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CAMILO EDUARDO ECHEVERRY NARANJO

THEORY OF LIGHT SCATTERING ON A PARABOLOID OF REVOLUTION

FORTALEZA

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Dissertação apresentada ao Curso de Mestrado Acadêmico em Física do Programa de Pós-Graduação em Física do Centro de Ciências da Universidade Federal do Ceará, como requisito parcial à obtenção do título de mestre em Física. Área de Concentração: Física da materia condensada.

Orientador: Prof. Dr. Carlos Lenz Cesar.

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A mis padres que me apoyaron en mis estudios y me permitieron estudiar en el exterior. Sin ellos esto no seria posible

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RESUMO

Uma teoria para espalhamento de luz em um parabolóide de revolução é desenvolvida pelo uso de vetores Hertz. A expansão de uma onda plana escalar viajando em qualquer direção foi encontrada em termos de funções separáveis que são soluções da equação escalar de Helmholtz. Este resultado foi usado para encontrar a expansão de ondas planas e um campo focalizado. A solução da equação parabólica de Helmholtz é expressa em termos de funções de Laguerre semelhantes à solução do átomo de hidrogênio em coordenadas parabólicas. Muitas identidades relacionadas a essas funções são demonstradas incluindo um teorema de multiplicação. Um método para resolver as condições de contorno também é apresentado no qual os coeficientes correspondentes dos campos espalhado e incidente são calculados.

Palavras-chave: espalhamento de luz; coordenadas parabolicas rotacionais; equação de Helmholtz; polinomios de Laguerre; NSOM.

ABSTRACT

A theory for light scattering on a paraboloid of revolution is developed by the use of Hertz vectors. The expansion of a scalar plane wave traveling on any direction was found in terms of separable functions which are solutions of the scalar Helmholtz equation. This result was used to find the expansion of plane waves and a focused field. The solution of the parabolic Helmholtz equation is expressed in terms of Laguerre functions similar to the solution of the hydrogen atom in parabolic coordinates. Many identities related to those functions are demonstrated including a multiplication theorem. A method for solving the boundary conditions is also presented in which the corresponding coefficients of the scattered and incident fields are calculated.

Keywords: light scattering; parabolic rotational coordinates; Helmholtz equation; Laguerre polynomials; NSOM.

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LIST OF ABBREVIATIONS AND ACRONYMS

BSC	Beam Shape Coefficients
GLMT	Generalized Lorentz-Mie Theory
PWE	Partial Wave Expansion
TENOM	Tip-Enhanced Near-field Optical Microscopy
VSWF	Vector Spherical Wave Functions

LIST OF SYMBOLS

$L_n^m(x)$	Associated Laguerre polynomials
$U_n^m(x)$	Second solution of the Associated Laguerre differential equation
$S_n^m(x)$	Pinney function of first kind
$V_n^m(x)$	Pinney function of second kind
$\vec{E}(\vec{r})$	Electric field
$\vec{H}(\vec{r})$	Electric field
$\vec{\pi}$	Hertz vector
$\vec{M}_{nm}(\vec{r})$	First expansion vector (for magnetic fields)
$\vec{N}_{nm}(\vec{r})$	Second expansion vector (for electric fields)

SUMMARY

1 INTRODUCCTION

In 1908 Gustav Mie published an article about light scattering of spherical particles while studying the optics of cloudy media(MIE, 1908). While the solution for light scattering on spherical particles of any size is commonly attributed to him, the Danish physicist Ludvig Lorenz solve this problem eighteen years earlier yet his work was largely ignored (A., 2013; GOUESBET, 2017; LILIENFELD, 2004). However both solutions consider only the scattering of plane waves by spherical particles, thus the theory is called Lorenz-Mie Theory.

The solution given by Mie and Lorenz is given in a expansion of Vector Spherical Wave Functions (VSWF) also called Partial Wave Expansion (PWE). The incident wave is expanded in VSWF and the scattered and internal fields are expressed in this expansion as well with unknown coefficients. These coefficients are then determined by the boundary conditions. Although the coefficients can be found, the calculation of this expansion was rather cumbersome due to the lack of computers by that time.

Due to the advent of lasers the plane wave model became unrealistic and a generalization of Lorenz-Mie scattering was needed. This generalization presented two problems: the calculation of the scattering and internal coefficients in terms of any incident beam coefficients and the calculation of the coefficients of the incident beam which are usually referred in the literature as Beam Shape Coefficients (BSC). The former was easily solved (NEVES; CESAR, 2019; GOUESBET, 2017; MOREIRA *et al.*, 2016) while the Latter could only be solved in special cases (NEVES *et al.*, 2006a; NEVES *et al.*, 2006b; NEVES; CESAR, 2019). The general procedure to calculate the BSC involves the Fourier transform of the beams and was developed by (MOREIRA *et al.*, 2016).

The theory of light scattering for other geometric shapes is not as well developed as the sphere. The other exactly solvable case is the plane wave scattering for an infinite cylinder (A., 2013). The cylinder case was treated with gaussian beams (GOUESBET, 1995; LOCK, 1997). The scattering by spheroidal particles was first attacked by Asano and Yamamoto (ASANO; YAMAMOTO, 1975). They run into a particular difficulty, two of the three separable functions obtained by solving the scalar Helmholtz equation in spheroidal coordinates depend on the magnitude of the wavevector k:

$$k = \frac{w}{c} n \tag{1.1}$$

where w is the angular frequency, c the speed of light and n the refractive index. This implies that it is no possible to eliminate the dependence in one variables after applying boundary conditions by factoring out the corresponding function since the index of refraction of the spheroidal particle is in general different from the medium. This problem was solved by expanding the corresponding function in terms of Legendre polynomial's. This expansion included the dependence on n only on the expansion coefficients. The price to pay was that the resulting system of algebraic equations involving the wave vector expansion coefficients became an infinite system of linear algebraic equations. The approximate solution can be obtained by truncating the series and choosing a sufficient number of equations.

Asano and Yamamoto used the Debye potentials to approach the problem of scattering by spheroidal particles, the same approach used in Generalized Lorentz-Mie Theory (GLMT). The Hertz vectors are used when dealing with infinite cylinders and a combination of Debye and Hertz vectors can also be used to treat the spheroidal particles case (LUK'YANCHUK *et al.*, 2015).

The literature on light scattering by a paraboloid of revolution is scarce. There appears to be a solution in a unreachable article in Russian (KLESHCHEV, 2012). An old and pretty much forgotten article on the subject is given by Horton "On the Diffraction of a Plane Electromagnetic Wave by a Paraboloid of Revolution"(HORTON; KARAL, 1951). This article relies heavily on other largely forgotten article "Laguerre Functions in the Mathematical Foundations of the Electromagnetic Theory of the Paraboloidal Reflector"written by Edmund Pinney (PINNEY, 1946). As the name implies Pinney developed this theory to apply it to a Paraboloidal Reflector (PINNEY, 1947). Pinney's article presents a second solution for the generalized Laguerre equation which was barely mentioned in an article (ZEPPENFELD, 2009) and in Bulchholz book (BUCHHOLZ, 1969), this solution is practically nonexistent in the literature.

This is a shame since the paraboloid can be used to model a sharp tip commonly used in Tip-Enhanced Near-field Optical Microscopy (TENOM) (HARTSCHUH, 2008). Thus the problem of light scattered by a paraboloid of revolution has an important application in a field that seeks for the best way to the amplify electromagnetic field near the tip. It's solution can be used to predict such enhancement. Moreover it can be used to predict resonances of various contributions to the field enhancement namely plasmon and anntena resonance which can be used to choose a laser wavelength that best exploit these resonances for a certain tip size radius.

Hoping that we can use this theory in TENOM experiments; the theory developed by Pinney is reviewed. Some results obtained by Bulchholz and Horton are also considered (HORTON; KARAL, 1951; BUCHHOLZ, 1969). New identities are obtained in particular an expansion obtained using a formula known as Hardy-Hille it is possible to expand a plane wave polarized along the z axis in terms of parabolic functions defined by Pinney:

$$\sum_{n=0}^{\infty}\binom{n+\alpha}{n}^{-1}L_n^{\alpha}(x)L_n^{\alpha}(y)w^n$$
$$=\Gamma(\alpha+1)\frac{e^{\frac{-(x+y)w}{1-w}}}{1-w}(-xyw)^{-\frac{\alpha}{2}}J_{\alpha}\left(\frac{2(-xyw)^{1/2}}{1-w}\right) \tag{1.2}$$

where the parabolic function defined by Pinney is

$$S_n^m(x)=x^{m/2}e^{x/2}L_n^m(x) \tag{1.3}$$

The Hardy-Hille formula can be manipulated to find

$$J_{\alpha}\left((xy)^{1/2}\sin(\theta)\right)e^{-\frac{(x+y)\cos(\theta)}{2}}$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\frac{\tan^{\alpha+2n}\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}(-1)^nS_n^{\alpha}(x)S_n^{\alpha}(y) \tag{1.4}$$

These results can be used to find the expansion of plane waves in terms of solutions of the Helmholtz equation in parabolic coordinates:

$$e^{i\vec{k}.\vec{r}}=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{i^m\varepsilon_m\Gamma(n+1)}{\Gamma(n+|m|+1)}\frac{tan^{2n+|m|}\left(\frac{\theta_k}{2}\right)}{cos^2\left(\frac{\theta_k}{2}\right)}(-1)^nS_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im(\varphi-\varphi_k)} \tag{1.5}$$

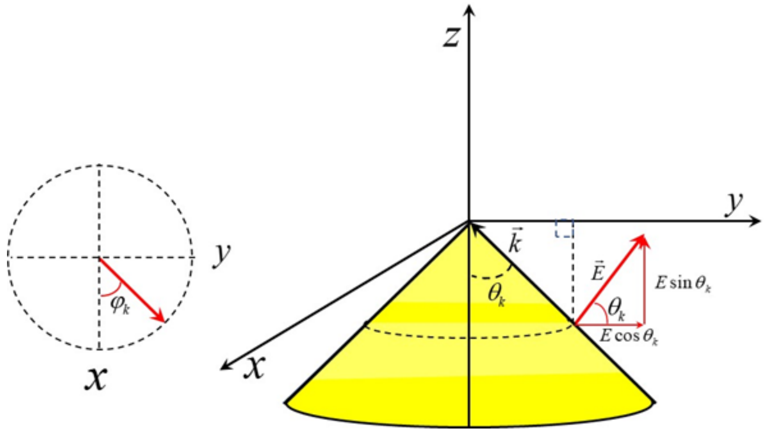
where:

$$\varepsilon_m=\begin{cases}1 & \text{if} m\geq 0\\ (-1)^m & \text{if} m<0\end{cases}$$

Even thought similar expansions are found by Buchholz (BUCHHOLZ, 1969); This particular expansion was not found on in the literature. Neither is the expansion of these plane waves in terms of Hertz vectors. Those two are the main theoretical contributions of this work.

The main problem treated is the light scattering by a paraboloid of revolution. A focused field is modeled as the sum of plane waves whose wavevector make an angle θ_k with the z axis as shown in the figure 1:

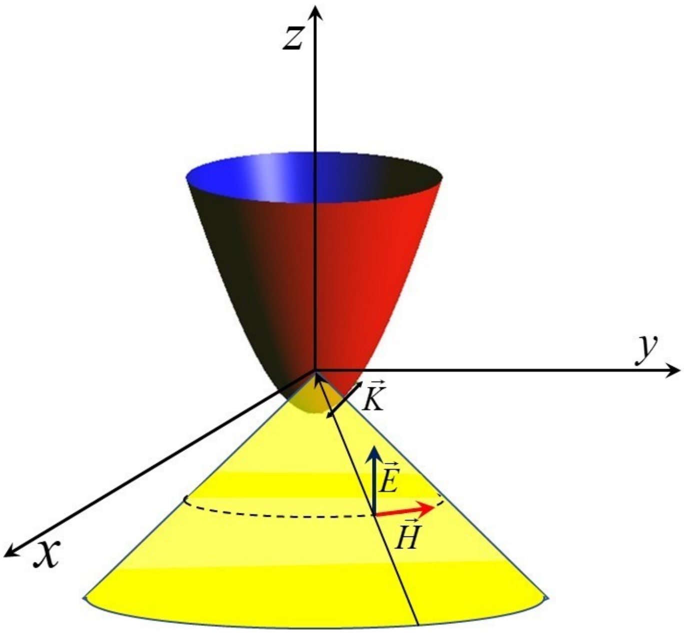
Figura 1 – Conic wave created by summing plane waves over all values of φ_k. Each plane wave has a wavevector making angle θ_k with the z axis and are traveling towards the origin.



Source: author.

By further summing along θ_k from zero to θ_{kf} the incident field represent a focused field exiting an objective Len. This field travels towards the focus of a paraboloid located at the origin as shown in figure 2

Figura 2 – Conic wave traveling towards a paraboloid of revolution.



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This work is organized as follows:

1. An introduction the mathematics used is presented. This includes an introduction to the parabolic rotational coordinates, Gamma function identities and the confluent hypergeometric function which are heavily used. A brief introduction to Maxwell's equations and the Hertz vectors is also treated.
2. The solutions of the scalar Helmholtz equation are presented. A new type of Bessel function is introduced along with the Whittaker and Pinney functions. A discussion of which functions are more suited to approach the scattering problem is presented while the demonstration of these solutions can be found in appendix A.
3. The solutions and properties of the Generalized Laguerre equation are presented. This includes the solution of the second kind which is barely mentioned in the literature. Demonstrations of the most unknown and useful properties of both solutions are presented. Although old, many results can be considered new and are not found anywhere else aside from Pinney's article. For this reason it is treated as one of the main cores of this work.
4. The Pinney functions are presented with some properties easily obtained from the Laguerre Polynomials. It can be considered a continuation of the last section.
5. The expansion of scalar plane waves is obtained as well as the field expansion of a plane wave traveling along the x axis and polarized along the axis of the paraboloid is obtained with the aid of the Hertz vectors.
6. A method for solving the scattering problem is presented. The resultant set of equations obtained after applying boundary conditions is transformed into a matrix $Mx = b$ problem. The calculation of the components of expansion vectors are found in appendix C.
7. The results of light enhancement on a paraboloidal tip illuminated by the focused field previously presented is shown with some remarks about the limits of the method used.

2 MATHEMATICAL PRELIMINARIES

2.1 Parabolic coordinates

The parabolic coordinates are given in (WILLATZEN; VOON, 2011) by the following transformation:

$$x = \sigma\tau cos\varphi$$
$$y = \sigma\tau sin\varphi$$
$$z = \frac{1}{2}(\tau^2 - \sigma^2) \tag{2.1}$$

with $0 \leq \sigma \leq \infty$, $0 \leq \tau \leq \infty$ and $0 \leq \varphi \leq 2\pi$. Another equivalent definition is:

$$x = 2\sqrt{\xi\eta} cos\varphi$$
$$y = 2\sqrt{\xi\eta} sin\varphi$$
$$z = \eta - \xi \tag{2.2}$$

While the former is more common nowdays (VOON; WILLATZEN, 2004; WILLATZEN; VOON, 2011) the latter was used by Edmund Pinney to develop a theory of wave functions in parabolic coordinates in terms of Laguerre functions in 1946 and the generalized Laguerre equation for both parabolic variables is obtained more naturally from it (PINNEY, 1946). Both transformations are related by $\xi = \sigma^2/2$ and $\eta = \tau^2/2$ and both approaches are equivalent. In this work the first and more common transformation is used due to the symmetry of the scale factors.

In practice it is desired that the plots depend on z and ρ (cylindrical coordinates) contrary to σ and τ. Thus the inverted system becomes:

$$\rho^2 = x^2 + y^2 = \sigma^2\tau^2 \quad \Longrightarrow \quad \sigma = \pm\frac{\rho}{\tau} \quad \Longrightarrow \quad \tau^2 - \sigma^2 = 2z \quad \Longrightarrow \quad \tau^2 - \frac{\rho^2}{\tau^2} = 2z$$
$$\tau^4 - 2z\tau^2 - \rho^2 = 0 \quad \Longrightarrow \quad \tau^2 = \pm\sqrt{z^2+\rho^2} + z \quad \Longrightarrow \quad \tau = \sqrt{\sqrt{z^2+\rho^2} + z}$$

$$\rho^2 = x^2 + y^2 = \sigma^2\tau^2 \quad \Longrightarrow \quad \tau = \pm\frac{\rho}{\sigma} \quad \Longrightarrow \quad \tau^2 - \sigma^2 = 2z \quad \Longrightarrow \quad \frac{\rho^2}{\sigma^2} - \sigma^2 = 2z$$

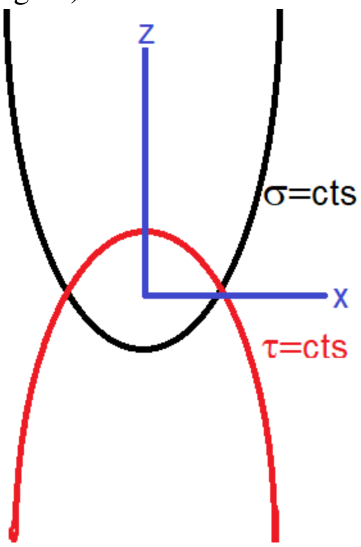
$$\sigma^4 + 2z\sigma^2 - \rho^2 = 0 \quad \Longrightarrow \quad \sigma^2 = \pm\sqrt{z^2+\rho^2} - z \quad \Longrightarrow \quad \sigma = \sqrt{\sqrt{z^2+\rho^2} - z}$$

therefore, avoiding signs that would lead to complex values of σ and τ we have

$$\sigma = \sqrt{\sqrt{z^2+\rho^2} - z} \quad and \quad \tau = \sqrt{\sqrt{z^2+\rho^2} + z} \tag{2.3}$$

For clarity and to avoid confusion[1] when we talk about parabolic coordinates we refer to either transformation shown above with the corresponding constant surfaces below:

Figura 3 – Surfaces of constant coordinates σ and τ in parabolic coordinates. both surfaces are paraboloids with the origin as their focus. The surface defined by the azimutual angle is the usual half-plane (not shown in the figure).



Source: author.

Note that $\sigma = cts$ defines a paraboloid upwards the positive z axis and $\tau = cts$ defines a paraboloid downwards the negative z axis. Both paraboloids have the focus as the origin. The scale factors, gradient, curl, Laplacian and other operators and quantities in parabolic coordinates are found in (WILLATZEN; VOON, 2011). These quantities are presented here for completeness.

The corresponding scale factors are:

[1] There are other types of parabolic coordinates, for instance, Paraboloid coordinates and cylindrical parabolic coordinates. (WILLATZEN; VOON, 2011)

$$h_\sigma = \left(\tau^2+\sigma^2\right)^{1/2} \tag{2.4}$$

$$h_\tau = \left(\tau^2+\sigma^2\right)^{1/2} \tag{2.5}$$

$$h_\varphi = \sigma\tau \tag{2.6}$$

From them the gradient operator is expressed as:

$$\nabla = \frac{\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\frac{\partial}{\partial\sigma} + \frac{\hat{\mathbf{e}}_\tau}{(\tau^2+\sigma^2)^{1/2}}\frac{\partial}{\partial\tau} + \frac{\hat{\mathbf{e}}_\varphi}{\sigma\tau}\frac{\partial}{\partial\varphi} \tag{2.7}$$

And the unitary vectors $\hat{\mathbf{e}}_i$:

$$\hat{\mathbf{e}}_i = \frac{1}{h_i}\left[\frac{\partial x}{\partial q_i}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial q_i}\hat{\mathbf{e}}_y + \frac{\partial z}{\partial q_i}\hat{\mathbf{e}}_z\right] \tag{2.8}$$

Which for future reference is written here:

$$\hat{\mathbf{e}}_\sigma = \frac{1}{(\tau^2+\sigma^2)^{1/2}}\left[\tau cos\varphi\hat{\mathbf{e}}_x + \tau sin\varphi\hat{\mathbf{e}}_y - \sigma\hat{\mathbf{e}}_z\right] \tag{2.9}$$

$$\hat{\mathbf{e}}_\tau = \frac{1}{(\tau^2+\sigma^2)^{1/2}}\left[\sigma cos\varphi\hat{\mathbf{e}}_x + \sigma sin\varphi\hat{\mathbf{e}}_y + \tau\hat{\mathbf{e}}_z\right] \tag{2.10}$$

$$\hat{\mathbf{e}}_\varphi = -sin\varphi\hat{\mathbf{e}}_x + cos\varphi\hat{\mathbf{e}}_y \tag{2.11}$$

From them it can be easily deduced:

$$\hat{\mathbf{e}}_x = \frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi \tag{2.12}$$

$$\hat{\mathbf{e}}_y = \frac{sin\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + cos\varphi\hat{\mathbf{e}}_\varphi \tag{2.13}$$

$$\hat{\mathbf{e}}_z = \frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}} \tag{2.14}$$

Finally the equations above can be used to express the vector $\vec{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ in parabolic coordinates by projecting $\vec{r}$ into each unitary vector:

$$\vec{r} = \frac{\sigma}{2}\left(\tau^2+\sigma^2\right)^{1/2}\hat{\mathbf{e}}_\sigma + \frac{\tau}{2}\left(\tau^2+\sigma^2\right)^{1/2}\hat{\mathbf{e}}_\tau = \frac{\sigma}{2}h_\sigma\hat{\mathbf{e}}_\sigma + \frac{\tau}{2}h_\tau\hat{\mathbf{e}}_\tau \tag{2.15}$$

The divergence and curl operators are respectively:

$$\nabla \cdot \vec{A} = \frac{1}{h_\sigma h_\tau h_\varphi}\left[\frac{\partial}{\partial \sigma}\left(A_\sigma h_\tau h_\varphi\right) + \frac{\partial}{\partial \tau}\left(A_\tau h_\sigma h_\varphi\right) + \frac{\partial}{\partial \varphi}\left(A_\varphi h_\sigma h_\tau\right)\right] \tag{2.16}$$

$$\nabla \times \vec{A} = \frac{1}{h_\sigma h_\tau h_\varphi}\begin{vmatrix} h_\tau \hat{\mathbf{e}}_\tau & h_\sigma \hat{\mathbf{e}}_\sigma & h_\varphi \hat{\mathbf{e}}_\varphi \\ \frac{\partial}{\partial \tau} & \frac{\partial}{\partial \sigma} & \frac{\partial}{\partial \varphi} \\ h_\tau A_\tau & h_\sigma A_\sigma & h_\varphi A_\varphi \end{vmatrix} \tag{2.17}$$

Note: There is a mistake in the way the curl operator is defined in (WILLATZEN; VOON, 2011). If we use the following definition:

$$\nabla \times \vec{A} = \frac{1}{h_\sigma h_\tau h_\varphi}\begin{vmatrix} h_\sigma \hat{\mathbf{e}}_\sigma & h_\tau \hat{\mathbf{e}}_\tau & h_\varphi \hat{\mathbf{e}}_\varphi \\ \frac{\partial}{\partial \sigma} & \frac{\partial}{\partial \tau} & \frac{\partial}{\partial \varphi} \\ h_\sigma A_\sigma & h_\tau A_\tau & h_\varphi A_\varphi \end{vmatrix} \tag{2.18}$$

we end up with a left handed rule. Calculation of $\vec{H} = -\frac{i}{kZ_0}\nabla \times \vec{E}$ with $\vec{E} = E_0 e^{ikx}\hat{\mathbf{e}}_z$ in parabolic coordinates with the above definition gives $Z\vec{H} = E_0 e^{ikx}\hat{\mathbf{e}}_y$ instead of $Z\vec{H} = -E_0 e^{ikx}\hat{\mathbf{e}}_y$. This is corrected by permuting the first and second column.

The last operator we are going to deal is the Laplacian:

$$\nabla^2 \psi = \frac{1}{\sigma^2 + \tau^2}\left[\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial \psi}{\partial \sigma}\right) + \frac{1}{\tau}\frac{\partial}{\partial \tau}\left(\tau \frac{\partial \psi}{\partial \tau}\right)\right] + \frac{1}{\sigma^2 \tau^2}\frac{\partial^2 \psi}{\partial \varphi^2} \tag{2.19}$$

2.2 Curvature radius on the tip of a paraboloid defined by $\sigma = \sigma_0$

Consider the surface $z = \frac{\rho^2}{2\sigma_0^2} - \frac{\sigma_0^2}{2}$ with minimum at $\rho = 0$ and $z = -\frac{\sigma_0^2}{2}$. We wish to find the radius of a circle contained in the tip of a paraboloid. In other words, the Taylor expansion of the circle around $\rho = 0$ must be equal to the paraboloid.

Circle: $(z - z_o)^2 + \rho^2 = R^2 \quad \rightarrow \quad z = z_o - \sqrt{R^2 - \rho^2}$. The negative sign is chosen because we known the minimum of the paraboloid is on the lower side of the circle whose center is on the z axis above the minimum $z = -z_0 = -\frac{\sigma_0^2}{2}$. The Taylor expansion must satisfy:

$$z = z(0) + \left.\frac{\mathrm{d}z}{\mathrm{d}\rho}\right|_{\rho=0}\rho + \left.\frac{\mathrm{d}^2 z}{\mathrm{d}\rho^2}\right|_{\rho=0}\rho^2 = \frac{\rho^2}{2\sigma_0^2} - \frac{\sigma_0^2}{2}$$

Since

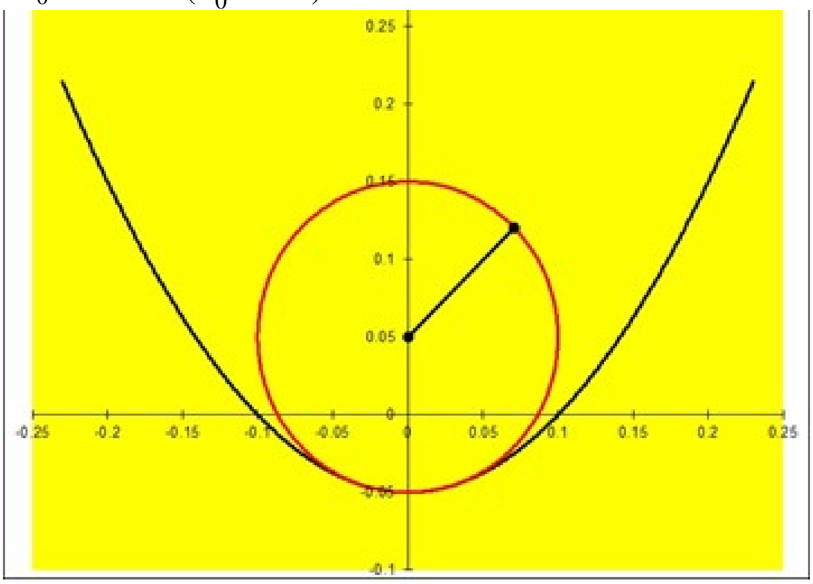
$$\frac{\mathrm{d}z}{\mathrm{d}\rho} = \frac{\rho}{\sqrt{R^2-\rho^2}}, \qquad and \qquad \frac{\mathrm{d}^2z}{\mathrm{d}\rho^2} = \frac{1}{\sqrt{R^2-\rho^2}} + \frac{\rho^2}{(R^2-\rho^2)^{3/2}}$$

then

$$z_0 - R + \frac{\rho^2}{2R} = \frac{\rho^2}{2\sigma_0^2} - \frac{\sigma_0^2}{2}$$

Which is satisfied by $R = \sigma_o^2$. The circle is centered at $(\rho, z) = (0, \frac{\sigma_0^2}{2})$ as can be seen in figure 4 with $\sigma_0^2 = 0.1$:

Figura 4 – Visual representation of the radius of curvature for a paraboloid defined by the constant coordinate $\sigma = \sigma_0 = 0.3162$ ($\sigma_0^2 = 0.1$)



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2.3 Gamma function

Due to its intensive use in the development of the Laguerre functions applied to parabolic coordinates we give a brief overview of the Gamma function. The Gamma function is defined as:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{2.20}$$

Which is valid for $z \in \mathbb{C}$ except for 0 and negative integers where the function diverges. It has the property $\Gamma(z+1) = z\Gamma(z)$ so it can be used as an analytic continuation of the factorial:

$$\Gamma(n+1) = n!, \qquad n \in \mathbb{Z}^+ \tag{2.21}$$

The two main properties we are going to use are the Euler's reflection formula:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{sin(\pi z)}, \qquad z \notin \mathbb{Z} \tag{2.22}$$

and:

$$\Gamma(a-b) = (-1)^{b-1}\frac{\Gamma(-a)\Gamma(1+a)}{\Gamma(b+1-a)}, \qquad b \in \mathbb{Z} \tag{2.23}$$

The last equation is consequence of Euler's reflection formula:

$$\begin{aligned}
&\frac{sin(\pi(c-d))}{\pi} = \frac{sin(\pi c)cos(\pi d) - sin(\pi d)cos(\pi c)}{\pi} \\
\Longrightarrow &\frac{1}{\Gamma(1-(c-d))\Gamma(c-d)} = -\frac{(-1)^c}{\Gamma(1-d)\Gamma(d)}, \qquad c \in \mathbb{Z} \\
\Longrightarrow &(-1)^{c+1}\Gamma(c-d) = \frac{\Gamma(1-d)\Gamma(d)}{\Gamma(1-c+d)} \\
\Longrightarrow &\Gamma(c-d) = (-1)^{-c-1}\frac{\Gamma(1-d)\Gamma(d)}{\Gamma(1-c+d)} \\
\Longrightarrow &\Gamma(a-b) = (-1)^{b-1}\frac{\Gamma(-a)\Gamma(1+a)}{\Gamma(b+1-a)}, \qquad c = -b, \qquad a = -d
\end{aligned}$$

The Gamma function has other properties and alternative definitions. However these are the only ones we need. The Gamma function can be used to define the Binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \tag{2.24}$$

So n and k do not necessary have to be integers.

The Pochhammer's symbol can also be defined in terms of the Gamma function:

$$(a)_k = (a)(a+1)(a+2)...(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \tag{2.25}$$

were $(a)_0 = 1$ by definition.

The Gamma function is always introduced in the first chapters of any book about special functions, for instance (ANDREWS RICHARD ASKEY, 1999).

2.4 Confluent hypergeometric function

The core of most demonstrations involving the Laguerre functions (including Laguerre polynomials as a special case) is the representation of them in terms of the confluent hypergeometric function or Kummer's function. The confluent hypergeometric function is the solution of the differential equation (DE):

$$z\frac{\mathrm{d}^2M}{\mathrm{d}z^2} + (b-z)\frac{\mathrm{d}M}{\mathrm{d}z} - aM = 0 \tag{2.26}$$

This DE can be solved by the Frobenius method. Suppose the solution can be expressed as:

$$M(z) = \sum_{n=0}^{\infty} a_n z^{n+r}, \qquad a_0 \neq 0 \tag{2.27}$$

Then:

$$\frac{\mathrm{d}M(z)}{\mathrm{d}z} = \sum_{n=0}^{\infty} (n+r)a_n z^{n+r-1} \tag{2.28}$$

$$\frac{\mathrm{d}^2M(z)}{\mathrm{d}z^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n z^{n+r-2} \tag{2.29}$$

implies:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n z^{n+r-1} + b(n+r)a_n z^{n+r-1}$$
$$\sum_{n=0}^{\infty} -(n+r)a_n z^{n+r} - a a_n z^{n+r} = 0$$
$$\Longrightarrow [r(r-1)+br]a_0 z^{r-1} + \sum_{n=1}^{\infty}(n+r)(n+r-1+b)a_n z^{n+r-1}$$
$$-\sum_{n=0}^{\infty}(n+r+a)a_n z^{n+r} = 0$$
$$\Longrightarrow [r(r-1)+br]a_0 z^{r-1}$$
$$+\sum_{n=0}^{\infty}[(n+1+r)(n+r+b)a_{n+1} - (n+r+a)a_n]z^{n+r} = 0$$

Since $a_0 \neq 0$ we have $r = 0$ or $r = 1-b$ from the indicial equation $r(r+b-1) = 0$. Also the recurrence relation gives:

$$a_{n+1} = \frac{(n+r+a)a_n}{(n+1+r)(n+r+b)} \tag{2.30}$$

The first solution comes from $r = 0$:

$$a_{n+1} = \frac{(n+a)a_n}{(n+1)(n+b)} \tag{2.31}$$

Writing the first coefficients we quickly find a pattern:

$$a_1 = \frac{(a)a_0}{(1)(b)}$$
$$a_2 = \frac{(1+a)}{(2)(1+b)}\frac{(a)a_0}{(1)(b)}$$
$$a_3 = \frac{(2+a)}{(3)(2+b)}\frac{(1+a)}{(2)(1+b)}\frac{(a)a_0}{(1)(b)}$$
$$a_n = \frac{(a)(a+1)(a+2)...(a+n-1)}{(b)(b+1)(b+2)...(b+n-1)n!}a_0$$
$$a_n = \frac{(a)_n}{(b)_n n!}a_0$$

we chose $a_0 = 1$ therefore:

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{2.32}$$

Note that it is not defined for b is 0 or negative integer.For the second solution we take $r = 1 - b$ so:

$$a_{n+1} = \frac{(n+1-b+a)a_n}{(n+2-b)(n+1)} \tag{2.33}$$

However if we define $\alpha = 1 - b + a$ and $\beta = 2 - b$:

$$a_{n+1} = \frac{(n+\alpha)a_n}{(n+\beta)(n+1)} \tag{2.34}$$

This is the same recursion formula for the first solution then:

$$U(a,b,z) = x^{1-b}M(a+1-b,2-b,z) = x^{1-b} \sum_{n=0}^{\infty} \frac{(a+1-b)_n}{(2-b)_n} \frac{z^n}{n!} \tag{2.35}$$

This solution is not defined for integer values of $b > 1$ just as $M(a,b,z)$ is not defined for values of $b < 1$. However they can be defined by taking a particular linear combination an example of this are the Laguerre functions which are treated later.

The confluent hypergeometric function is related to many special functions like Bessel and Legendre (WILLATZEN; VOON, 2011; OLVER *et al.*, 2010; BUCHHOLZ, 1969; ABRAMOWITZ; STEGUN, 1970).For properties related to the confluent hypergeometric functions the reader may consult the handbooks (OLVER *et al.*, 2010; ABRAMOWITZ; STEGUN, 1970). For a more specific study (BUCHHOLZ, 1969) seems to be quite famous. The Whittaker functions which are more commonly used as solutions to the Helmholtz equations in parabolic coordinates are also treated in (BUCHHOLZ, 1969).

