# Depolarization's Dynamic: Exponential and $q$-Exponential Decay

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***Abstract*— Light polarization is an important property that has being extensively used for quantum communication purposes. However, light polarization is also fragile and during propagation in a noisy channel the light becomes partially or even completely unpolarized, making impossible the realization of quantum protocols without polarization control. In this direction, the present work studies the dynamic of the depolarization of some noisy channels found in the literature identifying when the polarization vanishes exponentially and *q*-exponentially.**

***Keywords*— light polarization, depolarization, noisy channel, q-exponential**

## I. Introduction

Quantum polarization is an important and useful property that permits the designing and implementation of quantum communication protocols in optical networks. For example, some quantum key distribution (QKD) setups encode the information in light polarization. On the other hand, in QKD setups where the information is coded in the phase of the light pulse, the polarization has to be controlled in order to permit good interference [1-4]. A light pulse can be polarized, partially polarized or unpolarized. The amount of polarization is measured by the degree of polarization, DOP. Classically, the DOP is measured or calculated using the Stokes parameters. However, the classical DOP is not a good measure of polarization for the highly quantum states used in quantum communication, like single-photon or two-photon pulses. In these cases, the quantum DOP has to be used. In what concerns quantum communication, the lower the DOP the higher is the error rate. Therefore, it is important to determine the dynamic of the depolarization in order to know how fast a noisy channel depolarizes an initially polarized light pulse. In this direction, the present work studies the dynamic of depolarization of some known noisy channels found in the literature aiming to identify if the polarization vanishes exponentially or $q$-exponentially, as well determining the amount of time required for the DOP to reach a given value. The $q$-exponential function provided by Tsallis is given by

$$e_q^{\,z} = \begin{cases} e^z & q = 1 \\ \left[1+(1-q)z\right]^{1/(1-q)} & q \neq 1 \;\&\; 1+(1-q)z \geq 0 \\ 0 & q \neq 1 \;\&\; 1+(1-q)z < 0 \end{cases} \quad (1)$$

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This work is outlined as follow: In Section II, the main concepts of quantum polarization used in this work are reviewed. In Section III, the depolarization caused by some channel models are analyzed. At last, the conclusions are drawn in Section IV.

## II. Mathematical Tools

When quantum polarization is considered, one has to use the quantum version of the Stokes parameters [5,6]:

$$\hat{S}_0 = \hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2, \quad (2.1)$$

$$\hat{S}_1 = \hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2, \quad (2.2)$$

$$\hat{S}_2 = \hat{a}_1^+\hat{a}_2 + \hat{a}_2^+\hat{a}_1, \quad (2.3)$$

$$\hat{S}_3 = i\left(\hat{a}_2^+\hat{a}_1 - \hat{a}_1^+\hat{a}_2\right), \quad (2.4)$$

$$\left[\hat{S}_2, \hat{S}_3\right] = i2\hat{S}_1. \quad (2.5)$$

In (2.1)-(2.5) $\hat{a}_1$ ($\hat{a}_1^\dagger$) is the annihilation (creation) operator of the horizontal mode while $\hat{a}_2$ ($\hat{a}_2^\dagger$) is the annihilation (creation) operator of the vertical mode. Equation (2.5) and its cyclic versions imply that it is not possible to know, with total accuracy, any pair of Stokes parameters simultaneously. Hence, quantum polarization cannot be represented by only a point on the Poincaré sphere. In order to apply a phase shift $\phi$ between two linearly polarized modes, the unitary operator $U_\phi = exp(i0.5\phi\hat{S}_1)$ is used. On the other hand, a geometric rotating of $\theta$ in the polarization is achieved by the application of the unitary operator $U_\theta = exp(i\theta\hat{S}_3)$.

