^S(x^i))| > |N^S(x^i)|$ and there exists $u^i \in N^{W^i}(N^S(x^i))$ such that $\Psi(u^i) \notin \Psi(N(x^i) \cup N(x^0))$, i.e., we can colour x^i with $\Psi(u^i)$.

Now, we colour $Q^i \setminus Q^*$, for every $i \in \{1, \cdots, p\}$. Let $v^i \in Q^i \setminus Q^*$, for some $i \in \{1, \cdots, p\}$, and note that, as $N(v^i) \cap (S \cup W^i) = \emptyset$, we must have:

(I) If $x^i \in N(v^i) \cap (L \cup W)$, then $x^i \in L_3$.

Thus, we can colour the connected components of $H[Q^i]$ separately. Let J^p be the vertex set of a connected component of $T^p[Q^p]$ and $J^i = \{x^i : x^p \in J^p\}$, for every $i \in \{1, \cdots, p-1\}$. Denote J^p/T by J. We will choose a root for $T[J]$, say r, and colour the uncoloured vertices in $J^1, \cdots, J^p$ from the roots $r^1, \cdots, r^p$ down to the leaves of the subtrees $T^1[J^1], \cdots, T^p[J^p]$. Let $\{w^1, \cdots, w^p\} \subseteq W$. If there exists $x^p \in Q^*$ coloured with $\Psi(w^p)$, then choose x as the root; otherwise, if there exists $x^i \in J^i$ such that $\Psi(x^i) \in \Psi(\{w^1, \cdots, w^p\})$, for some $i \in \{1, \cdots, p\}$, then choose x as the root; finally, choose any vertex of J. Let r be the chosen root. As we said before, we start by colouring $r^1, \cdots, r^p$. So, suppose that $r^i, \cdots, r^p$ are still uncoloured (note that $i \in \{1, i_\alpha + 1\}$). As $V^0 \cap L_e = \emptyset$, we know that $|M(r^0) \cap \Psi(\{w^1, \cdots, w^p\})| \geq p - i + 1$. Also, by the choice of the root and by (c3) and (I), we know that $\Psi(x^j) \notin \Psi(\{w^1, \cdots, w^p\})$, for all $x^j \in N(r^j)$, for all $j \in \{i, \cdots, p\}$. Thus, we can colour $r^i, \cdots, r^p$

with colours from $M(r^0) \cap \Psi(w^1, \cdots, w^p)$. Now, let $t \in J$ be such that some t^i is still uncoloured, for some $i \in \{1, \cdots, p\}$, and $x^1, \cdots, x^p$ are all coloured, where $x \in J$ is the parent of t in $T[J]$. By the choice of the root and by (I), we know that x^i is the only neighbour of t^i that may be coloured with some colour in $\Psi(\{w^1, \cdots, w^p\})$, for all $t^i \in L_3 \setminus Q^*$. Also, analogously as before, we know that there are as much colours in $M(t^0) \cap \Psi(\{w^1, \cdots, w^p\})$ as uncoloured vertices in $\{t^1, \cdots, t^p\}$. Thus, if there are at least two uncoloured vertices in $\{t^1, \cdots, t^p\}$, we can make a proper attribution of colours from $M(t^0) \cap \Psi(\{w^1, \cdots, w^p\})$ to the uncoloured vertices in $\{t^1, \cdots, t^p\}$. So, suppose otherwise, i.e., that the only uncoloured vertex in $\{t^1, \cdots, t^p\}$ is t^p. If there exists $y^p \in N(t^p) \setminus \{x^p\}$ such that $\Psi(y^p) \neq \emptyset$, by (c3), (I) and the choice of the root, we know that $y^p \in Q^*$ and $\Psi(y^p) = \Psi(v^p)$, for some $v^p \in W^p \setminus \{w^p\}$. Thus, as t^0 has at most $p-1$ neighbours coloured with colours in $\Psi(W \setminus W^0)$, there must exist a colour $c \in (M(t^0) \cap \Psi(\{w^1, \cdots, w^p, v^1, \cdots, v^p\})) \setminus \Psi(\{x^p, y^p\})$. By (c1) and (I), we know that $c \notin \Psi(N(t^p))$; thus, we colour t^p with c. Now, if y^0 and x^p are the only coloured neighbours of y^p, as $|W \setminus W^0| \geq p+1$ (remember that $|W^1| \geq 2$), by an analogous argument we know that $|M(t^0)| \geq 2$; thus, there exists a colour $c \in M(t^0) \setminus \{\Psi(x^p)\}$ with which we can colour t^p.

Finally, we colour L_4. Note that links of Type 1 always have an extremity in T^0 and if $x^i \in L_4$, then x^i is within a link of Type 1. Now, let $x^i \in L_4$, $i \in \{1, \cdots, p\}$. As x^i is not an internal link vertex and $x^0 \notin W$, we must have $N^{W^i}(x^i) = \{v^i\}$. Also, if there exists $y^i \in N^L(x^i)$, then, as $x^i \notin L_1$, we have that $y^0 \notin W$ and, consequently, there exists $w^i \in N^{W^i}(y^i)$, a contradiction to the fact that $x^i \notin L_I$. So, $N^L(x^i) = \{x^0\}$, for all $x^i \in L_4$, $i \in \{1, \cdots, p\}$. Now, let $v^0 \in W$ such that $N^{L_4}(v^0) \neq \emptyset$. We start by colouring $N^{L_4}(v^0)$ with the colours in $M(v^0)$; then we colour $N^{L_4}(v^i)$, for all $v^i \in W$, $i \in \{1, \cdots, p\}$ (note that at least v^1 is in W). Observe that if $x^0 \in N^{L_4}(v^0)$, then $x^i \in N^{L_4}(v^i)$, i.e., $|N^{L_4}(v^0)| \geq |N^{L_4}(v^i)|$, for every $v^i \in W^i$. So, we prove that $|N^{L_4}(v^0)| \geq 2$ and, consequently, we can use Lemma 6.3 to colour $N^{L_4}(v^i)$, for all $v^i \in W$, $i \in \{1, \cdots, p\}$. First, note that, as $v^1 \in W$, if $x^0 \in N(v^0) \setminus (W \cup L_I)$, then $x^0 \in L_4$. So, denote by q the value $|N^{L_4}(v^0)|$; we have that $d(v^0) = |N^{V^0}(v^0) \cap (L_4 \cup L_I \cup W)| + p = |N^{L_4}(v^0)| + |N^{V^0}(v^0) \cap (L_I \cup W)| + p \leq q + p + |W^0| - 1$. By Equation 6.21, we also know that $d(v^0) \geq m(H) + p - 2$. Thus: $m(H) + p - 2 \leq q + p + |W^0| - 1 \Rightarrow q \geq |W \setminus W^0| - 1$. As $|W^1| \geq 2$, we have that $|W \setminus W^0| > p \geq 2$ and, consequently, $q \geq 2$. $\square$

Chapter 7

Perspectives

In this chapter, we summarize the results presented in this thesis and discuss about some questions left open.