2.5 Maxwell's equations and Hertz's vectors

The Maxwell's equations in sourceless non magnetic media are:

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{H} = 0$$
$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$
$$\nabla \times \vec{H} = \varepsilon \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Assuming the fields have a time dependence e^{-iwt} any field can be expressed as a Fourier transform:

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} \vec{E}(\vec{r},w) e^{-iwt} dw \tag{2.36}$$

with inverse:

$$\vec{E}(\vec{r},w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E}(\vec{r},t) e^{iwt} dt \tag{2.37}$$

Therefore applying the Fourier transform to the Maxwell equations:

$$\nabla \cdot \vec{E} = 0 \tag{2.38}$$
$$\nabla \cdot \vec{H} = 0 \tag{2.39}$$
$$\vec{E} = \frac{i}{k} \nabla \times Z\vec{H} \tag{2.40}$$
$$Z\vec{H} = -\frac{i}{k} \nabla \times \vec{E} \tag{2.41}$$

where $Z = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is the vacuum impedance and $k = \frac{w}{c} n$ where n is the refractive index. These are the Maxwell's equations in the frequency domain.

The solution of wave scattering problems in electromagnetism can be separated in two parts:

1. The expansion of the incident field in terms of orthogonal vectors Which are solutions of the Maxwell equations
2. The application of the boundary conditions on such orthogonal vectors

In sourceless media two mutually orthogonal vectors are enough to describe any arbitrary field (A., 2013).

Although there are many ways of expanding fields based on scalar potentials (KLESHCHEV, 2012) the Hertz vectors are the only ones presented in this work. Whittaker proved in 1904 that the electromagnetic fields in sourceless media can be described by the derivatives of two scalar potential functions (WHITTAKER, 1904). The Hertz vectors are just an example.

The magnetic Hertz vector $\vec{\pi}_m(\vec{r},w)$ is defined in such way so that the usual vector potential satisfies the Coulomb gauge constraint $\nabla \cdot \vec{A}_C = 0$:

$$\vec{A}_C = \nabla \times \vec{\pi}_m \tag{2.42}$$

$\vec{\pi}_m$ also satisfies the vector Helmholtz equation:

$$\nabla^2 \vec{\pi}_m + k^2 \vec{\pi}_m = 0 \tag{2.43}$$

Since the Laplacian and the curl operator commute $\vec{A}_C$ also satisfies the Helmholtz equation. Therefore $\vec{A}_C$ is a solution of the Maxwell's equations. The electric and magnetic field are then represented by:

$$\vec{E} = iw\nabla \times \vec{\pi}_m, \qquad \mu_0 \vec{H} = \nabla \times \nabla \times \vec{\pi}_m \tag{2.44}$$

A particular choice of $\vec{\pi}_m$ is $\psi(\vec{r},w)\hat{\mathbf{s}}$ where $\psi(\vec{r},w)$ is a solution of the scalar Helmholtz equation and $\hat{\mathbf{s}}$ is a constant unit vector. $\hat{\mathbf{s}}$ is defined as a constant vector to ensure $\vec{\pi}_m$ is solution to the vector Helmholtz equation. The position vector $\vec{r}$ can also be used and $\vec{\pi}_m$ would still be a solution to the Helmholtz equation. Unfortunately this is not true in general for unitary vectors representing other system of coordinates like $\hat{\mathbf{e}}_\sigma$ and $\hat{\mathbf{e}}_\tau$.

The electric Hertz vector $\vec{\pi}_e$ is defined in a similar manner. This time the vector potential satisfies the Lorenz gauge and:

$$\vec{A}_L = -\frac{iw}{c^2}\vec{\pi}_e, \qquad \varphi_L = -\nabla \cdot \vec{\pi}_e \tag{2.45}$$

The most relevant feature in this case is that the electric field is proportional to $\nabla \times \nabla \times \vec{\pi}_e$ and the magnetic to $\nabla \times \vec{\pi}_e$. In general the electric and magnetic field can be described by the vectors:

$$\vec{M} \quad \alpha \quad \nabla \times \vec{\pi}$$
$$\vec{N} \quad \alpha \quad \nabla \times \nabla \times \vec{\pi}$$

with the vector $\vec{\pi} = \psi(\vec{r}, w)\hat{\mathbf{s}}$. Both $\vec{M}$ and $\vec{N}$ are orthogonal vectors since $\nabla \times \vec{\pi}$ satisfies the Maxwell's equations and can represent real fields. Thus the problem is reduce to find a the solution of the scalar Helmholtz and choose a unitary vector $\hat{\mathbf{s}}$ which best simplifies the system. The choice $\vec{\pi} = \psi(\vec{r}, w)\vec{r}$ leads to the Debye potentials and $\nabla \times \vec{\pi}$ is proportional to the angular momentum operator acting on the scalar potential $\psi(\vec{r}, w)$. This particular choice was used in the light scattering by a sphere first by Lorenz, then by Mie. The relation between the angular momentum operator and the solutions of the Helmholtz equation along with the fact that only the radial function depends on the properties of the material greatly simplifies the problem in spherical coordinates (NEVES; CESAR, 2019). This is no longer the case in other coordinates.

3 HELMHOLTZ EQUATION IN PARABOLIC COORDINATES

3.1 Possible approaches

The solution to the Helmholtz equation in parabolic coordinates is one of the cornerstones of this work. It is a difficult one, there are three known ways to approach it, two provided by (WILLATZEN; VOON, 2011) which are briefly discussed and one provided by (PINNEY, 1946) which is presented in this work.

The Helmholtz equation in parabolic coordinates is:

$$\frac{1}{\sigma^2+\tau^2}\left[\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\psi}{\partial\sigma}\right)+\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\psi}{\partial\tau}\right)\right]+\frac{1}{\sigma^2\tau^2}\frac{\partial^2\psi}{\partial\varphi^2}+k^2\psi=0 \tag{3.1}$$

we define $\psi = S(\sigma)T(\tau)\Phi(\varphi)$ and divide the Helmholtz equation by this quantity:

$$\frac{1}{\sigma^2+\tau^2}\left[\frac{1}{\sigma S}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\frac{1}{\tau T}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)\right]+\frac{1}{\sigma^2\tau^2\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+k^2=0 \tag{3.2}$$

Let $\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=-m^2$ then:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+m^2\Phi=0 \tag{3.3}$$

whose solutions are:

$$\Phi(\varphi)=Asin(m\varphi)+Bcos(m\varphi) \tag{3.4}$$

m can only take positive integer values. The reason for positives values is going to be clear later. For the other functions we have:

$$\frac{1}{\sigma^2+\tau^2}\left[\frac{1}{\sigma S}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\frac{1}{\tau T}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)\right]-\frac{m^2}{\sigma^2\tau^2}+k^2=0 \tag{3.5}$$

Multiplying by $\sigma^2+\tau^2$ and separating the terms containing only its respective variable:

$$\left[\frac{1}{\sigma S}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]+\left[\frac{1}{\tau T}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)+\tau^2k^2-\frac{m^2}{\tau^2}\right]=0 \tag{3.6}$$

This is the turning point, we can follow the Pinney approach by making the substitutions $\xi = \sigma^2/2$ and $\eta = \tau^2/2$. For now we follow the more common approach (WILLATZEN; VOON, 2011). Define the first bracket as $-q^2$ and the second as q^2. This leads to:

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\left[q^2+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]S=0 \tag{3.7}$$

$$\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)-\left[q^2-\tau^2k^2+\frac{m^2}{\tau^2}\right]T=0 \tag{3.8}$$

These equations are known as Bessel wave equations and their solutions are the called Bessel wave functions by (WILLATZEN; VOON, 2011):

$$S(\sigma)=CJ_m(q,k,\sigma)+DJ_{-m}(q,k,\sigma) \tag{3.9}$$

$$T(\tau)=EJ_m(q,k,i\tau)+FJ_{-m}(q,k,i\tau) \tag{3.10}$$

Similar to the Bessel functions, if m is an integer, J_m and J_{-m} are not independent functions and a Bessel wave function of second kind need to be defined according to (WILLATZEN; VOON, 2011; VOON; WILLATZEN, 2004). These functions are obtained by solving the differential equations by the Frobenius method and have the advantage that are both real and converges rapidly (VOON; WILLATZEN, 2004). However The recursion formulas obtained for the coefficients by the Frobenius method gives a three term recursion formula nearly impossible to solve. As a consequence no explicit series solution was given by (WILLATZEN; VOON, 2011; VOON; WILLATZEN, 2004) so each term of the series must be obtain recursively. Further attempts to find properties of these functions by this method are limited and no other information about these functions was found aside from (WILLATZEN; VOON, 2011; VOON; WILLATZEN, 2004).

The second approach given by (WILLATZEN; VOON, 2011) is to transform the Bessel wave equations into the Whittaker equations by the following transformations: Recall:

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\left[q^2+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]S=0 \tag{3.11}$$

Let $S(\sigma)=V(\upsilon)/\sqrt{\upsilon}$ with $\sigma^2=\upsilon$. This transformation leads to:

$$\frac{\mathrm{d}^2V}{\mathrm{d}\upsilon^2}+\left[\frac{k^2}{4}+\frac{q^2}{4\upsilon}+\frac{\frac{1}{4}-\frac{m^2}{4}}{\upsilon^2}\right]V=0 \tag{3.12}$$

By defining $\alpha = q^2/4ik$,$\mu = m/2$ and with the change of variables $z = ik\upsilon$ the whittaker equation is obtained:

$$\frac{d^2V}{dz^2} + \left[-\frac{1}{4} + \frac{\alpha}{z} + \frac{\frac{1}{4} - \mu^2}{\upsilon^2} \right] V = 0 \tag{3.13}$$

With solutions:

$$M_{\alpha,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} M(\frac{1}{2} + \mu - \alpha, 1 + 2\mu, z) \tag{3.14}$$

$$W_{\alpha,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} U(\frac{1}{2} + \mu - \alpha, 1 + 2\mu, z) \tag{3.15}$$

Where $M(a,b,z)$ and $U(a,b,z)$ are the confluent hypergeometric functions of first and second kind respectively (WILLATZEN; VOON, 2011). These are more manageable in the sense that they have series and integral representations known (OLVER *et al.*, 2010; ABRAMOWITZ; STEGUN, 1970) therefore an asymptotic expression can be obtained which allows us to chose the corresponding functions satisfying certain boundary conditions. Although this may work, finding the possible values of the separation constant for a given set of boundary conditions can be a titanic task; a priori it can take any real value. There is however a way to restraint the values this separation constant may take.

By making $q/2 = ik(m+1+2n)$ an suitable change of variables Pinney gives the solution in terms of Laguerre polynomials:

$$S_n^m(x) = x^{\frac{m}{2}} e^{-\frac{x}{2}} L_n^m(x) \tag{3.16}$$

$$V_n^m(x) = x^{\frac{m}{2}} e^{-\frac{x}{2}} U_n^m(x) \tag{3.17}$$

where $x = \pm ik\sigma^2$ or $x = \mp ik\tau^2$. $L_n^m(x)$ are the Associated Laguerre polynomials and $U_n^m(x)$ is a second linearly independant solution of the Generalized Laguerre differential equation. Such solution is found nearly exclusively on Pinney's articles (PINNEY, 1946; PINNEY, 1947) and a is briefly mentioned by Buchholz (BUCHHOLZ, 1969). The properties of Laguerre functions are well known and can be used to easily obtain the expansions of plane waves in parabolic coordinates. The extensive demonstration of this solution can be found on appendix A while the properties of Laguerre functions and the second solution are disscused in the next section.

4 GENERALIZED LAGUERRE FUNCTIONS

The generalized Laguerre differential equation is:

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \tag{4.1}$$

Whose solutions of first kind are the Laguerre polynomials $L_n^\alpha(x)$ with $\alpha > -1$. It is possible to use the Frobenius method to obtain the solutions $M(-n, \alpha+1, x)$ (confluent hypergeometric function of first kind) and $U(-n, \alpha+1, x)$ (confluent hypergeometric function of second kind). While $M(-n, \alpha+1, x)$ is proportional to the Laguerre polynomials, the factor it is lacking (which depends on n and α) affect the recurrence relations satisfied by the Laguerre polynomials. Besides $U(-n, \alpha+1, x)$ is not defined for positive integer values of α which is the case in our application. For this reasons it is more convenient to solve the differential equation by other methods like the generating function:

$$\frac{1}{(1-t)^{\alpha+1}} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n^\alpha(x) \tag{4.2}$$

or the contour integral:

$$L_n^\alpha(x) = \frac{1}{2\pi i} \oint_C \frac{e^{-tx/(1-t)}}{(1-t)^{\alpha+1} t^{n+1}} dt \tag{4.3}$$

where the contour C encloses the origin without enclosing the singularity $t = 1$. with this contour integral the Rodrigues formula can be obtained:

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{-x} x^{n+\alpha} \right) \tag{4.4}$$

Using the general Leibniz rule for the n derivative of a product of functions:

$$(fg)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)} g^{(k)} \tag{4.5}$$

The Laguerre Polynomials take the closed form:

$$
\begin{aligned}
L_n^{\alpha}(x) &= \frac{x^{-\alpha}e^{x}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{\mathrm{d}^k(e^{-x})}{\mathrm{d}x^k}\frac{\mathrm{d}^{n-k}(x^{n+\alpha})}{\mathrm{d}x^{n-k}} \\
&= x^{-\alpha}e^{x}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}(-1)^k e^{-x}\frac{(n+\alpha)!}{(n+\alpha-(n-k))!}x^{n+\alpha-(n-k)} \\
&= \sum_{k=0}^{n}\frac{(-1)^k(n+\alpha)!}{(n-k)!(n+\alpha-(n-k))!}\frac{x^k}{k!} \\
L_n^{\alpha}(x) &= \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}
\end{aligned}
$$

4.1 Properties of the Laguerre polynomials

Most of the properties presented here are not demonstrated, many of them can be found on Mathematica library of functions and articles like (PINNEY, 1946). The objective of this section is to summarize all important properties which are used in this work or can be useful in future works. Let ν and α be real positive numbers and z be a complex number. The Laguerre polynomials satisfy the following recurrence relations:

$$\frac{\mathrm{d}^n L_\nu^{\alpha}(z)}{\mathrm{d}z^n} = (-1)^n L_{\nu-n}^{\alpha+n}(z) \tag{4.6}$$

$$z\frac{\mathrm{d}L_\nu^{\alpha}(z)}{\mathrm{d}z} = -\alpha L_\nu^{\alpha}(z) + (\nu+\alpha)L_\nu^{\alpha-1}(z) \tag{4.7}$$

$$\frac{\mathrm{d}L_\nu^{\alpha}(z)}{\mathrm{d}z} = -L_\nu^{\alpha+1}(z) + L_\nu^{\alpha}(z) \tag{4.8}$$

$$z\frac{\mathrm{d}L_\nu^{\alpha}(z)}{\mathrm{d}z} = \nu L_\nu^{\alpha}(z) - (\nu+\alpha)\alpha L_{\nu-1}^{\alpha}(z) \tag{4.9}$$

$$z\frac{\mathrm{d}L_\nu^{\alpha}(z)}{\mathrm{d}z} = (\nu+1)L_{\nu+1}^{\alpha}(z) + (z-\alpha-\nu-1)L_\nu^{\alpha}(z) \tag{4.10}$$

$$L_\nu^{\alpha}(z) = L_\nu^{\alpha+1}(z) - L_{\nu-1}^{\alpha+1}(z) \tag{4.11}$$

$$zL_\nu^{\alpha}(z) = -(\nu+1)L_{\nu+1}^{\alpha-1}(z) + (\nu+\alpha)L_\nu^{\alpha-1}(z) \tag{4.12}$$

$$(\nu+\alpha)L_\nu^{\alpha-1}(z) - (z-\alpha)L_\nu^{\alpha}(z) + zL_\nu^{\alpha+1}(z) = 0 \tag{4.13}$$

$$(\nu+\alpha)L_{\nu-1}^{\alpha}(z) + (z-\alpha-2\nu-1)L_\nu^{\alpha}(z) + (\nu+1)L_{\nu+1}^{\alpha}(z) = 0 \tag{4.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{\alpha}L_\nu^{\alpha}(z)) = (\alpha+\nu)z^{\alpha-1}L_\nu^{\alpha-1}(z) \tag{4.15}$$

They can be shown by using the Laguerre function used by Pinney:

$$L_\nu^{\alpha}(z) = -\frac{sin(\pi\nu)}{\pi}\Gamma(\nu+\alpha+1)\sum_{p=0}^{\infty}\frac{\Gamma(p-\nu)}{\Gamma(p+\alpha+1)}\frac{z^p}{p!} \tag{4.16}$$

The Laguerre function defined this way is rarely seen nowadays. However by using properties of the Gamma function, it can be shown it is equivalent to:

$$L_\nu^\alpha(z) = \binom{\nu+\alpha}{\nu} M(-\nu, \alpha+1, z) \tag{4.17}$$

where $M(a,b,z)$ is the confluent hypergeometric function of first kind. Unlike Pinney we used this definition instead to demonstrate most of the results obtained by him (PINNEY, 1946). It will be shown that the Laguerre polynomials can be defined this way when ν is a positive integer. Then eq (4.17) can be used as an analytic continuation of the Laguerre polynomials.

For completeness we are going to prove that eq (4.16) is equivalent to eq (4.17). By Euler's reflection formula:

$$-\frac{sin(\pi\nu)}{\pi} = \frac{sin(-\pi\nu)}{\pi} = \frac{1}{\Gamma(-\nu)\Gamma(1+\nu)}$$

Therefore:

$$\begin{aligned} L_\nu^\alpha(z) &= \frac{\Gamma(\nu+\alpha+1)}{\Gamma(-\nu)\Gamma(1+\nu)} \sum_{p=0}^{\infty} \frac{\Gamma(p-\nu)}{\Gamma(p+\alpha+1)} \frac{z^p}{p!} \\ &= \frac{\Gamma(\nu+\alpha+1)}{\Gamma(1+\nu)} \sum_{p=0}^{\infty} \frac{(-\nu)_p}{\Gamma(p+\alpha+1)} \frac{z^p}{p!} \\ &= \frac{\Gamma(\nu+\alpha+1)}{\Gamma(1+\nu)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \sum_{p=0}^{\infty} \frac{(-\nu)_p}{\Gamma(p+\alpha+1)} \frac{z^p}{p!} \\ &= \frac{\Gamma(\nu+\alpha+1)}{\Gamma(1+\nu)\Gamma(\alpha+1)} \sum_{p=0}^{\infty} \frac{(-\nu)_p}{(\alpha+1)_p} \frac{z^p}{p!} \\ &= \binom{\nu+\alpha}{\nu} M(-\nu, \alpha+1, z) \end{aligned}$$

Besides the Laguerre polynomials satisfy an orthogonal relation with respect to a weight function $x^\alpha e^{-x}$ in the real positive interval $(0,\infty)$:

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m} \tag{4.18}$$

4.2 Relationship with the confluent hypergeometric function of first and second kind

The Laguerre polynomials are related to the confluent hypergeometric function or Kummer's function of first kind $M(a,b,x)$. This relation is useful because we can extend L_n^α to real values of n and it can be used to find a second solution to the generalized Laguerre equation. We start by the Laguerre closed form:

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)}(-1)^k \frac{x^k}{k!} \tag{4.19}$$

Multiplying and dividing by $\Gamma(n+1)\Gamma(\alpha+1)$:

$$L_n^{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)}(-1)^k \frac{x^k}{k!}$$

Now we define the Pochhammer Symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)...(a+n-1)$ as the rising factorial. This implies:

$$L_n^{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{\Gamma(n+1)}{\Gamma(n-k+1)(\alpha+1)_k}(-1)^k \frac{x^k}{k!}$$

Now recall the following property of the Gamma function:

$$\Gamma(a-b) = (-1)^{b-1} \frac{\Gamma(-a)\Gamma(1+a)}{\Gamma(b+1-a)}$$

From which we conclude:

$$\Gamma(n-(k-1)) = (-1)^{k-2} \frac{\Gamma(-n)\Gamma(n+1)}{\Gamma((k-1)+1-n)} = (-1)^k \frac{\Gamma(-n)\Gamma(n+1)}{\Gamma(-n+k)}$$

Then:

$$L_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{\Gamma(-n+k)}{\Gamma(-n)(\alpha+1)_k} \frac{x^k}{k!}$$

or

$$L_n^{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}$$

We used the definition of the Pochhammer symbol in the last step. The confluent hypergeometric function is:

$$M(a,b,x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!} \tag{4.20}$$

The terms $(-n)_{k=n+1}$ of the Pochhammer symbol and onwards ($k > n$) would contain the factor $(-n+n) = 0$. This allows to change the sum from $k = 0$ to ∞ and finally

$$L_n^{\alpha}(x) = \binom{n+\alpha}{n} M(-n, \alpha+1, x) \tag{4.21}$$

Given a confluent geometric function of first kind $M(a,b,x)$, the second kind is given as $U(a,b,x) = x^{1-b}M(a+1-b, 2-b, x)$. They are linearly independant for most values of a and b, integer values of b are one exception. Nevertheless we can guess the form of second solution for the Laguerre equation by setting $a = -n$ and $b = \alpha+1$. $U(a,b,x)$ takes the form:

$$U(-n, \alpha+1, x) = x^{-\alpha} M(-n-\alpha, 1-\alpha, x) \tag{4.22}$$

This suggest:

$$x^{-\alpha} L_{n+\alpha}^{-\alpha}(x) = \binom{n}{n+\alpha} x^{-\alpha} M(-n-\alpha, 1-\alpha, x) \tag{4.23}$$

is a second linearly independant solution for non positive integer values of α. Pinney defined $T_n^{\alpha}(x) = x^{-\alpha} L_{n+\alpha}^{-\alpha}(x)$ and used the property:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\alpha} L_n^{\alpha}) = (n+\alpha) x^{\alpha-1} L_n^{\alpha-1} \tag{4.24}$$

to find:

$$\frac{\mathrm{d}}{\mathrm{d}x}(T_n^{\alpha}) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\alpha} L_{n+\alpha}^{-\alpha}\right) = n x^{-\alpha-1} L_{n+\alpha}^{-\alpha-1} = n T_{n-1}^{\alpha+1} \tag{4.25}$$

Therefore:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(T_n^{\alpha}) = n(n-1) T_{n-2}^{\alpha+2} \tag{4.26}$$

and:

$$
\begin{aligned}
& z\frac{\mathrm{d}^2}{\mathrm{d}x^2}(T_n^\alpha)+(\alpha+1-z)\frac{\mathrm{d}}{\mathrm{d}x}(T_n^\alpha)+nT_n^\alpha \\
& = xn(n-1)T_{n-2}^{\alpha+2}+(\alpha+1-x)nT_{n-1}^{\alpha+1}+nT_n^\alpha \\
& = (n-1)x^{-\alpha-1}L_{n+2+\alpha-2}^{-\alpha-2}+(\alpha+1-x)x^{-\alpha-1}L_{n+1+\alpha-1}^{-\alpha-1}+x^{-\alpha}L_{n+\alpha}^{-\alpha} \\
& = x^{-\alpha-1}\left[(n-1)L_{n+2+\alpha-2}^{-\alpha-2}+(\alpha+1-x)L_{n+1+\alpha-1}^{-\alpha-1}+xL_{n+\alpha}^{-\alpha}\right]
\end{aligned}
$$

The Laguerre polynomials satisfy (PINNEY, 1946):

$$(n+\alpha)L_n^{\alpha-1}-(x+\alpha)L_n^\alpha+xL_n^{\alpha+1}=0 \tag{4.27}$$

Which is a famous recurrence relation relating shifted values of α. Set $\alpha\to-\alpha-1$ and $n\to n+\alpha$:

$$(n-1)L_{n+\alpha}^{-\alpha-2}-(x-\alpha-1)L_{n+\alpha}^{-\alpha-1}+xL_{n+\alpha}^{-\alpha}=0 \tag{4.28}$$

So $T_n^\alpha(x)=x^{-\alpha}L_{n+\alpha}^{-\alpha}(x)$ satisfies:

$$z\frac{\mathrm{d}^2}{\mathrm{d}x^2}(T_n^\alpha)+(\alpha+1-z)\frac{\mathrm{d}}{\mathrm{d}x}(T_n^\alpha)+nT_n^\alpha=0 \tag{4.29}$$

and is a second solution to the Generalized Laguerre equation. However it is not linearly independant of L_n^α for integer values of α as shall be shown below.

4.3 The Laguerre function of second kind $U_n^\nu(x)$

In the Gabor Szego's book on Orthogonal Polynomials a generalization of eq (4.19) to negative integers of α (SZEGO, 1939) is shown.

$$L_n^{-k}(x)=(-x)^k\frac{(n-k)!}{n!}\sum_{\nu=0}^{n-k}\binom{n}{n-k-\nu}\frac{(-x)^\nu}{\nu!},\quad (n\geq k),\quad (k\in\mathbb{Z}^+) \tag{4.30}$$

Or

$$L_n^{-k}(x) = (-x)^k \frac{(n-k)!}{n!} L_{n-k}^k(x), \quad (n \geq k), \quad (k \in \mathbb{Z}^+) \tag{4.31}$$

Recall

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

Multipliying and dividing by $n!$

$$L_n^{-k}(x) = \sum_{\nu=0}^{n} \binom{n-k}{n-\nu} \frac{(-x)^{\nu}}{\nu!} = \frac{(n-k)!}{n!} \sum_{\nu=0}^{n} \frac{n!}{(n-\nu)!(\nu-k)!} \frac{(-x)^{\nu}}{\nu!}$$

Shifting indices $\sum_{\nu=0}^{n} f(\nu) = \sum_{\nu=-k}^{n-k} f(\nu+k)$:

$$L_n^{-k}(x) = \frac{(n-k)!}{n!} \sum_{\nu=-k}^{n-k} \frac{n!}{(n-k-\nu)!\nu!} \frac{(-x)^{\nu+k}}{(\nu+k)!}$$

The factorial is not defined for negative integers while the Gamma function goes to infinity as the function approaches to these values so the contribution to the sum of these terms can be set to zero:

$$L_n^{-k}(x) = (-x)^k \frac{(n-k)!}{n!} \sum_{\nu=0}^{n-k} \frac{n!}{(n-k-\nu)!(\nu+k)!} \frac{(-x)^{\nu}}{\nu!}$$

Or:

$$L_n^{-k}(x) = (-x)^k \frac{(n-k)!}{n!} L_{n-k}^k(x) \tag{4.32}$$

It is known that in general the functions $L_n^{-k}(x)$ do not belong to a Hilbert space of square integral functions like $L_n^k(x)$. However, for $n \geq k$ they do (EVERITT *et al.*, 2004). In fact the previous formula has a striking consequence; replacing $n- > n+k$:

$$L_{n+k}^{-k}(x) = (-x)^k \frac{n!}{(n+k)!} L_n^k(x) \tag{4.33}$$

Which is valid for $n \geq 0$ and $k \in \mathbb{Z}^+$ and allows us to obtain another formula for $L_n^k(x)$:

$$L_n^k(x) = \frac{(n+k)!}{n!}(-x)^{-k} L_{n+k}^{-k}(x) \tag{4.34}$$

Clearly this implies that the function $x^{-k} L_{n+k}^{-k}(x)$ is another solution to the Laguerre equation which is not linearly independant of $L_n^k(x)$ for integer values of k. To be more precise:

$$L_n^k(x) = (-1)^{-k} T_n^k(x) = (-1)^{-k} \frac{\Gamma(n+k+1)}{\Gamma(n+1)} x^{-k} L_{n+k}^{-k}(x) \tag{4.35}$$

We shall demonstrate that $T_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} x^{-\alpha} L_{n+\alpha}^{-\alpha}(x)$ is reduced to $(-1)^\alpha L_n^\alpha(x)$ when α becomes an integer by using the representation of the Laguerre function in terms of the confluent hypergeometric function. This is done not only for completeness but because we are going to use some of the results to define the Laguerre function of second kind as a limit when $\alpha -> m$ with m integer. In terms of the confluent hypergeometric function $M(a,b,x)$ $T_n^\alpha(x)$ takes the form:

$$\begin{aligned}
T_n^\alpha(x) &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} x^{-\alpha} L_{n+\alpha}^{-\alpha}(x) \\
&= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(1-\alpha)} x^{-\alpha} M(-n-\alpha, 1-\alpha, x) \\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{(-n-\alpha)_k}{(1-\alpha)_k} \frac{x^k}{k!} \\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{(-n-\alpha)_k \Gamma(1-\alpha)}{\Gamma(k+1-\alpha)} \frac{x^k}{k!} \\
&= x^{-\alpha} \sum_{k=0}^{\infty} \frac{(-n-\alpha)_k}{\Gamma(k+1-\alpha)} \frac{x^k}{k!}
\end{aligned}$$

where we have used $(1-\alpha)_k = \Gamma(k+1-\alpha)/\Gamma(1-\alpha)$. The previous result is valid for real values of α. If we take the limit when α approaches an integer value m the denominator tends to infinity or minus infinity when $k < m$ so the contribution of the terms with $k < m$ becomes negligible so the series may start with the value $k = m$. Then:

$$\begin{aligned} T_n^m(x) &= \sum_{k=m}^{\infty} \frac{(-n-m)_k}{\Gamma(k+1-m)} \frac{x^{k-m}}{\Gamma(k+1)} \\ &= \sum_{k=m}^{\infty} \frac{\Gamma(-n-m+k)}{\Gamma(-n-m)\Gamma(k+1-m)} \frac{x^{k-m}}{\Gamma(k+1)} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(-n+k)}{\Gamma(-n-m)\Gamma(k+1)} \frac{x^{k}}{\Gamma(k+m+1)} \end{aligned}$$

we have used the definition of the Pochhammer's symbol again and shifted the index k. From the ever useful property of the Gamma function:

$$\begin{aligned} \Gamma(a-b) &= (-1)^{b-1} \frac{\Gamma(-a)\Gamma(1+a)}{\Gamma(b+1-a)} \\ \Gamma(n-(k-1)) &= (-1)^{k-2} \frac{\Gamma(-n)\Gamma(1+n)}{\Gamma(k-n)} \\ \Gamma(n-(-m-1)) &= (-1)^{-m-2} \frac{\Gamma(-n)\Gamma(1+n)}{\Gamma(-m-n)} \\ \Longrightarrow \frac{\Gamma(k-n)}{\Gamma(-m-n)} &= (-1)^{k-2}(-1)^{m+2} \frac{\Gamma(n+m+1)}{\Gamma(n-k+1)} \end{aligned}$$

Allows us to express $T_n^m(x)$ as:

$$T_n^m(x) = (-1)^m \sum_{k=0}^{\infty} \frac{\Gamma(n+m+1)}{\Gamma(n-k+1)\Gamma(k+1)} \frac{(-x)^k}{\Gamma(k+m+1)} \tag{4.36}$$

(4.37)

The denominator tends to infinity when $k > n$ so the sum is truncated:

$$T_n^m(x) = (-1)^m \sum_{k=0}^{n} \frac{\Gamma(n+m+1)}{\Gamma(n-k+1)\Gamma(k+m+1)} \frac{(-x)^k}{\Gamma(k+1)} \tag{4.38}$$

$$T_n^m(x) = (-1)^m L_n^m(x) \qquad m \in \mathbb{Z}^+ \tag{4.39}$$

We known for the Bessel wave functions approach that the functions $J_m(q,k,x)$ and $J_{-m}(q,k,x)$ are linearly independent for real values of m but are linearly dependant if m is an integer similar to the Bessel functions. In the case of the Bessel functions a linearly independent solution can be found as:

$$N_\nu(x) = \lim \nu \to m \frac{Cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{Sin(\nu\pi)}, \quad m \in \mathbb{Z} \tag{4.40}$$

Using the fact that $J_{-m}(x) = (-1)^m J_m(x)$. Pinney used the same method to define a Laguerre function of second kind by noting that $T_n^\alpha(x) = x^{-\alpha} L_{n+\alpha}^{-\alpha}(x)$ is a solution to the Laguerre differential equation. He defined:

$$U_n^\alpha(x) = \pm i Csc(\alpha\pi) \left[e^{\mp i\pi\alpha} L_n^\alpha(x) - \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} x^{-\alpha} L_{n+\alpha}^{-\alpha}(x) \right] \tag{4.41}$$

Where the upper and lower signs are taken for $0 < arg(x) < \pi$ and $-\pi < arg(x) < 0$ respectively and the limit $\alpha \to m$ is taken for integer values of α since $U_n^\alpha(x)$ is undetermined for those values. For $\alpha = m$ integer we get a $0/0$ therefore we can apply L'hopital rule. Before taking the limit it is convenient to calculate a few derivatives.

$$\frac{\partial}{\partial\alpha} Ln(\Gamma(\alpha+n+1)) = \frac{\Gamma'(\alpha+n+1)}{\Gamma(\alpha+n+1)} = \psi^{(0)}(\alpha+n+1) \tag{4.42}$$

Where $\psi^{(0)}(x)$ is the Diagamma function. Therefore we have:

$$\frac{\partial}{\partial\alpha}\Gamma(\alpha+n+1) = \Gamma(\alpha+n+1)\psi^{(0)}(\alpha+n+1) \tag{4.43}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{\Gamma(\alpha+k+1)}\right) = -\frac{\psi^{(0)}(\alpha+k+1)}{\Gamma(\alpha+k+1)} \tag{4.44}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{\Gamma(k-\alpha+1)}\right) = \frac{\psi^{(0)}(k-\alpha+1)}{\Gamma(k-\alpha+1)} \tag{4.45}$$

$$\tag{4.46}$$

Also from the definition of the Pochhammer's symbol in terms of Gamma functions:

$$\frac{\partial}{\partial x}(x)_k = \frac{\partial}{\partial x}\frac{\Gamma(x+k)}{\Gamma(x)} = (x)_k(\psi^{(0)}(x+k) - \psi^{(0)}(x)) \tag{4.47}$$

The Digamma function also diverges when evaluated on the negative integers. However the Pochhammer's symbol does not have singularities and is continous for any real value of x. It follows from it's primary definition $(x)_k = 1(x)(x+1)(x+2)...(x+k-1)$ with $(x)_0 = 1$. We are going to show that $(x)_k[\psi^{(0)}(x+k) - \psi^{(0)}(x)]$ actually converges for negative integers. We start with the property of the Gamma function:

$$\Gamma(x+1) = x\Gamma(x) \tag{4.48}$$

It's derivative gives:

$$\Gamma'(x+1) = x\Gamma'(x) + \Gamma(x) \tag{4.49}$$

Dividing by $\Gamma(x+1) = x\Gamma(x)$:

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{1}{x} \tag{4.50}$$

$$\psi^{(0)}(x+1) = \psi^{(0)}(x) + \frac{1}{x} \tag{4.51}$$

Now consider the sum:

$$\begin{aligned}
&[\psi^{(0)}(x+1) - \psi^{(0)}(x)] + [\psi^{(0)}(x+2) - \psi^{(0)}(x+1)] + \\
&[\psi^{(0)}(x+3) - \psi^{(0)}(x+2)] + ... + [\psi^{(0)}(x+k) - \psi^{(0)}(x+k-1)] \\
&= \psi^{(0)}(x+k) - \psi^{(0)}(x) = \sum_{j=0}^{k-1} \frac{1}{x+j}
\end{aligned}$$