Classically, a light pulse is unpolarized if its Stokes parameters vanish. When quantum light is considered, that condition (in average) is necessary but not sufficient. From a quantum optics point of view, a light beam can be considered unpolarized if its observable properties remain unchanged after an application of a geometric rotating and/or a phase shift between the two linearly polarized components. These conditions are mathematically described by [7-9]:

$$\left[\rho, \hat{S}_3\right] = \left[\rho, \hat{S}_1\right] = 0. \quad (3)$$

The most general form of an unpolarized state is [7-9]:

$$\rho = \sum_n p_n \frac{1}{n+1}\sum_{k=0}^{n} |k\rangle|n-k\rangle\langle k|\langle n-k|. \quad (4)$$

The first attempt to quantify how much polarized a quantum light is, was proposed in [10] using the $Q$ function based on SU(2) coherent states:

$$Q(\theta,\varphi)=\sum_{n=0}^{\infty}\frac{n+1}{4\pi}\langle n,\theta,\varphi|\rho|n,\theta,\varphi\rangle, \tag{5}$$

$$|n,\theta,\varphi\rangle=\sum_{m=0}^{n}\binom{n}{m}^{1/2}\left[\sin\left(\frac{\theta}{2}\right)\right]^{n-m}\left[\cos\left(\frac{\theta}{2}\right)\right]^{m}e^{-im\varphi}|m\rangle|n-m\rangle. \tag{6}$$

The quantum degree of polarization is then given by

$$D_Q=4\pi\int\left[Q(\theta,\varphi)-\frac{1}{4\pi}\right]^2\sin(\theta)\,d\theta d\varphi, \tag{7}$$

$$P_Q=\frac{D_Q}{1+D_Q},\quad 0\le P_Q\le 1. \tag{8}$$

In eq. (7) $1/(4\pi)$ is the $Q$ function of the unpolarized light. As can be seen, $P_Q$ is only a normalization of the distance between the pseudo-distribution ($Q$ function) of the light whose polarization one wants to measure and the pseudo-distribution of the unpolarized light. Hence, one can also consider $D_Q$ as a DOP.

Another quantum DOP is the minimal distance between the density matrix of the light whose polarization one wishes to measure and the density matrix of an unpolarized light. The distance commonly used is the Hilbert-Schmidt metric: $D_{HS}(\rho_1,\rho_2) = \mathrm{Tr}[(\rho_1-\rho_2)^2]$. Thus, the quantum DOP is defined as [11]:

$$P_{HS}(\rho)=\min_{\rho_{unp}\in U}D_{HS}(\rho,\rho_{unp})=\min_{\rho_{unp}\in U}Tr\left[(\rho-\rho_{unp})^2\right], \tag{9}$$

where $U$ is the set of all possible unpolarized light. Now using (4) in (9) one can get the following expression for $P_{HS}$:

$$\begin{aligned}P_{HS}(\rho)&=Tr(\rho^2)-\sum_{n=0}^{\infty}\frac{\left[\sum_{k=0}^{n}\langle k,n-k|\rho|k,n-k\rangle\right]^2}{n+1}=\\&=Tr(\rho^2)-Tr(\rho_{unp}^{opt\,2}).\end{aligned} \tag{10}$$

Equation (10) shows that the quantum DOP depends on how much pure is the quantum light state (measured by $\mathrm{Tr}(\rho^2)$) and the total photon number distribution.

## III. Dynamic of the Depolarization

The light depolarization depends on the channel's properties and on the quantum light state at the channel's input. For example, for the same noisy channel, coherent light, single-photon and two-photon pulses will experiment different dynamics of depolarization. Let us start by considering the depolarization of a single-photon pulse. A general single-photon light state can be written in the following way:

$$\rho=(1-\xi)\frac{\left[|01\rangle\langle 01|+|10\rangle\langle 10|\right]_{HV}}{2}+\xi|\psi\rangle\langle\psi|, \tag{11}$$

$$|\psi\rangle=\cos(\lambda)|10\rangle_{HV}+e^{i\mu}\sin(\lambda)|01\rangle_{HV}. \tag{12}$$

The average value of the Stokes parameters $\rho$ in eq. (10) are $\langle\hat{S}_0\rangle=1$, $\langle\hat{S}_1\rangle=\xi cos(2\lambda)$, $\langle\hat{S}_2\rangle=\xi sin(2\lambda)cos(\mu)$, $\langle\hat{S}_2\rangle=\xi sin(2\lambda)sin(\mu)$, and its classical DOP, $P_{class}$, is

$$P_{class}=\frac{\sqrt{\langle S_1\rangle^2+\langle S_2\rangle^2+\langle S_3\rangle^2}}{\langle S_0\rangle}=\xi. \tag{13}$$