We generalized the result on trees by Irving and Manlove to the cacti with m-degree at least 7. We conjecture that if G is a cactus that has a good set and G is not anomalous, then $\chi_b(G) = m(G)$. Given the results presented in Chapter 3, it remains to prove the conjecture for cacti with m-degree at most 6. We believe that a non-constructive approach towards proving this conjecture is more tangible. For now, we have proven that if G is a minimal counter-example for this conjecture, then $|D(G)| = m(G)$, $d(v) = m(G) - 1$, for all $v \in D(G)$, $G \subseteq D(G) \cup N(D(G))$ and, for all $(u, v) \in E(G)$, at least one between u and v is a dense vertex. This proof is shown in Appendix A. The next step to obtain a complete proof of our conjecture would be to prove that such a minimal counter-example contains either an encircled vertex or an encircled pair of vertices.

Regarding outerplanar graphs, we give a polynomial-time algorithm to find an optimal b-colouring of G with either $m(G) - 1$ or $m(G)$ colours, where G is an outerplanar graph with girth at least 8. By the construction given in Section 5.1, we know that this result cannot be generalized to series-parallel graphs. However, we can still ask: (1) does $\chi_b(G) \geq m(G) - 1$ hold for every outerplanar graph G?; and also (2) does $\chi_b(G) \geq m(G) - 1$ hold for every graph G with girth at least 8? We recall Conjecture 2.12: *If G is a graph that does not have a $K_{2,3}$ as subgraph, not necessarily induced, and $G \neq C_3 \square C_3$, then $\chi_b(G) \geq m(G) - 1$.* Remark that if this conjecture holds, then the answer to both questions is yes. Also, we remark that a partial answer to Question (2) is given in Theorem 2.19: we prove that if G has girth at least 8 and does not have a good set, then $\chi_b(G) = m(G) - 1$.

Motivated by the conjecture on graphs with no $K_{2,3}$ as subgraph, we investigated the b-chromatic number of the cartesian products of trees by cycles, paths and stars. Let T be a tree. We proved that if G is a path of length greater than 4, or a cycle of length greater than 3 or a star $K_{1,p}$, $p \geq 2$, then $\chi_b(G \,\square\, T) \geq m(G \,\square\, T) - 1$. We also give polynomial-time algorithms to find optimal b-colourings of those graphs. Observe that the star $K_{1,2}$ is also a P_3. Also, we believe that we can easily prove that $\chi_b(T \,\square\, P_2) \geq m(T \,\square\, P_2) - 1$ using the results presented in Chapter 6. Thus, it remains to find the b-chromatic number of the cartesian products $T \,\square\, P_4$ and $T \,\square\, C_3$. We remark that an important aspect of the algorithms presented in Chapter 6 is the existence of an algorithm that colours the internal link vertices of T^i preserving certain properties (mainly Lemma 2.16). An interesting question would be to try to prove these results for graph classes where we can do the same. For instance, if G has girth at least 11, does $\chi_b(G \square P_k) \geq m(G \square P_k) - 1$ also hold for $k \geq 5$?

We have also investigated the b-chromatic number of block graphs and found that the difference $m(G) - \chi_b(G)$ is arbitrarily large, even if G is a claw-free block graph (i.e., the line graph of a tree). This is the only graph class attacked in this thesis for which we could not provide a polynomial-time algorithm to find its b-chromatic number. Nevertheless, we obtained the following results for block graphs: the fixed parameter decision problem is polynomially solvable; given a subset W of cardinality k such that $d(u) \geq k - 1$, for all $k \in W$, we proved that the difficulty in obtaining a b-colouring with basis W lies on the existence of a special type of vertex, called side vertex; and we showed a special case where we can decide if $W \subseteq D_k(G)$ can be the basis of a b-colouring of G with k colours, G being a claw-free block graph. So, the complexity of determining the b-chromatic number of a block graph remains open. We remark that the problem is NP-hard for distance-hereditary chordal graphs [17], which is a super class of block graphs. Also, in personal communication, Leonardo Sampaio, currently a graduate student at INRIA Sophia-Antipolis, France, showed us a proof that the problem is NP-hard for line graphs. Thus, maybe a good start would be to answer the question for block graphs which are also line graphs (claw-free block graphs).

Finally, we mention that the problem of finding the b-chromatic number of a graph with bounded treewidth is still open. Actually, although it is known that computing the b-chromatic number of a chordal graph is NP-hard [17], the problem of finding the b-chromatic number of chordal graphs with bounded clique number is still open (note that these graphs also have bounded treewidth). In [14], Faik gives a construction of interval graphs for which the difference $m(G) - \chi_b(G)$ is arbitrarily large. However, in his construction, the difference $m(G) - \chi_b(G)$ increases as the treewidth of the graph increases. Thus, it does not apply for graphs with bounded treewidth, i.e., it is not known wether the difference $m(G) - \chi_b(G)$ is bounded or not for graphs with bounded treewidth. No further results on the b-chromatic number of interval graphs were found.

Appendices

Appendix A

Minimal m-defective cacti

In the following, we denote by $H-e$ the graph obtained from H by removing the edge e.

We say that G is an *m-defective* graph if $\chi_b(G) < m(G)$ and $m(G) = m$. We say that G is *minimal m-defective* if G is m-defective and every proper subgraph H of G is not m-defective, i.e., either $m(H) < m$ or $m(H) = m$ and $\chi_b(H) = m(H)$.

Let G be a minimal m-defective cactus. The main results shown here are related to describing unnecessary vertices and edges in G. To be more precise, one result is to prove the following theorem.

Theorem A.1. *If G is a minimal m-defective cactus and $m \geq 4$, then $|D(G)| = m$ and $d(u) = m-1$, for every $u \in D(G)$.*

From Theorem A.1, we know that the dense vertices in G are incident to just enough edges for them to be dense and there are just enough vertices in $D(G)$ so that $m(G) = m$. With this idea in mind, one could expect that there were no edges between vertices not in $D(G)$. This is false due to the existence of the anomalous graphs presented in Figure 3.2 and in Figure 3.3. If H is one of the graphs in these figures, then consider $H' = H - (u, v)$. Since H' is small, one can check that $\chi_b(H') = m(H')$. Actually, for any b-colouring of H' with $m(H')$ colours, u and v have the same colour. This implies that H is $m(H)$-defective. To see that H is minimal, note that for any $e \in E(H)$, either $e = uv$ and $\chi_b(H - e) = m(H)$ or $e \neq uv$ and $m(H - e) < m(H)$. We prove that the anomalous cacti are the only minimal m-defective cacti having an edge not incident to a dense vertex.

Theorem A.2. *Let G be a minimal m-defective cactus with $m \geq 4$. If there exists an edge $(u, v) \in E(G)$ such that $u, v \notin D(G)$, then G is isomorphic to a graph in Figure 3.2 or in Figure 3.3.*

Throughout this section, we use many techniques based on recolouring a previously defined colouring. To simplify the presentation of these results, we define a recolouring function. Let ψ be a colouring of a graph G, A be a subset of $V(G)$ and c and c' be two colours in ψ. Define $\psi(A, c \leftrightarrow c')$ as the colouring obtained from ψ by exchanging in A the colours c and c'.