Therefore:

$$\frac{\partial}{\partial x}(x)_k = (x)_k \sum_{j=0}^{k-1} \frac{1}{x+j} \tag{4.52}$$

Apparently it diverges when $k = 1 + |x|$ if $x \in \mathbb{Z}^-$. However in that case the only nonzero term of the sum is:

$$Lim_{x\to -n} \frac{(x)_{|x|+1}}{x+|x|} = (-n)_n Lim_{x\to -n} \frac{x+|x|}{x+|x|} = (-n)_n \tag{4.53}$$

with $n \in \mathbb{Z}^+$. In fact each term of the sum will always cancel out with a factor included in the Pochhammer's symbol so the derivative never goes to infinity. For $k = 0$ we have $\frac{\partial}{\partial x}(x)_k = 0$

As was previously shown:

$$T_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} x^{-\alpha} L_{n+\alpha}^{-\alpha}(x) = x^{-\alpha} \sum_{k=0}^{\infty} \frac{(-n-\alpha)_k}{\Gamma(k+1-\alpha)} \frac{x^k}{k!} \tag{4.54}$$

It's derivative with respect to α is:

$$\begin{aligned}
\frac{\partial}{\partial\alpha} T_n^\alpha(x) = & -x^{-\alpha} Ln(x) \sum_{k=0}^{\infty} \frac{(-n-\alpha)_k}{\Gamma(k+1-\alpha)} \frac{x^k}{k!} \\
& + x^{-\alpha} \sum_{k=0}^{\infty} (-n-\alpha)_k \frac{\psi^{(0)}(k-\alpha+1)}{\Gamma(k-\alpha+1)} \frac{x^k}{k!} \\
& + x^{-\alpha} \sum_{k=0}^{\infty} \sum_{j=0}^{k-1} \frac{1}{-n-\alpha+j} \frac{(-n-\alpha)_k}{\Gamma(k+1-\alpha)} \frac{x^k}{k!}
\end{aligned}$$

The first series with $x^{-\alpha}$ becomes $(-1)^m L_n^m(x)$ when taking the limit $\alpha \to m$. The third is truncated from the Bottom where the first non-vanishing term starts with $k = \alpha$ or $k = 1$ if $\alpha \leq 1$ since $\frac{\partial}{\partial x}(x)_k = 0$ for $k = 0$. The second is also truncated by the Pochhammer's symbol:

$$\begin{aligned}
\frac{\partial}{\partial\alpha} T_n^\alpha(x)|_{\alpha=m} = & -Ln(x)(-1)^m L_n^m(x) \\
& + x^{-m} \sum_{k=0}^{n+m} (-n-m)_k \frac{\psi^{(0)}(k-m+1)}{\Gamma(k-m+1)} \frac{x^k}{k!} \\
& + x^{-m} \sum_{k=Max(1,m)}^{\infty} \sum_{j=0}^{k-1} \frac{1}{-n-m+j} \frac{(-n-m)_k}{\Gamma(k+1-m)} \frac{x^k}{k!}
\end{aligned}$$

We still need to show that the limit:

$$Lim_{\alpha \to m} \frac{\psi^{(0)}(k-\alpha+1)}{\Gamma(k-\alpha+1)} = L < \infty \tag{4.55}$$

For this purpose consider the natural log of the reflection formula:

$$Ln(\Gamma(z)) + Ln(\Gamma(1-z)) = Ln(\pi) - Ln(\sin(\pi z))$$

The derivative with respect to z gives:

$$\psi^{(0)}(z) - \psi^{(0)}(1-z) = -\frac{\pi}{\sin(\pi z)} \cos(\pi z)$$

By using the reflection formula again and dividing by $\Gamma(z)$:

$$\frac{\psi^{(0)}(z)}{\Gamma(z)} = \frac{\psi^{(0)}(1-z)}{\Gamma(z)} - \cos(\pi z)\Gamma(1-z)$$

If $z = -n$:

$$\frac{\psi^{(0)}(-n)}{\Gamma(-n)} = \frac{\psi^{(0)}(1+n)}{\Gamma(-n)} - \cos(\pi n)\Gamma(1+n)$$

The first term on the right goes to zero if n is a positive integer since the denominator goes to infinity while the nominator is finite, then:

$$\frac{\psi^{(0)}(-n)}{\Gamma(-n)} = -(-1)^n n! \qquad n \in \mathbb{Z}^+$$

The next step is:

$$\begin{aligned}\frac{\partial}{\partial\alpha}\left(e^{\mp i\pi\alpha} L_n^{\alpha}(x)\right) = &\mp i\pi e^{\mp i\pi\alpha} L_n^{\alpha}(x) \\ &+ \psi^{(0)}(n+\alpha+1) e^{\mp i\pi\alpha} L_n^{\alpha}(x) \\ &- e^{\mp i\pi\alpha} \sum_{k=0}^{n} \frac{\Gamma(n+1+\alpha)\psi^{(0)}(k+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)} \frac{(-x)^k}{k!}\end{aligned}$$

The closed form of the Laguerre polynomials and previous derivatives were used. We are interested in the case $\alpha = m$:

$$\begin{aligned}\frac{\partial}{\partial\alpha}\left(e^{\mp i\pi\alpha} L_n^{\alpha}(x)\right)|_{\alpha=m} = &\mp i\pi(-1)^m L_n^m(x) \\ &+ (-1)^m \psi^{(0)}(n+m+1) L_n^m(x) \\ &- (-1)^m \sum_{k=0}^{n} \frac{\Gamma(n+1+m)\psi^{(0)}(k+m+1)}{\Gamma(n-k+1)\Gamma(m+k+1)} \frac{(-x)^k}{k!}\end{aligned}$$

Finally by L'Hopital rule:

$$U_n^m(x) = Lim_{\alpha\to m} U_n^\alpha(x) = \pm\frac{i}{\pi}(-1)^{-m}[\mp i\pi(-1)^m L_n^m(x)$$
$$+(-1)^m \psi^{(0)}(n+m+1)L_n^m(x)$$
$$-(-1)^m \sum_{k=0}^{n} \frac{\Gamma(n+1+m)\psi^{(0)}(k+m+1)}{\Gamma(n-k+1)\Gamma(m+k+1)}\frac{(-x)^k}{k!}$$
$$+(-1)^m Ln(x)L_n^m(x)$$
$$-x^{-m}\sum_{k=0}^{n+m}(-n-m)_k \frac{\psi^{(0)}(k-m+1)}{\Gamma(k-m+1)}\frac{x^k}{k!}$$
$$-x^{-m}\sum_{k=Max(1,m)}^{\infty}\sum_{j=0}^{k-1}\frac{1}{-n-m+j}\frac{(-n-m)_k}{\Gamma(k+1-m)}\frac{x^k}{k!}]$$

Which simplifies to:

$$U_n^m(x) = Lim_{\alpha\to m} U_n^\alpha(x) = \pm\frac{i}{\pi}[\mp i\pi L_n^m(x)$$
$$+\psi^{(0)}(n+m+1)L_n^m(x)$$
$$-\sum_{k=0}^{n} \frac{\Gamma(n+1+m)\psi^{(0)}(k+m+1)}{\Gamma(n-k+1)\Gamma(m+k+1)}\frac{(-x)^k}{k!}$$
$$+Ln(x)L_n^m(x)$$
$$-(-x)^{-m}\sum_{k=0}^{n+m}(-n-m)_k \frac{\psi^{(0)}(k-m+1)}{\Gamma(k-m+1)}\frac{x^k}{k!}$$
$$-(-x)^{-m}\sum_{k=Max(1,m)}^{\infty}\sum_{j=0}^{k-1}\frac{1}{-n-m+j}\frac{(-n-m)_k}{\Gamma(k+1-m)}\frac{x^k}{k!}]$$

Where the upper and lower signs are taken for $0 < arg(x) < \pi$ and $-\pi < arg(x) < 0$ respectively.

For real values of x the second solution may be defined as:

$$U_n^\alpha(x) = csc(\alpha\pi)\left[cos(\pi\alpha)L_n^\alpha(x) - \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}x^{-\alpha}L_{n+\alpha}^{-\alpha}(x)\right] \tag{4.56}$$

Taking the limit $\alpha \to m$:

$$U_n^m(x) = Lim_{\alpha \to m} U_n^{\alpha}(x) = \frac{1}{\pi}\Bigg[\psi^{(0)}(n+m+1)L_n^m(x)$$
$$-\sum_{k=0}^{n} \frac{\Gamma(n+1+m)\psi^{(0)}(k+m+1)}{\Gamma(n-k+1)\Gamma(m+k+1)} \frac{(-x)^k}{k!}$$
$$+Ln(x)L_n^m(x)$$
$$-(-x)^{-m} \sum_{k=0}^{n+m} (-n-m)_k \frac{\psi^{(0)}(k-m+1)}{\Gamma(k-m+1)} \frac{x^k}{k!}$$
$$-(-x)^{-m} \sum_{k=Max(1,m)}^{\infty} \sum_{j=0}^{k-1} \frac{1}{-n-m+j} \frac{(-n-m)_k}{\Gamma(k+1-m)} \frac{x^k}{k!}\Bigg]$$

4.4 Asymptotic expressions

In this section, it is convenient to think that the indices ν and α are real numbers. The Laguerre Polynomials can be considered as a special case of the Laguerre function:

$$L_\nu^\alpha(z) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)} M(-\nu, \alpha+1, z) \tag{4.57}$$

When $\nu \in \mathbb{Z}$. This is convenient because we can use an asymptotic expression for the confluent hypergeometric function for large $|z|$ to (see (ABRAMOWITZ; STEGUN, 1970)):

$$M(a,b,z) \approx \Gamma(b)\left[\frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)} + \frac{e^z z^{a-b}}{\Gamma(a)}\right], \qquad |z| >> 1 \tag{4.58}$$

where the upper sign is taken if $-\pi/2 < arg(z) < 3\pi/2$ and the lower if $-3\pi/2 < arg(z) < -\pi/2$.

Therefore:

$$L_\nu^\alpha(z) \approx \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)} \Gamma(\alpha+1)\left[\frac{e^{\mp i\pi\nu} z^{\nu}}{\Gamma(\nu+\alpha+1)} + \frac{e^z z^{-\nu-\alpha-1}}{\Gamma(-\nu)}\right], \qquad |z| >> 1$$
$$L_\nu^\alpha(z) \approx \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(-\nu)} e^z z^{-\nu-\alpha-1} + \frac{e^{\mp i\pi\nu} z^{\nu}}{\Gamma(\nu+1)}, \qquad |z| >> 1$$

By the use of Euler's reflection formula:

$$\Gamma(1-\nu)\Gamma(\nu) = \frac{\pi}{sin(\pi\nu)}, \nu \notin \mathbb{Z}$$

we obtain the same expression found by Pinney (eq 2.11 of (PINNEY, 1946)):

$$L_{\nu}^{\alpha}(z) \approx -\frac{sin(\pi\nu)}{\pi}\Gamma(\nu+\alpha+1)e^{z}z^{-\nu-\alpha-1} + \frac{e^{\mp i\pi\nu}z^{\nu}}{\Gamma(\nu+1)}, \qquad |z| >> 1 \tag{4.59}$$

Do note however, that Pinney used the convention $0 < arg(z) < \pi$ for the upper sign and $-\pi < arg(z) < 0$ for the lower unlike (ABRAMOWITZ; STEGUN, 1970). Since Pinney convention is most suited for our needs, we stick to it. Note that when ν is an integer:

$$L_{n}^{\alpha}(z) \approx \frac{(-1)^{n}z^{n}}{n!} \tag{4.60}$$

Which is just highest power term of the Laguerre polynomial $L_{n}^{\alpha}(z)$. The asymptotic expression (4.59) can be used to obtain an asymptotic expression for $U_{\nu}^{\alpha}(z)$. Recall by definition:

$$U_{\nu}^{\alpha}(z) = \pm icsc(\alpha\pi)\left[e^{\mp i\pi\alpha}L_{\nu}^{\alpha}(z) - \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)}z^{-\alpha}L_{\nu+\alpha}^{-\alpha}(z)\right] \tag{4.61}$$

Where the upper and lower signs follows the convention $0 < arg(z) < \pi$ for the upper sign and $-\pi < arg(z) < 0$ for the lower. we need to find an asymptotic expression for $\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)}z^{-\alpha}L_{\nu+\alpha}^{-\alpha}(z)$. First by (4.59):

$$L_{\nu+\alpha}^{-\alpha}(z) \approx -\frac{sin(\pi(\nu+\alpha))}{\pi}\Gamma(\nu+1)e^{z}z^{-\nu-1} + \frac{e^{\mp i\pi(\nu+\alpha)}z^{\nu+\alpha}}{\Gamma(\nu+\alpha+1)}, \qquad |z| >> 1$$

Then:

$$\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)}z^{-\alpha}L_{\nu+\alpha}^{-\alpha}(z) \approx -\frac{sin(\pi(\nu+\alpha))}{\pi}\Gamma(\nu+\alpha+1)e^{z}z^{-\nu-\alpha-1} + \frac{e^{\mp i\pi(\nu+\alpha)}z^{\nu}}{\Gamma(\nu+1)}, \qquad |z| >> 1$$

and:

$$\begin{aligned} U_{\nu}^{\alpha}(z) &\approx \pm icsc(\alpha\pi)[-e^{\mp i\pi\alpha}\frac{sin(\pi\nu)}{\pi}\Gamma(\nu+\alpha+1)e^{z}z^{-\nu-\alpha-1} \\ &\quad + \frac{sin(\pi(\nu+\alpha))}{\pi}\Gamma(\nu+\alpha+1)e^{z}z^{-\nu-\alpha-1}] \\ U_{\nu}^{\alpha}(z) &\approx \pm iCsc(\alpha\pi)\Gamma(\nu+\alpha+1)e^{z}z^{-\nu-\alpha-1}\left[\frac{sin(\pi(\nu+\alpha))}{\pi} - \frac{sin(\pi\nu)}{\pi}e^{\mp i\pi\alpha}\right] \end{aligned}$$

The expression in brackets can be simplified:

$$csc(\alpha\pi)[sin(\pi\nu)cos(\pi\alpha)+sin(\pi\alpha)cos(\pi\nu)-sin(\pi\nu)cos(\pi\alpha)\pm isin(\pi\nu)sin(\pi\alpha)]$$
$$=csc(\alpha\pi)[sin(\pi\alpha)cos(\pi\nu)\pm isin(\pi\nu)sin(\pi\alpha)]$$
$$=cos(\pi\nu)\pm isin(\pi\nu)$$

Which leaves:

$$U_\nu^\alpha(z)\approx\pm\frac{i}{\pi}\Gamma(\nu+\alpha+1)e^z z^{-\nu-\alpha-1}[cos(\pi\nu)\pm isin(\pi\nu)] \tag{4.62}$$

For integer values of $\nu=n$:

$$U_n^\alpha(z)\approx\pm\frac{i}{\pi}(-1)^n\Gamma(n+\alpha+1)e^z z^{-n-\alpha-1} \tag{4.63}$$

4.5 Auxiliary functions $A_n^m(x,y)$ and $B_n^m(x,y)$

One of the main difficulties of scattering problems with parabolic symmetry is expressing each component of the Hertz vectors in a manner in which the functions involved are separable (PINNEY, 1947; HORTON; KARAL, 1951). For this purpose Pinney defined the Functions:

$$A_n^m(x,y)=\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[L_{n+1}^m(x)L_n^m(y)-L_n^m(x)L_{n+1}^m(y)\right] \tag{4.64}$$
$$B_n^m(x,y)=\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[U_{n+1}^m(x)L_n^m(y)-U_n^m(x)L_{n+1}^m(y)\right] \tag{4.65}$$

(4.66)

These functions can be expressed as a sum of $L_n^m(x)L_n^m(y)$ and $U_n^m(x)L_n^m(y)$ functions respectively, with a common factor $(y-x)$ which is conveniently canceled out in the calculation of one of the components of the Hertz vectors. We begin with the auxiliary $A_n^m(x,y)$ function defined as:

$$A_n^m(x,y)=\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[L_{n+1}^m(x)L_n^m(y)-L_n^m(x)L_{n+1}^m(y)\right]$$

From this definition:

$$A_n^m(x,y) = \frac{\Gamma(n+1)(n+1)}{\Gamma(m+n+1)}\left[L_{n+1}^m(x)L_n^m(y) - L_n^m(x)L_{n+1}^m(y)\right]$$
$$A_{n-1}^m(x,y) = \frac{\Gamma(n+1)(m+n)}{\Gamma(m+n+1)}\left[L_n^m(x)L_{n-1}^m(y) - L_{n-1}^m(x)L_n^m(y)\right]$$

Next from the identity:

$$(n+m)L_{n-1}^m(z) + (z-m-2n-1)L_n^m(z) + (n+1)L_{n+1}^m(z) = 0$$
$$(n+m)L_{n-1}^m(x)L_n^m(y) + (x-m-2n-1)L_n^m(x)L_n^m(y) + (n+1)L_{n+1}^m(x)L_n^m(y) = 0$$
$$(n+m)L_{n-1}^m(y)L_n^m(x) + (y-m-2n-1)L_n^m(y)L_n^m(x) + (n+1)L_{n+1}^m(y)L_n^m(x) = 0$$

The difference of the last equations leads to:

$$(x-y)L_n^m(x)L_n^m(y) - (n+m)[L_{n-1}^m(y)L_n^m(x) - L_{n-1}^m(x)L_n^m(y)]$$
$$+ (n+1)[L_{n+1}^m(x)L_n^m(y) - L_{n+1}^m(y)L_n^m(x)] = 0$$

Which is simply:

$$(x-y)L_n^m(x)L_n^m(y) + \frac{\Gamma(m+n+1)}{\Gamma(n+1)}[A_n^m(x,y) - A_{n-1}^m(x,y)] = 0 \tag{4.67}$$

Then:

$$[A_n^m(x,y) - A_{n-1}^m(x,y)] = (y-x)\frac{\Gamma(n+1)}{\Gamma(m+n+1)}L_n^m(x)L_n^m(y) \tag{4.68}$$

Finally consider the sum:

$$[A_n^m(x,y) - A_{n-1}^m(x,y)] + [A_{n+1}^m(x,y) - A_n^m(x,y)] + [A_{n+2}^m(x,y) - A_{n+1}^m(x,y)]$$
$$+ \ldots + [A_{n+l}^m(x,y) - A_{n+l-1}^m(x,y)] = A_{n+l}^m(x,y) - A_{n-1}^m(x,y)$$
$$= (y-x)\sum_{p=0}^{l}\frac{\Gamma(n+p+1)}{\Gamma(m+n+p+1)}L_{n+p}^m(x)L_{n+p}^m(y)$$

Set $n=0$ and since $A^m_{-1}(x,y)=0$:

$$A^m_l(x,y)=(y-x)\sum_{p=0}^{l}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}L^m_p(x)L^m_p(y) \tag{4.69}$$

The $B^m_n(x,y)$ follows a similar procedure. Since the function $U^m_n(x)$ satisfies the same recursion formulas as $L^m_n(x)$, then:

$$(x-y)U^m_n(x)L^m_n(y)+\frac{\Gamma(m+n+1)}{\Gamma(n+1)}[B^m_n(x,y)-B^m_{n-1}(x,y)]=0 \tag{4.70}$$

now consider the sum:

$$\begin{aligned}
&[B^m_n(x,y)-B^m_{n-1}(x,y)]+[B^m_{n+1}(x,y)-B^m_n(x,y)]+[B^m_{n+2}(x,y)-B^m_{n+1}(x,y)]\\
&+\ldots+[B^m_{n+l}(x,y)-B^m_{n+l-1}(x,y)]=B^m_{n+l}(x,y)-B^m_{n-1}(x,y)\\
&=(y-x)\sum_{p=0}^{l}\frac{\Gamma(n+p+1)}{\Gamma(m+n+p+1)}U^m_{n+p}(x)L^m_{n+p}(y)
\end{aligned}$$

Care must be taken from this point, for $B^m_{-1}(x,y)\neq 0$. By definition:

$$\begin{aligned}
B^m_{-1}(x,y)&=\frac{\Gamma(1)}{\Gamma(m)}\left[U^m_0(x)L^m_{-1}(y)-U^m_{-1}(x)L^m_0(y)\right]\\
&=-\frac{1}{\Gamma(m)}U^m_{-1}(x)
\end{aligned}$$

As usual $L^m_{-1}(y)=0$ and $L^m_0(y)=1$. From the definition of $U^\alpha_n(x)$ with α non-integer:

$$U^\alpha_n(x)=\pm icsc(\pi\alpha)\left[e^{\mp i\pi\alpha}L^\alpha_n(x)-T^\alpha_n(x)\right] \tag{4.71}$$

where

$$T^\alpha_n(x)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}x^{-\alpha}L^{-\alpha}_{n+\alpha}(x)=x^{-\alpha}\sum_{k=0}^{\infty}\frac{(-n-\alpha)_k}{\Gamma(k+1-\alpha)}\frac{x^k}{k!} \tag{4.72}$$

This implies:

$$
\begin{aligned}
U_{-1}^{\alpha}(x) &= \mp icsc(\pi\alpha)T_{-1}^{\alpha}(x) \\
&= \mp icsc(\pi\alpha)x^{-\alpha}\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{\Gamma(k+1-\alpha)}\frac{x^k}{k!} \\
&= \mp\frac{i}{\pi}\Gamma(1-\alpha)\Gamma(\alpha)x^{-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(k+1-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1-\alpha)}\frac{x^k}{k!} \\
&= \mp\frac{i}{\pi}\Gamma(\alpha)x^{-\alpha}\sum_{k=0}^{\infty}\frac{x^k}{k!} \\
&= \mp\frac{i}{\pi}\Gamma(\alpha)x^{-\alpha}e^x
\end{aligned}
$$

Euler's reflection formula was used to get rid of the $csc(\pi\alpha)$. The previous formula should be valid for $\alpha = m$ with m being a positive integer then:

$$
\begin{aligned}
U_{-1}^{m}(x) &= \mp\frac{i}{\pi}\Gamma(m)x^{-m}e^x \\
\implies B_{-1}^{m}(x,y) &= -\frac{1}{\Gamma(m)}U_{-1}^{m}(x) \\
\implies B_{-1}^{m}(x,y) &= \pm\frac{i}{\pi}x^{-m}e^x
\end{aligned}
$$

As a consequence:

$$
\begin{aligned}
B_{n+l}^{m}(x,y) &= \pm\frac{i}{\pi}x^{-m}e^x + (y-x)\sum_{p=0}^{l}\frac{\Gamma(n+p+1)}{\Gamma(m+n+p+1)}U_{n+p}^{m}(x)L_{n+p}^{m}(y) \\
B_{l}^{m}(x,y) &= \pm\frac{i}{\pi}x^{-m}e^x + (y-x)\sum_{p=0}^{l}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}U_{p}^{m}(x)L_{p}^{m}(y)
\end{aligned}
$$

Pinney sorted this out by extending the sum to infinity and used $B_n^m = 0$ as $n \to \infty$:

$$
-B_{n-1}^{m}(x,y) = (y-x)\sum_{p=0}^{\infty}\frac{\Gamma(n+p+1)}{\Gamma(m+n+p+1)}U_{n+p}^{m}(x)L_{n+p}^{m}(y)
$$

and finally:

$$
B_{n}^{m}(x,y) = -(y-x)\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}U_{p}^{m}(x)L_{p}^{m}(y)
$$

Provided $|y|^2 sin|\frac{1}{2}arg(y)| \leq |x|^2 sin|\frac{1}{2}arg(x)|$ and excluding the origin in the intervals. To summarize:

$$A_n^m(x,y) = \frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[L_{n+1}^m(x)L_n^m(y) - L_n^m(x)L_{n+1}^m(y)\right] \tag{4.73}$$

$$B_n^m(x,y) = \frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[U_{n+1}^m(x)L_n^m(y) - U_n^m(x)L_{n+1}^m(y)\right] \tag{4.74}$$

$$A_n^m(x,y) = (y-x)\sum_{p=0}^{n}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}L_p^m(x)L_p^m(y) \tag{4.75}$$

$$B_n^m(x,y) \approx -(y-x)\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}U_p^m(x)L_p^m(y) \tag{4.76}$$

$$B_n^m(x,y) = \pm\frac{i}{\pi}x^{-m}e^x + (y-x)\sum_{p=0}^{n}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}U_p^m(x)L_p^m(y) \tag{4.77}$$

Note: The upper sign is taken when $0 < arg(x) < \pi$ and the lower when $-\pi < arg(x) < 0$.

Note: It can be shown that if

$$B_n^m(x,y) = -(y-x)\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}U_p^m(x)L_p^m(y)$$

is true, it implies the wronskian is zero so care must be taken if this relation is used as an aproximation.

4.6 Wronskian

The previous auxiliary functions can be used to easily calculate the Wronskian of $L_n^\alpha(x)$ and $U_n^\alpha(x)$.

$$W(L_n^\alpha(x), U_n^\alpha(x)) = L_n^\alpha(x)\frac{\mathrm{d}U_n^\alpha(x)}{\mathrm{d}x} - \frac{\mathrm{d}L_n^\alpha(x)}{\mathrm{d}x}U_n^\alpha(x) \tag{4.78}$$

From:

$$\frac{\mathrm{d}L_n^\alpha(x)}{\mathrm{d}x} = -L_{n-1}^{\alpha+1}(x)$$
$$L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x)$$

It is obtained:

$$
\begin{aligned}
W(L_n^{\alpha}(x),U_n^{\alpha}(x)) &= L_{n-1}^{\alpha+1}(x)U_n^{\alpha+1}(x)-L_n^{\alpha+1}(x)U_{n-1}^{\alpha+1}(x)\\
&= \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}B_{n-1}^{\alpha+1}(x,x)\\
&= \pm\frac{i}{\pi}\frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}x^{-\alpha-1}e^{x}
\end{aligned}
$$

which is non-zero for every x provided $\alpha > -1$.

4.7 Derivatives in terms of auxiliary functions

To simplify the calculation of the expansion vectors components and the application of the boundary conditions the following derivatives were used by (HORTON; KARAL, 1951):

$$
\begin{aligned}
&\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)U_n^m(x)L_n^m(y) = x\frac{\partial U_n^m}{\partial x}L_n^m(y)-U_n^m(x)y\frac{\partial L_n^m}{\partial y}\\
&= nU_n^m(x)L_n^m(y)-(m+n)U_{n-1}^m(x)L_n^m(y)-nU_n^m(x)L_n^m(y)+(m+n)U_n^m(x)L_{n-1}^m(y)\\
&= (m+n)\left[U_n^m(x)L_{n-1}^m(y)-U_{n-1}^m(x)L_n^m(y)\right] = \frac{\Gamma(m+n+1)}{\Gamma(n+1)}B_{n-1}^m(x,y)
\end{aligned}
$$

The identity $x\frac{\partial L_n^m}{\partial x} = nL_n^m(x)-(m+n)L_{n-1}^m(x)$ was used. An analogous procedure gives:

$$
\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y) = \frac{\Gamma(m+n+1)}{\Gamma(n+1)}A_{n-1}^m(x,y)
$$

Now consider:

$$
\begin{aligned}
&xy\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)U_n^m(x)L_n^m(y)\\
&= y[-mU_n^m(x)L_n^m(y)+(m+n)U_n^{m-1}(x)L_n^m(y)]-x[-mU_n^m(x)L_n^m(y)+(m+n)U_n^m(x)L_n^{m-1}(y)]\\
&= -m(y-x)U_n^m(x)L_n^m(y)+(m+n)\left[yU_n^{m-1}(x)L_n^m(y)-xU_n^m(x)L_n^{m-1}(y)\right]\\
&= -m(y-x)U_n^m(x)L_n^m(y)+(m+n)[-(n+1)U_n^{m-1}(x)L_{n+1}^{m-1}(y)+(m+n)U_n^{m-1}(x)L_n^{m-1}(y)\\
&+(n+1)U_{n+1}^{m-1}(x)L_n^{m-1}(y)-(m+n)U_n^{m-1}(x)L_n^{m-1}(y)]\\
&= -m(y-x)U_n^m(x)L_n^m(y)+(m+n)(n+1)[U_{n+1}^{m-1}(x)L_n^{m-1}(y)-U_n^{m-1}(x)L_{n+1}^{m-1}(y)]\\
&= -m(y-x)U_n^m(x)L_n^m(y)+\frac{\Gamma(n+m+1)}{\Gamma(n+1)}B_n^{m-1}(x,y)
\end{aligned}
$$

The following relations were used:

$$x\frac{\mathrm{d}L_n^m}{\mathrm{d}x} = -mL_n^m(x) + (n+m)L_n^{m-1}(x)$$
$$xL_n^m = -(n+1)L_{n+1}^{m-1}(x) + (m+n)L_n^{m-1}(x)$$
$$B_n^{m-1}(x,y) = \frac{\Gamma(n+1)}{\Gamma(n+m+1)}(n+1)(n+m)[U_{n+1}^{m-1}(x)L_n^{m-1}(y) - U_n^{m-1}(x)L_{n+1}^{m-1}(y)]$$

By analogy:

$$\begin{aligned} &xy\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y) \\ &= -m(y-x)L_n^m(x)L_n^m(y) + \frac{\Gamma(n+m+1)}{\Gamma(n+1)}A_n^{m-1}(x,y) \end{aligned}$$

The Last type of derivative encountered is:

$$\begin{aligned} &\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_n^m(x)L_n^m(y) = -U_n^{m+1}(x)L_n^m(y) + U_n^m(x)L_n^m(y) \\ &+ U_n^m(x)L_n^{m+1}(y) - U_n^m(x)L_n^m(y) \\ &= -U_n^{m+1}(x)[L_n^{m+1}(y) - L_{n-1}^{m+1}(y)] + [U_n^{m+1}(x) - U_{n-1}^{m+1}(x)]L_n^{m+1}(y) \\ &= \frac{\Gamma(n+m+1)}{\Gamma(n+1)}B_{n-1}^{m+1}(x,y) \end{aligned}$$

In this case the following relations were used:

$$\frac{\mathrm{d}L_n^m}{\mathrm{d}x} = -L_n^{m+1}(x) + L_n^m(x)$$
$$L_n^m = L_n^{m+1}(x) - L_{n-1}^{m+1}(x)$$
$$B_{n-1}^{m+1}(x,y) = \frac{\Gamma(n+1)}{\Gamma(n+m+1)}[U_n^{m+1}(x)L_{n-1}^{m+1}(y) - U_{n-1}^{m+1}(x)L_n^{m+1}(y)]$$

To summarize:

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y)=\frac{\Gamma(m+n+1)}{\Gamma(n+1)}A_{n-1}^m(x,y) \tag{4.79}$$

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)U_n^m(x)L_n^m(y)=\frac{\Gamma(m+n+1)}{\Gamma(n+1)}B_{n-1}^m(x,y) \tag{4.80}$$

$$xy\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y)=-m(y-x)L_n^m(x)L_n^m(y)$$
$$+\frac{\Gamma(n+m+1)}{\Gamma(n+1)}A_n^{m-1}(x,y) \tag{4.81}$$

$$xy\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)L_n^m(x)U_n^m(y)=-m(y-x)U_n^m(x)L_n^m(y)$$
$$+\frac{\Gamma(n+m+1)}{\Gamma(n+1)}B_n^{m-1}(x,y) \tag{4.82}$$

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y)=\frac{\Gamma(n+m+1)}{\Gamma(n+1)}A_{n-1}^{m+1}(x,y) \tag{4.83}$$

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)U_n^m(x)L_n^m(y)=\frac{\Gamma(n+m+1)}{\Gamma(n+1)}B_{n-1}^{m+1}(x,y) \tag{4.84}$$

5 PROPERTIES OF THE PINNEY FUNCTIONS S_n^α AND V_n^α

From the definitions:

$$S_n^\alpha(z) = z^{\frac{\alpha}{2}} e^{-\frac{z}{2}} L_n^\alpha(z) \quad (5.1)$$

$$V_n^\alpha(z) = z^{\frac{\alpha}{2}} e^{-\frac{z}{2}} U_n^\alpha(z) \quad (5.2)$$

And the properties of the Laguerre functions; the following properties can be deduced:

$$z^{1/2} S_\nu^\alpha(z) = S_\nu^{\alpha+1}(z) - S_{\nu-1}^{\alpha+1}(z) \quad (5.3)$$

$$z^{1/2} S_\nu^\alpha(z) = -(\nu+1) S_{\nu+1}^{\alpha-1}(z) + (\nu+\alpha) S_\nu^{\alpha-1}(z) \quad (5.4)$$

$$(\nu+\alpha) S_\nu^{\alpha-1}(z) - (z-\alpha) z^{-1/2} S_\nu^\alpha(z) + S_\nu^{\alpha+1}(z) = 0 \quad (5.5)$$

$$(\nu+\alpha) S_{\nu-1}^\alpha(z) + (z-\alpha-2\nu-1) S_\nu^\alpha(z) + (\nu+1) S_{\nu+1}^\alpha(z) = 0 \quad (5.6)$$

$$2z \frac{\mathrm{d}S_\nu^\alpha}{\mathrm{d}z} = (\alpha+2\nu-z) S_\nu^\alpha(z) - 2(\nu+\alpha) S_{\nu-1}^\alpha(z) \quad (5.7)$$

$$(5.8)$$

For $\alpha = -k$ with $k \in \mathbb{Z}$:

$$S_n^{-k}(z) = (-1)^k \frac{(n-k)!}{n!} S_{n-k}^k(z) \quad (5.9)$$

This follows from:

$$L_n^{-k}(z) = (-z)^k \frac{(n-k)!}{n!} L_{n-k}^k(z), \quad (n \geq k), \quad (k \in \mathbb{Z}^+) \quad (5.10)$$

multiplying both sides by $z^{-k/2} e^{-z/2}$.