On the other hand, the quantum DOP using (7)-(10) are given by:

$$P_{HS}=\frac{1}{2}e_{1/2}^{-2(1-\xi)}, \tag{14}$$

$$P_Q=\frac{1}{3}\xi^2 e_2^{-\frac{1}{3}\xi^2}. \tag{15}$$

The exact dynamic of the depolarization depends on how $\xi$ decreases toward zero during light propagation, however, the $q$-exponential behavior is clear in eqs. (14)-(15). Now, using the quantum noisy channel described in [10], one has the following DOP

$$D_Q(t)=\frac{3}{4}e^{-16\nu t}e_0^{\frac{1}{15}e^{-32\nu t}}, \tag{16}$$

where ν is a parameter that models the velocity of the depolarization. For the same channel, the DOP for the states $|\beta_2\rangle = (|2,0\rangle+|0,2\rangle)_{HV}/2^{1/2}$ and $|\beta_3\rangle = (|3,0\rangle+|0,3\rangle)_{HV}/2^{1/2}$ are, respectively,

$$D_Q(t)=\frac{1}{5}e^{-48\nu t}, \tag{17}$$

$$D_Q(t)=\frac{1}{5}e^{-48\nu t}e_0^{\frac{2}{7}e^{-48\nu t}}. \tag{18}$$

As it can be noted, $|\beta_2\rangle$ and $|\beta_3\rangle$ have the same initial polarization state, they are linearly polarized in $\pi/4$. However, $|\beta_2\rangle$ suffers an exponential depolarization while $|\beta_3\rangle$ suffers an $q$-exponential depolarization.

Now, using the channel discussed in [12] and the DOP proposed by them, one has the following DOPs when the input states are, respectively, the states $|HH\rangle$ and $(|HV\rangle+|VH\rangle)/2^{1/2}$:

$$P(t)=\frac{1}{2}e^{-2\gamma_1 t}e_0^{2e^{-2(\gamma_2-\gamma_1)t}}, \tag{19}$$

$$P(t)=\frac{1}{2}e^{-(\gamma_1+\gamma_2)t}. \tag{20}$$

In (19)-(20) $\gamma_1$ and $\gamma_2$ are, respectively, the decoherence rates in channels 1 and 2 (each photon is sent through a different channel). One may note that both states $|HH\rangle$ and $(|HV\rangle+|VH\rangle)/2^{1/2}$ are two-photon states, however, the first one is disentangled while the last one is maximally entangled, showing the dynamic of the depolarization is not only dependent on the photon number distribution but also on the entanglement of the input state. The disentangled state experiments a $q$-exponential depolarization dynamic while the entangled state experiments an exponential depolarization dynamic.

One maybe interested in determining the channel parameter's value that will provide the maximal depolarization acceptable. This means to invert the equation that models the dynamic. When the dynamic is exponential, this task is trivially realized by using the logarithm function. On the other hand, the inversion of the $q$-exponential dynamic may require a more complex mathematical tool (that depends on the value of $q$). Equations (15), (16), (18) and (19) can be inverted using the Lambert-Tsallis $W_q$ function, the solution of $W_q(z)\exp_q[W_q(z)] = z$ [13]. For example, if

$$D=Ae^{-\alpha t}e_0^{\frac{B}{A}e^{-(\beta-\alpha)t}} \tag{21}$$

then

$$t=-\frac{1}{\beta-\alpha}\log\left(\frac{A}{B}\frac{\alpha}{\beta-\alpha}W_{1-\frac{\alpha}{\beta-\alpha}}\left(\frac{\beta-\alpha}{\alpha}\left(\frac{D}{A}\right)^{(\beta-\alpha)/\alpha}\right)\right), \qquad (22)$$

where $q = 1 - \alpha/(\beta-\alpha)$). For example, the inverse of (16) is

$$t=-\frac{1}{32\nu}\log\left\{\frac{15}{2}W_{1/2}\left[\frac{32}{135}D_Q^2\right]\right\}. \qquad (23)$$

Now, let us consider a coherent state propagating in a polarization maintaining (PM) optical fiber. In this case the input state is $|\alpha,0\rangle_{HV}$ (one of the axes of the fiber is considered to be the horizontal polarization) and the optical power is proportional to the mean photon number $|\alpha|^2$. The DOP of $|\alpha,0\rangle_{HV}$ is given by

$$P_Q=1-\frac{4|\alpha|^2}{1+2|\alpha|^2\left(1+|\alpha|^2\right)-e^{-2|\alpha|^2}}. \qquad (24)$$