Lemma A.3. *If G is minimal m-defective, then any vertex not in $D(G)$ is adjacent to at least one vertex in $D(G)$.*

Proof: By contradiction, suppose that $v \notin D(G)$ and v is not adjacent to a vertex in $D(G)$. Consider the graph G' obtained by deleting v from G. Since no vertex in $D(G)$ changed its degree in G', then $m(G') = m(G) = m$. This implies that $\chi_b(G') = m$ as G is minimal m-defective and G' is a proper subgraph of G. A b-colouring ψ of G' with m colours can be extended to G by colouring v with a colour not used in its neighbourhood, which can be done since $d(v) < m - 1$. □

Lemma A.4. *Let G be a minimal m-defective cactus, $w \in D(G)$ and C be a component of $G \setminus \{w\}$. If C does not contain dense vertices, then $|V(C)| = 1$ and $d(w) = m - 1$.*

Proof: By contradiction, suppose that C does not contain vertices in $D(G)$. Lemma A.3 implies that any vertex in C is adjacent to w. Now, Lemma 3.1 implies that there are at most two vertices in C. If C contains an edge uv, let G' be obtained from G by deleting the edge uv. Since no vertex in $D(G)$ changed its degree in G', then $m(G') = m(G) = m$. This implies that $\chi_b(G') = m$ as G is minimal m-defective and G' is a proper subgraph of G. Let ψ be a b-colouring of G' with m colours. If $\psi(u) \neq \psi(v)$, then ψ is also a b-colouring of G. Otherwise, we can recolour u to any other colour in ψ as w is adjacent to v with the same colour as u. In any case, we get a b-colouring of G with m colours. If $|V(C)| = 1$ and $d(w) \geq m$, then we can use a similar argument as before to show that a b-colouring of $G \setminus C$ with m colours can be extended to a b-colouring of G by colouring the unique vertex in C with a colour different from w to get a contradiction. Thus, $d(w) = m - 1$. □

Lemma A.5. *If G is a minimal m-defective cactus, $m \geq 4$, then $|D(G)| = m$.*

Proof: Let $u, v \in D(G)$ have maximum distance among pairs of dense vertices. Let C be a component of $G \setminus \{u\}$ that contains at least one neighbour of u in G. Since $d(u) \geq 3$, there are at least two such components C by Lemma 3.1. Thus, consider that C does not contain v. We know that C does not contain any dense vertex as, otherwise, u and v would not have maximum distance among pairs of vertices in $D(G)$. Lemma A.4 tells us that $d(u) = m - 1$ and u has a unique neighbour w in C. Let $H = G \setminus \{w\}$. Note that $m(H) \leq m(G) = m$. By contradiction, suppose that $m(H) = m(G)$; hence $\chi_b(H) = m$ and there is a b-colouring ψ of H with m colours. We can build a b-colouring of G with m colours from ψ by giving a colour to w different from $\psi(u)$ to get a contradiction. So, $m(H) < m$ and, since u was the only vertex in $D(G)$ whose degree changed, we have that $|D(G \setminus \{w\})| = m - 1$ which implies the lemma. □

Lemma A.6. *Let $m \geq 4$, G be a minimal m-defective cactus and $w \in D(G)$. If $d(w) = m - 1$, then $N(w) \setminus D(G)$ is a stable set.*

Proof: By contradiction, suppose e is an edge betwen two neighbours of w not in $D(G)$. Note that $m(G - e) = m(G) = m$ and, since G is minimal m-defective, we have $\chi_b(G - e) = m$. Therefore, let ψ be a b-colouring of $G - e$ with m colours. Note that Lemma A.5 implies that w is a b-vertex of ψ. Since $d(w) = m - 1$, all neighbours of w have distinct colours and ψ is also a b-colouring of G. □

Lemma A.7. *Let $m \geq 4$, G be a minimal m-defective cactus, u and v be two vertices of $D(G)$ with $d(u) = d(v) = m - 1$ and C be a component of $G \setminus \{u, v\}$. If C contains neighbours of both u and v and C contains no dense vertices, then $|V(C)| = 1$.*

Proof: Suppose that C contains neighbours of both u and v and C contains no dense vertices. Lemma A.6 implies that no edge in C has both endpoints in $N(u)$ or $N(v)$. If u and v have a common neighbour in C, then $|V(C)| = 1$. Thus, consider that u and v have no common neighbour in C. By Lemma A.3, any vertex in C is adjacent to either u or v. This fact together with Lemma 3.1 implies that C has at most four vertices. Actually, as C is connected and G is a cactus, it is easy to see that C has at most three vertices. By contradiction, suppose that e is an edge of C with endpoints u' and v' such that $u' \in N(u)$ and $v' \in N(v)$. As before, note that $m(G-e) = m$ which implies that $\chi_b(G-e) = m$. Let ψ be a b-colouring of $G-e$ with m colours. Since ψ cannot be a b-colouring of G, then $\psi(u') = \psi(v') = c$. Let w be the b-vertex coloured c. The fact that C has at most three vertices implies that either u or v has only one neighbour in C. Without loss of generality, suppose that v has only one neighbour r in C. Let B_v be the connected component of $G \setminus N[u]$ containing v and $B_u = G \setminus B_v$. If w is in B_u, then $\psi(B_v, c \leftrightarrow \psi(u))$ is a b-colouring of G with m colours. Thus, consider that w is in B_v. If u is not adjacent to a vertex in C with colour $\psi(v)$, then $\psi(B_v, c \leftrightarrow \psi(v))$ is a b-colouring of G with m colours. Thus, u has another neighbour in C with colour $\psi(v)$. Note that in this case, r is adjacent to v and to two neighbours in C which implies $d(r) = 3$. So, as $r \notin D(G)$, we have $m \geq 5$. If there is a colour c' not in the set $\{c, \psi(v), \psi(u)\}$ whose b-vertex is in B_v, then $\psi(B_v, c \leftrightarrow c')$ is a b-colouring of G with m colours. Thus, consider that the only b-vertices in B_v are v and w. Let $B_v^- = B_v \setminus \{r\}$. Observe that, since $m \geq 5$, there is a component of $G[B_v^-] - v$ that does not contain w and, by Lemma A.4, we know that such a component contains a single vertex ℓ. In this case, $\psi(N(v), c \leftrightarrow \psi(\ell))$ is a b-colouring of G with m colours, a contradiction. Therefore, u and v have at least one common neighbour in C which, in turn, implies the lemma. □

Proof of Theorem A.2 Let G be a minimal m-defective cactus and let e be an edge between u and v such that $u, v \notin D(G)$. Let $H = G - e$. Since $m(H) = m(G) = m$ and G is minimal m-defective, then $\chi_b(H) = m$. Thus, suppose that ψ is a b-colouring of H with m colours. Note that Lemma A.5 implies that each colour class in ψ has precisely one dense vertex and this vertex is a b-vertex. If $\psi(u) \neq \psi(v)$, then we get a contradiction as ψ is also a b-colouring of G. Thus, assume that $\psi(u) = \psi(v)$ and let w be the b-vertex of this colour class. Throughout the proof of this theorem, we build a contradiction by constructing a b-colouring φ from ψ with m colours such that $\varphi(u) \neq \varphi(v)$.

We first consider the case in which u and v are in two different components of H. Let C_v be the vertex set of the component of v and, without loss of generality, consider that w is not in C_v. By Lemma A.3, there must exist $w' \in N(u) \cap D(G)$ (hence, $w' \notin C_v$). Then, $\psi(C_v, \psi(w') \leftrightarrow \psi(w))$ is a b-colouring of G with m colours such that u and v have different colours, a contradiction.