Also from the orthogonality of the Laguerre polynomials:

$$\int_0^\infty S_n^\alpha(x) S_m^\alpha(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m} \quad (5.11)$$

Its derivative is:

$$\frac{\mathrm{d}S_n^m}{\mathrm{d}z} = \left(\frac{m}{2z} - \frac{1}{2}\right) S_n^m(z) - z^{-1/2} S_{n-1}^{m+1}(z) \tag{5.12}$$

In terms of Laguerre Polynomials is:

$$\frac{\mathrm{d}S_n^m}{\mathrm{d}z} = \frac{z^{\frac{m}{2}}}{2} e^{-\frac{z}{2}} \left[\left(\frac{m}{z} - 1\right) L_n^m(z) - 2L_{n-1}^{m+1}(z) \right] \tag{5.13}$$

It may be convenient to write it in terms of only one index m. For this purpose we can use the following identities (PINNEY, 1946):

$$z\frac{\mathrm{d}L_n^m}{\mathrm{d}z} = nL_n^m(z) - (n+m)L_{n-1}^m(z) \tag{5.14}$$

$$\frac{\mathrm{d}L_n^m}{\mathrm{d}z} = -L_{n-1}^{m+1}(z) \tag{5.15}$$

Then:

$$\frac{\mathrm{d}S_n^m}{\mathrm{d}z} = \frac{z^{\frac{m}{2}}}{2} e^{-\frac{z}{2}} \left[\left(\frac{m+2n}{z} - 1\right) L_n^m(z) - 2\frac{(n+m)}{z} L_{n-1}^m(z) \right] \tag{5.16}$$

Which simplifies to:

$$\frac{\mathrm{d}S_n^m}{\mathrm{d}z} = \frac{1}{2} \left[\left(\frac{m+2n}{z} - 1\right) S_n^m(z) - 2\frac{(n+m)}{z} S_{n-1}^m(z) \right] \tag{5.17}$$

Note that this relation seems to imply that the derivative diverges when $z \to 0$. To avoid this problem the derivative in terms of the Laguerre polynomials is more useful.

5.1 Series representation of Pinney function $S_n^m(y)$

Let $f(y)$ and $g(y)$ be functions of y with a series representation:

$$f(y) = \sum_{p=0}^{\infty} a_p y^p$$

$$g(y) = \sum_{l=0}^{\infty} b_l y^l$$

Then:

$$h(y) = \sum_{p=0}^{\infty}\sum_{l=0}^{\infty} a_p b_l y^{p+l} = (a_0 b_0)y^0 + (a_1 b_0 + a_0 b_1)y^1$$
$$+ (a_2 b_0 + a_1 b_1 + a_0 b_2)y^2 + \ldots + \left(\sum_{p=0}^{l} a_{l-p} b_p\right) y^l + \ldots$$
$$h(y) = \sum_{l=0}^{\infty} c_l y^l, \qquad c_l = \sum_{p=0}^{l} a_{l-p} b_p$$

Now let:

$$f(y) = e^{-y/2} = \sum_{p=0}^{\infty} \left(\frac{-1}{2}\right)^p \frac{y^p}{p!}$$
$$g(y) = L_n^m(y) = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)} \sum_{l=0}^{\infty} \frac{(-n)_l}{(m+1)_l} \frac{y^l}{l!}$$

So $S_n^m(y)$ has the following series expansion:

$$S_n^m(y) = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)} y^{\frac{m}{2}} \sum_{p=0}^{\infty} c_p(n,m) y^p \tag{5.18}$$
$$c_p(n,m) = \sum_{l=0}^{p} \left(\frac{-1}{2}\right)^l \left(\frac{(-n)_{p-l}}{(m+1)_{p-l}}\right) \frac{1}{l!(p-l)!} \tag{5.19}$$

5.2 Asymptotic forms of S_n^m and V_n^m

Recall

$$L_\nu^\alpha(z) \approx -\frac{sin(\pi\nu)}{\pi}\Gamma(\nu+\alpha+1)e^z z^{-\nu-\alpha-1} + \frac{e^{\mp i\pi\nu} z^\nu}{\Gamma(\nu+1)}, \qquad |z| >> 1 \tag{5.20}$$
$$U_\nu^\alpha(z) \approx \pm\frac{i}{\pi}\Gamma(\nu+\alpha+1)e^z z^{-\nu-\alpha-1}[cos(\pi\nu) \pm isin(\pi\nu)], \qquad |z| >> 1 \tag{5.21}$$

so:

$$S_\nu^\alpha(x) \approx x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \left[-\frac{sin(\pi\nu)}{\pi}\Gamma(\nu+\alpha+1)e^x x^{-\nu-\alpha-1} + \frac{e^{\mp i\pi\nu} x^\nu}{\Gamma(\nu+1)} \right] \tag{5.22}$$
$$V_\nu^\alpha(x) \approx x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \left[\pm\frac{i}{\pi}\Gamma(\nu+\alpha+1)e^x x^{-\nu-\alpha-1}[cos(\pi\nu) \pm isin(\pi\nu)] \right] \tag{5.23}$$

when $|x| >> 1$. For $\nu = n$ positive integer:

$$S_n^\alpha(x) \approx x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \left[\frac{(-1)^n x^n}{n!} \right] \tag{5.24}$$

$$V_n^\alpha(x) \approx x^{\frac{\alpha}{2}} e^{\frac{x}{2}} \left[\pm \frac{i(-1)^n}{\pi} \Gamma(n+\alpha+1) x^{-n-\alpha-1} \right] \tag{5.25}$$

By taking a look on the exponential term and knowing that x is either $\pm ik\sigma^2$ or $\pm ik\tau^2$ in our problem, we can find which combination of $S_n^m(x)$ and $V_n^m(x)$ functions gives waves traveling in the z± direction ($e^{\pm ikz} = e^{\pm ik\frac{\tau^2-\sigma^2}{2}}$) and toward and inward the origin r± ($e^{\pm ikr} = e^{\pm ik\frac{\tau^2+\sigma^2}{2}}$):

- $S_n^m(-ik\sigma^2)V_n^m(+ik\tau^2) \to e^{ikr}$
- $V_n^m(+ik\sigma^2)S_n^m(-ik\tau^2) \to e^{ikr}$
- $S_n^m(+ik\sigma^2)V_n^m(-ik\tau^2) \to e^{-ikr}$
- $V_n^m(-ik\sigma^2)S_n^m(+ik\tau^2) \to e^{-ikr}$
- $S_n^m(+ik\sigma^2)S_n^m(-ik\tau^2) \to e^{ikz}$
- $V_n^m(-ik\sigma^2)V_n^m(+ik\tau^2) \to e^{ikz}$
- $S_n^m(-ik\sigma^2)S_n^m(+ik\tau^2) \to e^{-ikz}$
- $V_n^m(+ik\sigma^2)V_n^m(-ik\tau^2) \to e^{-ikz}$

5.3 Derivative relations for S_n^m and V_n^m functions in terms of Auxiliary functions

We define auxiliary functions just like we defined the the auxiliary functions for Laguerre functions. Naturally we are going to use the results from that section. First we consider the derivative:

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) S_n^m(x) S_n^m(y) \\
&= \frac{x^{\frac{m}{2}}}{2} e^{-\frac{x}{2}} \left[\left(\frac{m}{x} - 1\right) L_n^m(x) + 2\frac{\partial L_n^m}{\partial x} \right] S_n^m(y) - S_n^m(x) \frac{y^{\frac{m}{2}}}{2} e^{-\frac{y}{2}} \left[\left(\frac{m}{y} - 1\right) L_n^m(y) + 2\frac{\partial L_n^m}{\partial y} \right] \\
&= \frac{(xy)^{\frac{m}{2}}}{2} e^{-\frac{(x+y)}{2}} \left[\left(\frac{m}{x} - 1\right) L_n^m(x) L_n^m(y) - \left(\frac{m}{y} - 1\right) L_n^m(x) L_n^m(y) + 2\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) L_n^m(x) L_n^m(y) \right] \\
&= \frac{(xy)^{\frac{m}{2}}}{2xy} e^{-\frac{(x+y)}{2}} \left[m(y-x) L_n^m(x) L_n^m(y) + 2xy \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) L_n^m(x) L_n^m(y) \right] \\
&= \frac{(xy)^{\frac{m}{2}}}{2xy} e^{-\frac{(x+y)}{2}} \left[-m(y-x) L_n^m(x) L_n^m(y) + 2\frac{\Gamma(n+m+1)}{\Gamma(n+1)} A_n^{m-1}(x,y) \right] \\
&= \frac{1}{2xy} \left[-m(y-x) S_n^m(x) S_n^m(y) + 2(xy)^{1/2} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} C_n^{m-1}(x,y) \right]
\end{aligned}
$$

where:

$$C_n^m(x,y) = \frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[S_{n+1}^m(x)S_n^m(y) - S_n^m(x)S_{n+1}^m(y)\right] \tag{5.26}$$

Also if:

$$D_n^m(x,y) = \frac{\Gamma(n+2)}{\Gamma(m+n+1)}\left[V_{n+1}^m(x)S_n^m(y) - V_n^m(x)S_{n+1}^m(y)\right] \tag{5.27}$$

then:

$$\begin{aligned}
&\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)\\
&= \frac{1}{2xy}\left[-m(y-x)V_n^m(x)S_n^m(y) + 2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}D_n^{m-1}(x,y)\right]
\end{aligned}$$

As a consequence:

$$\begin{aligned}
&xy\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)\\
&= \frac{1}{2}\left[-m(y-x)S_n^m(x)S_n^m(y) + 2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}C_n^{m-1}(x,y)\right]\\
&xy\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)\\
&= \frac{1}{2}\left[-m(y-x)V_n^m(x)S_n^m(y) + 2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}D_n^{m-1}(x,y)\right]
\end{aligned}$$

One may also write (It's not going to be used):

$$\begin{aligned}
&\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)\\
&= \frac{(xy)^{\frac{m}{2}}}{2xy}e^{-\frac{(x+y)}{2}}\left[m(y-x)L_n^m(x)L_n^m(y) + 2xy\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y)\right]\\
&= \frac{(xy)^{\frac{m}{2}}}{2xy}e^{-\frac{(x+y)}{2}}\left[m(y-x)L_n^m(x)L_n^m(y) + 2xy\frac{\Gamma(n+m+1)}{\Gamma(n+1)}A_{n-1}^{m+1}(x,y)\right]\\
&= \frac{1}{2xy}\left[m(y-x)S_n^m(x)S_n^m(y) + 2(xy)^{1/2}C_{n-1}^{m+1}(x,y)\right]
\end{aligned}$$

and:

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$
$$=\frac{1}{2xy}\left[m(y-x)V_n^m(x)S_n^m(y)+2(xy)^{1/2}D_{n-1}^{m+1}(x,y)\right]$$

The remaining relations involve:

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)$$
$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$

The first one gives:

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)$$
$$=\frac{x^{\frac{m}{2}}}{2}e^{-\frac{x}{2}}x\left[\left(\frac{m}{x}-1\right)L_n^m(x)+2\frac{\partial L_n^m}{\partial x}\right]S_n^m(y)-S_n^m(x)\frac{y^{\frac{m}{2}}}{2}e^{-\frac{y}{2}}y\left[\left(\frac{m}{y}-1\right)L_n^m(y)+2\frac{\partial L_n^m}{\partial y}\right]$$
$$=\frac{(xy)^{\frac{m}{2}}}{2}e^{-\frac{(x+y)}{2}}\left[(y-x)L_n^m(x)L_n^m(y)+2\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)L_n^m(x)L_n^m(y)\right]$$
$$=\frac{(xy)^{\frac{m}{2}}}{2}e^{-\frac{(x+y)}{2}}\left[(y-x)L_n^m(x)L_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}A_{n-1}^m(x,y)\right]$$
$$=\frac{1}{2}\left[(y-x)S_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}C_{n-1}^m(x,y)\right]$$

while the second:

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$
$$=\frac{1}{2}\left[(y-x)V_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}D_{n-1}^m(x,y)\right]$$

To summarize:

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)$$
$$=\frac{1}{2}\left[(y-x)S_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}C_{n-1}^m(x,y)\right] \tag{5.28}$$

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$
$$=\frac{1}{2}\left[(y-x)V_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}D_{n-1}^m(x,y)\right] \tag{5.29}$$

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)$$
$$=\frac{1}{2xy}\left[-m(y-x)S_n^m(x)S_n^m(y)+2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}C_n^{m-1}(x,y)\right] \tag{5.30}$$

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$
$$=\frac{1}{2xy}\left[-m(y-x)V_n^m(x)S_n^m(y)+2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}D_n^{m-1}(x,y)\right] \tag{5.31}$$

5.4 Relationship with the Bessel function of first kind

In order to expand plane waves in terms of Laguerre functions the following relation is required (SZEGO, 1939):

$$\sum_{n=0}^{\infty}\binom{n+\alpha}{n}^{-1}L_n^\alpha(x)L_n^\alpha(y)w^n$$
$$=\Gamma(\alpha+1)\frac{e^{\frac{-(x+y)w}{1-w}}}{1-w}(-xyw)^{-\frac{\alpha}{2}}J_\alpha\left(\frac{2(-xyw)^{1/2}}{1-w}\right) \tag{5.32}$$

where J_α is the Bessel function of first kind. The series converge if $|w|<1$ Luckily we can let it approach as close as -1 as we can. Set $w=-1$ (we deal with the convergence problem later), $x=ik\sigma^2$ and $y=-ik\tau^2$:

$$\sum_{n=0}^{\infty}\binom{n+\alpha}{n}^{-1}L_n^\alpha(ik\sigma^2)L_n^\alpha(-ik\tau^2)(-1)^n$$
$$=\Gamma(\alpha+1)\frac{e^{-ik(\tau^2-\sigma^2)/2}}{2}(ik\sigma^2)^{-\frac{\alpha}{2}}(-ik\tau^2)^{-\frac{\alpha}{2}}J_\alpha(k\sigma\tau)$$

In terms of $S_n^\alpha(x)=x^{\alpha/2}e^{-x/2}L_n^\alpha(x)$:

$$\sum_{n=0}^{\infty} \binom{n+\alpha}{n}^{-1} S_n^{\alpha}(ik\sigma^2) S_n^{\alpha}(-ik\tau^2)(-1)^n$$
$$= \frac{\Gamma(\alpha+1)}{2} J_{\alpha}(k\sigma\tau)$$

Then:

$$J_{\alpha}(k\sigma\tau) = \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \binom{n+\alpha}{n}^{-1} S_n^{\alpha}(ik\sigma^2) S_n^{\alpha}(-ik\tau^2)(-1)^n$$

More specifically:

$$J_{\alpha}(k\sigma\tau) = \sum_{n=0}^{\infty} \frac{2(-1)^n \Gamma(n+1)}{\Gamma(n+\alpha+1)} S_n^{\alpha}(ik\sigma^2) S_n^{\alpha}(-ik\tau^2) \tag{5.33}$$

Since $w = -1$ does not satisfy $|w| < 1$ this series actually diverges. To make sure it converges, eq (5.33) must be replaced by

$$J_{\alpha}(k\sigma\tau) = \lim_{\delta\to 0^+} \sum_{n=0}^{\infty} \frac{(2-\delta)(-1+\delta)^n \Gamma(n+1)}{\Gamma(n+\alpha+1)} S_n^{\alpha}(ik\sigma^2) S_n^{\alpha}(-ik\tau^2) \tag{5.34}$$

It is obvious that for $\alpha = l$ with l a non negative integer the equation holds. Now let $\alpha = -l$ with l a non negative integer:

$$J_{-l}(k\sigma\tau) = \sum_{n=0}^{\infty} \frac{2(-1)^n n!}{(n-l)!} S_n^{-l}(ik\sigma^2) S_n^{-l}(-ik\tau^2)$$

By using the closed form of the Laguerre polynomials it can be shown that:

$$L_n^{-l}(x) < \infty, \qquad \forall \quad l > n \in \mathbb{Z}^+ \ \& \ n > 0$$

and

$$L_0^{-l}(x) = 1 \qquad \forall \quad l \in \mathbb{Z}^+$$

In other words the Laguerre polynomials are finite. This implies that the sum may start with $n = l$ since the term in the denominator $(n-l)$ goes to infinity:

$$J_{-l}(k\sigma\tau) = \sum_{n=l}^{\infty} \frac{2(-1)^n n!}{(n-l)!} S_n^{-l}(ik\sigma^2) S_n^{-l}(-ik\tau^2)$$

Now recall $S_n^{-l}(x) = (-1)^l \frac{(n-l)!}{n!} S_{n-l}^l(x)$ for $n \geq l$:

$$J_{-l}(k\sigma\tau) = \sum_{n=l}^{\infty} \frac{2(-1)^{n+2l}(n-l)!}{n!} S_{n-l}^l(ik\sigma^2) S_{n-l}^l(-ik\tau^2)$$

We can shift indices with $\sum_{n=l}^{p} f(n) = \sum_{n=0}^{p-l} f(n+l)$:

$$J_{-l}(k\sigma\tau) = \sum_{n=0}^{\infty} \frac{2(-1)^{n+l} n!}{(n+l)!} S_n^l(ik\sigma^2) S_n^l(-ik\tau^2) \tag{5.35}$$

From this we recover the famous property $J_{-l}(x) = (-1)^l J_l(x)$ for l integer.

Note that this is not a demonstration but a hint. The demonstration should use the Hardy-Hille formula (5.32) instead and show explicitly that $L_n^{-l}(x) < \infty$ for all $l > n \in \mathbb{Z}^+$ & $n > 0$

6 EXPANSION IN PARABOLIC COORDINATES

From the solution of the Helmholtz equation the Debye potential in parabolic coordinates takes the form:

$$\psi_{n,m} = (A_{n,m}S_n^m(\pm ik\sigma^2) + B_{n,m}V_n^m(\pm ik\sigma^2))$$
$$\times (C_{n,m}S_n^m(\mp ik\tau^2) + D_{n,m}V_n^m(\mp ik\tau^2))(E_{n,m}e^{im\varphi} + F_{n,m}e^{-im\varphi}) \quad (6.1)$$

At first glance the $\pm$ sign can be taken arbitrarily. However, by requiring the potential to meet certain boundary conditions we are left with only one choice. For instance, the scattered wave on a paraboloid defined by $\sigma = \sigma_0$ the only combinations of waves that gives outgoing waves at the infinity are $S_n^m(-ik\sigma^2)V_n^m(+ik\tau^2)$ and $V_n^m(+ik\sigma^2)S_n^m(-ik\tau^2)$ as was pointed out in the asymptotic form of the Pinney functions in the last section. However, to evaluate the potential on the negative z axis outside the paraboloid τ must be equal to zero. So the corresponding combination of functions have to be regular at $\sigma > \sigma_0$ and $\tau \geq 0$ which leave us only with the combination $V_n^m(+ik\sigma^2)S_n^m(-ik\tau^2)$ since $V_n^m(+ik\tau^2)$ diverges as $\tau \to 0$. Since this example is the focus of this research we take the $+$ sign for the variable σ and the $-$ sign for τ for the rest of this paper/work. Also the temporal dependance on (e^{-iwt}) is assumed but not shown for clarity.

6.1 Expansion of scalar plane waves in parabolic coordinates

Since every plave wave has definite values at any point in the space including the origin, the expansion is made with:

$$\psi_{n,m} = S_n^m(ik\sigma^2)S_n^m(-ik\tau^2)(A_{n,m}e^{im\varphi} + B_{n,m}e^{-im\varphi})$$

6.1.1 Plane waves traveling along the x^+ axis

For plane wave traveling on the x positive direction ($x = \sigma\tau cos(\varphi)$):

$$e^{ik\sigma\tau cos(\varphi)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} S_n^m(ik\sigma^2)S_n^m(-ik\tau^2)(A_{n,m}e^{im\varphi} + B_{n,m}e^{-im\varphi}) \quad (6.2)$$

Multiply by $e^{-il\varphi}$ and suppose that $l > 0$ is a non-negative integer. Then divide by 2π and integrate φ from 0 to 2π. Using the definition of kroneker delta:

$$\delta_{m,l} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-l)\varphi} d\varphi$$

The term containing $B_{n,m}$ is zero since neither m or l take negative values, then:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\sigma\tau cos(\varphi)-il\varphi} d\varphi = \sum_{n=0}^{\infty} S_n^l(ik\sigma^2) S_n^l(-ik\tau^2) A_{n,l} \tag{6.3}$$

The term on the right hand can be identified as the integral form of the Bessel function of first kind(see (JACKSON, 1999) problem 3.16):

$$J_l(k\sigma\tau) = \frac{1}{2\pi i^l}\int_0^{2\pi} e^{ik\sigma\tau cos(\varphi)-il\varphi} d\varphi \tag{6.4}$$

So:

$$i^l J_l(k\sigma\tau) = \sum_{n=0}^{\infty} S_n^l(ik\sigma^2) S_n^l(-ik\tau^2) A_{n,l} \tag{6.5}$$

Previously it was shown that:

$$J_l(k\sigma\tau) = \lim_{\delta\to 0^+} \sum_{n=0}^{\infty} \frac{(2-\delta)(-1+\delta)^n n!}{(n+l)!} S_n^l(ik\sigma^2) S_n^l(-ik\tau^2) \tag{6.6}$$

For $l \geq 0$. Thus:

$$A_{n,l} = \lim_{\delta\to 0^+} i^l \frac{(2-\delta)(-1+\delta)^n n!}{(n+l)!} \tag{6.7}$$

An analogous procedure, multiplying by $e^{il\varphi}$ and assuming $l > 0$ is a non-negative integer gives:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\sigma\tau cos(\varphi)+il\varphi} d\varphi = \sum_{n=0}^{\infty} S_n^l(ik\sigma^2) S_n^l(-ik\tau^2) B_{n,l} \tag{6.8}$$

Then:

$$i^{-l} J_{-l}(k\sigma\tau) = \sum_{n=0}^{\infty} S_n^l(ik\sigma^2) S_n^l(-ik\tau^2) B_{n,l} \tag{6.9}$$

Since $J_{-l} = (-1)^l J_l$:

$$B_{n,l} = \lim_{\delta \to 0^+} i^{-l}(-1)^l \frac{(2-\delta)(-1+\delta)^n n!}{(n+l)!} \tag{6.10}$$

Finally:

$$\begin{aligned} e^{ik\sigma\tau cos(\varphi)} = & \lim_{\delta \to 0^+} \sum_{n=0}^{\infty} S_n^0(ik\sigma^2) S_n^0(A_{n,0} + B_{n,0}) \\ & + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(2-\delta)(-1+\delta)^n n!}{(n+m)!} S_n^m(ik\sigma^2) S_n^m(-ik\tau^2)(i^m e^{im\varphi} + i^{-m}(-1)^m e^{-im\varphi}) \end{aligned} \tag{6.11}$$

The case $l = 0$ has to be treated differently since both exponentials contribute in this case:

$$J_0(k\sigma\tau) = S_n^0(ik\sigma^2) S_n^0(A_{n,0} + B_{n,0}) \tag{6.12}$$

thus:

$$\begin{aligned} e^{ik\sigma\tau cos(\varphi)} = & \lim_{\delta \to 0^+} \sum_{n=0}^{\infty} \frac{(2-\delta)(-1+\delta)^n n!}{(n+0)!} S_n^0(ik\sigma^2) S_n^0 \\ & + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(2-\delta)(-1+\delta)^n n!}{(n+m)!} S_n^m(ik\sigma^2) S_n^m(-ik\tau^2)(i^m e^{im\varphi} + i^{-m}(-1)^m e^{-im\varphi}) \end{aligned} \tag{6.13}$$

To check the validity of this result we can compare it with the Jacobi-Anger expansion (JACKSON, 1999):

$$e^{ik\sigma\tau cos(\varphi)} = \sum_{m=-\infty}^{\infty} i^m J_m(k\sigma\tau) e^{im\varphi} \tag{6.14}$$

For this purpuse lets take the sum:

$$\lim_{\delta \to 0^+} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(2-\delta)(-1+\delta)^n n!}{(n+m)!} S_n^m(ik\sigma^2) S_n^m(-ik\tau^2)(i^m e^{im\varphi} + i^{-m}(-1)^m e^{-im\varphi}) \tag{6.15}$$

We can separate the sum containing $e^{im\varphi}$ and $e^{-im\varphi}$. In particular:

$$\lim_{\delta\to 0^+}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2-\delta)(-1+\delta)^n n!}{(n+m)!}S_n^m(ik\sigma^2)S_n^m(-ik\tau^2)i^{-m}(-1)^m e^{-im\varphi} \quad (6.16)$$

This sum can be rearranged as follows:

$$\lim_{\delta\to 0^+}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{-1}\frac{(2-\delta)(-1+\delta)^n n!}{(n-m)!}S_n^{-m}(ik\sigma^2)S_n^{-m}(-ik\tau^2)i^{m}(-1)^{-m} e^{im\varphi} \quad (6.17)$$

Which is the same as:

$$\lim_{\delta\to 0^+}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{-1}\frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)i^{m}(-1)^{|m|} e^{im\varphi} \quad (6.18)$$

Then we can encapsulate the whole result as:

$$e^{ik\sigma\tau cos(\varphi)} = \sum_{m=-\infty}^{\infty} i^m\left(\lim_{\delta\to 0^+}\varepsilon_m\sum_{n=0}^{\infty}\frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)e^{im\varphi} \quad (6.19)$$

where $\varepsilon_m = (-1)^{|m|}$ for negative values of m and 1 for positives. The term in parenthesis is just:

$$J_m(k\sigma\tau) = \lim_{\delta\to 0^+}\varepsilon_m\sum_{n=0}^{\infty}\frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2) \quad (6.20)$$

For any integer value of m.

6.1.2 Plane Waves traveling along the y^+ axis

For plane waves traveling on the y positive direction we can repeat the same procedure, note that $y = \sigma\tau sin(\varphi) = \sigma\tau cos(\varphi - \pi/2)$ then:

$$e^{ik\sigma\tau sin(\varphi)} = \lim_{\delta\to 0^+}\sum_{m=-\infty}^{\infty} i^m\left(\varepsilon_m\sum_{n=0}^{\infty}\frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)e^{im(\varphi-\pi/2)} \quad (6.21)$$

6.1.3 Plane Waves traveling along the z^+ axis

Note that:

$$e^{ikz} = e^{ik(\frac{\tau^2-\sigma^2}{2})} = S_0^0(ik\sigma^2)S_0^0(-ik\tau^2) \quad (6.22)$$

6.1.4 Plane Waves traveling on any direction

we can consider a plane wave traveling on an arbitrary direction by:

$$e^{i\vec{k}.\vec{r}} = e^{ik\sigma\tau sin(\theta_k)cos(\varphi-\varphi_k)+ik(\frac{\tau^2-\sigma^2}{2})cos(\theta_k)} \tag{6.23}$$

$$= e^{ik(\frac{\tau^2-\sigma^2}{2})cos(\theta_k)} \sum_{m=-\infty}^{\infty} i^m J_m(k\sigma\tau sin(\theta_k))e^{im(\varphi-\varphi_k)} \tag{6.24}$$

Clearly if $\theta_k = 0$ we recover the result for a plane wave traveling on the z axis ($J_m(0) = \delta_{m,0}$). However from the expansion of $J_m(k\sigma\tau sin(\theta_k))$ without the limit (5.33) this is not the case:

$$\sum_{m=-\infty}^{\infty} i^m J_m(0) = \sum_{m=-\infty}^{\infty} i^m \varepsilon_m \sum_{n=0}^{\infty} \frac{2(-1)^n n!}{(n+|m|)!} S_n^{|m|}(0) S_n^{|m|}(0) = \sum_{n=0}^{\infty} \frac{2(-1)^n n!}{(n)!}$$

$$= \sum_{n=0}^{\infty} 2(-1)^n \tag{6.25}$$

Which clearly diverges. However note that the geometric progression:

$$\sum_{n=0}^{k-1} ar^n = \frac{a(1-r^k)}{1-r} \tag{6.26}$$

converges when $k \to \infty$:

$$\sum_{n=0}^{k-1} ar^n = \frac{a}{1-r} \tag{6.27}$$

provided $|r| < 1$. In particular if $a = (2-\delta)$ and $r = (-1+\delta)$:

$$Lim_{\delta\to 0} \sum_{n=0}^{\infty} (2-\delta)(-1+\delta)^n = Lim_{\delta\to 0} \frac{(2-\delta)}{1-(-1+\delta)} = 1$$

which means we can solve the convergence issue by replacing $2(-1)^n$ with $(2-\delta)(-1+\delta)^n$ with δ sufficiently small as was done in the previous section. This is what we would have obtained if we have taken the limit $w \to -1$ in the Hardy Hille formula:

$$\sum_{n=0}^{\infty} \binom{n+\alpha}{n}^{-1} L_n^{\alpha}(x) L_n^{\alpha}(y) w^n$$

$$= \Gamma(\alpha+1) \frac{e^{\frac{-(x+y)w}{1-w}}}{1-w} (-xyw)^{-\frac{\alpha}{2}} J_\alpha\left(\frac{2(-xyw)^{1/2}}{1-w}\right) \tag{6.28}$$

The expansion of $e^{i\vec{k}.\vec{r}}$ is not entirely satisfactory. The component along the z direction should be completely included in the expansion. That is to say, it should be possible to write:

$$e^{i\vec{k}.\vec{r}} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} A_{n,m} S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \tag{6.29}$$

First lets write $w- > -w$ in the Hardy-Hille Formula:

$$\sum_{n=0}^{\infty} \binom{n+\alpha}{n}^{-1} L_n^{\alpha}(x) L_n^{\alpha}(y)(-w)^n$$
$$= \Gamma(\alpha+1)\frac{e^{\frac{(x+y)w}{1+w}}}{1+w}(xyw)^{-\frac{\alpha}{2}} J_{\alpha}\left(\frac{2(xyw)^{1/2}}{1+w}\right) \tag{6.30}$$

Now we require that:

$$\frac{w}{1+w} = sin^2\left(\frac{\theta}{2}\right) = \frac{1-cos(\theta)}{2} \tag{6.31}$$

From which we get:

$$w = tan^2\left(\frac{\theta}{2}\right) \tag{6.32}$$
$$\frac{2w^{1/2}}{1+w} = \frac{2tan\left(\frac{\theta}{2}\right)}{sec^2\left(\frac{\theta}{2}\right)} = \frac{2sin\left(\frac{\theta}{2}\right)cos^2\left(\frac{\theta}{2}\right)}{cos\left(\frac{\theta}{2}\right)} = sin(\theta) \tag{6.33}$$

Then:

$$J_{\alpha}\left((xy)^{1/2} sin(\theta)\right) e^{-\frac{(x+y)cos(\theta)}{2}} e^{\frac{(x+y)}{2}}$$
$$= \frac{sec^2\left(\frac{\theta}{2}\right) tan^{\alpha}\left(\frac{\theta}{2}\right)}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \binom{n+\alpha}{n}^{-1} (xy)^{\frac{\alpha}{2}} L_n^{\alpha}(x) L_n^{\alpha}(y)(-1)^n tan^{2n}\left(\frac{\theta}{2}\right) \tag{6.34}$$

Which can be written as:

$$J_{\alpha}\left((xy)^{1/2} sin(\theta)\right) e^{-\frac{(x+y)cos(\theta)}{2}}$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{tan^{\alpha+2n}\left(\frac{\theta}{2}\right)}{cos^2\left(\frac{\theta}{2}\right)} (-1)^n S_n^{\alpha}(x) S_n^{\alpha}(y) \tag{6.35}$$

Only valid for $\theta < \pi/2$ since $|w| = |tan^2(\frac{\theta}{2})| < 1$. We can use this result to generalize the expansion of the plane waves:

$$e^{i\vec{k}.\vec{r}} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} \frac{i^m \varepsilon_m \Gamma(n+1)}{\Gamma(n+|m|+1)} \frac{tan^{2n+|m|}\left(\frac{\theta_k}{2}\right)}{cos^2\left(\frac{\theta_k}{2}\right)} (-1)^n S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im(\varphi-\varphi_k)} \quad (6.36)$$

Note that the wave vector variables are written in spherical coordinates. Moreover for $\theta_k = 0$ and for $\theta_k \to \pi/2$ we recover the previous results for a wave traveling along z axis and perpendicular to the z axis respectively. However this expansion can only represent plane waves traveling from the origin to the superior hemisphere defined by $k = cte$. To consider the lower hemisphere we can redefine the intervals in spherical coordinates.

The usual convention to represent a point in spherical coordinates is:

$$k \in [0, \infty)$$
$$\theta_k \in [0, \pi]$$
$$\varphi_k \in [0, 2\pi]$$

Since maximum value θ_k can take in the expansion is $\pi/2$ It may be necessary to use an alternative convention to represent a point in spherical coordinates:

$$k \in (-\infty, \infty)$$
$$\theta_k \in [0, \frac{\pi}{2})$$
$$\varphi_k \in [0, 2\pi]$$

To see how negative values of k affect the position of the wavevector $\vec{k}$, we express $\vec{k}$ in Cartesian coordinates:

$$\vec{k} = (ksin(\theta_k)cos(\varphi_k), ksin(\theta_k)sin(\varphi_k), kcos(\theta_k)) \quad (6.37)$$

Replacing $k \to -k$ it is evident that negative values of k just represent the the same wavevector $\vec{k}$ in the opposite direction. Besides $S_n^{|m|}(-ik\sigma^2)S_n^{|m|}(ik\tau^2)e^{im(\varphi-\varphi_k)}$ is also a solution of the Helmholtz equation; so there is no problem in assigning negative values of k in the expansion.

6.2 Field expansion of plane waves in parabolic coordinates in terms of the Hertz vector $\vec{\pi} = \psi\hat{\mathbf{z}}$

Consider a plane wave traveling on the x+ direction polarized along the z axis. The corresponding fields are:

$$\vec{E}(\vec{r}) = E_0 e^{ikx}\hat{\mathbf{e}}_z \tag{6.38}$$

$$Z\vec{H}(\vec{r}) = -E_0 e^{ikx}\hat{\mathbf{e}}_y \tag{6.39}$$

where Z is the vacuum impedance. This can be easily check because the fields are related by:

$$\vec{E}(\vec{r}) = \frac{i}{k} Z\nabla \times H \tag{6.40}$$

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}\nabla \times E \tag{6.41}$$

This is valid for parabolic coordinates. Recall $x = \sigma\tau cos(\varphi)$ and:

$$\hat{\mathbf{e}}_x = \frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi \tag{6.42}$$

$$\hat{\mathbf{e}}_y = \frac{sin\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + cos\varphi\hat{\mathbf{e}}_\varphi \tag{6.43}$$

$$\hat{\mathbf{e}}_z = \frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}} \tag{6.44}$$

From which we get:

$$\vec{E}(\vec{r}) = E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right] \tag{6.45}$$

$$Z\vec{H}(\vec{r}) = -E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{sin\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + cos\varphi\hat{\mathbf{e}}_\varphi\right] \tag{6.46}$$

The magnetic field should be obtained from two different ways by taking the curl of $\vec{E}$. The first is to directly take the rotational of $\vec{E}$ shown above. The second is expanding the exponential $e^{ik\sigma\tau cos(\varphi)}$ in terms of solutions of the scalar Helmholtz equation in $\vec{E}$ and then take the rotational.