As one can note in eq. (24), the polarization depends on the optical power, hence, the depolarization will be caused by the optical loss of the PM fiber. Since we are interested in the low photon number regime, we can use $\exp(x) \sim 1 + x$ in (24), obtaining

$$P_Q \sim 1-e_2^{-\frac{|\alpha|^2}{2}}. \qquad (25)$$

Thus, the depolarization dynamic of the coherent state in the low photon number regime, due to optical losses, is approximately $q$-exponential. The coherent state becomes less polarized when the optical power decreases because the state $|0,0\rangle_{HV}$ is completely unpolarized.

## IV. CONCLUSIONS

As observed in the cases here described, two types of depolarization dynamic were considered: exponential decay, eqs. (16) and (19) and $q$-exponential decay, eqs. (13), (14), (15), (17) and (18). The exponential decay is trivial and it does not deserve much explanation. The $q$-exponential decay, by its turn, is more complex and it may lead someone to believe, for example, that polarization sudden death may occur. However, observing eq. (22), one can easily see that, when $D$ tends to 0, $t$ tends to $\infty$ ($W_q(0) = 0$ for any $q$). Hence, the polarization sudden death does not occur for the examples here considered.

## REFERENCES


[1] V. Scarani, H. Bechmann-Pasquinucci, N. J. Cerf, M. Dušek, N. Lütkenhaus, and M. Peev, The security of practical quantum key distribution, Rev. Mod. Phys. 81, 1301–1350, 2009.

[2] K. Inoue, Differential phase-shift quantum key distribution systems, IEEE Sel. Top. in Quant. Elec., 21, 3, 6600207, 2015.

[3] F. Laudenbach, C. Pacher, C.-H. F. Fung, A. Poppe, M. Peev, B. Schrenk, M. Hentschel, P. Walther, H. Hübel, Continuous-Variable Quantum Key Distribution with Gaussian Modulation — The Theory of Practical Implementations, Adv. Quantum Technol.,1, 1800011/1-37, 2018.

[4] Y.-P. Li, W. Chen, F.-X. Wang, Z.-Q. Yin, L. Zhang, H. Liu, S. Wang, D.-Y. He, Z. Zhou, G.-C. Guo, Z.-F. Han, Experimental realization of a reference-frame independent decoy BB84 quantum key distribution based on Sagnac interferometer, Optics Letters, 44, 18, 4523-4526, 2019.

[5] A. S. Chirkin, A. A. Orlov, and D. Y. Paraschuk, Quantum theory of two-mode interactions in optically anisotropic media with cubic nonlinearities: Generation of quadrature- and polarization-squeezed light, Quantum Electron., 23, 10, 870, 1993.

[6] P. Usachev, J. Söderholm, G. Björk, A. Trifonov, Experimental verification of differences between classical and quantum polarization properties, Opt. Commun., 193, 161-173, 2001.

[7] G. S. Agarwal, J. Lehner, H. Paul, Invariances for states of light and their quasi-distributions,, Opt. Commun., 129, 369-372,1996.

[8] J. Söderholm, G. Björk, A. Trifonov, Unpolarized light in quantum optics, Opt. and Spectroscopy, 91, 532–534, 2001.

[9] J. Lehner, U. Leonhardt, H. Paul, Unpolarized light: Classical and quantum states, Phys. Rev. A, 53, 2727-2735, 1996.

[10] A. Luis, Degree of polarization in quantum optics, Phys. Rev. A, 66, 13806/1-8, 2002.

[11] A. B. Klimov, L. L. Sánchez-soto, E. C. Yustas, J. Söderholm, and G. Björk, Distance-based degrees of polarization for a quantum field, Phys. Rev. A, 72, 033813/1-6, 2005.

[12] A. B. Klimov, J. L. Romero, L. L. Sanchez-Soto, A simple quantum model for light depolarization, Opt. Soc. Am. B, 23, 1, 126-133, 2006.

[13] G. B. da Silva, R. V. Ramos, The Lambert-Tsallis $W_q$ function, Phys. A: Statistical Mechanics and its Applications, 525, 164-170, 2019.