Now, suppose that u and v are in the same component of H. As G is a cactus and $(u,v) \in E(G)$, we know that there is a unique path P between u and v in H. Number P from u to v, $x_0 = u, x_1, \cdots, x_q, x_{q+1} = v$. For $i = 0, \cdots, q+1$, let C_i be the vertex set of the component of $H \setminus E(P)$ containing x_i and $R_i = V(H) \setminus C_i$ be the remaining vertices. Also, for each $i \in [1,q]$, let $C_{i,u}$ be the vertex set of the component of $H \setminus \{x_i\}$ containing u; define $C_{i,v}$ analogously. Without loss of generality, suppose that the distance from u to w is not greater than the distance from v to w in H, i.e., $dist_H(u,w) \leq dist_H(v,w)$. Note that this implies that w is not in $C_{q,v}$. Let y be a b-vertex adjacent to u according to Lemma A.3. Observe that if there exists $x_i \in P \setminus D(G)$ such that $\psi(x_i) \neq \psi(w)$ and $C_{i,v}$ does not contain w and the b-vertex of a colour $c \neq \psi(x_i), \psi(w)$, then $\psi(C_{i,v}, c \leftrightarrow \psi(w))$ is a b-colouring of H with m colours such that u and v have different colours, a contradiction. Thus, the following holds:

(*) For all $x_i \in P \setminus D(G)$ such that $\psi(x_i) \neq \psi(w)$, $C_{i,v}$ either contains w or the b-vertex of every colour different from $\psi(x_i)$.

Observe that the same is analogously valid for $C_{i,u}$. Now, observe that if there exists $x_i \in P \cap D(G)$ such that $C_{i,u}$ contains w and a b-vertex w' such that $\psi(w') \neq \psi(x_{i-1}), \psi(x_{i+1})$, then $\psi(C_{i,u}, \psi(w') \leftrightarrow \psi(w)$ is a b-colouring of H with m colours where u and v have different colours, a contradiction. Thus, the following also holds:

(**) If $x_i \in P \cap D(G)$ and $w \in C_{i,u}$, then $C_{i,u} \cap D(G) \subseteq \{w, w', w''\}$, where $\psi(w') = \psi(x_{i-1})$ and $\psi(w'') = \psi(x_{i+1})$.

Consider first the case that x_q is not a b-vertex. Trivially, $\psi(x_q) \neq \psi(w)$, as $v \in N(x_q)$. By (*) and the fact that $w \notin C_{q,v}$, we have that $\psi(x_q) = \psi(y)$ and the only b-vertices in R_v are w and y. If x_{q-1} is not a b-vertex, choose a colour class c not in the set $\{\psi(w), \psi(y), \psi(x_{q-1})\}$. One such colour class exists as $m \geq 4$. We know that the b-vertex of colour class c is in C_v; thus, it is not in C_q. Note that $\sigma = \psi(C_q, c \leftrightarrow \psi(y))$ is a b-colouring of H having a vertex x_q that contradicts (*). So, consider that x_{q-1} is a b-vertex. Since the only two b-vertices in R_v are w and y, x_{q-1} is adjacent to x_q and $\psi(x_q) = \psi(y)$, then $x_{q-1} = w$ (and, consequently, $q > 2$). Let c be a colour not in the set $\{\psi(w), \psi(x_{q-2}), \psi(y)\}$ (exists as $m \geq 4$). We have that $C_{q-2,v}$ contains all b-vertices other than y. We get a contradiction from $\varphi = \psi(C_{q-2,v}, c \leftrightarrow \psi(w))$. Observe that if $x_{q-2} = y$, then it is adjacent to u and φ is a b-colouring as y is a b-vertex in φ.

Now, consider that x_q is a b-vertex and that $\psi(x_{q-1}) = \psi(w)$. By (**) and the existence of $y \in N(u) \cap D(G)$, we have that $q = 1$ and $x_1 = y$. Since u cannot be adjacent to w, the only b-vertex adjacent to u is y. As $d_H(u) \leq m - 3$ and $v \in N(y)$, we can obtain a b-colouring of H with m colours by changing the colour of u in ψ to some colour it is not adjacent to, a contradiction.

Now, we consider the case that x_q is a b-vertex but $\psi(x_{q-1}) \neq \psi(w)$ (thus, $q > 1$). Let x_i be the vertex the closest to v in $P \setminus \{v\}$ that is not a b-vertex. Since u is not a b-vertex, such a vertex x_i exists. Note that if $i < q - 2$ or $0 < i = q - 2$, then x_q contradicts (**). Thus, $i \in \{q - 2, q - 1\}$ and if $i = q - 2$, then $i = 0$ (i.e., $x_i = u$). Also, if $i = 0$, then $q = 2$ (as $q > 1$ and $i \geq q - 2$). Let $B_v = C_{i,v}$ and $B_u = V(H) \setminus B_v$. We now consider the possibilities of whether w is in B_v or B_u and whether x_i equals to u or not.

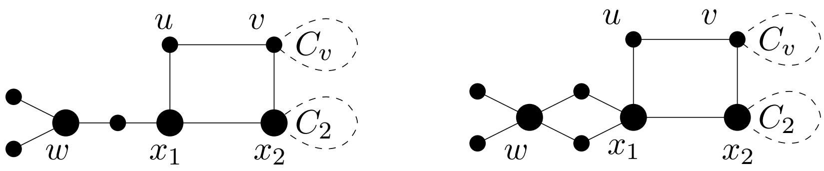
First, suppose that $w \in B_u$ and $u = x_i$. Then, $q = 2$ and, as x_2 satisfies (**), we know that $x_1 = y$ and $C_{2,u} \cap D(G) = \{w, y\}$. Thus, the only b-vertex adjacent to u is y. Since $d_H(u) \leq m-3$ and u is adjacent to y, then there is a colour c not in $\{\psi(w), \psi(y), \psi(x_2)\}$ such that u has no neighbour coloured c. We know that the b-vertex coloured c is not in $C_{2,u}$. Therefore, we get a contradiction from $\varphi = \psi(C_1 \cup \{u\}, c \leftrightarrow \psi(w))$.

Now, suppose that $w \in B_u$ and $x_i \neq u$. Then, $i = q - 1$, $C_{q,u} \cap D(G) = \{w, y\}$ and $\psi(x_i) = \psi(y)$. As $x_q \notin C_{i,u}$ and $\psi(x_q) \neq \psi(x_i)$, by (*) we have $w \in C_{i,u}$. Also, as $w \notin C_{i-1}, v$ and $\psi(x_{i_1} \neq \psi(y)$, then, by (*) and the fact that $C_{i,u} \cap D(G) = \{w, y\}$, we have $\psi(x_{i-1}) = \psi(w)$. We proceed by considering whether $i = 1$ or $i > 1$ and whether w is in C_{i-1} or not. If $i > 1$ and $w \in C_{i-1}$, then let c be a colour not in $\{\psi(w), \psi(y), \psi(x_{i-2})\}$. We get a contradiction from $\varphi = \psi(C_{i-1,u}, c \leftrightarrow \psi(w))$. Note that this colouring works if $x_{i-2} = y$, as u is adjacent to y. If $i > 1$ and $w \notin C_{i-1}$, then note that x_{i-1} is not a b-vertex. Thus, as $u \in N(y)$ is also coloured with $\psi(w)$ and y is the only possible b-vertex adjacent to x_{i-1}, we can change the colour of x_{i-1} to a colour it is not adjacent to and, then, treat this case as if $\psi(x_{i-1}) \neq \psi(w)$. If $i = 1$, then $y \in C_u$ as u is adjacent to x_1 in P and x_1 is not a b-vertex. If w is not in C_u, let c be a colour not in $\{\psi(w), \psi(y)\}$. We get a contradiction from $\varphi = \psi(C_u, \psi(w) \leftrightarrow c)$. Thus, consider that w and y are in C_u. Let H_w be the component that contains w in $H[C_u] \setminus \{y\}$. From Lemma 3.1, y has at most two neighbours in H_w. Since u is adjacent to x_1 and y, $\psi(x_1) = \psi(y)$ and $d_H(u) \leq m-3$, then u is not adjacent to at least three colour classes different from $\psi(w)$. Therefore, there is a colour c other than $\psi(w)$ such that u has no neighbour coloured c and y has no neighbour coloured c in H_w. Thus, φ

is obtained from $\psi(C_u \setminus V(H_w), c \leftrightarrow \psi(w))$ by colouring u with colour c.