6.2.1 First method

The magnetic field can be obtained as:

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}\nabla \times E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]$$

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}E_0\frac{1}{h_\tau h_\sigma h_\varphi}\begin{vmatrix} h_\tau\hat{\mathbf{e}}_\tau & h_\sigma\hat{\mathbf{e}}_\sigma & h_\varphi\hat{\mathbf{e}}_\varphi \\ \frac{\partial}{\partial\tau} & \frac{\partial}{\partial\sigma} & \frac{\partial}{\partial\varphi} \\ h_\tau\frac{\tau e^{ik\sigma\tau cos(\varphi)}}{(\tau^2+\sigma^2)^{1/2}} & h_\sigma\frac{(-\sigma)e^{ik\sigma\tau cos(\varphi)}}{(\tau^2+\sigma^2)^{1/2}} & 0 \end{vmatrix}$$

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}E_0\frac{1}{h_\tau h_\sigma h_\varphi}[h_\tau\frac{\partial}{\partial\varphi}\left(\sigma e^{ik\sigma\tau cos(\varphi)}\right)\hat{\mathbf{e}}_\tau + h_\sigma\frac{\partial}{\partial\varphi}\left(\tau e^{ik\sigma\tau cos(\varphi)}\right)\hat{\mathbf{e}}_\sigma$$
$$+ h_\varphi(-\sigma\frac{\partial}{\partial\tau}\left(e^{ik\sigma\tau cos(\varphi)}\right) - \tau\frac{\partial}{\partial\sigma}\left(e^{ik\sigma\tau cos(\varphi)}\right))\hat{\mathbf{e}}_\varphi]$$

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}E_0\frac{1}{h_\tau h_\sigma h_\varphi}[-h_\tau ik\tau\sigma^2 sin(\varphi)e^{ik\sigma\tau cos(\varphi)}\hat{\mathbf{e}}_\tau - h_\sigma ik\sigma\tau^2 sin(\varphi)e^{ik\sigma\tau cos(\varphi)}\hat{\mathbf{e}}_\sigma$$
$$- h_\varphi(ik\sigma^2 cos(\varphi)e^{ik\sigma cos(\varphi)} + ik\tau^2 cos(\varphi)e^{ik\sigma\tau cos(\varphi)})\hat{\mathbf{e}}_\varphi]$$

$$Z\vec{H}(\vec{r}) = -E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{sin\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + cos\varphi\hat{\mathbf{e}}_\varphi\right]$$

In the whole process we used $h_\sigma = h_\tau = (\tau^2 - \sigma^2)^{1/2}$ and $h_\varphi = \sigma\tau$.

6.2.2 Second method

The magnetic field can be obtained as:

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}\nabla \times E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]$$

Recall the expansion of the plane wave:

$$e^{ik\sigma\tau cos(\varphi)} = \lim_{\delta\to 0^+}\sum_{m=-\infty}^{\infty} i^m\left(\varepsilon_m\sum_{n=0}^{\infty}\frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)e^{im\varphi} \quad (6.47)$$

$$\varepsilon_m = \begin{cases} 1 & \text{if} m \geq 0 \\ (-1)^m & \text{if} m < 0 \end{cases} \quad (6.48)$$

We define

$$G_{n,m}^{(TM)} = \lim_{\delta \to 0^+} i^m \varepsilon_m \frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!} \tag{6.49}$$

So the Magnetic field takes the form:

$$Z\vec{H}(\vec{r}) = -\frac{i}{k} E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} G_{n,m}^{(TM)} \nabla \times S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \left[\frac{\tau \hat{\mathbf{e}}_\tau - \sigma \hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}} \right]$$

Solving the curl:

$$\begin{aligned}
&\nabla \times S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \left[\frac{\tau \hat{\mathbf{e}}_\tau - \sigma \hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}} \right] \\
&= \frac{1}{h_\tau h_\sigma h_\varphi} \Big[h_\tau (\sigma S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) \frac{\partial e^{im\varphi}}{\partial \varphi}) \hat{\mathbf{e}}_\tau \\
&+ h_\sigma (\tau S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) \frac{\partial e^{im\varphi}}{\partial \varphi}) \hat{\mathbf{e}}_\sigma \\
&- h_\varphi \left(\sigma \frac{\partial}{\partial \tau} \left(S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \right) + \tau \frac{\partial}{\partial \sigma} \left(S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \right) \right) \hat{\mathbf{e_\varphi}} \Big] \\
&= \frac{im S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi}}{(\tau^2+\sigma^2)^{1/2} \sigma\tau} (\sigma \hat{\mathbf{e}}_\tau + \tau \hat{\boldsymbol{\sigma}}) \\
&- \frac{2ik\tau\sigma e^{im\varphi}}{(\tau^2+\sigma^2)} \left(\frac{\partial}{\partial x} \left(S_n^{|m|}(x) S_n^{|m|}(y) \right) - \frac{\partial}{\partial y} \left(S_n^{|m|}(x) S_n^{|m|}(y) \right) \right) \hat{\mathbf{e}}_\varphi
\end{aligned}$$

where $x = ik\sigma^2$ and $y = -ik\tau^2$. For clarity lets define:

$$\frac{-k}{i} \vec{M}_{n,m}(\vec{r}) = \nabla \times S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \left[\frac{\tau \hat{\mathbf{e}}_\tau - \sigma \hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}} \right] \tag{6.50}$$

$$\begin{aligned}
&= \frac{im S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi}}{(\tau^2+\sigma^2)^{1/2} \sigma\tau} (\sigma \hat{\mathbf{e}}_\tau + \tau \hat{\boldsymbol{\sigma}}) \\
&- \frac{2ik\tau\sigma e^{im\varphi}}{(\tau^2+\sigma^2)} \left(\frac{\partial}{\partial x} \left(S_n^{|m|}(x) S_n^{|m|}(y) \right) - \frac{\partial}{\partial y} \left(S_n^{|m|}(x) S_n^{|m|}(y) \right) \right) \hat{\mathbf{e}}_\varphi
\end{aligned} \tag{6.51}$$

Then

$$Z\vec{H}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} G_{n,m}^{(TM)} \vec{M}_{n,m}(\vec{r})$$

What was done now is to take the electric field as the electric Hertz vector Given $Z\vec{H}(\vec{r})$ we can find the electric field as:

$$\vec{E}(\vec{r}) = \frac{i}{k}\nabla \times Z\vec{H}$$
$$= E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)} \frac{i}{k}\nabla \times \vec{M}_{n,m}(\vec{r})$$

Again we define:

$$\frac{k}{i}\vec{N}_{n,m}(\vec{r}) = \nabla \times \vec{M}_{n,m}(\vec{r}) \tag{6.52}$$

So

$$\vec{E}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)} \vec{N}_{n,m}(\vec{r}) \tag{6.53}$$

The general expansion would take the form:

$$\vec{E}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TE)} \vec{M}_{n,m}(\vec{r}) + G_{n,m}^{(TM)} \vec{N}_{n,m}(\vec{r}) \tag{6.54}$$

$$Z\vec{H}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)} \vec{M}_{n,m}(\vec{r}) - G_{n,m}^{(TE)} \vec{N}_{n,m}(\vec{r}) \tag{6.55}$$

However since we took the electric field $\vec{E}(\vec{r})$ as the electric Hertz vector we know that $G_{n,m}^{(TE)} = 0$ then we recover our result. The (TM) and (TE) refer to transverse magnetic and transverse electric to the Hertz vector respectively.The importance of this method is that we can immediately determine the beam shape coefficients just from knowing the expansion of the scalar plane waves. To calculate $\vec{N}_{n,m}(\vec{r})$ we need to calculate:

$$\vec{N}_{n,m}(\vec{r}) = \frac{1}{k^2}\nabla \times \nabla \times S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]$$

$$\vec{N}_{n,m}(\vec{r}) = \frac{1}{k^2 h_\tau h_\sigma h_\varphi}\begin{vmatrix} h_\tau\hat{\mathbf{e}}_\tau & h_\sigma\hat{\mathbf{e}}_\sigma & h_\varphi\hat{\mathbf{e}}_\varphi \\ \frac{\partial}{\partial\tau} & \frac{\partial}{\partial\sigma} & \frac{\partial}{\partial\varphi} \\ \frac{-k}{i}h_\tau M_\tau & \frac{-k}{i}h_\sigma M_\sigma & \frac{-k}{i}h_\varphi M_\varphi \end{vmatrix}$$

In components:

$$(\vec{N}_{n,m}(\vec{r}))_\tau = \frac{1}{k^2 h_\sigma h_\varphi}\left[\frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\varphi M_\varphi\right) - \frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\sigma M_\sigma\right)\right] \quad (6.56)$$

$$(\vec{N}_{n,m}(\vec{r}))_\sigma = \frac{-1}{k^2 h_\tau h_\varphi}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\varphi M_\varphi\right) - \frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\tau M_\tau\right)\right] \quad (6.57)$$

$$(\vec{N}_{n,m}(\vec{r}))_\varphi = \frac{1}{k^2 h_\tau h_\sigma}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\sigma M_\sigma\right) - \frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\tau M_\tau\right)\right] \quad (6.58)$$

with

$$\frac{-k}{i}h_\sigma M_\sigma = \frac{im}{\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$

$$\frac{-k}{i}h_\tau M_\tau = \frac{im}{\tau}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$

$$\frac{-k}{i}h_\varphi M_\varphi = \frac{-2ik\sigma^2\tau^2 e^{im\varphi}}{\tau^2+\sigma^2}\left[\frac{\partial}{\partial x}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right) - \frac{\partial}{\partial y}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right)\right]$$

Again $x = ik\sigma^2$ and $y = -ik\tau^2$. The orthogonal functions $\vec{M}_{n,m}(\vec{r})$ and $\vec{N}_{n,m}(\vec{r})$ have components where the dependence on the variables are not expressed as a product. This leads to difficulties when applying boundary conditions ($\sigma = \sigma_0$). The components M_φ, N_τ and N_φ can be rearranged in a separable way. This is done on the Appendix B.

6.2.3 Divergence of the plane wave Expansion

An easy calculation of the divergence of the fields:

$$\vec{E}(\vec{r}) = E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right] \quad (6.59)$$

$$Z\vec{H}(\vec{r}) = -E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{sin\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + cos\varphi\hat{\mathbf{e}}_\varphi\right] \quad (6.60)$$

shows that they are divergenless. As a consequence their expansion should also be divergenless. This is clear for the vectors obtained from a curl since $\nabla\cdot\nabla\times A = 0$ but is not obvious for the expansion:

$$\begin{aligned}\vec{E}(\vec{r}) &= E_0 e^{ik\sigma\tau cos(\varphi)}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]\\ &= \sum_{m=-\infty}^{\infty}\left(\sum_{n=0}^{\infty} G_{n,m}^{(TM)} S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)e^{im\varphi}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]\end{aligned}$$

with

$$G_{n,m}^{(TM)} = Lim_{\delta\to 0}\left(i^m \varepsilon_m \frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}\right) \tag{6.61}$$

The divergence of the Electric field is:

$$\begin{aligned}\nabla\cdot\vec{E} &= \frac{1}{h_\sigma h_\tau h_\varphi}\left[\frac{\partial}{\partial\sigma}\left(E_\sigma h_\tau h_\varphi\right)+\frac{\partial}{\partial\tau}\left(E_\tau h_\sigma h_\varphi\right)+\frac{\partial}{\partial\varphi}\left(E_\varphi h_\sigma h_\tau\right)\right]\\ &= \frac{1}{h_\sigma h_\tau}\left[\frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma E_\sigma h_\tau)+\frac{1}{\tau}\frac{\partial}{\partial\tau}(\tau E_\tau h_\sigma)\right]\end{aligned}$$

The term in brackets is:

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)} e^{im\varphi}\left[-\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma^2 S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)+\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau^2 S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)\right]$$

Note that:

$$\begin{aligned}&-\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma^2 f(\sigma,\tau)\right)+\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau^2 f(\sigma,\tau)\right)\\ &= -\frac{1}{\sigma}\left[2\sigma f(\sigma,\tau)+\sigma^2\frac{\partial}{\partial\sigma}(f(\sigma,\tau))\right]+\frac{1}{\tau}\left[2\tau f(\sigma,\tau)+\tau^2\frac{\partial}{\partial\tau}(f(\sigma,\tau))\right]\\ &= \tau\frac{\partial}{\partial\tau}f(\sigma,\tau)-\sigma\frac{\partial}{\partial\sigma}f(\sigma,\tau)\end{aligned}$$

Now consider only the sum on n:

$$\begin{aligned}&\sum_{n=0}^{\infty} G_{n,m}^{(TM)}\left[\tau\frac{\partial}{\partial\tau}\left(S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)-\sigma\frac{\partial}{\partial\sigma}\left(S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\right)\right]\\ &= 2\sum_{n=0}^{\infty} G_{n,m}^{(TM)}\left[y\frac{\partial}{\partial y}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right)-x\frac{\partial}{\partial x}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right)\right]\end{aligned}$$

We did the change of variables $x = ik\sigma^2$ and $y = -ik\tau^2$. This type of derivative was found to equal to:

$$
\begin{aligned}
&-\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)\\
&=\frac{-1}{2}\left[(y-x)S_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}C_{n-1}^m(x,y)\right]\\
&=\frac{-(y-x)}{2}\left[S_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}\sum_{p=0}^{n-1}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}S_p^m(x)S_p^m(y)\right]
\end{aligned}
$$

Now if we take the series:

$$
\sum_{n=0}^{\infty}G_{n,m}^{(TM)}\left[S_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}\sum_{p=0}^{n-1}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}S_p^m(x)S_p^m(y)\right]
$$

And rearrange it we get:

$$
\sum_{n=0}^{\infty}\left[G_{n,m}^{(TM)}S_n^m(x)S_n^m(y)+2\tilde{G}_{n+1,m}^{(TM)}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}S_n^{|m|}(x)S_n^{|m|}(y)\right]
$$

with:

$$
\tilde{G}_{n+1,m}^{(TM)}=\sum_{p=n+1}^{\infty}G_{p,m}\frac{\Gamma(|m|+p+1)}{\Gamma(p+1)}
$$

Series Rearrangement is treated with care in Appendix B. It was used by Horton (HORTON; KARAL, 1951) to solve the system of equation obtained from the boundary conditions. It can also be used to prove obvious results such as this divergence and that the φ component of:

$$
\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}G_{n,m}^{(TM)}\vec{N}_{n,m}(\vec{r})_{\varphi}=0
$$

Note that the expansion is divergenless if:

$$
G_{n,m}^{(TM)}+2\tilde{G}_{n+1,m}^{(TM)}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}=0
$$

In fact:

$$G_{n,m}^{(TM)} + 2\tilde{G}_{n+1,m}^{(TM)} \frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}$$
$$= Lim_{\delta\to 0}\left[i^m \varepsilon_m \frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!} + 2\frac{n!}{(n+|m|)!}\sum_{n+1}^{\infty}\left(i^m \varepsilon_m \frac{(2-\delta)(-1+\delta)^p p!}{(p+|m|)!}\right)\frac{(p+|m|)!}{p!}\right]$$
$$= i^m \varepsilon_m \frac{n!}{(n+|m|)!} Lim_{\delta\to 0}\left[(2-\delta)(-1+\delta)^n + 2\sum_{n+1}^{\infty}(2-\delta)(-1+\delta)^p\right]$$

The sum can be seen as a geometric progression of the type:

$$\sum_{p=m}^{k} ar^p = \frac{a(r^m - r^{k+1})}{1-r} \tag{6.62}$$

with $r = (-1+\delta)$, $a = (2-\delta)$ and $k \to \infty$:

$$\sum_{p=n+1}^{\infty}(2-\delta)(-1+\delta)^p = \frac{(2-\delta)(-1+\delta)^{n+1}}{(1-(-1+\delta))} = (-1+\delta)^{n+1} \tag{6.63}$$

Therefore:

$$G_{n,m}^{(TM)} + 2\tilde{G}_{n+1,m}^{(TM)} \frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}$$
$$= i^m \varepsilon_m \frac{n!}{(n+|m|)!} Lim_{\delta\to 0}\left[(2-\delta)(-1+\delta)^n + 2(-1+\delta)^{n+1}\right] = 0$$

We have an expansion of the electric field in terms of vector functions of the type $\psi(\sigma,\tau,\varphi)\hat{\mathbf{e}}_z$ which are not divergenless yet the sum is divergenless. By using the Maxwell's equations the Hertz vectors have been transformed from an expansion of non-divergenless vector functions (Not solutions of Maxwell's equations) into an expansion of divergenless vector functions (solutions of Maxwell's equations). The downside is that the components of these vectors are not always expressed in a separable manner, that is, as a function $M(\sigma)N(\tau)\Phi(\varphi)$ so that the boundary conditions can easily be applied.

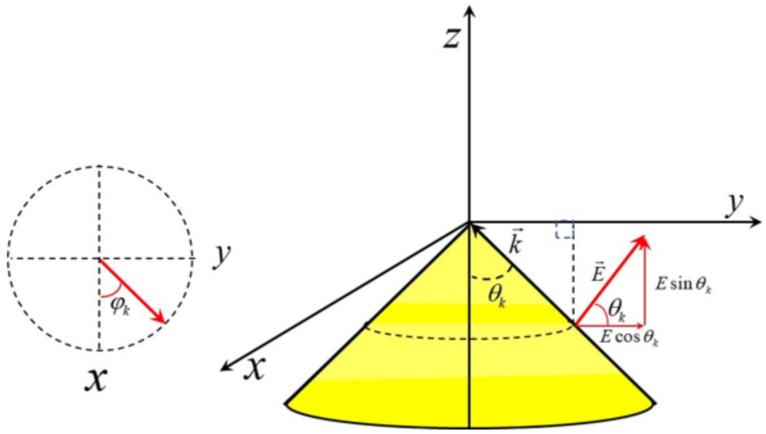
6.3 Field expansion of a focalized beam

Consider a plane wave whose wavevector makes an angle $\theta_k < \frac{\pi}{2}$ with the z axis like the one shown on figure 5:

$$e^{i\vec{k}.\vec{r}}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z)$$
$$= e^{ik\cos(\theta_k)(\tau^2-\sigma^2)/2+ik\sigma\tau\cos(\varphi-\varphi_k)\sin(\theta_k)}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z) \quad (6.64)$$

with $\hat{\mathbf{e}}_\rho = \cos(\varphi_k)\hat{\mathbf{e}}_x + \sin(\varphi_k)\hat{\mathbf{e}}_y$.

Figura 5 – Plane wave making an angle θ_k with the z axis. A focused beam can be reproduced by summing over all values of φ_k.



Source: author.

To construct the focalized beam we sum plane waves with the same amplitude and and angle θ_k but with different values of φ_k:

$$\vec{E}(\vec{r}) = E_0 \int_0^{2\pi} e^{ik\cos(\theta_k)(\tau^2-\sigma^2)/2+ik\sigma\tau\cos(\varphi-\varphi_k)\sin(\theta_k)}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z)d\varphi_k \quad (6.65)$$

which can be written as a column vector:

$$\vec{E}(\vec{r}) = E_0 e^{ik_z z}\begin{pmatrix} \cos(\theta_k)\int_0^{2\pi} e^{ik_\rho\sigma\tau\cos(\varphi-\varphi_k)}\cos(\varphi_k)d\varphi_k \\ \cos(\theta_k)\int_0^{2\pi} e^{ik_\rho\sigma\tau\cos(\varphi-\varphi_k)}\sin(\varphi_k)d\varphi_k \\ \sin(\theta_k)\int_0^{2\pi} e^{ik_\rho\sigma\tau\cos(\varphi-\varphi_k)}d\varphi_k \end{pmatrix} \quad (6.66)$$

The integral on the z component is just the Bessel function $2\pi J_0(k_\rho\sigma\tau)$ ($k_\rho = k\sin(\theta_k)$). So we are left with the integrals:

$$I_x = \int_0^{2\pi} e^{ik_\rho\sigma\tau\cos(\varphi-\varphi_k)}\cos(\varphi_k)d\varphi_k \quad (6.67)$$
$$I_y = \int_0^{2\pi} e^{ik_\rho\sigma\tau\cos(\varphi-\varphi_k)}\sin(\varphi_k)d\varphi_k \quad (6.68)$$

Upon substitution $\varphi_k - \varphi = \phi$:

$$I_x = \int_{-\varphi}^{2\pi-\varphi} e^{ik_\rho \sigma\tau\cos(\phi)} \cos(\phi+\varphi) d\phi \tag{6.69}$$

$$I_y = \int_{-\varphi}^{2\pi-\varphi} e^{ik_\rho \sigma\tau\cos(\phi)} \sin(\phi+\varphi) d\phi \tag{6.70}$$

Defining:

$$I_1 = \int_{-\varphi}^{2\pi-\varphi} e^{ik_\rho \sigma\tau\cos(\phi)} \cos(\phi) d\phi \tag{6.71}$$

$$I_2 = \int_{-\varphi}^{2\pi-\varphi} e^{ik_\rho \sigma\tau\cos(\phi)} \sin(\phi) d\phi \tag{6.72}$$

we have

$$I_x = \cos(\varphi) I_1 - \sin(\varphi) I_2 \tag{6.73}$$

$$I_y = \cos(\varphi) I_2 + \sin(\varphi) I_1 \tag{6.74}$$

Notice that:

$$I_1 = \int_0^{2\pi} e^{ik_\rho \sigma\tau\cos(\phi)} \cos(\phi) d\phi = \int_{-\pi}^{\pi} e^{ik_\rho \sigma\tau\cos(\phi)} \cos(\phi) d\phi$$
$$I_2 = \int_0^{2\pi} e^{ik_\rho \sigma\tau\cos(\phi)} \sin(\phi) d\phi = \int_{-\pi}^{\pi} e^{ik_\rho \sigma\tau\cos(\phi)} \sin(\phi) d\phi$$

I_2 can be broken into:

$$I_2 = \int_{-\pi}^{0} e^{ik_\rho \sigma\tau\cos(\phi)} \sin(\phi) d\phi + \int_0^{\pi} e^{ik_\rho \sigma\tau\cos(\phi)} \sin(\phi) d\phi$$

The substitution $\phi = \alpha - \pi$ in the first integral leads to:

$$\int_0^{\pi} e^{ik_\rho \sigma\tau\cos(\alpha-\pi)} \sin(\alpha-\pi) d\alpha = -\int_0^{\pi} e^{-ik_\rho \sigma\tau\cos(\alpha)} \sin(\alpha) d\alpha$$

Therefore:

$$I_2 = \int_0^{\pi} (e^{ik_\rho \sigma\tau\cos(\phi)} - e^{-ik_\rho \sigma\tau\cos(\phi)}) \sin(\phi) d\phi = 2i \int_0^{\pi} \sin\big(k_\rho \sigma\tau\cos(\phi)\big) \sin(\phi) d\phi$$

Now

$$I_2 = 2i\left[\int_0^{\pi/2} \sin\big(k_\rho \sigma\tau\cos(\phi)\big)\sin(\phi)d\phi + \int_{\pi/2}^{\pi} \sin\big(k_\rho \sigma\tau\cos(\phi)\big)\sin(\phi)d\phi\right]$$

Making the substitution $\phi = \pi - \alpha$ on the right hand integral leads to:

$$\int_{\pi/2}^{0} \sin\big(k_\rho \sigma\tau\cos(\alpha)\big)\sin(\pi-\alpha)d\alpha = -\int_0^{\pi/2} \sin\big(k_\rho \sigma\tau\cos(\alpha)\big)\sin(\alpha)d\alpha$$

So $I_2 = 0$. For I_1 we have:

$$I_1 = \int_0^{2\pi} e^{ik_\rho\sigma\tau\cos(\phi)}\cos(\phi)d\phi = -i\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi} e^{ix\cos(\phi)}d\phi = -i\frac{\mathrm{d}}{\mathrm{d}x}2\pi J_0(x)$$
$$I_1 = -2\pi i\frac{\mathrm{d}J_0(x)}{\mathrm{d}x}$$

where $x = k_\rho\sigma\tau$. Now by the following properties of the Bessel functions:

$$2\frac{\mathrm{d}J_m(x)}{\mathrm{d}x} = J_{m-1}(x) - J_{m+1}(x) \tag{6.75}$$

$$J_{-m}(x) = (-1)^{-m}J_m(x) \tag{6.76}$$

The integral becomes

$$I_1 = 2\pi i J_1(x) = 2\pi i J_1(k_\rho\sigma\tau)$$

Therefore

$$I_x = 2\pi i J_1(k_\rho\sigma\tau)\cos(\varphi) \tag{6.77}$$

$$I_y = 2\pi i J_1(k_\rho\sigma\tau)\sin(\varphi) \tag{6.78}$$

In summary the field becomes:

$$\vec{E}(\vec{r}) = 2\pi E_0 e^{ik_z z}\begin{pmatrix} \cos(\theta_k)iJ_1(k_\rho\rho)\cos(\varphi) \\ \cos(\theta_k)iJ_1(k_\rho\rho)\sin(\varphi) \\ \sin(\theta_k)J_0(k_\rho\rho) \end{pmatrix} \tag{6.79}$$

or

$$\vec{E}(\vec{r}) = 2\pi E_0 e^{ik\cos(\theta_k)\frac{(\tau^2-\sigma^2)}{2}}\left[-i\cos(\theta_k)J_1(k\cos(\theta_k)\sigma\tau)\hat{\mathbf{e}}_\rho + \sin(\theta_k)J_0(k\sin(\theta_k)\sigma\tau)\hat{\mathbf{e}}_z\right] \quad (6.80)$$

If the region of interest is the z axis, then only non-zero component of the field lies in the z axis. In that case it is possible to find an expansion of in terms of elementary solutions of the Helmholtz equation the same process used to find the expansion of plane wave described in the previous section can be used. Luckily:

$$\begin{aligned} &J_\alpha\left((xy)^{1/2}\sin(\theta)\right)e^{-\frac{(x+y)\cos(\theta)}{2}} \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\frac{\tan^{\alpha+2n}\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}(-1)^n S_n^\alpha(x)S_n^\alpha(y) \end{aligned} \quad (6.81)$$

This expression was found in the expansion of scalar plane waves traveling on any direction. With $x = ik\sigma^2$ and $y = -ik\tau^2$ we have:

$$\begin{aligned} &e^{ik\cos(\theta_k)(\tau^2-\sigma^2)/2}J_0(k\sin(\theta_k)\sigma\tau) \\ &= \sum_{n=0}^{\infty}\frac{\tan^{2n}\left(\frac{\theta_k}{2}\right)}{\cos^2\left(\frac{\theta_k}{2}\right)}(-1)^n S_n^0(ik\sigma^2)S_n^0(-ik\tau^2) \end{aligned} \quad (6.82)$$

Therefore:

$$\vec{E}(\vec{r}) = E_0\sin(\theta_k)e^{ik\sin(\theta_k)(\tau^2-\sigma^2)/2}J_0(k\sin(\theta_k)\sigma\tau)\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right] \quad (6.83)$$

$$\vec{E}(\vec{r}) = E_0\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}G_{n,m}^{(TM)}(\theta_k)\vec{N}_{n,m}(k,\vec{r}) \quad (6.84)$$

with the coefficients being:

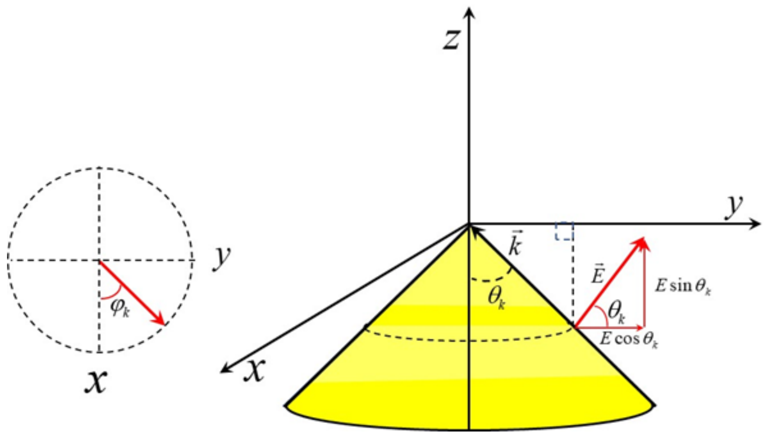
$$G_{n,m}^{(TM)}(\theta_k) = \delta_{0,m}\sin(\theta_k)\frac{\tan^{2n}\left(\frac{\theta_k}{2}\right)}{\cos^2\left(\frac{\theta_k}{2}\right)}(-1)^n \quad (6.85)$$

7 LIGHT SCATTERING ON A PARABOLOID OF REVOLUTION

The theory developed until now allows us to treat scattering problems if the Hertz vectors use $\hat{\mathbf{z}}$ as the unitary constant vector. In principle this was done because any different unitary vector would involve derivatives not previously calculated and it is not known if it's possible to rearrange the resulting components of the Hertz vectors in a similar fashion as was done by (HORTON; KARAL, 1951). Besides the expansion coefficients of a plane wave polarized along the $\hat{\mathbf{z}}$ direction and for the axial component of a highly focused beam traveling along the $\hat{\mathbf{z}}$ direction was found. The former has a problem, since the coefficients depend on a limit to converge, getting accurate results would require the calculation of a large number of coefficients The latter is the most interesting case for light enhancement applications. Moreover the latter is more realistic and can be more easily reproduced in the laboratory.

The general problem of light scattering on a paraboloid of revolution is treated first for a dielectric paraboloid. We are dealing with the incident focused beam discussed in the previous section, therefore the problem has azimuthal simmetry as shown in the figure:

Figura 6 – Focused beam created by summing plane waves along φ.



Source: author.

7.1 General scattering by the use of the Hertz vector $\vec{\pi} = \psi\hat{\mathbf{z}}$

Following the tradition, we call the coefficients of the incident wave Beam Shape Coefficients (BSC) and denote them with the letter G:

$$\vec{E}_{inc}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} G_{n,m}^{(TE)} \vec{M}_{n,m}^{(s)}(k_1, \vec{r}) + G_{n,m}^{(TM)} \vec{N}_{n,m}^{(s)}(k_1, \vec{r}) \tag{7.1}$$

$$Z_1 \vec{H}_{inc}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} G_{n,m}^{(TM)} \vec{M}_{n,m}^{(s)}(k_1, \vec{r}) - G_{n,m}^{(TE)} \vec{N}_{n,m}^{(s)}(k_1, \vec{r}) \tag{7.2}$$

Here however, $G_{n,m}^{(TM)}$ and $G_{n,m}^{(TE)}$ means transverse to the electric Hertz vector and transverse to the magnetic Hertz vector respectively. The (s) superscript means that the vector involves the functions $S_n^m(ik\sigma^2)S_n^m(-ik\tau^2)$ which are finite at the origin. In contrast the (v) superscript means that the vector involves the functions $V_n^m(ik\sigma^2)S_n^m(-ik\tau^2)$ which represent an outgoing wave towards infinity at high values of τ and σ. Then the scattered field is given by:

$$\vec{E}_{scat}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} C_{n,m} \vec{M}_{n,m}^{(v)}(k_1, \vec{r}) + D_{n,m} \vec{N}_{n,m}^{(v)}(k_1, \vec{r}) \tag{7.3}$$

$$Z_1 \vec{H}_{scat}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} D_{n,m} \vec{M}_{n,m}^{(v)}(k_1, \vec{r}) - C_{n,m} \vec{N}_{n,m}^{(v)}(k_1, \vec{r}) \tag{7.4}$$

For the internal field we have:

$$\vec{E}_{int}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} A_{n,m} \vec{M}_{n,m}^{(s)}(k_2, \vec{r}) + B_{n,m} \vec{N}_{n,m}^{(s)}(k_2, \vec{r}) \tag{7.5}$$

$$Z_2 \vec{H}_{int}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} B_{n,m} \vec{M}_{n,m}^{(s)}(k_2, \vec{r}) - A_{n,m} \vec{N}_{n,m}^{(s)}(k_2, \vec{r}) \tag{7.6}$$

k_1 and k_2 are the magnitude of the wave vector outside and inside the paraboloid respectively.

We wish to apply the boundary conditions on the surface defined by $\sigma = \sigma_0$ which is a paraboloid of revolution upwards the positive z axis. Since no sources are present this implies:

$$[\vec{E}_{inc}(\vec{r}) + \vec{E}_{scat}(\vec{r}) - \vec{E}_{int}(\vec{r})] \times \hat{\mathbf{e}}_\sigma|_{\sigma=\sigma_0} = 0 \tag{7.7}$$

$$[\vec{H}_{inc}(\vec{r}) + \vec{H}_{scat}(\vec{r}) - \vec{H}_{int}(\vec{r})] \times \hat{\mathbf{e}}_\sigma|_{\sigma=\sigma_0} = 0 \tag{7.8}$$

Both expansions of incident fields found previously only have $G_{n,m}^{(TM)}$ as non-zero coefficients. Thus the boundary conditions reduces to:

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[G_{n,m}^{(TM)}\vec{N}_{n,m}^{(s)}(k_1,\vec{r})+D_{n,m}\vec{N}_{n,m}^{(v)}(k_1,\vec{r})-B_{n,m}\vec{N}_{n,m}^{(s)}(k_2,\vec{r})\right]\times\hat{\mathbf{e}}_\sigma|_{\sigma=\sigma_0}=0 \tag{7.9}$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\frac{G_{n,m}^{(TM)}}{Z_1}\vec{M}_{n,m}^{(s)}(k_1,\vec{r})+\frac{D_{n,m}}{Z_1}\vec{M}_{n,m}^{(v)}(k_1,\vec{r})-\frac{B_{n,m}}{Z_2}\vec{M}_{n,m}^{(s)}(k_2,\vec{r})\right]\times\hat{\mathbf{e}}_\sigma|_{\sigma=\sigma_0}=0 \tag{7.10}$$

If the problem has azimuthal symmetry, then:

$$\sum_{n=0}^{\infty}\left[G_{n,0}^{(TM)}\vec{N}_{n,0}^{(s)}(k_1,\vec{r})+D_{n,0}\vec{N}_{n,0}^{(v)}(k_1,\vec{r})-B_{n,0}\vec{N}_{n,0}^{(s)}(k_2,\vec{r})\right]\times\hat{\mathbf{e}}_\sigma|_{\sigma=\sigma_0}=0 \tag{7.11}$$

$$\sum_{n=0}^{\infty}\left[k_1G_{n,0}^{(TM)}\vec{M}_{n,0}^{(s)}(k_1,\vec{r})+k_1D_{n,0}\vec{M}_{n,0}^{(v)}(k_1,\vec{r})-k_2B_{n,0}\vec{M}_{n,0}^{(s)}(k_2,\vec{r})\right]\times\hat{\mathbf{e}}_\sigma|_{\sigma=\sigma_0}=0 \tag{7.12}$$

Here we made use of $Z_n=\frac{\mu_0\omega}{k_n}$ for non-magnetic media.

From Appendix C we bring the components of the functions M and N. The components of the $\vec{M}_{n,0}(k,\vec{r})$ vectors are:

$$(\vec{M}_{n,0}(\vec{r}))_\sigma=0 \tag{7.13}$$

$$(\vec{M}_{n,0}(\vec{r}))_\tau=0 \tag{7.14}$$

$$(\vec{M}_{n,0}(\vec{r}))_\varphi=\frac{-2ni}{k\sigma\tau(x-y)}\Bigg((y-x)S_n^0(x)S_n^0(y)-yS_{n-1}^0(x)S_n^0(y)$$
$$+xS_n^0(x)S_{n-1}^0(y)\Bigg) \tag{7.15}$$

The components of the $\vec{N}_{n,0}(k,\vec{r})$ vectors are:

$$(\vec{N}_{n,0}(k,\vec{r}))_\varphi=0 \tag{7.16}$$

$$(\vec{N}_{n,0}(\vec{r}))_\tau=\frac{-2ni}{k\tau h_\sigma(x-y)^2}\Bigg((x-y)^2\left[\frac{2n}{x}\left(S_n^0(x)-S_{n-1}^0(x)\right)-S_n^0(x)\right]S_n^0(y)$$
$$+(x-y)\,y\left[\frac{2(n-1)}{x}\left(S_{n-1}^0(x)-S_{n-2}^0(x)\right)-S_{n-1}^0(x)\right]S_n^0(y)$$
$$-(x-y)\,S_{n-1}^0(y)\left[(2n-x)\,S_n^0(x)-2nS_{n-1}^0(x)\right]+2y\left(S_{n-1}^0(y)S_n^0(x)-S_{n-1}^0(x)S_n^0(y)\right)\Bigg) \tag{7.17}$$

$$
\begin{aligned}
(\vec{N}_{n,0}(\vec{r}))_\sigma = \frac{2ni}{k\sigma h_\sigma (x-y)^2} \Bigg(& 2x\left[S^0_{n-1}(y)S^0_n(x) - S^0_{n-1}(x)S^0_n(y)\right] \\
& - (x-y)\,S^0_{n-1}(x)\left[2n\left(S^0_n(y) - S^0_{n-1}(y)\right) - yS^0_n(y)\right] - (x-y)^2 S^0_n(x)\left[\frac{2n}{y}\left(S^0_n(y) - S^0_{n-1}(y)\right) - S^0_n(y)\right] \\
& + (x-y)\,x\left[\frac{2(n-1)}{y}\left(S^0_{n-1}(y) - S^0_{n-2}(y)\right) - S^0_{n-1}(y)\right] S^0_n(x) \Bigg) \qquad (7.18)
\end{aligned}
$$

Here $x = ik\sigma^2$ and $y = -ik\tau^2$, is easier to apply recurrence relations and derivate with this substitution. These components are for the incident and internal field, for the scattered we just replace any function $S^{|m|}_n(x)$ with the corresponding $V^{|m|}_n(x)$.

To evaluate the electric field in the z axis below the paraboloid ($\tau = 0$) we also need:

$$
(\vec{N}_{n,0}(x, y=0))_\sigma = \frac{-4n}{x^2}\left(S^0_n(x) - S^0_{n-1}(x) + (x)S^0_n(x)\right) \qquad (7.19)
$$

or

$$
(\vec{N}_{n,0}(x, y=0))_\sigma = \frac{-4n}{x^2}\left(V^0_n(x) - V^0_{n-1}(x) + (x)V^0_n(x)\right) \qquad (7.20)
$$

for the scattered field.

> Note: Here $x_{0,1} = ik_1\sigma_0^2$, $y_1 = -ik_1\tau^2$, $x_{0,2} = ik_2\sigma_0^2$ and $y_2 = -ik_2\tau^2$ with k_1 and k_2 being the magnitude of the wave vector outside and inside the paraboloid respectively.

7.1.0.1 Solving boundary condition equations

The only non-zero component of the $\vec{M}$ vector field is the φ component and the only component of $\vec{N}$ which contributes is the τ component. Therefore the boundary conditions become

$$
\sum_{n=1}^{\infty} B_{n,0}\vec{N}^{(s)}_{n,0}(k_2,\sigma_0,\tau)_\tau - D_{n,0}\vec{N}^{(v)}_{n,0}(k_1,\sigma_0,\tau)_\tau = \sum_{n=1}^{\infty} G^{(TM)}_{n,0}\vec{N}^{(s)}_{n,0}(k_1,\sigma_0,\tau)_\tau \qquad (7.21)
$$

$$
\sum_{n=1}^{\infty} k_2 B_{n,0}\vec{M}^{(s)}_{n,0}(k_2,\sigma_0,\tau)_\varphi - k_1 D_{n,0}\vec{M}^{(v)}_{n,0}(k_1,\sigma_0,\tau)_\varphi = \sum_{n=1}^{\infty} k_1 G^{(TM)}_{n,0}\vec{M}^{(s)}_{n,0}(k_1,\sigma_0,\tau)_\varphi \qquad (7.22)
$$

$n = 0$ do not contribute. Since these equations must hold for any value of τ or y_i, We can use the series expansion of $S_n^{|m|}(y)$ to get rid of τ.

$$S_n^m(-ik_{1,2}\tau^2) = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}(-ik_{1,2}\tau^2)^{\frac{m}{2}} \sum_{p=0}^{\infty} c_p(n,m)(-ik_{1,2}\tau^2)^p$$
$$c_p(n,m) = \sum_{l=0}^{p} \left(\frac{-1}{2}\right)^l \left(\frac{(-n)_{p-l}}{(m+1)_{p-l}}\right) \frac{1}{l!(p-l)!}$$

$k_{1,2}$ means either k_1 or k_2. With azimuthal symmetry:

$$S_n^0(-ik_{1,2}\tau^2) = \sum_{p=0}^{\infty} c_p(n,0)(-ik_{1,2}\tau^2)^p$$
$$c_p(n,0) = \sum_{l=0}^{p} \left(\frac{-1}{2}\right)^l \left(\frac{(-n)_{p-l}}{(p-l)!}\right) \frac{1}{l!(p-l)!}$$

Here we used $(1)_n = \Gamma(1+n)/\Gamma(1) = n!$. The $\vec{M}_\varphi$ component becomes:

$$(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2ni}{k\sigma\tau(x-y)} \Big((y-x)S_n^0(x)S_n^0(y) - yS_{n-1}^0(x)S_n^0(y) + xS_n^0(x)S_{n-1}^0(y) \Big)$$

$$(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2ni}{k\sigma\tau(x-y)} \sum_{p=0}^{\infty} \Big((y-x)S_n^0(x)c_p(n,0)y^p - yS_{n-1}^0(x)c_p(n,0)y^p + xS_n^0(x)c_p(n-1,0)y^p \Big)$$

$$(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2ni}{k\sigma\tau(x-y)} \sum_{p=0}^{\infty} \Big(S_n^0(x)c_p(n,0)y^{p+1} - xS_n^0(x)c_p(n,0)y^p - S_{n-1}^0(x)c_p(n,0)y^{p+1} + xS_n^0(x)c_p(n-1,0)y^p \Big)$$

Defining $c_{-1}(n,m) = 0$ allows us to write:

$$(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2ni}{k\sigma\tau(x-y)} \sum_{p=0}^{\infty} \Big(S_n^0(x)c_{p-1}(n,0) - xS_n^0(x)c_p(n,0) - S_{n-1}^0(x)c_{p-1}(n,0) + xS_n^0(x)c_p(n-1,0) \Big) y^p \tag{7.23}$$

The $\vec{N}_\tau$ component is much more complex

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{-2ni}{k\tau h_\sigma (x-y)^2}\Bigg((x-y)^2\left[\frac{2n}{x}\left(S_n^0(x)-S_{n-1}^0(x)\right)-S_n^0(x)\right]S_n^0(y) + (x-y)\,y\left[\frac{2(n-1)}{x}\left(S_{n-1}^0(x)-S_{n-2}^0(x)\right)-S_{n-1}^0(x)\right]S_n^0(y) - (x-y)\,S_{n-1}^0(y)\left[(2n-x)\,S_n^0(x)-2nS_{n-1}^0(x)\right]+2y\left(S_{n-1}^0(y)S_n^0(x)-S_{n-1}^0(x)S_n^0(y)\right)\Bigg)$$

let

$$(fs)_n(x) = \frac{2n}{x}\left(S_n^0(x)-S_{n-1}^0(x)\right)-S_n^0(x) \tag{7.24}$$

$$(fv)_n(x) = \frac{2n}{x}\left(V_n^0(x)-V_{n-1}^0(x)\right)-V_n^0(x) \tag{7.25}$$

$$(hs)_n(x) = (2n-x)S_n^0(x)-2nS_{n-1}^0(x) \tag{7.26}$$

$$(hv)_n(x) = (2n-x)V_n^0(x)-2nV_{n-1}^0(x) \tag{7.27}$$

then

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{-2ni}{k\tau h_\sigma (x-y)^2}\Bigg((x-y)^2(fs)_n(x)S_n^0(y)+(x-y)\,y(fs)_{n-1}(x)S_n^0(y) - (x-y)\,S_{n-1}^0(y)(hs)_n(x)+2y\left(S_{n-1}^0(y)S_n^0(x)-S_{n-1}^0(x)S_n^0(y)\right)\Bigg)$$

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{-2ni}{k\tau h_\sigma (x-y)^2}\Bigg((x^2-2xy+y^2)(fs)_n(x)S_n^0(y)+(xy-y^2)(fs)_{n-1}(x)S_n^0(y) - (x-y)S_{n-1}^0(y)(hs)_n(x)+2yS_{n-1}^0(y)S_n^0(x)-2yS_{n-1}^0(x)S_n^0(y)\Bigg)$$

By recycling the same trick used on $\vec{M}_\varphi$

$$S_n^0(y) = \sum_{p=0}^{\infty} c_p(n,0)y^p$$

$$yS_n^0(y) = \sum_{p=0}^{\infty} c_p(n,0)y^{p+1} = \sum_{p=0}^{\infty} c_{p-1}(n,0)y^p$$

$$y^2S_n^0(y) = \sum_{p=0}^{\infty} c_p(n,0)y^{p+2} = \sum_{p=0}^{\infty} c_{p-2}(n,0)y^p$$

$$c_{p-1}(n,0) = c_{p-2}(n,0) = 0$$

The component becomes

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{-2ni}{k\tau h_\sigma (x-y)^2} \sum_{p=0}^{\infty} \Bigg((x^2 c_p(n,0) - 2xc_{p-1}(n,0) + c_{p-2}(n,0))(fs)_n(x)$$
$$+ (xc_{p-1}(n,0) - c_{p-2}(n,0))(fs)_{n-1}(x) - (xc_p(n-1,0) - c_{p-1}(n-1,0))(hs)_n(x)$$
$$+ 2c_{p-1}(n-1,0)S_n^0(x) - 2c_{p-1}(n,0)S_{n-1}^0(x) \Bigg) y^p$$

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{2ni}{k^3\tau(\sigma^2+\tau^2)^{\frac{5}{2}}} \sum_{p=0}^{\infty} \Bigg((x^2 c_p(n,0) - 2xc_{p-1}(n,0) + c_{p-2}(n,0))(fs)_n(x)$$
$$+ (xc_{p-1}(n,0) - c_{p-2}(n,0))(fs)_{n-1}(x) - (xc_p(n-1,0) - c_{p-1}(n-1,0))(hs)_n(x)$$
$$+ 2c_{p-1}(n-1,0)S_n^0(x) - 2c_{p-1}(n,0)S_{n-1}^0(x) \Bigg) y^p$$

To simplify calculations let

$$(NS)(n,k,\boldsymbol{\sigma},p) = \Bigg((x^2 c_p(n,0) - 2xc_{p-1}(n,0) + c_{p-2}(n,0))(fs)_n(x) + (xc_{p-1}(n,0) - c_{p-2}(n,0))(fs)_{n-1}(x$$
$$- (xc_p(n-1,0) - c_{p-1}(n-1,0))(hs)_n(x) + 2c_{p-1}(n-1,0)S_n^0(x) - 2c_{p-1}(n,0)S_{n-1}^0(x) \Bigg) \tag{7.28}$$

and

$$(MS)(n,k,\boldsymbol{\sigma},p) = \Bigg(S_n^0(x)c_{p-1}(n,0) - xS_n^0(x)c_p(n,0)$$
$$- S_{n-1}^0(x)c_{p-1}(n,0) + xS_n^0(x)c_p(n-1,0) \Bigg) \tag{7.29}$$

Therefore

$$(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2n}{k^2\boldsymbol{\sigma}\tau(\boldsymbol{\sigma}^2+\tau^2)} \sum_{p=0}^{\infty} (MS)(n,k,\boldsymbol{\sigma},p)y^p \tag{7.30}$$

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{2ni}{k^3\tau(\boldsymbol{\sigma}^2+\tau^2)^{\frac{5}{2}}} \sum_{p=0}^{\infty} (NS)(n,k,\boldsymbol{\sigma},p)y^p \tag{7.31}$$

For the Scattered field

$$(\vec{M}_{n,0}(\vec{r}))_{\varphi} = \frac{-2n}{k^2\sigma\tau(\sigma^2+\tau^2)} \sum_{p=0}^{\infty} (MV)(n,k,\sigma,p)y^p \tag{7.32}$$

$$(\vec{N}_{n,0}(\vec{r}))_{\tau} = \frac{2ni}{k^3\tau(\sigma^2+\tau^2)^{\frac{5}{2}}} \sum_{p=0}^{\infty} (NV)(n,k,\sigma,p)y^p \tag{7.33}$$

with

$$(NV)(n,k,\sigma,p) = \Big((x^2c_p(n,0) - 2xc_{p-1}(n,0) + c_{p-2}(n,0))(fv)_n(x) + (xc_{p-1}(n,0) - c_{p-2}(n,0))(fv)_{n-1}(\\ - (xc_p(n-1,0) - c_{p-1}(n-1,0))(hv)_n(x) + 2c_{p-1}(n-1,0)V_n^0(x) - 2c_{p-1}(n,0)V_{n-1}^0(x) \Big) \tag{7.34}$$

$$(MV)(n,k,\sigma,p) = \Big(V_n^0(x)c_{p-1}(n,0) - xV_n^0(x)c_p(n,0) \\ - V_{n-1}^0(x)c_{p-1}(n,0) + xV_n^0(x)c_p(n-1,0) \Big) \tag{7.35}$$

Replacing on the boundary conditions

$$\sum_{n=1}^{\infty} B_{n,0}\vec{N}_{n,0}^{(s)}(k_2,\sigma_0,\tau)_{\tau} - D_{n,0}\vec{N}_{n,0}^{(v)}(k_1,\sigma_0,\tau)_{\tau} = \sum_{n=1}^{\infty} G_{n,0}^{(TM)}\vec{N}_{n,0}^{(s)}(k_1,\sigma_0,\tau)_{\tau}$$

$$\sum_{n=1}^{\infty} k_2B_{n,0}\vec{M}_{n,0}^{(s)}(k_2,\sigma_0,\tau)_{\varphi} - k_1D_{n,0}\vec{M}_{n,0}^{(v)}(k_1,\sigma_0,\tau)_{\varphi} = \sum_{n=1}^{\infty} k_1G_{n,0}^{(TM)}\vec{M}_{n,0}^{(s)}(k_1,\sigma_0,\tau)_{\varphi}$$

$$\sum_{n=1}^{\infty}\left(\frac{B_{n,0}}{k_2^3}\sum_{p=0}^{\infty}(NS)(n,k_2,\sigma_0,p)y^p - \frac{D_{n,0}}{k_1^3}\sum_{p=0}^{\infty}(NV)(n,k_1,\sigma_0,p)y^p\right) = \sum_{n=1}^{\infty}\frac{G_{n,0}^{(TM)}}{k_1^3}\sum_{p=0}^{\infty}(NS)(n,k_1,\sigma_0,p)y^p$$

$$\sum_{n=1}^{\infty}\left(\frac{B_{n,0}}{k_2}\sum_{p=0}^{\infty}(MS)(n,k_2,\sigma_0,p)y^p - \frac{D_{n,0}}{k_1}\sum_{p=0}^{\infty}(MV)(n,k_1,\sigma_0,p)y^p\right) = \sum_{n=1}^{\infty}\frac{G_{n,0}^{(TM)}}{k_1}\sum_{p=0}^{\infty}(MS)(n,k_1,\sigma_0,p)y^p$$

By the linear independence of y^p

$$\sum_{n=1}^{\infty}\frac{B_{n,0}}{k_2^3}(NS)(n,k_2,\sigma_0,p)k_2^p - \frac{D_{n,0}}{k_1^3}(NV)(n,k_1,\sigma_0,p)k_1^p = \sum_{n=1}^{\infty}\frac{G_{n,0}^{(TM)}}{k_1^3}(NS)(n,k_1,\sigma_0,p)k_1^p$$

$$\sum_{n=1}^{\infty}\frac{B_{n,0}}{k_2}(MS)(n,k_2,\sigma_0,p)k_2^p - \frac{D_{n,0}}{k_1}(MV)(n,k_1,\sigma_0,p)k_1^p = \sum_{n=1}^{\infty}\frac{G_{n,0}^{(TM)}}{k_1}(MS)(n,k_1,\sigma_0,p)k_1^p$$

which must be valid for every value of p. This system is solved by truncating the sum on n and solving it like a matrix problem $M.X = b$ where X is a vector containing the coefficients $B_{n,0}$ and $D_{n,0}$. To consider Non-dielectric materials, a complex index of refraction may be used ($n_2 = n + ik$ in $k_2 = \frac{2\pi}{\lambda} n_2$)

8 LIGHT ENHANCEMENT AT THE TIP OF A PARABOLOID

8.1 Numeric limitations

The main motivation to build this theory is to apply it to Near Field Scanning Microscopy. For this purpose an attempt to reproduce some of the results of the article (NOVOTNY *et al.*, 1997) was considered. The article solves the Maxwell's equations using the multiple multipole method (MMP) in two different cases: case a) a plane wave polarized perpendicular to the axis of the tip and case b) a plane wave polarized parallel to the axis of the tip. The article also gives the necessary information to reproduce its results, namely wavelength ($\lambda = 810nm$) the dielectric constants ($\varepsilon_1 = 1.77$ for water and $\varepsilon_2 = -24.9 + i1.57$ for the gold tip at the incident wavelength) and the tip radius (5 nm). The article also gives an estimate value of the enhancement at the tip of $|\frac{E_{scat}}{E_{inc}}|^2 \approx 3000$ for the case b). However the expansion of a plane wave polarized along the axis of the paraboloid is taken at the limit of its convergence. As such the enhancement defined by:

$$I_{enh} = \left| \frac{\vec{E}_{scat} + \vec{E}_{inc}}{\vec{E}_{inc}} \right|^2 \tag{8.1}$$

Gives unreasonable high values at the tip of the paraboloid. The expansion

$$e^{i\vec{k}.\vec{r}} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{i^m \varepsilon_m \Gamma(n+1)}{\Gamma(n+|m|+1)} \frac{tan^{2n+|m|}\left(\frac{\theta_k}{2}\right)}{cos^2\left(\frac{\theta_k}{2}\right)} (-1)^n S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im(\varphi-\varphi_k)}$$

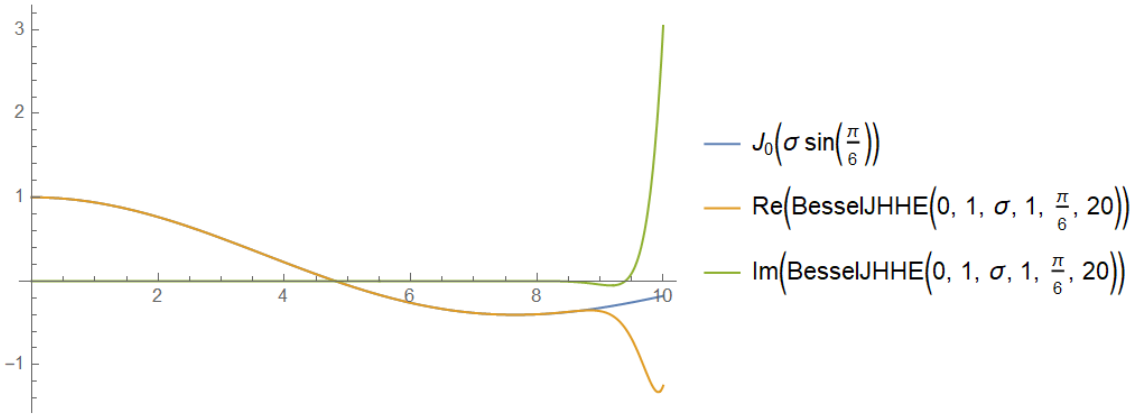
converges slowly on n as $\theta \to \pi/2$. This can be seen by numerically testing the relation

$$J_m(k\sigma\tau\sin(\theta)) = e^{-\frac{ik(\tau^2-\sigma^2)\cos(\theta)}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+m+1)} \frac{\tan^{m+2n}\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)} (-1)^n S_n^m(ik\sigma^2) S_n^m(ik\tau^2) \tag{8.2}$$

figures 7, 8 and 9 show this behavior

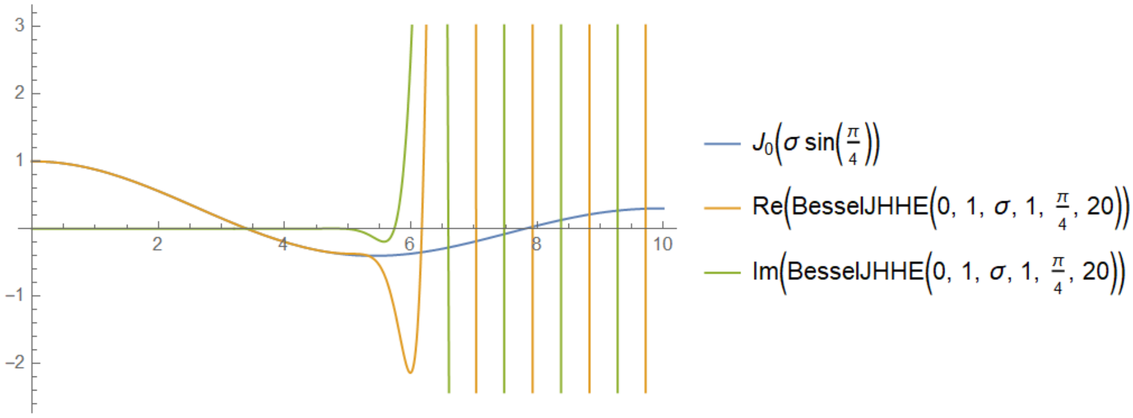
Clearly the expansion converges more slowly as the angle increases. Moreover The imaginary part of the expansion is zero and the real part coincides with the Bessel function as expected in the region of convergence. The last important thing to note is that when the

Figura 7 – Plot of the function $J_0(k\sigma\tau\sin(\theta))$ vs σ with $k = \tau = 1$ and $\theta_k = \pi/6$. The function BesselJHHE$(m,k,\sigma,\tau,\theta_k,N)$ is the right hand of the equation 8.2. Here $N = 20$ is the number of terms used in the expansion.



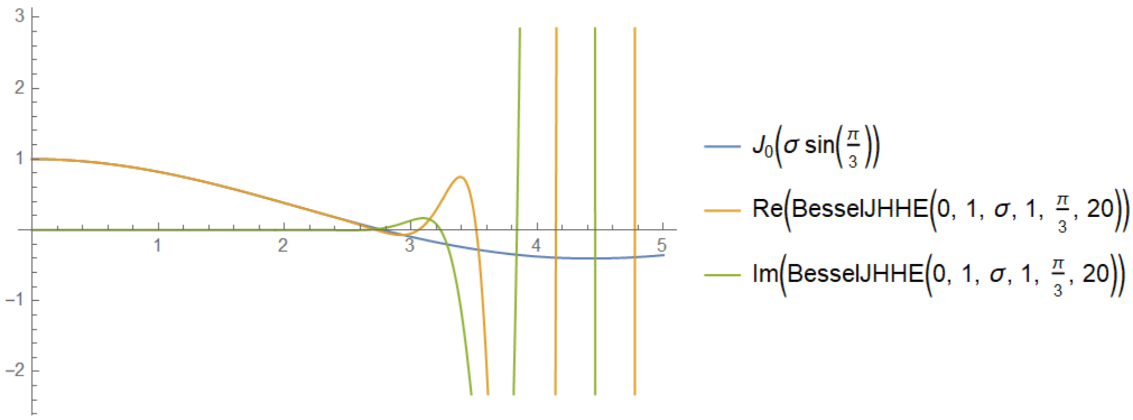
Source: Elaborated by the author using the program Wolfram Mathematica (INC.,).

Figura 8 – Plot of the function $J_0(k\sigma\tau\sin(\theta))$ vs σ with $k = \tau = 1$ and $\theta_k = \pi/4$. The function BesselJHHE$(m,k,\sigma,\tau,\theta_k,N)$ is the right hand of the equation 8.2. Here $N = 20$ is the number of terms used in the expansion.



Source: Elaborated by the author using the program Wolfram Mathematica (INC.,).

Figura 9 – Plot of the function $J_0(k\sigma\tau\sin(\theta))$ vs σ with $k = \tau = 1$ and $\theta_k = \pi/3$. The function BesselJHHE$(m,k,\sigma,\tau,\theta_k,N)$ is the right hand of the equation 8.2. Here $N = 20$ is the number of terms used in the expansion.



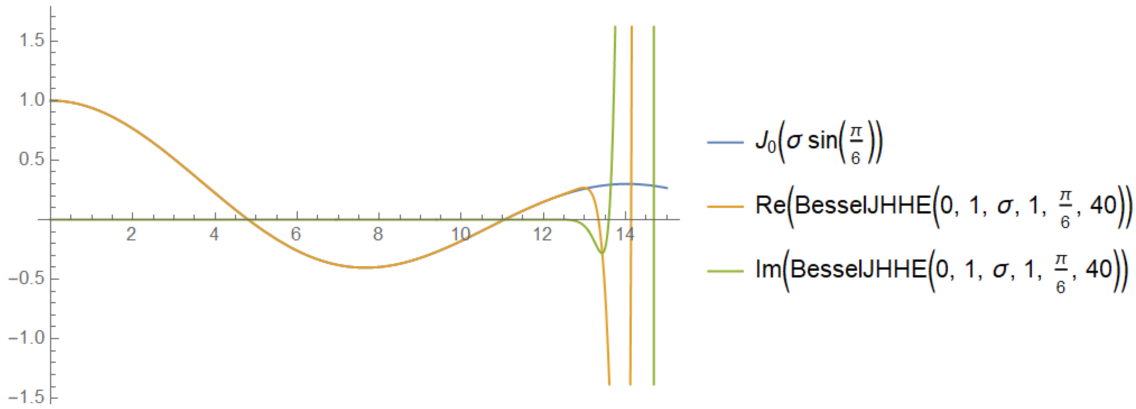
Source: Elaborated by the author using the program Wolfram Mathematica (INC.,).

expansion starts to deviate from its expected value it diverges rapidly. Therefore it is expected that the scattered field presents the same behavior should N not be big enough.

For practical purposes it is required that $ik\sigma^2$ and/or $ik\tau^2$ are low enough. At the negative z axis just below the paraboloid $\tau = 0$ so only the values of k and σ limit our calculations. Let $\sigma = 10nm^{1/2}$ and $k = \frac{2\pi}{\lambda}$, with $\lambda = 800$ then the product $k\sigma^2 = \frac{2\pi 100}{800} < 1$. In general for low values of σ (near the tip) $k\sigma^2 < 1$.

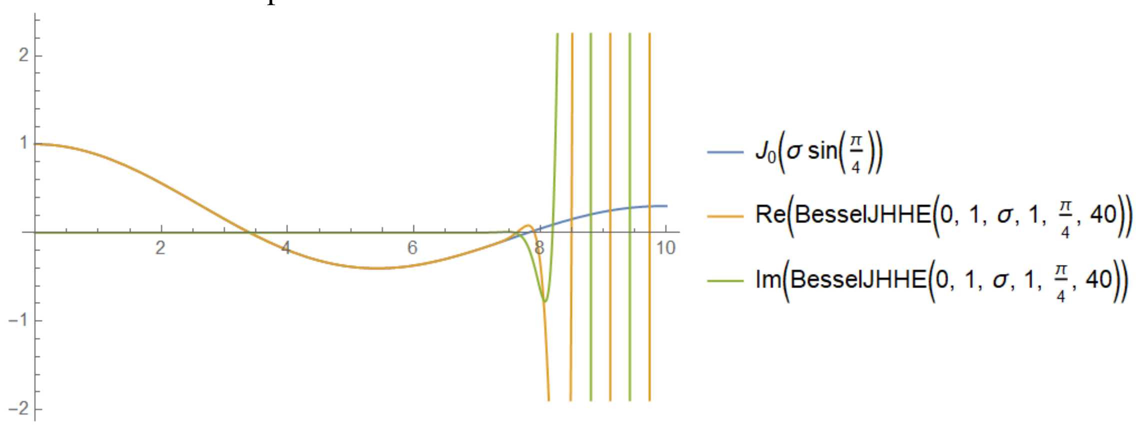
The expansion can be improved by taking more terms in the sum as shown in figures 10, 11 and 12

Figura 10 – Plot of the function $J_0(k\sigma\tau\sin(\theta))$ vs σ with $k = \tau = 1$ and $\theta_k = \pi/6$. The function BesselJHHE$(m, k, \sigma, \tau, \theta_k, N)$ is the right hand of the equation 8.2. Here $N = 40$ is the number of terms used in the expansion.



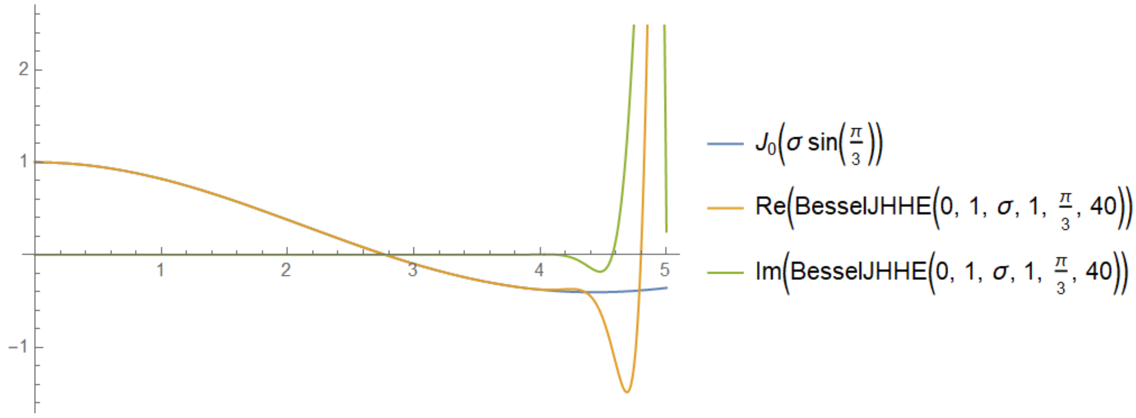
Source: Elaborated by the author using the program Wolfram Mathematica (INC.,).

Figura 11 – Plot of the function $J_0(k\sigma\tau\sin(\theta))$ vs σ with $k = \tau = 1$ and $\theta_k = \pi/4$. The function BesselJHHE$(m, k, \sigma, \tau, \theta_k, N)$ is the right hand of the equation 8.2. Here $N = 40$ is the number of terms used in the expansion.



Source: Elaborated by the author using the program Wolfram Mathematica (INC.,).

Figura 12 – Plot of the function $J_0(k\sigma\tau\sin(\theta))$ vs σ with $k=\tau=1$ and $\theta_k=\pi/3$. The function BesselJHHE$(m,k,\sigma,\tau,\theta_k,N)$ is the right hand of the equation 8.2. Here $N=40$ is the number of terms used in the expansion.



Source: Elaborated by the author using the program Wolfram Mathematica (INC.,).

8.2 Calculation of the light enhancement

Both the scattered and incident field are represented only by the vector fields $\vec{N}^{(v)}_{n,m}(\vec{r})$ and $\vec{N}^{(s)}_{n,m}(\vec{r})$ respectively. To evaluate those fields on the negative z axis below the paraboloid we require that $\tau=0$ which implies two things: only the index $m=0$ (the radial component of the focused beam is going to be ignored since 1. It is known experimentally that it does not contributes much to the enhancement and 2. at the z axis is equal to zero) is going to appear and the only non-zero component of the field is $\vec{N}_{n,m}(\vec{r})_\sigma$. The calculation of these components is done in appendix C along with their evaluation on $\tau=0$. The incident field is the one shown on eq (6.80) integrated from $\theta_k = 0 \rightarrow \frac{\pi}{4}$ to represent a field obtained by a beam with radial polarization passing through an objective len of high numerical aperture.

With all these information in mind the parameters defined for the implementation are shown on table 1 whose values were taken from (POLYANSKIY, 2008-2022a; POLYANSKIY, 2008-2022b):

Tabela 1 – Refractive index (n+ik) of gold (Au) and water at different wavelengths

	Gold		Water at 25 °C	
λ (nm)	n	k	n	k
810	0.15659	4.9908	1.3290	1.4780×10^{-7}
600	0.24873	3.0740	1.3320	1.0900×10^{-8}
400	1.4684	1.9530	1.3390	1.8600×10^{-9}

Source: data taken from (POLYANSKIY, 2008-2022a; POLYANSKIY, 2008-2022b).

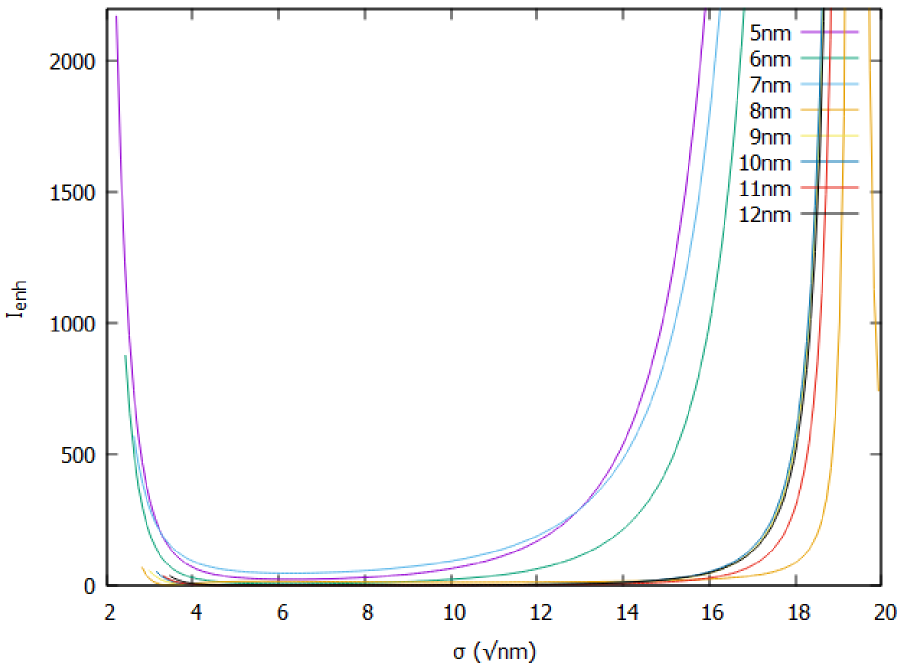
we also have $\sigma_0=\sqrt{R}$ with $R=5,6,7,..,12nm$

The units used are nanometers. Using the procedure shown in the last section 20 coefficients were calculated for the Scattered and internal field (with $m = 0$ and n from $0 \rightarrow 39$). The implementation is done in Mathematica since this is only a test and Mathematica already has most of the functions necessary implemented; hence it simplifies the implementation.