Finally, suppose that $w \in B_v$. In this case, we show properties of G that imply that G is isomorphic to a graph in Figure 3.2 or Figure 3.3. We know that if $x_i \neq u$, then $i = q-1$ and, in this case, w is closer to v than to u, a contradiction. So, $i = 0$ and, consequently, $q = 2$. Also, as $dist_H(u,w) \leq dist_H(v,w)$, we have that $w \in C_1$, and as x_2 satisfies (**), we have that $x_1 = y$ and $C_{2,u} \cap D(G) = \{w, y\}$. Thus, there is no b-vertex in B_u and, by Lemma A.3, we have that $B_u = \{u\}$. If $d_G(y) \geq m$, then note that $m(G \setminus \{u\}) = m(G) = m$ and, therefore, $\chi_b(G \setminus \{u\}) = m$. If σ is a b-colouring of $G \setminus \{u\}$ with m colours, then we can extend σ to a b-colouring of G by giving a colour to u not in $\{\sigma(y), \sigma(v)\}$. Therefore, $d_G(y) = m-1$. Note that this implies that y is not adjacent to a vertex with colour $\psi(w)$ in C_y as all of its neighbours must have distinct colours and y is adjacent to u. In particular, y is not adjacent to w. If there is a component Q of $G[C_1] \setminus \{y\}$ that does not contain w, then Q contains no dense vertices. Lemma A.4 tells us that $|V(Q)| = 1$. Let r be the neighbour of y in Q. We obtain from ψ a b-colouring of H with m colours where u and v have different colours by exchanging the colour between r and u, a contradiction. Thus, suppose that $G[C_1] \setminus \{y\}$ is connected. Lemma 3.1 implies that y has at most two neighbours in $G[C_1]$. Therefore, $d(y) \leq 4$ which implies that $m \leq 5$ as $y \in D(G)$. By Lemma 3.1 and the fact that $d(w) \geq 3$, we have that there exists a component R of $G[C_1] \setminus \{w\}$ that does not contain y. Thus, Lemma A.4 implies that $|V(R)| = 1$ and $d(w) = m-1$. Now, we apply Lemma A.7 to conclude that the neighbours of y not in $\{u, x_2\}$ are also neighbours of w. Therefore, G has the structure of Figure A.1(a) if $m = 4$ and of Figure A.1(b) if $m = 5$. Now, to get the complete structure of G, we have to describe the behaviour of $C_v = C_{q+1}$ and C_2. Note that $D(G) \setminus \{w, y, x_2\}$ has either one or two vertices depending on whether m equals to four or five. Moreover, they are either in C_2 or in C_v.

First, consider that there are no dense vertices in $C_2 \setminus \{x_2\}$. In this case, as $4 \leq m \leq 5$, $d_G(v) \geq 3$ and $v \notin D(G)$, we have that $m = 5$ and $d_G(v) = 3$. Thus, by Lemma A.4, $d_G(x_2) = m-1 = 4$ and x_2 has two neighbours in C_2 with degree one. Now, let r_1 and r_2 be the two dense vertices in $D(G) \setminus \{w, y, x_2\}$; we analyse the structure of C_v. First, note that, by Lemma 3.1 and the facts that $d_G(v) = 3$ and $d_G(r_1) \geq 4$, we have that there is a component of $G \setminus \{r_1\}$ that contains neither v nor r_2. Thus, by Lemma A.4, $d_G(r_1) = m-1 = 4$; analogously, $d_G(r_2) = m-1 = 4$. Now, consider the two colourings $\sigma_1 = \psi(N(p), \psi(r_1) \leftrightarrow \psi(v))$ and $\sigma_2 = \psi(N(p), \psi(r_2) \leftrightarrow \psi(v))$



(a) Scheme for G when $m = 4$. (b) Scheme for G when $m = 5$.

Figure A.1: Schemes for an anomalous cactus.

and observe that if v is not adjacent to a vertex in $\{r_1, r_2\}$, then either σ_1 or σ_2 is a b-colouring of G with m colours where u and v have different colours, a contradiction. So, suppose, without loss of generality, that v is adjacent to r_1. Finally, note that if r_1 is not adjacent to r_2, by Lemma A.7, the neighbour of r_1 with colour $\psi(r_2)$ has degree one. Then, we get a contradiction from $\varphi = \psi(N(r_1) \cup N(x_2), \psi(v) \leftrightarrow \psi(r_2))$. Thus, Lemma A.6 implies that the cactus graphs that satisfy these properties are isomorphic to the anomalous graphs in Figure 3.3(a) or Figure 3.3(b).

Now, consider there is precisely one dense vertex, r, in $C_2 \setminus \{x_2\}$. We consider two cases based on whether $m = 4$ or $m = 5$. If $m = 4$, then vertices not in $D(G)$ have degree at most two in G; so, $d_G(v) = 2$. If there exists a neighbour x of x_2 in C_2 that is not adjacent to r, then we recolour x with $\psi(w)$ and v with $\psi(r)$, obtaining a b-colouring of H with m colours where u and v have different colours, a contradiction. So, $N(x_2) \cap C_2 \subseteq N(r)$ and, as x_2 must have a neighbour coloured with $\psi(r)$, we must have $(x_2, r) \in E(G)$. Now, if $d(x_2) > 3$, let $x \in N(r) \cap N(x_2)$; if $\psi(x) = \psi(w)$, just recolour v with $\psi(y)$; otherwise, exchange the colours $\psi(y)$ and $\psi(w)$ in $C_2 \cup \{v\}$. We get a contradiction as the obtained b-colouring has m colours and u and v have different colours. So, $d(x_2) = 3$ and Lemmas A.4 and A.6 give us that any cactus that satisfies these properties is isomorphic to the anomalous graph in Figure 3.2. Now, consider the case $m = 5$ and let r' be the dense vertex in C_v. First, note that if $x_2 \notin N(r)$, by Lemma A.7, as x_2 must have a neighbour coloured with $\psi(r)$, there exists $x \in N(x_2) \cap C_2$ isolated in $C_2 \setminus \{x_2\}$; thus, we can exchange the colours of x and v, a contradiction. So, $x_2 \in N(r)$. Also, by Lemma A.4 and the fact that at most one neighbour of x_2 is connected to r not through x_2, we have that $d_G(x_2) = m - 1$. Now, consider C_v. If v is not adjacent to r', then Lemma A.3 implies that all vertices in $C_v \setminus \{v\}$ are adjacent to r'. Therefore, the neighbour of v in C_v is not coloured $\psi(r')$.

We get a contradiction from $\varphi = \psi(C_2 \cup \{v\}, \psi(w) \leftrightarrow \psi(r'))$. Therefore, by Lemmas A.4 and A.6, any cactus graph that satisfies these properties is isomorphic to the anomalous graph in Figure 3.3(c) or Figure 3.3(d).