The behavior of the light enhancement along the z axis is shown in the next figures. Do note however, that the method used gives a badly conditioned matrix and Mathematica gave an alert of possible numeric issues in all calculations. Nevertheless the behavior of the enhancement is more or less as expected.

Figure 13 shows the behavior of the field enhancement calculated from the tip ($z = -\frac{R}{2}$ where R is the tip radius) to a sigma value of $\sigma = 20\ \sqrt{nm}$ which is about $z = -200$ nm .

Figura 13 – Field enhancement along the negative z axis by a focused beam shown on eq (6.80) integrating θ_k from 0 to $\frac{\pi}{4}$. Each line corresponds to a different tip-radius and starts at $\sigma_0 = \sqrt{R}$ with R being the tip-radius. The wavelentgh used is 810 nm while the tip is assumed to be made of gold submerged in water.

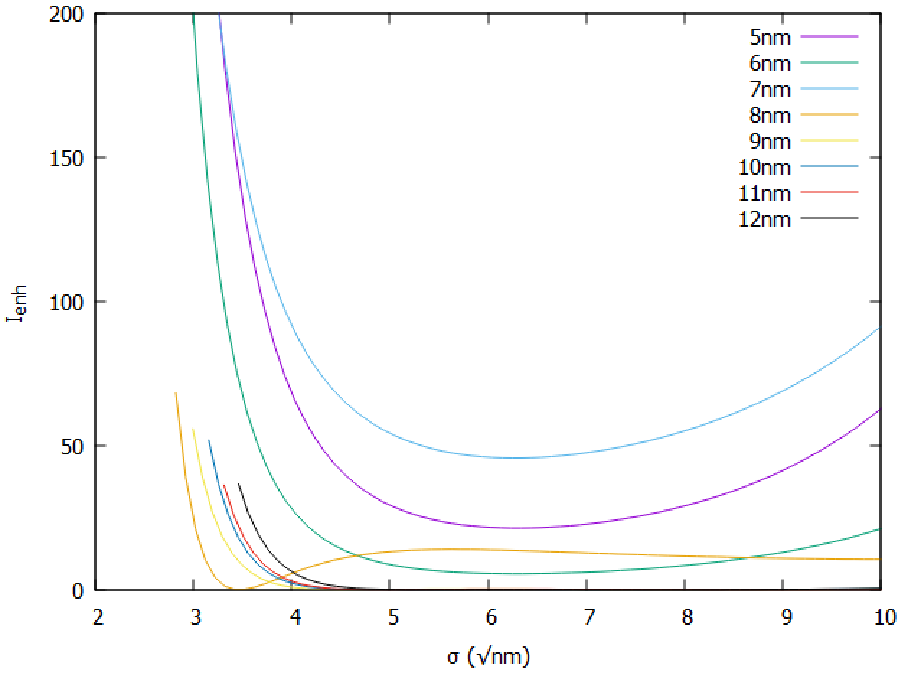


Source: Elaborated by the author using the program gnuplot (WILLIAMS *et al.*, 2013)

The behavior is as expected, the intensity enhancement decreases as we get far away from the tip up until a point where the incident field expansion starts to diverge start to diverge. There is also a huge increase in the field enhancement between 8 and 5 nm tip radius. This can be appreciated in figure 14:

Despite the convergence issue for bigger values of σ the field enhancement values at

Figura 14 – Field enhancement along the negative z axis by a focused beam shown on eq (6.80) integrating θ_k from 0 to $\frac{\pi}{4}$. Each line corresponds to a different tip-radius and starts at $\sigma_0 = \sqrt{R}$ with R being the tip-radius. The wavelentgh used is 810 nm while the tip is assumed to be made of gold submerged in water.



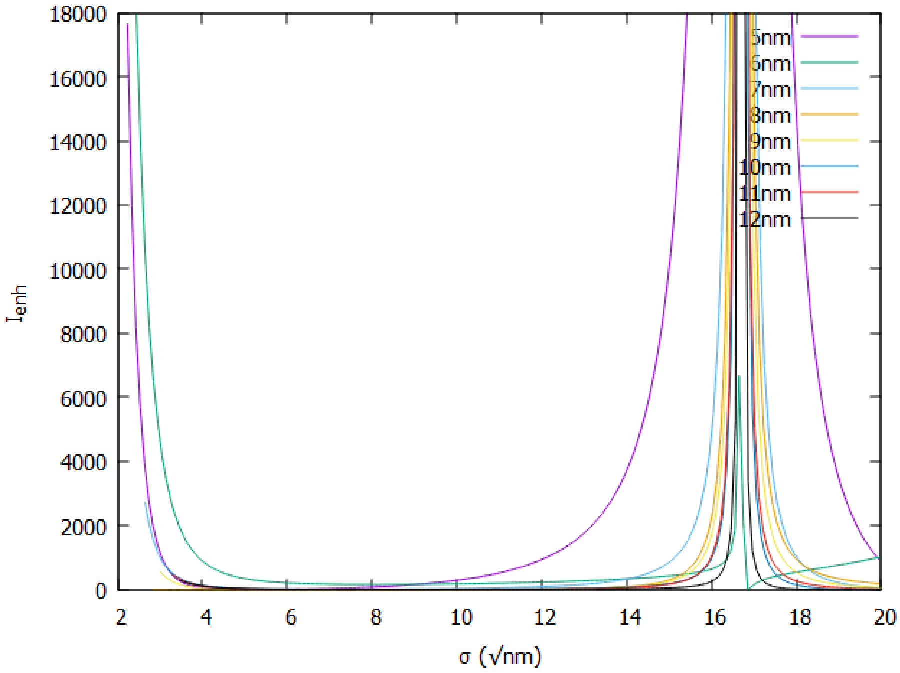
Source: Elaborated by the author using the program gnuplot (WILLIAMS *et al.*, 2013)

the tip are similar to (THOMAS *et al.*, 2015).

Similar results are obtained for $\lambda = 600$ nm as seen in the figure 15. There is a big increment in the field enhancement values at $\lambda = 600$ nm compared with $\lambda = 810$ nm. The closer we get to the plasma frequency the bigger the enhancement.

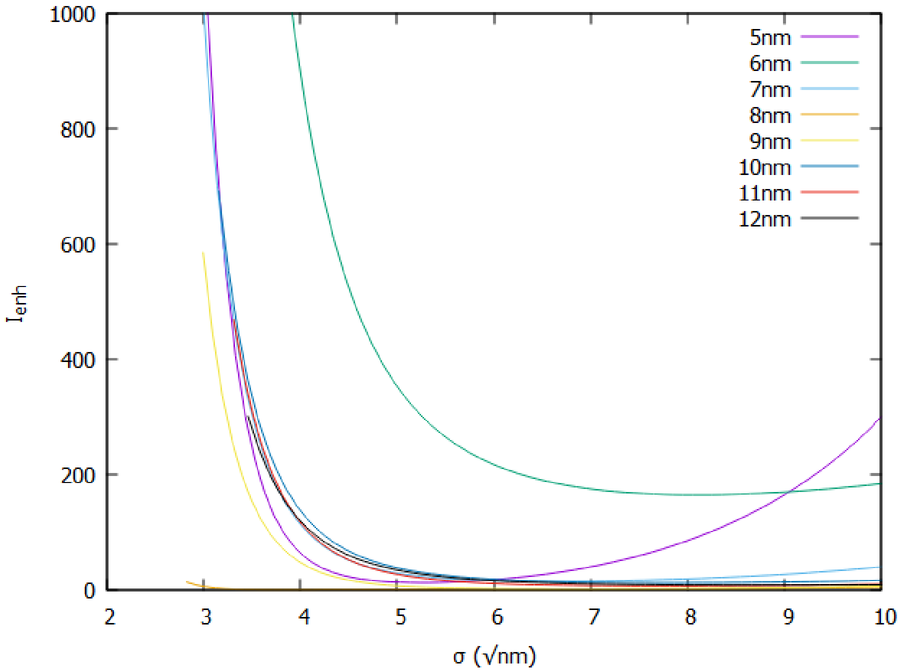
Past the plasma frequency and for low values of λ numeric errors may shift the behavior as conductor and dielectric. For $\lambda = 400$ nm and lower values, the method becomes too inaccurate. Figure 17 shows these results.

Figura 15 – Field enhancement along the negative z axis by a focused beam shown on eq (6.80) integrating θ_k from 0 to $\frac{\pi}{4}$. Each line corresponds to a different tip-radius and starts at $\sigma_0 = \sqrt{R}$ with R being the tip-radius. The wavelentgh used is 600 nm while the tip is assumed to be made of gold submerged in water. T



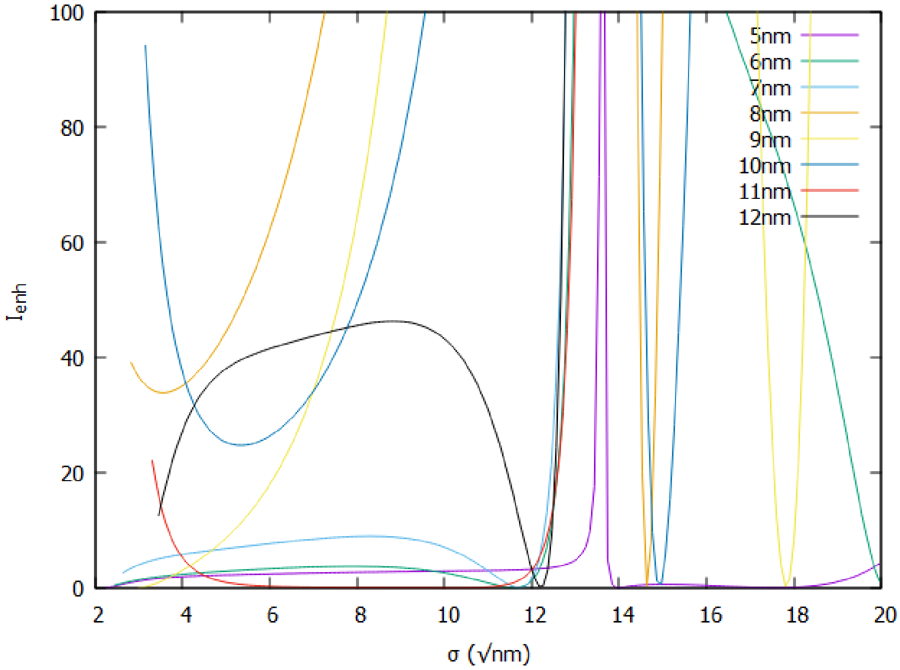
Source: Elaborated by the author using the program gnuplot (WILLIAMS *et al.*, 2013)

Figura 16 – Field enhancement along the negative z axis by a focused beam shown on eq (6.80) integrating θ_k from 0 to $\frac{\pi}{4}$. Each line corresponds to a different tip-radius and starts at $\sigma_0 = \sqrt{R}$ with R being the tip-radius. The wavelentgh used is 600 nm while the tip is assumed to be made of gold submerged in water.



Source: Elaborated by the author using the program gnuplot (WILLIAMS *et al.*, 2013)

Figura 17 – Field enhancement along the negative z axis by a focused beam shown on eq (6.80) integrating θ_k from 0 to $\frac{\pi}{4}$. Each line corresponds to a different tip-radius and starts at $\sigma_0 = \sqrt{R}$ with R being the tip-radius. The wavelentgh used is 400 nm while the tip is assumed to be made of gold submerged in water.



Source: Elaborated by the author using the program gnuplot (WILLIAMS *et al.*, 2013)

9 CONCLUSIONS AND PERSPECTIVES

The theory of light scattering on a paraboloid of revolution was further developed and verified from its state of the art which was not taken seriously since Horton's article in 1951(HORTON; KARAL, 1951).

The expansion of scalar plane waves in terms of solutions of the Helmholtz equation in parabolic was found. This expansion opened the possibility to obtain an analytical expansion of incident fields in terms of Hertz Vectors.

Although a method for solving the boundary conditions is presented. The problem of solving the boundary conditions is still open both analytically and numerically. More than 40 coefficients are necessary if the solution is required to be valid in the far-field.

Up to the author knowledge, this work is the closest to an analytical solution to calculate the field enhancement at the tip of gold probes.

For treating more general incident fields new Hertz vectors may be defined. In particular

$$e^{i\vec{k}.\vec{r}}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z)$$
$$= e^{ik\cos(\theta_k)(\tau^2-\sigma^2)/2+ik\sigma\tau\cos(\varphi-\varphi_k)\sin(\theta_k)}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z) \tag{9.1}$$

may be expanded with the help of

$$e^{i\vec{k}.\vec{r}} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} \frac{i^m \varepsilon_m \Gamma(n+1)}{\Gamma(n+|m|+1)} \frac{tan^{2n+|m|}\left(\frac{\theta_k}{2}\right)}{cos^2\left(\frac{\theta_k}{2}\right)}(-1)^n S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im(\varphi-\varphi_k)}$$

$$G_{n,m}(\theta_k) = \frac{i^m \varepsilon_m \Gamma(n+1)}{\Gamma(n+|m|+1)} \frac{tan^{2n+|m|}\left(\frac{\theta_k}{2}\right)}{cos^2\left(\frac{\theta_k}{2}\right)}(-1)^n e^{-im\varphi_k}$$

To express the field as

$$e^{i\vec{k}.\vec{r}}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}(\theta_k) S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}(\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z) \tag{9.2}$$

By using Maxwells equations

$$\vec{E}(\vec{r}) = \frac{i}{k} Z \nabla \times H \tag{9.3}$$

$$Z\vec{H}(\vec{r}) = -\frac{i}{k} \nabla \times E \tag{9.4}$$

The new Hertz vectors may be defined as

$$\frac{-k}{i} \vec{M}_{n,m}(\vec{r}) = \nabla \times S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} (\cos(\theta_k)\hat{\mathbf{e}}_\rho + \sin(\theta_k)\hat{\mathbf{e}}_z) \tag{9.5}$$

$$\frac{k}{i} \vec{N}_{n,m}(\vec{r}) = \nabla \times \vec{M}_{n,m}(\vec{r}) \tag{9.6}$$

Should an analytical solution for the boundary conditions be found for these Hertz vectors, the radial component of the focused beam may be taken into account as well. This problem is too large and complicated to be treated in this work. Therefore it is left as a perspective.

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APPENDIX A – SOLUTION OF THE SCALAR HELMHOLTZ EQUATION

The parabolic coordinates are given in (WILLATZEN; VOON, 2011) by the following transformation:

$$\begin{aligned} x &= \sigma\tau cos\varphi \\ y &= \sigma\tau sin\varphi \\ z &= \frac{1}{2}(\tau^2 - \sigma^2) \end{aligned} \tag{A.1}$$

with $0 \leq \sigma \leq \infty$, $0 \leq \tau \leq \infty$ and $0 \leq \varphi \leq 2\pi$.

The Helmholtz equation in parabolic coordinates is:

$$\frac{1}{\sigma^2+\tau^2}\left[\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial\psi}{\partial\sigma}\right)+\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\psi}{\partial\tau}\right)\right]+\frac{1}{\sigma^2\tau^2}\frac{\partial^2\psi}{\partial\varphi^2}+k^2\psi=0 \tag{A.2}$$

we define $\psi = S(\sigma)T(\tau)\Phi(\varphi)$ and divide the Helmholtz equation by this quantity:

$$\frac{1}{\sigma^2+\tau^2}\left[\frac{1}{\sigma S}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\frac{1}{\tau T}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)\right]+\frac{1}{\sigma^2\tau^2\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+k^2=0 \tag{A.3}$$

Let $\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = -m^2$ then:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+m^2\Phi=0 \tag{A.4}$$

whose solutions are:

$$\Phi(\varphi) = Asin(m\varphi)+Bcos(m\varphi) \tag{A.5}$$

or

$$\Phi(\varphi) = Ae^{im\varphi}+Be^{-im\varphi} \tag{A.6}$$

It is assumed that m can only take positive integer values. While the reason for positives values is going to be clear later the integer values are required for the function to be one-valued. For parabolic variables we have:

$$\frac{1}{\sigma^2+\tau^2}\left[\frac{1}{\sigma S}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\frac{1}{\tau T}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)\right]-\frac{m^2}{\sigma^2\tau^2}+k^2=0 \tag{A.7}$$

Multiplying by $\sigma^2+\tau^2$ and separating the terms containing only its respective variable:

$$\left[\frac{1}{\sigma S}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]+\left[\frac{1}{\tau T}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)+\tau^2k^2-\frac{m^2}{\tau^2}\right]=0 \tag{A.8}$$

Define the first bracket as $-q^2$ and the second as q^2. This leads to:

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\left[q^2+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]S=0 \tag{A.9}$$

$$\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)-\left[q^2-\tau^2k^2+\frac{m^2}{\tau^2}\right]T=0 \tag{A.10}$$

The solutions are essentially the same. Given the solution for $S(\sigma)$ the Solution for $T(\tau)$ can be found replacing $q^2\to-q^2$ or $T(\tau)=S(-\tau)$.

A.1 Parabolic functions in terms of Whittaker functions

The most popular solution is given in terms of Whittaker functions. Consider:

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right)+\left[q^2+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]S=0 \tag{A.11}$$

Let $S(\sigma)=V(\upsilon)/\sqrt{\upsilon}$ with $\sigma^2=\upsilon$. This transformation leads to:

$$\frac{\mathrm{d}^2V}{\mathrm{d}\upsilon^2}+\left[\frac{k^2}{4}+\frac{q^2}{4\upsilon}+\frac{\frac{1}{4}-\frac{m^2}{4}}{\upsilon^2}\right]V=0 \tag{A.12}$$

By defining $\alpha=q^2/4ik,\mu=m/2$ and with the change of variables $z=ik\upsilon$ the whittaker equation is obtained:

$$\frac{\mathrm{d}^2V}{\mathrm{d}z^2}+\left[-\frac{1}{4}+\frac{\alpha}{z}+\frac{\frac{1}{4}-\mu^2}{\upsilon^2}\right]V=0 \tag{A.13}$$

The solutions of the previous differential equation are known as Whittaker functions:

$$M_{\alpha,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} M(\frac{1}{2} + \mu - \alpha, 1 + 2\mu, z) \tag{A.14}$$

$$W_{\alpha,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} U(\frac{1}{2} + \mu - \alpha, 1 + 2\mu, z) \tag{A.15}$$

Where $M(a,b,z)$ and $U(a,b,z)$ are the confluent hypergeometric functions of first and second kind respectively(WILLATZEN; VOON, 2011). Therefore:

$$S(\sigma) = \frac{1}{\sqrt{ik\sigma^2}} M_{q^2/4ik,\frac{m}{2}}(ik\sigma^2) \tag{A.16}$$

$$V(\sigma) = \frac{1}{\sqrt{ik\sigma^2}} W_{q^2/4ik,\frac{m}{2}}(ik\sigma^2) \tag{A.17}$$

Are the first and second kind solutions respectively. Finally, note that the function of first kind can be written as:

$$f(z) = e^{-\frac{z}{2}} z^{m/2} M(\frac{1}{2} + m/2 - q^2/4ik, 1 + m, z) \tag{A.18}$$

with $z = ik\sigma^2$.

The downside of this solution is that we do not known what values $\alpha = q^2/4ik$ may take. The solution given by Pinney in terms of Laguerre polynomials gives the value for the separation constant(PINNEY, 1946):

$$\frac{q^2}{2} = ik(m+1+2n) \tag{A.19}$$

This implies that:

$$\frac{q^2}{4ik} = \frac{1}{2}(m+1+2n) \tag{A.20}$$

Therefore:

$$f(z) = e^{-\frac{z}{2}} z^{m/2} M(-n, 1+m, z) \tag{A.21}$$

With n being an integer $M(-n, 1+m, z)$ is proportional to the Generalized Laguerre polynomials. Thus, the solution found by Pinney and the solution found in terms of Whittaker functions are essentially the same.

A.2 Parabolic functions in terms of Generalized Laguerre polynomials

Recall the differential equations:

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}S}{\mathrm{d}\sigma}\right) + \left[q^2 + \sigma^2 k^2 - \frac{m^2}{\sigma^2}\right]S = 0 \tag{A.22}$$

$$\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau\frac{\mathrm{d}T}{\mathrm{d}\tau}\right) - [q^2 - \tau^2 k^2 + \frac{m^2}{\tau^2}]T = 0 \tag{A.23}$$

Following the Pinney approach to this problem let $\xi = \sigma^2/2$ and $\eta = \tau^2/2$. For the first equation:

$$\frac{\mathrm{d}S}{\mathrm{d}\sigma} = \frac{\mathrm{d}S}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\sigma} = \frac{\mathrm{d}S}{\mathrm{d}\xi}\sigma$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma^2\frac{\mathrm{d}S}{\mathrm{d}\xi}\right) = 2\sigma\frac{\mathrm{d}S}{\mathrm{d}\xi} + \sigma^2\frac{\mathrm{d}^2S}{\mathrm{d}\xi^2}\sigma = \sigma\left(2\frac{\mathrm{d}S}{\mathrm{d}\xi} + \sigma^2\frac{\mathrm{d}S}{\mathrm{d}\xi}\right)$$

Then

$$2\frac{\mathrm{d}S}{\mathrm{d}\xi}+\sigma^2\frac{\mathrm{d}^2S}{\mathrm{d}\xi^2}+\left[q^2+\sigma^2k^2-\frac{m^2}{\sigma^2}\right]S=0$$
$$2\xi\frac{\mathrm{d}^2S}{\mathrm{d}\xi^2}+2\frac{\mathrm{d}S}{\mathrm{d}\xi}+\left[q^2+2\xi k^2-\frac{m^2}{2\xi}\right]S=0$$

Finally

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}S}{\mathrm{d}\xi}\right)+\left[h+\xi k^2-\frac{m^2}{4\xi}\right]S=0 \tag{A.24}$$

In the last step we redefined the separation constant $\frac{q^2}{2}=h$. An analogous procedure for the other variable gives

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}T}{\mathrm{d}\eta}\right)+\left[-h+\eta k^2-\frac{m^2}{4\eta}\right]T=0 \tag{A.25}$$

Note that the change of variable $u\rightarrow-\eta$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u\frac{\mathrm{d}N}{\mathrm{d}u}\right)+\left[h+uk^2-\frac{m^2}{4u}\right]N=0 \tag{A.26}$$

Which is the same equation satisfied by $M(\xi)$. Then the solution can be expressed as:

$$\psi(\xi,\eta,\varphi)=S(\pm\xi)S(\mp\eta)\begin{bmatrix}Sin(m\varphi)\\Cos(m\varphi)\end{bmatrix} \tag{A.27}$$

Now the problem is reduced to find the solution to the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}S}{\mathrm{d}x}\right)+\left[h+xk^2-\frac{m^2}{4x}\right]S=0 \tag{A.28}$$

Now let $S(x)=x^{\frac{m}{2}}e^{-ikx}\Phi_\nu^m(x)$:

$$\begin{aligned}
&\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{m}{2}x^{\frac{m}{2}-1}e^{-ikx}\Phi_\nu^m(x) - ikx^{\frac{m}{2}}e^{-ikx}\Phi_\nu^m(x) + x^{\frac{m}{2}}e^{-ikx}\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} \\
&\frac{\mathrm{d}S}{\mathrm{d}x} = x^{\frac{m}{2}}e^{-ikx}\left[\left(\frac{m}{2x} - ik\right)\Phi_\nu^m(x) + \frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right] \\
&x\frac{\mathrm{d}S}{\mathrm{d}x} = x^{\frac{m}{2}}e^{-ikx}\left[\left(\frac{m}{2} - ikx\right)\Phi_\nu^m(x) + x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right] \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}S}{\mathrm{d}x}\right) = x^{\frac{m}{2}}e^{-ikx}\left(\frac{m}{2x} - ik\right)\left[\left(\frac{m}{2} - ikx\right)\Phi_\nu^m(x) + x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right] \\
&+ x^{\frac{m}{2}}e^{-ikx}\left[-ik\Phi_\nu^m(x) + \left(\frac{m}{2} - ikx\right)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right)\right] \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}S}{\mathrm{d}x}\right) = x^{\frac{m}{2}}e^{-ikx}\left\{\left(\frac{m^2}{4x} - ikm - k^2x\right)\Phi_\nu^m(x) + \left(\frac{m}{2} - ikx\right)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right\} \\
&+ x^{\frac{m}{2}}e^{-ikx}\left\{-ik\Phi_\nu^m(x) + \left(\frac{m}{2} - ikx\right)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right)\right\} \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}S}{\mathrm{d}x}\right) = [\left(\frac{m^2}{4x} - ikm - k^2x - ik\right)\Phi_\nu^m(x) + 2\left(\frac{m}{2} - ikx\right)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} \\
&+ \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right)]x^{\frac{m}{2}}e^{-ikx}
\end{aligned}$$

Inserting the last equation into the differential equation and factorizing $x^{\frac{m}{2}}e^{-ikx}$:

$$\begin{aligned}
&\left(\frac{m^2}{4x} - ikm - k^2x - ik\right)\Phi_\nu^m(x) + \left(h + xk^2 - \frac{m^2}{4x}\right)\Phi_\nu^m(x) \\
&+ (m - 2ikx)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right) = 0
\end{aligned}$$

Canceling terms:

$$(h - ikm - ik)\Phi_\nu^m(x) + (m - 2ikx)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x}\right) = 0$$

Defining $h = ik(m + 1 + 2\nu)$ we obtain:

$$x\frac{\mathrm{d}^2\Phi_\nu^m(x)}{\mathrm{d}x^2} + (m + 1 - 2ikx)\frac{\mathrm{d}\Phi_\nu^m(x)}{\mathrm{d}x} + 2ik\nu\Phi_\nu^m(x) = 0$$

Finally upon the last change of variable $u = 2ikx$:

$$\begin{aligned}
&\frac{\mathrm{d}\Phi_\nu^m}{\mathrm{d}x} = \frac{\mathrm{d}\Phi_\nu^m}{\mathrm{d}u}2ik \\
&\frac{\mathrm{d}^2\Phi_\nu^m}{\mathrm{d}x^2} = (2ik)^2\frac{\mathrm{d}^2\Phi_\nu^m}{\mathrm{d}u^2} = \frac{2iku}{x}\frac{\mathrm{d}^2\Phi_\nu^m}{\mathrm{d}u^2}
\end{aligned}$$

Then:

$$2iku\frac{\mathrm{d}^2\Phi_\nu^m}{\mathrm{d}u^2}+(m+1-u)2ik\frac{\mathrm{d}\Phi_\nu^m}{\mathrm{d}u}+2ik\nu\Phi_\nu^m=0$$
$$u\frac{\mathrm{d}^2\Phi_\nu^m}{\mathrm{d}u^2}+(m+1-u)\frac{\mathrm{d}\Phi_\nu^m}{\mathrm{d}u}+\nu\Phi_\nu^m=0 \tag{A.29}$$

The last equation is the Generalized Laguerre equation for $m\in\mathbb{R}$ or the associeted Laguerre equation for $m\in\mathbb{Z}$ which is our case. The solutions take the form:

$$S_n^m(x)=x^{\frac{m}{2}}e^{-ikx}L_n^m(2ikx) \tag{A.30}$$
$$V_n^m(x)=x^{\frac{m}{2}}e^{-ikx}U_n^m(2ikx) \tag{A.31}$$

By noting that $2ik$ is just a constant a more practical definition is:

$$S_n^m(u)=u^{\frac{m}{2}}e^{-\frac{u}{2}}L_n^m(u) \tag{A.32}$$
$$V_n^m(u)=u^{\frac{m}{2}}e^{-\frac{u}{2}}U_n^m(u) \tag{A.33}$$

Where $u=\pm 2ikx$ and x can be either $\xi=\sigma^2/2$ or $\eta=\tau^2/2$. The function $U_n^m(u)$ is the second solution of the Generalized Laguerre differential equation to be found and defined in section 4.

APPENDIX B – SERIES REARRANGEMENT

This section is devoted to rearrange series which involve components of the functions $\vec{M}_{n,m}(\vec{r})$ and $\vec{N}_{n,m}(\vec{r})$ in such way that the boundary conditions lead to simple relations of the coefficients involved. The following identities are used:

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)$$
$$=\frac{1}{2}\left[(y-x)S_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}C_{n-1}^m(x,y)\right]$$
$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$
$$=\frac{1}{2}\left[(y-x)V_n^m(x)S_n^m(y)+2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}D_{n-1}^m(x,y)\right]$$
$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)$$
$$=\frac{1}{2xy}\left[-m(y-x)S_n^m(x)S_n^m(y)+2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}C_n^{m-1}(x,y)\right]$$
$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)$$
$$=\frac{1}{2xy}\left[-m(y-x)V_n^m(x)S_n^m(y)+2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}D_n^{m-1}(x,y)\right]$$

With

$$C_n^m(x,y)=(xy)^{m/2}e^{-(x+y)/2}A_n^m(x,y)=(y-x)\sum_{p=0}^{n}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}S_p^m(x)S_p^m(y)$$
$$D_n^m(x,y)=(xy)^{m/2}e^{-(x+y)/2}B_n^m(x,y)=-(y-x)\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m+p+1)}V_p^m(x)S_p^m(y)$$

Then:

$$(xy)^{1/2}C_n^{m-1}(x,y)=(xy)^{m/2}e^{-(x+y)/2}A_n^{m-1}(x,y)$$
$$(xy)^{1/2}D_n^{m-1}(x,y)=(xy)^{m/2}e^{-(x+y)/2}B_n^{m-1}(x,y)$$

B.1 Rearranging M_φ for incident and internal fields

For a moment we are going to ignore the sum over n and m and the expansion coefficient. Also for simplicity we ignore the absolute value of m but recover it at the end of the demonstration. With this in mind recall M_φ is equal to:

$$(M_{n,m})_\varphi = \frac{-2\sigma\tau e^{im\varphi}}{\tau^2+\sigma^2}\left[\frac{\partial}{\partial x}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right) - \frac{\partial}{\partial y}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right)\right] \tag{B.1}$$

where $x = ik\sigma^2$ and $y = -ik\tau^2$.

$$\begin{aligned}
&\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)\\
&=\frac{1}{2xy}\left[-m(y-x)S_n^m(x)S_n^m(y)+2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}C_n^{m-1}(x,y)\right]\\
&=\frac{(y-x)}{2xy}\left[-mS_n^m(x)S_n^m(y)+2(xy)^{m/2}e^{-(x+y)/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\sum_{p=0}^{n}\frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}L_p^{m-1}(x)L_p^{m-1}(y)\right]
\end{aligned}$$

Since $x = ik\sigma^2$ and $y = -ik\tau^2$ then $(y-x)/xy = -ik(\tau^2+\sigma^2)/k^2\sigma^2\tau^2$. Which gives:

$$(M_{n,m})_\varphi = \frac{ie^{im\varphi}(xy)^{m/2}e^{-(x+y)/2}}{k\sigma\tau}\left[-mL_n^m(x)L_n^m(y)+2\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\sum_{p=0}^{n}\frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}L_p^{m-1}(x)L_p^{m-1}(y)\right]$$

Now consider the series:

$$\sum_{n=0}^{\infty}A_{n,m}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\sum_{p=0}^{n}\frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}L_p^{m-1}(x)L_p^{m-1}(y) \tag{B.2}$$

Expanding:

$$\begin{aligned}
&A_{0,m}\frac{\Gamma(m+1)}{\Gamma(1)}\frac{\Gamma(1)}{\Gamma(m-1+1)}L_0^{m-1}(x)L_0^{m-1}(y)\\
&+A_{1,m}\frac{\Gamma(1+m+1)}{\Gamma(1+1)}\left(\frac{\Gamma(1)}{\Gamma(m-1+1)}L_0^{m-1}(x)L_0^{m-1}(y)+\frac{\Gamma(1+1)}{\Gamma(1+m-1+1)}L_1^{m-1}(x)L_1^{m-1}(y)\right)\\
&+A_{2,m}\frac{\Gamma(2+m+1)}{\Gamma(2+1)}\left(\frac{\Gamma(1)}{\Gamma(m-1+1)}L_0^{m-1}(x)L_0^{m-1}(y)+\frac{\Gamma(1+1)}{\Gamma(1+m-1+1)}L_1^{m-1}(x)L_1^{m-1}(y)\right.\\
&\left.+\frac{\Gamma(2+1)}{\Gamma(2+m-1+1)}L_2^{m-1}(x)L_2^{m-1}(y)\right)\\
&+\ldots
\end{aligned}$$

And reorganizing:

$$\sum_{n=0}^{\infty}\left(\sum_{p=n}^{\infty}A_{p,m}\frac{\Gamma(p+m+1)}{\Gamma(p+1)}\right)\frac{\Gamma(n+1)}{\Gamma(m-1+n+1)}L_n^{m-1}(x)L_n^{m-1}(y) \tag{B.3}$$

Which can be expressed as:

$$\sum_{n=0}^{\infty}\tilde{A}_{n,m}\frac{\Gamma(n+1)}{\Gamma(m+n+1)}(n+m)L_n^{m-1}(x)L_n^{m-1}(y) \tag{B.4}$$

where

$$\tilde{A}_{n,m}=\sum_{p=n}^{\infty}A_{p,m}\frac{\Gamma(p+m+1)}{\Gamma(p+1)} \tag{B.5}$$

Horton defined:

$$B_{n,m}=\sum_{p=n}^{\infty}A_{p,m}\frac{\Gamma(p+m+1)}{\Gamma(p+1)}\frac{\Gamma(n+1)}{\Gamma(m+n+1)}=\sum_{p=n}^{\infty}A_{p,m}\frac{(p+m)!n!}{(n+m)!p!} \tag{B.6}$$

instead, treating the same problem(HORTON; KARAL, 1951). However the inclusion of the factor $\frac{\Gamma(n+1)}{\Gamma(m+n+1)}$ in the new coefficient leads Horton to define similar coefficients which are not the same type as (B.6).