Finally, consider there are two dense vertices in $C_2 \setminus \{x_2\}$ other than x_2. Note that this implies $m = 5$. Moreover, Lemma A.3 implies $C_v = \{v\}$. Let r_1 and r_2 be the two dense vertices in $D(G) \setminus \{w, x_1, x_2\}$. First, note that if there exists a neighbour of x_2 isolated in $C_2 \setminus \{x_2\}$, say x, then: if $\psi(x) = \psi(w)$, just change the colour of v to $\psi(x_1)$; otherwise, exchange the colours of x and v. In both cases, we obtain a b-colouring of G with m colours, a contradiction. So, suppose that every neighbour of x_2 in C_2 is within a component of $C_2 \setminus \{x_2\}$ containing r_1 and/or r_2. Let $y_1, y_2 \in N(x_2)$ such that $\psi(y_i) = \psi(r_i)$, $i = 1, 2$. Suppose, first, that $r_i \notin N(x_2)$, $i = 1, 2$. By Lemma A.7, we have that $y_1 \in N(r_2)$ and $y_2 \in N(r_1)$. If r_1 and r_2 are in distinc connected components of $C_2 \setminus \{x_2\}$, say r_1 is in component Q, then $\psi(Q \cup \{v\}, \psi(r_2) \leftrightarrow \psi(w))$ gives us a contradiction. Otherwise, by Lemma A.7, there exists $x \in N(r_1) \cap N(r_2)$: if $\psi(x) \neq \psi(w)$, then $\psi(N(r_1) \cup \{v\}, \psi(r_2) \leftrightarrow \psi(w))$ gives us a contradiction; otherwise, we can apply $\psi(N(r_1) \cup N(r_2), \psi(x) \leftrightarrow \psi(w))$ and consider again the case where $\psi(x) \neq \psi(w)$. So, suppose that $r_1 \in N(x_2)$. By analogous arguments, we can suppose that r_1 and r_2 are in the same connected component of $C_2 \setminus \{x_2\}$. Thus, by Lemmas 3.1 and A.4 and the facts that x_2 has no isolated neighbours in $C_2 \setminus \{x_2\}$ and $d(r_i) \geq 4$, we have that $d_G(x_2) = d_G(r_1) = d_G(r_2) = 4$. If $r_2 \in N(x)$, by Lemmas A.4, A.6 and A.7, we obtain a structure as the one in Figures 3.3(e), Figure 3.3(f) or Figure 3.3(g). Otherwise, we know that there exists $x \in N(x_2) \cap N(r_1)$. Also, as $d_G(r_1) = 4$ and x must be coloured with $\psi(r_2)$, we have that $r_2 \notin N(r_1)$; so, $N(r_1) \cap N(r_2) \neq \emptyset$. So, let $z \in N(r_1) \cap N(r_2)$. If z is unique, then we can consider that $\psi(z) = \psi(y)$, otherwise, it suffices to apply $\psi(N(r_1 \cup N(r_2), \psi(y) \leftrightarrow \psi(w))$. So, $\psi(N(r_1) \cup \{v\}, \psi(r_2) \leftrightarrow \psi(w)$ gives us a contradiction. Thus, r_1 and r_2 have two common neighbours and the structure of any cactus graph that satisfies these properties is isomorphic to the anomalous graph in Figure 3.3(h). □

Proof of Theorem A.1 Let $m \geq 4$ and G be a minimal m-defective cactus. If G is anomalous, then G is isomorphic to a graph in Figure 3.2 or in Figure 3.3 which satisfies the theorem. So, suppose G is not anomalous. By Lemma A.5, we know that $|D(G)| = m$; thus, it remains to prove that $d_G(u) = m-1$, for all $u \in D(G)$. Suppose the contrary and let $u \in D(G)$ such that $d_G(u) \geq m$. By Lemma A.4, we know that there is no isolated vertex in $N(u)$. Also, as G is not anomalous and by Theorem A.2, we have that there are no edges between non-dense vertices, i.e., (I) $|N(v) \cap D(G)| \geq 2$, for all $v \in N(u) \setminus D(G)$. Now, let $v \in N(u) \setminus D(G)$ (there exists as $d(u) \geq m$ and $|D(G)| = m$). Denote by C_v the set of vertices of the connected component of $G \setminus \{u\}$ containing v and by C_u the set $V(G) \setminus C_v$. By Lemma 3.1, we know that u has at most one neighbour in $C_v \setminus \{v\}$; if such a neighbour exists, denote it by v'. Also, note that, as $d_G(u) \geq 4$ and by (I), we have that $|C_u \cap D(G)| \geq 1$. Suppose that $|C_u \cap D(G)| = 1$. Then, as $m \geq 4$, we have that $|C_v \cap D(G)| \geq 2$. Also, as $d_G(u) \geq 4$ and u has at most two neighbours in C_v, we have that u must have at least one neighbour x in $C_u \setminus D(G)$. In this case we consider x instead of v, i.e., we can suppose that (II) $|C_u \cap D(G)| \geq 2$. Now, let $H = G \setminus \{(u, v)\}$. Trivially, $m(H) = m$ and, consequently, $\chi_b(H) = m$. Let ψ be a b-colouring of H with m colours. If $\psi(v) \neq \psi(u)$, then ψ is also a b-colouring of G with m colours, a contradiction. Otherwise, let $z \in C_u \cap D(G)$ such that $\psi(z) \neq \psi(v')$ (exists by (II)); observe that $\psi(C_v, \psi(z) \leftrightarrow \psi(u)$ is also a b-colouring of G with m colours, a contradiction. □

Appendix B

Résumés des Chapitres

Dans cet annexe, nous faisons un résumé de chaque chapitre de résultats de cette thèse, donnant une idée générale des méthodes utilisés dans les preuves.

Si G n'a pas un ensemble bon, G est dit pivoté.

B.1 Chapitre 3

On dit que G est un *cactus* si G ne contient pas de deux cycles qui partagent une arrête. Dans ce chapitre, nous montrons que si G est un cactus et $m(G) \geq 7$, donc le nombre b-chromatic de G est au moins $m(G) - 1$. La preuve ressemble à celle des arbres: nous montrons comment trouver un ensemble des sommets denses de G qui peut jouer le rôle de basis d'une b-coloration en $m(G)$ couleurs, s'il en existe un, puis, si ce n'est pas le case, on obtient une b-coloration de G avec $m(G) - 1$ couleurs, sinon, on obtient une b-coloration de G avec $m(G)$ couleurs. Plus précisement, nous prouvons les suivants principaux résultats:

Théorème B.1. *Soit G un cactus avec $|D(G)| > m(G)$ et soit $W \subseteq D(G)$ un sous-ensemble de cardinalité $m(G) + 1$ contenant tous les sommets dont le degré supérieur à $m(G) - 1$. Alors, G n'est pas un ensemble bon si et seulement si $|M(G)| = m(G) + 1$ et:*

(I) la structure de W est représentée à la Figure B.1 ou à la Figure B.2, ou

(II) il existe des sommets $u, v \in W$ de degré $m(G) - 1$ et $w \notin W$ tels que $\langle u, v, w \rangle$ est un cycle et chaque sommet de W est adjacent à u ou à v.

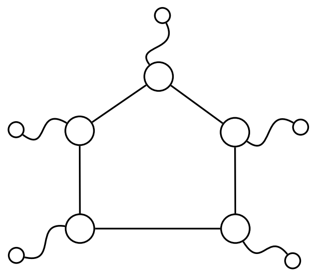


Figure B.1: Dans ce graphe, $m(G) = 4$, W est représenté par les sommets plus gros et $d(u) = 3$, pour tous $u \in W$.

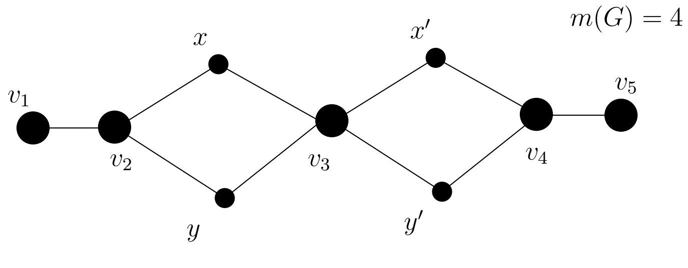


Figure B.2: Dans ce graphe, $m(G) = 4$, W est représenté par les sommets plus gros et $d(v_2) = d(v_4) = 3$.

Initielement, nous voulions prouver que tout cactus G ayant un ensemble bon pourrait être b-coloré en $m(G)$ couleurs. Malheureusement, ce n'est pas vrai à cause de l'existence des graphes sur les Figures B.3 et B.4. Nous disons que ces graphs sont *anormaux*.

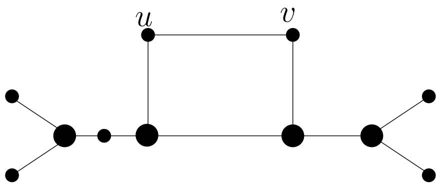


Figure B.3: Graphes anormaux avec 4 sommets denses.

Finalement, nous avons prouvés les deux theorèmes suivants concernant le nombre b-chromatic des cactus.

Théorème B.2. *Si G est un cactus et G n'a pas un ensemble bon ou G est anormal, alors $\chi_b(G) = m(G) - 1$.*

Théorème B.3. *Si G est un cactus qui a un ensemble bon et $m(G) \geq 7$, alors $\chi_b(G) = m(G)$.*

Nous avons, donc, généralisé le résultat sur les arbres par Irving et Manlove pour les cactus avec m-degré au moins 7. Nous donnons aussi un algorithme qui trouve une b-coloration optimale d'un tel cactus. Nous conjecturons que si G a un ensemble bon et G n'est pas anormal, alors $\chi_b(G) = m(G)$. Il reste à prouver cette conjecture pour les valeurs de m-degré 4, 5 et 6. Remarquez que, si cela est vrai, alors $\chi_b(G) \geq m(G) - 1$, pour tout cactus G. Dans l'Annexe A, nous montrons que, étant donné un entier m, si G est un cactus minimal tel que $m(G) = m$ et $\chi_b(G) < m$, alors $|D(G)| = m(G)$, $d(v) = m(G) - 1$, pour tout $v \in D(G)$, et $G \subseteq D(G) \cup N(D(G))$. De plus, si $(u, v) \in E(G)$, pour $u, v \in V(G) \setminus D(G)$, alors G est un cactus anormal minimal (i.e., G est un des graphes réprésenté sur les Figures B.3 et B.4).

B.2 Chapitre 4

Un graphe G est dit *planaire extérieur* si G a un plongement sur le plan de sorte que deux arêtes ne se croisent pas et tous les sommets sont situés sur

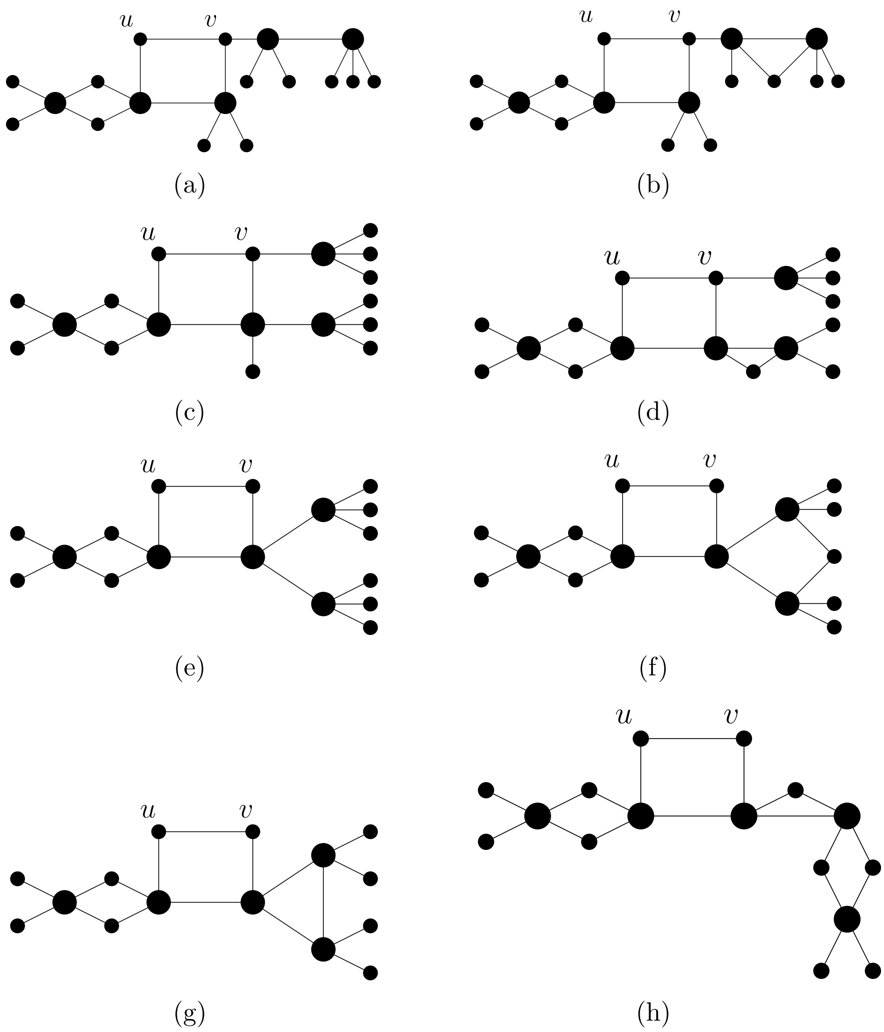


Figure B.4: Graphes anormaux avec 5 sommets denses.

la même face. La *maille* d'un graphe G est la longueur du plus petit cycle de G. Dans ce chapitre, nous considérons les planaires extérieurs dont la maille est au moins 8. Notre résultat principal est le suivant:

Théorème B.4. *Soit G un graphe planaire extérieur de maille au moins 8. Alors $\chi_b(G) \geq m(G) - 1$. En outre, nous pouvons déterminer la valeur de $\chi_b(G)$ (et une b-coloration avec $\chi_b(G)$ couleurs) en temps polynomial.*

Considerons G comme un graphe planaire extérieur de maille au moins 8. La preuve a la même structure de la preuve de Irving et Manlove pour les arbres:

1. Nous prouvons que G est pivoté (n'a pas un ensemble bon) si et seulement si $|D(G)| = m(G)$ et $D(G)$ encercle un sommet de $V(G) \setminus D(G)$. En même temps, la preuve donne aussi un algorithme pour trouver un ensemble bon de G, si un existe.

2. Nous montrons comment colorier G avec $m(G) - 1$ couleurs, dans le cas où G est pivoté.

3. Si G a un ensemble bon W, nous montrons comment obtenir une b-coloration avec $m(G)$ couleurs qui a W comme base.

En fait, les preuves des 2 premiers étapes n'utilisent pas le fait que G est planaire extérieur. Donc, il est vrai pour un graphe quelconque dont la maille est au moins 8.

Notez que tout cactus est également un graphe planaire extérieur; donc, ce résultat généralise le résultat présenté dans le Chapitre 3, mais seulement pour les cactus de maille au moins 8 (et, donc, est une autre généralisations du résultat par Irving et Manlove sur les arbres). La complexité de la preuve présentée dans le chapitre 3 nous indique qu'il pourrait nécessiter un effort beaucoup plus élevé pour généraliser les résultats présentés dans ce chapitre pour les graphes planaires extérieurs général. En outre, comme nous avons souligné dans la Section 5.1, ce résultat ne peut pas être généralisé pour les graphes série-parallèle, qui est une superclasse des graphes planaires extérieurs. D'autre part, nous avons prouvé que un graphe G général de maille au moins 8 qui soit pivoté a nombre b-chromatic égal a $m(G) - 1$. Jusqu'à présent, nous n'avons pas trouvé un exemple d'un graphe non-pivoté de maille au moins 8 qui ne peut pas être b-coloré avec $m(G)$ couleurs. Nous rappelons que si G a maille au moins 11 et G est non-pivoté, alors $\chi_b(G) = m(G)$ (Corollaire 2.18).

B.3 Chapitre 5

Un graphe bloc est un graphe dont les blocs sont des cliques. Dans ce chapitre, nous construisons un graphe bloc qui a la différence $m(G) - \chi_b(G)$ arbitrairement grande. La construction peut aussi produire de graphes bloc sans griffes (qui sont les graphes lignes des arbres) et des graphes série-parallèles (qui sont une superclasse des graphes planaires extérieurs). Comme conséquence, nous savons que la structure d'arbre n'aide pas toujours à borner la différence $m(G)-\chi_b(G)$ et que le résultat présenté dans le Chapitre 4 ne peut pas être généralisé pour les graphes série-parallèles.

Nous avons aussi étudier les problèmes de décision suivants:
k,`b-Coloration Fixé`
-Entrée: graphe G
-Question: existe-t-il une b-coloration de G avec k couleurs?

k,`b-Coloration`
-Entrée: graphe G, entier positif k
-Question: existe-t-il une b-coloration de G avec k couleurs?

Si $u \in V(G)$ a le degré au moins $k-1$, nous disons que u est k-dense et nous notons l'ensemble des sommets k-dense de G par $D_k(G)$. Nous avons généraliser la notion de sommets encerclé pour couvrir des cliques entières qui peuvent être encerclés. Si $W \subseteq D_k(G)$ de cardinalité k ne "encercle pas de clique", nous disons que W est un *ensemble non-bloqué*. Finalement, $u \in V(G) \setminus W$ est un *sommet latéral de* W si u est un sommet de liaison de W mais qui n'est pas contenu dans un lien induit.

Nous avons prouvé que k,`b-Coloration Fixé` peut être résolu en temps polynomial. Quant au problème k,`b-Coloration`, nous avons analysé la possibilité d'obtenir une b-coloration avec k couleurs quand on nous donne un ensemble non-bloqué W. Nous avons découvert que ce que difficulte la construction d'une b-coloration de G qui a W comme base sont les sommets latéraux de W. Plus précisement, nous avons prouvé que:

Théorème B.5. *Soit G un graphe bloc et $W \subseteq D_k(G)$ un ensemble non-bloqué, $k \in [\omega(G)+1, m(G)]$. Si W n'a pas de sommets latéraux, alors il existe une b-coloration de G avec k couleurs.*

Comme conséquence, nous avons les corollaires suivants.

Corollaire B.1. *Soit G un graphe bloc et k un entier positif, $k \in [\omega(G) + 1, m(G)]$. S'il existe $W \subseteq D_k(G)$ de cardinalité k tel que W est un stable, alors il y a une b-coloration de G en k couleurs.*

Corollaire B.2. *Soit G un graphe bloc et k un entier positif, $k \in [\omega(G) + 1, m(G)]$. Si $|D_k(G)| > \Delta^2 + \Delta$, alors il y a une b-coloration de G en k couleurs.*

Corollaire B.3. *Soit G un graphe bloc et notons $m(G)$ par m. Si $|D(G)| > m^2 + m$, alors $\chi_b(G) = m$.*

Finalement, nous avons étudié la possibilité de l'existence d'au plus un sommet latéral de W dans chaque bloc de G. Nous voudrions répondre à la question: "étant donné un tel ensemble W, est-ce qu'on peut construire une b-coloration de G avec $|W|$ couleurs en utilisant W"? En fait, ce n'est pas toujours possible; observez, par exemple, le graphe de la Figure B.5. Si W a une structure comme présentée dans la figure, nous l'appelons *nid*. Nous avons prouvé que les nids sont les seules exceptions à cette question quand le graphe d'entrée est un graph bloc sans griffes.

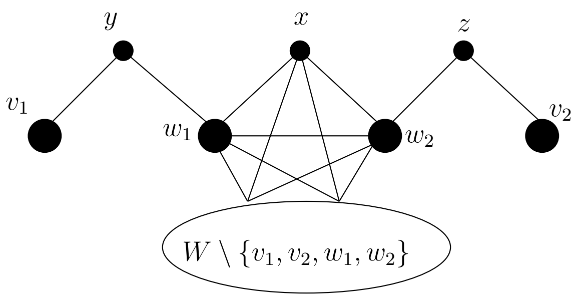


Figure B.5: Representation d'un nid.

Théorème B.6. *Soit G un graphe bloc sans griffes et $W \subseteq D_k(G)$ un ensemble non-bloqué, $k \in [\omega(G) + 1, m(G)]$. Si chaque bloc de G a au plus un sommet latéral de W et W n'est pas une nichée, alors il existe une b-coloration de G en k couleurs.*

B.4 Chapitre 6

Nous avons prouvé que si T est un arbre et G est une chaîne de longueur supérieure à 4, ou un cycle de longueur supérieure à 3 ou une étoile, alors

$\chi_b(G\square T) \geq m(G\square T)-1$. Nous donnons également des algorithmes en temps polynomial pour trouver une b-coloration optimale de ces graphes. Notons H le graphe $G\square T$. L'idée générale pour la construction d'une b-coloration de H est la même que pour les planaires extérieurs: nous trouvons un ensemble bon W, colorons chaque sommet de W avec une couleur différente, puis nous colorons les sommets de liasion de W et, finalement, nous étendons cette precolation à une b-coloration de H avec $m(H)$ couleurs. On remarque qu'un aspect important de trouver cette b-coloration est l'existence d'un algorithme qui colore les liens interne a T^i en n'utilisant que des couleurs des sommets de $W \cap V(T^i)$, pour chaque copie T^i de T. Une question intéressante serait d'essayer de prouver ces résultats pour certaines classes de graphes où nous pouvons faire la même chose. Par exemple, si G est planaire extérieur, est-ce que $\chi_b(G \square P_k) \geq m(G \square P_k) - 1$ est valide?

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