The rearranging is not over yet, the expansion needs to be expressed in terms of $L_n^m(y)$ functions to facilitate the application of boundary conditions. This can be achieved by using the identity $L_n^m(y)=L_n^{m+1}(y)-L_{n-1}^{m+1}(y)$. Then the series take the form: This is still not good enough. The expansion needs to be expressed in terms of $L_n^m(y)$ functions in order to apply boundary conditions. This can be achieved by using the identity $L_n^m(y)=L_n^{m+1}(y)-L_{n-1}^{m+1}(y)$. Then the series take the form:

$$\sum_{n=0}^{\infty}\tilde{A}_{n,m}\frac{\Gamma(n+1)}{\Gamma(m+n+1)}(n+m)L_n^{m-1}(x)[L_n^m(y)-L_{n-1}^m(y)]$$

We need to rearrange the term:

$$-\sum_{n=0}^{\infty}\tilde{A}_{n,m}\frac{\Gamma(n+1)}{\Gamma(m+n+1)}(n+m)L_n^{m-1}(x)L_{n-1}^m(y)$$

since $L^m_{-1}(y) = 0$:

$$-\sum_{n=1}^{\infty} \tilde{A}_{n,m} \frac{\Gamma(n+1)}{\Gamma(m+n+1)} (n+m) L_n^{m-1}(x) L_{n-1}^m(y)$$

Shifting indices:

$$-\sum_{n=0}^{\infty} \tilde{A}_{n+1,m} \frac{\Gamma(n+2)}{\Gamma(m+n+2)} (n+m+1) L_{n+1}^{m-1}(x) L_n^m(y)$$
$$= -\sum_{n=0}^{\infty} \tilde{A}_{n+1,m} \frac{\Gamma(n+1)}{\Gamma(m+n+1)} (n+1) L_{n+1}^{m-1}(x) L_n^m(y)$$

So the series can be written as:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(m+n+1)} L_n^m(y) [\tilde{A}_{n,m}(n+m) L_n^{m-1}(x) - \tilde{A}_{n+1,m}(n+1) L_{n+1}^{m-1}(x)]$$

By definition:

$$\tilde{A}_{n,m} = A_{n,m} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} + \tilde{A}_{n+1,m} \tag{B.7}$$

then:

$$\sum_{n=0}^{\infty} A_{n,m}(n+m) L_n^{m-1}(x) L_n^m(y) + \frac{\Gamma(n+1)}{\Gamma(m+n+1)} \tilde{A}_{n+1,m} L_n^m(y) [(n+m) L_n^{m-1}(x) - (n+1) L_{n+1}^{m-1}(x)]$$

So the M_φ component of the incident and internal field takes the form:

$$M_\varphi = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{i e^{im\varphi} (xy)^{|m|/2} e^{-(x+y)/2}}{k\sigma\tau} \Bigg[-|m| A_{n,m} L_n^{|m|}(x) L_n^{|m|}(y) + 2A_{n,m}(n+|m|) L_n^{|m|-1}(x) L_n^{|m|}(y)$$
$$+ 2\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)} \tilde{A}_{n+1,m} L_n^{|m|}(y) [(n+|m|) L_n^{|m|-1}(x) - (n+1) L_{n+1}^{|m|-1}(x)] \Bigg] \tag{B.8}$$

We make use of $L_n^{m-1}(x) = L_n^m(x) - L_{n-1}^m(x)$ to write everything in terms of Pinney functions:

$$M_\varphi = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{ie^{im\varphi}}{k\sigma\tau}\Bigg[-|m|A_{n,m}S_n^{|m|}(x)S_n^{|m|}(y)+2A_{n,m}(n+|m|)[S_n^{|m|}(x)-S_{n-1}^{|m|}(x)]S_n^{|m|}(y)$$
$$+2\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}\tilde{A}_{n+1,m}S_n^{|m|}(y)[(n+|m|)(S_n^{|m|}(x)-S_{n-1}^{|m|}(x))$$
$$-(n+1)(S_{n+1}^{|m|}(x)-S_n^{|m|}(x))]\Bigg]$$

Note that we can factor all the dependence on y:

$$M_\varphi = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{ie^{im\varphi}}{k\sigma\tau}\Bigg[A_{n,m}\Big(-|m|S_n^{|m|}(x)+2(n+|m|)[S_n^{|m|}(x)-S_{n-1}^{|m|}(x)]\Big)$$
$$+2\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}\tilde{A}_{n+1,m}[(n+|m|)(S_n^{|m|}(x)-S_{n-1}^{|m|}(x))$$
$$-(n+1)(S_{n+1}^{|m|}(x)-S_n^{|m|}(x))]\Bigg]S_n^{|m|}(y) \tag{B.9}$$

The previous formula can be used if the refractive index of the medium and the paraboloid are the same. When they are not there is no advantage of using it. For a more general procedure:

$$M_\varphi = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{ie^{im\varphi}}{k\sigma\tau}\Bigg[-A_{n,m}|m|S_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{A}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}(n+|m|)$$
$$\times[S_n^{|m|}(x)-S_{n-1}^{|m|}(x)][S_n^{|m|}(y)-S_{n-1}^{|m|}(y)]\Bigg] \tag{B.10}$$

is enough. where $\tilde{A}_{n,m}=\sum_{p=n}^{\infty}A_{p,m}\frac{\Gamma(p+|m|+1)}{\Gamma(p+1)}$. Equation (B.10) is just eq (B.9) without rearranging the series a second time on the term $S_{n-1}^{|m|}(y)$. Also with eq (B.10) we can immediately determine N_τ and N_σ:

$$N_\tau = \frac{-1}{k^2h_\sigma h_\varphi}\Bigg[\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\partial}{\partial\sigma}\Bigg(-A_{n,m}|m|S_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{A}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}(n+|m|)$$
$$\times[S_n^{|m|}(x)-S_{n-1}^{|m|}(x)][S_n^{|m|}(y)-S_{n-1}^{|m|}(y)]\Bigg)e^{im\varphi}-A_{n,m}\frac{m^2}{\sigma}S_n^{|m|}(x)S_n^{|m|}(y)e^{im\varphi}\Bigg] \tag{B.11}$$

$$N_\sigma = \frac{1}{k^2h_\tau h_\varphi}\Bigg[\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\partial}{\partial\tau}\Bigg(-A_{n,m}|m|S_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{A}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}(n+|m|)$$
$$\times[S_n^{|m|}(x)-S_{n-1}^{|m|}(x)][S_n^{|m|}(y)-S_{n-1}^{|m|}(y)]\Bigg)e^{im\varphi}-A_{n,m}\frac{m^2}{\tau}S_n^{|m|}(x)S_n^{|m|}(y)e^{im\varphi}\Bigg] \tag{B.12}$$

N_σ is not relevant when applying boundary conditions and N_τ can be left as it is or replace the term in [] with the term in [] in eq (B.9) to factor $S_n^{|m|}(y)$. Nevertheless with eq (B.9) we managed to separate variables at the cost of a second coefficient involving sums of the first coefficient.

Although not obvious, M_φ is finite at the origin for $m = 0$. the constant term of $L_n(x)$ is always 1 so $L_n(x) - L_{n-1}(x)$ has no constant term for $n > 0$ and for $n = 0$ the term $(n + |m|) = 0$ makes the contribution null. In short, the denominator $1/k\sigma\tau$ always cancel with each term of the series so the series is finite at the origin. Something similar should happen for N_τ and N_σ but the manipulation is rather large since both h_σ (or h_τ) and h_φ must be dealt with. Since those factors do not interfere with the system of equations resulting when applying boundary conditions no effort is made to take them out.

B.2 Rearranging M_φ for the scattered field

For a moment we are going to ignore the sum over n and m and the expansion coefficient. Also for simplicity we ignore the absolute value of m but recover it at the end of the demonstration. With this in mind recall M_φ is equal to:

$$(M_{n,m})_\varphi = \frac{-2\sigma\tau e^{im\varphi}}{\tau^2+\sigma^2}\left[\frac{\partial}{\partial x}\left(V_n^{|m|}(x)S_n^{|m|}(y)\right) - \frac{\partial}{\partial y}\left(V_n^{|m|}(x)S_n^{|m|}(y)\right)\right] \tag{B.13}$$

Recall:

$$\begin{aligned}&\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)V_n^m(x)S_n^m(y)\\ &= \frac{1}{2xy}\left[-m(y-x)V_n^m(x)S_n^m(y) + 2(xy)^{1/2}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}D_n^{m-1}(x,y)\right]\end{aligned}$$

where:

$$\begin{aligned}(xy)^{1/2}D_n^{m-1}(x,y) &= (xy)^{m/2}e^{-(x+y)/2}B_n^{m-1}(x,y)\\ &= -(y-x)(xy)^{m/2}e^{-(x+y)/2}\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}U_p^{m-1}(x)L_p^{m-1}(y)\end{aligned}$$

then:

$$(M_{n,m})_\varphi = \frac{ie^{im\varphi}(xy)^{m/2}e^{-(x+y)/2}}{k\sigma\tau}$$
$$\times\left[-mU_n^m(x)L_n^m(y) - 2\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}U_p^{m-1}(x)L_p^{m-1}(y)\right]$$

Now consider the series:

$$-\sum_{n=0}^{\infty}B_{n,m}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\sum_{p=n+1}^{\infty}\frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}U_p^{m-1}(x)L_p^{m-1}(y) \tag{B.14}$$

Expanding:

$$-B_{0,m}\frac{\Gamma(0+m+1)}{\Gamma(0+1)}\left[\frac{\Gamma(1+1)}{\Gamma(m-1+1+1)}U_1^{m-1}(x)L_1^{m-1}(y) + \frac{\Gamma(2+1)}{\Gamma(m-1+2+1)}U_2^{m-1}(x)L_2^{m-1}(y)\right.$$
$$\left. + .. + \frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}U_p^{m-1}(x)L_p^{m-1}(y)\right]$$
$$-B_{1,m}\frac{\Gamma(1+m+1)}{\Gamma(1+1)}\left[\frac{\Gamma(2+1)}{\Gamma(m-1+2+1)}U_2^{m-1}(x)L_2^{m-1}(y) + \frac{\Gamma(3+1)}{\Gamma(m-1+3+1)}U_3^{m-1}(x)L_3^{m-1}(y)\right.$$
$$\left. + .. + \frac{\Gamma(p+1)}{\Gamma(m-1+p+1)}U_p^{m-1}(x)L_p^{m-1}(y)\right] - \ldots$$

Rearranging the sum:

$$-\sum_{n=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(m-1+n+1)}U_n^{m-1}(x)L_n^{m-1}(y)\left(\sum_{p=0}^{n-1}B_{p,m}\frac{\Gamma(p+m+1)}{\Gamma(p+1)}\right)$$

Defining:

$$\tilde{B}_{n,m} = -\sum_{p=0}^{n-1}B_{p,m}\frac{\Gamma(p+m+1)}{\Gamma(p+1)}$$
$$\tilde{B}_{0,m} = 0$$

The series can be expressed as:

$$\sum_{n=0}^{\infty}\tilde{B}_{n,m}\frac{\Gamma(n+1)}{\Gamma(m+n+1)}(m+n)U_n^{m-1}(x)L_n^{m-1}(y)$$

From here onwards the procedure is exactly the same as in the case of incident and internal field:

$$M_\varphi = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{ie^{im\varphi}}{k\sigma\tau}\Bigg[-B_{n,m}|m|V_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{B}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}(n+|m|)$$
$$\times[V_n^{|m|}(x)-V_{n-1}^{|m|}(x)][S_n^{|m|}(y)-S_{n-1}^{|m|}(y)]\Bigg] \tag{B.15}$$

Where $\tilde{B}_{n,m} = -\sum_{p=0}^{n-1} B_{p,m}\frac{\Gamma(p+|m|+1)}{\Gamma(p+1)}$ with $\tilde{B}_{0,m} = 0$. Also with eq (B.15) we can immediately determine N_τ and N_σ:

$$N_\tau = \frac{-1}{k^2 h_\sigma h_\varphi}\Bigg[\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\partial}{\partial\sigma}\Bigg(-B_{n,m}|m|V_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{B}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}(n+|m|)$$
$$\times[V_n^{|m|}(x)-V_{n-1}^{|m|}(x)][S_n^{|m|}(y)-S_{n-1}^{|m|}(y)]\Bigg)e^{im\varphi}-B_{n,m}\frac{m^2}{\sigma}V_n^{|m|}(x)S_n^{|m|}(y)e^{im\varphi}\Bigg] \tag{B.16}$$

$$N_\sigma = \frac{1}{k^2 h_\tau h_\varphi}\Bigg[\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\partial}{\partial\tau}\Bigg(-B_{n,m}|m|V_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{B}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}(n+|m|)$$
$$\times[V_n^{|m|}(x)-V_{n-1}^{|m|}(x)][S_n^{|m|}(y)-S_{n-1}^{|m|}(y)]\Bigg)e^{im\varphi}-B_{n,m}\frac{m^2}{\tau}V_n^{|m|}(x)S_n^{|m|}(y)e^{im\varphi}\Bigg] \tag{B.17}$$

We can still rearrange (B.15) to factor the term $S_{n-1}^m(y)$. Although the procedure is a little different the result is the same as in (B.9) replacing $\tilde{A}_{n+1,m}\to\tilde{B}_{n,m}$ and $A_{n,m}\to B_{n,m}$:

$$M_\varphi = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{ie^{im\varphi}}{k\sigma\tau}\Bigg[B_{n,m}\Big(-|m|V_n^{|m|}(x)+2(n+|m|)[V_n^{|m|}(x)-V_{n-1}^{|m|}(x)]\Big)$$
$$+2\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}\tilde{B}_{n,m}[(n+|m|)(V_n^{|m|}(x)-V_{n-1}^{|m|}(x))$$
$$-(n+1)(V_{n+1}^{|m|}(x)-V_n^{|m|}(x))]\Bigg]S_n^{|m|}(y) \tag{B.18}$$

Again the only advantage of using eq (B.18) is applying boundary conditions when the refractive index of the paraboloid is the same as the medium. Also with eq (B.15) is easier to evaluate $\tau=0$ when $m=0$ since $(S_n^0(y)-S_{n-1}^0(y))/\tau$ has finite values when $\tau=0$ for $n>0$ and the expansion term when $n=0, m=0$ is null.

B.3 Rearranging N_φ for incident and internal fields

By definition:

$$N_\varphi = \frac{1}{k^2 h_\tau h_\sigma}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\sigma M_\sigma\right) - \frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\tau M_\tau\right)\right]$$

with:

$$\begin{aligned}\frac{-k}{i}h_\sigma(\vec{M}_{n,m})_\sigma &= \frac{im}{\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}\\ \frac{-k}{i}h_\tau(\vec{M}_{n,m})_\tau &= \frac{im}{\tau}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}\end{aligned}$$

which gives:

$$\begin{aligned}(N_{n,m})_\varphi &= \frac{2}{k^2 h_\tau h_\sigma}\frac{ime^{im\varphi}}{\sigma\tau}\\ &\times\left[y\frac{\partial}{\partial y}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right) - x\frac{\partial}{\partial x}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right)\right]\end{aligned}$$

since $x = ik\sigma^2$ and $y = -ik\tau^2$ implies:

$$\begin{aligned}\frac{1}{\sigma}\frac{\partial}{\partial\tau} &= \frac{\tau}{\tau\sigma}\frac{\partial y}{\partial\tau}\frac{\partial}{\partial y} = \frac{2y}{\sigma\tau}\frac{\partial}{\partial y}\\ \frac{1}{\tau}\frac{\partial}{\partial\sigma} &= \frac{\sigma}{\sigma\tau}\frac{\partial x}{\partial\sigma}\frac{\partial}{\partial x} = \frac{2x}{\sigma\tau}\frac{\partial}{\partial x}\end{aligned}$$

Now recall:

$$\begin{aligned}&\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)S_n^m(x)S_n^m(y)\\ &= \frac{1}{2}\left[(y-x)S_n^m(x)S_n^m(y) + 2\frac{\Gamma(m+n+1)}{\Gamma(n+1)}C_{n-1}^m(x,y)\right]\end{aligned}$$

Therefore:

$$(N_{n,m})_\varphi = \frac{-1}{k^2(\tau^2+\sigma^2)} \frac{ime^{im\varphi}}{\sigma\tau}$$
$$\times \left[(y-x)S_n^{|m|}(x)S_n^{|m|}(y) + 2\frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} C_{n-1}^{|m|}(x,y)\right]$$
$$(N_{n,m})_\varphi = \frac{-(y-x)}{k^2(\tau^2+\sigma^2)} \frac{ime^{im\varphi}}{\sigma\tau}$$
$$\times \left[S_n^{|m|}(x)S_n^{|m|}(y) + 2\frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} \sum_{p=0}^{n-1} \frac{\Gamma(p+1)}{\Gamma(|m|+p+1)} S_p^{|m|}(x)S_p^{|m|}(y)\right]$$
$$(N_{n,m})_\varphi = -\frac{me^{im\varphi}}{k\sigma\tau}$$
$$\times \left[S_n^{|m|}(x)S_n^{|m|}(y) + 2\frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} \sum_{p=0}^{n-1} \frac{\Gamma(p+1)}{\Gamma(|m|+p+1)} S_p^{|m|}(x)S_p^{|m|}(y)\right]$$

We have used $(y-x) = -ik(\tau^2+\sigma^2)$ and $\sum_{p=0}^{n-1}() = 0$ for $n = 0$. Now consider the series:

$$\sum_{n=0}^{\infty} A_{n,m} \frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} \sum_{p=0}^{n-1} \frac{\Gamma(p+1)}{\Gamma(|m|+p+1)} S_p^{|m|}(x)S_p^{|m|}(y) \tag{B.19}$$

Expanding:

$$A_{0,m}\frac{\Gamma(|m|+0+1)}{\Gamma(0+1)}0$$
$$+A_{1,m}\frac{\Gamma(|m|+1+1)}{\Gamma(1+1)}\left(\frac{\Gamma(0+1)}{\Gamma(m+0+1)}S_0^{|m|}(x)S_0^{|m|}(y)\right)$$
$$+A_{2,m}\frac{\Gamma(|m|+2+1)}{\Gamma(2+1)}\left(\frac{\Gamma(0+1)}{\Gamma(m+0+1)}S_0^{|m|}(x)S_0^{|m|}(y) + \frac{\Gamma(1+1)}{\Gamma(m+1+1)}S_1^{|m|}(x)S_1^{|m|}(y)\right)$$
$$+A_{3,m}\frac{\Gamma(|m|+3+1)}{\Gamma(3+1)}\left(\frac{\Gamma(0+1)}{\Gamma(m+0+1)}S_0^{|m|}(x)S_0^{|m|}(y) + \frac{\Gamma(1+1)}{\Gamma(m+1+1)}S_1^{|m|}(x)S_1^{|m|}(y)\right.$$
$$\left.+\frac{\Gamma(2+1)}{\Gamma(m+2+1)}S_2^{|m|}(x)S_2^{|m|}(y)\right)$$
$$+\dots$$

And reorganizing:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(|m|+n+1)} S_n^{|m|}(x)S_n^{|m|}(y) \sum_{p=n+1}^{\infty} A_{p,m}\frac{\Gamma(|m|+p+1)}{\Gamma(p+1)}$$
$$= \sum_{n=0}^{\infty} \tilde{A}_{n+1,m} \frac{\Gamma(n+1)}{\Gamma(|m|+n+1)} S_n^{|m|}(x)S_n^{|m|}(y)$$

Finally:

$$N_\varphi = \frac{-1}{k\sigma\tau} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} m S_n^{|m|}(x) S_n^{|m|}(y) e^{im\varphi} \times \left[A_{n,m} + 2\tilde{A}_{n+1,m} \frac{\Gamma(n+1)}{\Gamma(|m|+n+1)} \right] \tag{B.20}$$

Unlike the previous components we can easily check this one. Since the electric field has no component along $\hat{\mathbf{e}}_\varphi$ we see that:

$$A_{n,m} + 2\tilde{A}_{n+1,m} \frac{\Gamma(n+1)}{\Gamma(|m|+n+1)} = 0$$

This condition is the same as the condition for the expansion of a plane wave polarized along the $\hat{\mathbf{z}}$ to be divergenless. As a consequence $(\vec{N}_{n,m}(\vec{r}))_\varphi = 0$ for that electric field as it should be.

B.4 Rearranging N_φ for the scattered field

By a similar procedure of the previous section we have:

$$(N_{n,m})_\varphi = \frac{-1}{k^2(\tau^2+\sigma^2)} \frac{ime^{im\varphi}}{\sigma\tau} \times \left[(y-x) V_n^{|m|}(x) S_n^{|m|}(y) + 2 \frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} D_{n-1}^{|m|}(x,y) \right]$$

$$(N_{n,m})_\varphi = \frac{-(y-x)}{k^2(\tau^2+\sigma^2)} \frac{ime^{im\varphi}}{\sigma\tau} \times \left[V_n^{|m|}(x) S_n^{|m|}(y) - 2 \frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} \sum_{p=n}^{\infty} \frac{\Gamma(p+1)}{\Gamma(m+p+1)} V_p^m(x) S_p^m(y) \right]$$

$$(N_{n,m})_\varphi = \frac{-1}{k\sigma\tau} m e^{im\varphi} \times \left[V_n^{|m|}(x) S_n^{|m|}(y) - 2 \frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} \sum_{p=n}^{\infty} \frac{\Gamma(p+1)}{\Gamma(m+p+1)} V_p^m(x) S_p^m(y) \right]$$

Again consider the series:

$$-\sum_{n=0}^{\infty} B_{n,m} \frac{\Gamma(|m|+n+1)}{\Gamma(n+1)} \sum_{p=n}^{\infty} \frac{\Gamma(p+1)}{\Gamma(|m|+p+1)} V_p^{|m|}(x) S_p^{|m|}(y) \tag{B.21}$$

Rearranging:

$$\sum_{n=0}^{\infty}\left(-\sum_{p=0}^{n}B_{p,m}\frac{\Gamma(|m|+p+1)}{\Gamma(n+1)}\right)\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}V_n^{|m|}(x)S_n^{|m|}(y)$$
$$=\sum_{n=0}^{\infty}\tilde{B}_{n+1,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}V_n^{|m|}(x)S_n^{|m|}(y)$$
$$=\sum_{n=0}^{\infty}-B_{n,m}V_n^{|m|}(x)S_n^{|m|}(y)+\tilde{B}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}V_n^{|m|}(x)S_n^{|m|}(y)$$

Plugging back:

$$(\vec{N}_{n,m}(\vec{r}))_{\varphi}=\frac{-1}{k\sigma\tau}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}me^{im\varphi}$$
$$\times\left[-B_{n,m}V_n^{|m|}(x)S_n^{|m|}(y)+2\tilde{B}_{n,m}\frac{\Gamma(n+1)}{\Gamma(|m|+n+1)}V_n^{|m|}(x)S_n^{|m|}(y)\right] \tag{B.22}$$

APPENDIX C – VECTOR COMPONENTS WITHOUT REARRANGEMENT

C.1 Special case $m = 0$

By definition:

$$\frac{-k}{i}\vec{M}_{n,m}(\vec{r}) = \nabla \times S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]$$

$$(\vec{M}_{n,m}(\vec{r}))_\sigma = \frac{m}{k\sigma h_\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$
$$(\vec{M}_{n,m}(\vec{r}))_\tau = \frac{m}{k\tau h_\tau}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$
$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-e^{im\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)}\Bigg((y-x)(|m|+2n)S_n^{|m|}(x)S_n^{|m|}(y) - 2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y) + 2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)\Bigg)$$

if $m = 0$ the the first vector field takes the form:

$$(\vec{M}_{n,0}(\vec{r}))_\sigma = 0 \tag{C.1}$$
$$(\vec{M}_{n,0}(\vec{r}))_\tau = 0 \tag{C.2}$$
$$(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2ni}{k\sigma\tau(x-y)}\Bigg((y-x)S_n^0(x)S_n^0(y) - yS_{n-1}^0(x)S_n^0(y) + xS_n^0(x)S_{n-1}^0(y)\Bigg) \tag{C.3}$$

Recall $h_\varphi = \sigma\tau$. The only non-zero component can be written as:

$$h_\varphi(\vec{M}_{n,0}(\vec{r}))_\varphi = \frac{-2ni}{k(x-y)}\Bigg((y-x)S_n^0(x)S_n^0(y) - yS_{n-1}^0(x)S_n^0(y) + xS_n^0(x)S_{n-1}^0(y)\Bigg) \tag{C.4}$$

From the second vector field we have:

$$(\vec{N}_{n,m}(\vec{r}))_\tau = \frac{1}{k^2 h_\sigma h_\varphi}\left[\frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\varphi M_\varphi\right) - \frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\sigma M_\sigma\right)\right]$$
$$(\vec{N}_{n,m}(\vec{r}))_\sigma = \frac{-1}{k^2 h_\tau h_\varphi}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\varphi M_\varphi\right) - \frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\tau M_\tau\right)\right]$$
$$(\vec{N}_{n,m}(\vec{r}))_\varphi = \frac{1}{k^2 h_\tau h_\sigma}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\sigma M_\sigma\right) - \frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\tau M_\tau\right)\right]$$

if $m = 0$ the the second vector field simplifies to:

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{i}{kh_\sigma h_\varphi}\left[\frac{\partial}{\partial\sigma}\big(h_\varphi M_\varphi\big)\right] = \frac{i(2ik\sigma)}{kh_\sigma h_\varphi}\left[\frac{\partial}{\partial x}\big(h_\varphi M_\varphi\big)\right] \tag{C.5}$$
$$(\vec{N}_{n,0}(\vec{r}))_\sigma = \frac{-i}{kh_\tau h_\varphi}\left[\frac{\partial}{\partial\tau}\big(h_\varphi M_\varphi\big)\right] = \frac{i(2ik\tau)}{kh_\tau h_\varphi}\left[\frac{\partial}{\partial y}\big(h_\varphi M_\varphi\big)\right] \tag{C.6}$$
$$(\vec{N}_{n,0}(\vec{r}))_\varphi = 0 \tag{C.7}$$

The usual change of variables $x = ik\sigma^2$ and $y = -ik\tau^2$ was made. The derivative with respect to x gives:

$$\begin{aligned}\frac{\partial}{\partial x}\Big(h_\varphi(\vec{M}_{n,0}(\vec{r}))_\varphi\Big) = {}& \frac{2ni}{k(x-y)^2}\Bigg((y-x)S_n^0(x)S_n^0(y) - yS_{n-1}^0(x)S_n^0(y) + xS_n^0(x)S_{n-1}^0(y)\Bigg)\\ & - \frac{2ni}{k(x-y)}\Bigg((y-x)\frac{\mathrm{d}}{\mathrm{d}x}\big(S_n^0(x)\big)S_n^0(y) - S_n^0(x)S_n^0(y)\\ & - y\frac{\mathrm{d}}{\mathrm{d}x}\big(S_{n-1}^0(x)\big)S_n^0(y) + S_n^0(x)S_{n-1}^0(y) + x\frac{\mathrm{d}}{\mathrm{d}x}\big(S_n^0(x)\big)S_{n-1}^0(y)\Bigg)\end{aligned}$$

Note that the first term in the first line cancels out with the second on the second line

$$\begin{aligned}\frac{\partial}{\partial x}\Big(h_\varphi(\vec{M}_{n,0}(\vec{r}))_\varphi\Big) = {}& \frac{2ni}{k(x-y)^2}\Bigg(-yS_{n-1}^0(x)S_n^0(y) + xS_n^0(x)S_{n-1}^0(y)\Bigg)\\ & - \frac{2ni}{k(x-y)}\Bigg((y-x)\frac{\mathrm{d}}{\mathrm{d}x}\big(S_n^0(x)\big)S_n^0(y)\\ & - y\frac{\mathrm{d}}{\mathrm{d}x}\big(S_{n-1}^0(x)\big)S_n^0(y) + S_n^0(x)S_{n-1}^0(y) + x\frac{\mathrm{d}}{\mathrm{d}x}\big(S_n^0(x)\big)S_{n-1}^0(y)\Bigg)\end{aligned}$$

Multiplying and dividing the second line by $(x-y)$:

$$\frac{\partial}{\partial x}\Big(h_\varphi(\vec{M}_{n,0}(\vec{r}))_\varphi\Big) = \frac{2ni}{k(x-y)^2}\Bigg(-yS^0_{n-1}(x)S^0_n(y) + xS^0_n(x)S^0_{n-1}(y) + (x-y)^2\frac{\mathrm{d}}{\mathrm{d}x}\big(S^0_n(x)\big)S^0_n(y)$$
$$+ y(x-y)\frac{\mathrm{d}}{\mathrm{d}x}\big(S^0_{n-1}(x)\big)S^0_n(y) - (x-y)S^0_n(x)S^0_{n-1}(y) - x(x-y)\frac{\mathrm{d}}{\mathrm{d}x}\big(S^0_n(x)\big)S^0_{n-1}(y)\Bigg)$$

Now we focus on what is inside the parenthesis

$$\Bigg(-yS^0_{n-1}(x)S^0_n(y) + xS^0_{n-1}(y)S^0_n(x) + (x-y)^2\partial_x\left(S^0_n(x)\right)S^0_n(y)$$
$$+ (x-y)\,y\partial_x\left(S^0_{n-1}(x)\right)S^0_n(y) - (x-y)\,xS^0_{n-1}(y)\partial_x\left(S^0_n(x)\right) - (x-y)\,S^0_{n-1}(y)S^0_n(x)\Bigg)$$

$$\Bigg((x-y)^2\partial_x\left(S^0_n(x)\right)S^0_n(y) + (x-y)\,y\partial_x\left(S^0_{n-1}(x)\right)S^0_n(y)$$
$$- (x-y)\,xS^0_{n-1}(y)\partial_x\left(S^0_n(x)\right) + y\left(S^0_{n-1}(y)S^0_n(x) - S^0_{n-1}(x)S^0_n(y)\right)\Bigg)$$

Recall

$$\frac{dS^0_n(z)}{dz} = \frac{1}{2}\left[\left(\frac{2n}{z}-1\right)S^0_n(z) - \frac{2n}{z}S^0_{n-1}(z)\right]$$

$$\Bigg((x-y)^2\frac{1}{2}\left[\left(\frac{2n}{x}-1\right)S^0_n(x) - \frac{2n}{x}S^0_{n-1}(x)\right]S^0_n(y)$$
$$+ (x-y)\,y\frac{1}{2}\left[\left(\frac{2(n-1)}{x}-1\right)S^0_{n-1}(x) - \frac{2(n-1)}{x}S^0_{n-2}(x)\right]S^0_n(y)$$
$$- (x-y)\,xS^0_{n-1}(y)\frac{1}{2}\left[\left(\frac{2n}{x}-1\right)S^0_n(x) - \frac{2n}{x}S^0_{n-1}(x)\right] + y\left(S^0_{n-1}(y)S^0_n(x) - S^0_{n-1}(x)S^0_n(y)\right)\Bigg)$$

To get rid of the ½ multiply and divide by 2

$$\frac{1}{2}\Bigg((x-y)^2\left[\left(\frac{2n}{x}-1\right)S^0_n(x) - \frac{2n}{x}S^0_{n-1}(x)\right]S^0_n(y)$$
$$+ (x-y)\,y\left[\left(\frac{2(n-1)}{x}-1\right)S^0_{n-1}(x) - \frac{2(n-1)}{x}S^0_{n-2}(x)\right]S^0_n(y)$$
$$- (x-y)\,S^0_{n-1}(y)\left[(2n-x)\,S^0_n(x) - 2nS^0_{n-1}(x)\right] + 2y\left(S^0_{n-1}(y)S^0_n(x) - S^0_{n-1}(x)S^0_n(y)\right)\Bigg)$$

so

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{-2ni}{kh_\sigma(x-y)^2\tau}\Bigg((x-y)^2\left[\left(\frac{2n}{x}-1\right)S_n^0(x)-\frac{2n}{x}S_{n-1}^0(x)\right]S_n^0(y)$$
$$+(x-y)\,y\left[\left(\frac{2(n-1)}{x}-1\right)S_{n-1}^0(x)-\frac{2(n-1)}{x}S_{n-2}^0(x)\right]S_n^0(y)$$
$$-(x-y)\,S_{n-1}^0(y)\left[(2n-x)\,S_n^0(x)-2nS_{n-1}^0(x)\right]+2y\left(S_{n-1}^0(y)S_n^0(x)-S_{n-1}^0(x)S_n^0(y)\right)\Bigg)$$

Simplifying:

$$(\vec{N}_{n,0}(\vec{r}))_\tau = \frac{-2ni}{k\tau h_\sigma(x-y)^2}\Bigg((x-y)^2\left[\frac{2n}{x}\left(S_n^0(x)-S_{n-1}^0(x)\right)-S_n^0(x)\right]S_n^0(y)$$
$$+(x-y)\,y\left[\frac{2(n-1)}{x}\left(S_{n-1}^0(x)-S_{n-2}^0(x)\right)-S_{n-1}^0(x)\right]S_n^0(y)$$
$$-(x-y)\,S_{n-1}^0(y)\left[(2n-x)\,S_n^0(x)-2nS_{n-1}^0(x)\right]+2y\left(S_{n-1}^0(y)S_n^0(x)-S_{n-1}^0(x)S_n^0(y)\right)\Bigg) \quad \text{(C.8)}$$

A similar calculation yields:

$$(\vec{N}_{n,0}(\vec{r}))_\sigma = \frac{2ni}{k\sigma h_\sigma(x-y)^2}\Bigg(2x\left[S_{n-1}^0(y)S_n^0(x)-S_{n-1}^0(x)S_n^0(y)\right]$$
$$-(x-y)\,S_{n-1}^0(x)\left[2n\left(S_n^0(y)-S_{n-1}^0(y)\right)-yS_n^0(y)\right]-(x-y)^2S_n^0(x)\left[\frac{2n}{y}\left(S_n^0(y)-S_{n-1}^0(y)\right)-S_n^0(y)\right]$$
$$+(x-y)\,x\left[\frac{2(n-1)}{y}\left(S_{n-1}^0(y)-S_{n-2}^0(y)\right)-S_{n-1}^0(y)\right]S_n^0(x)\Bigg) \quad \text{(C.9)}$$

C.1.1 Evaluation at y=0

We require to evaluate terms like
$\frac{2n}{y}\left(S_n^0(y)-S_{n-1}^0(y)\right)$ and $\frac{2(n-1)}{y}\left(S_{n-1}^0(y)-S_{n-2}^0(y)\right)$ at $y=0$
Which is equivalent to

$$\lim_{y\to 0}\frac{\left(S_n^0(y)-S_{n-1}^0(y)\right)}{y} = \lim_{y\to 0}\frac{e^{-y/2}\left(L_n(y)-L_{n-1}(y)\right)}{y} = \lim_{y\to 0}\frac{L_n(y)-L_{n-1}(y)}{y} = \frac{1-1}{0} = \frac{0}{0}$$

As long as $n \geq 1$.

So by L'Hopital rule

$$\lim_{y\to 0}\frac{L_n(y)-L_{n-1}(y)}{y}=-\lim_{y\to 0}\frac{L^1_{n-1}(y)-L^1_{n-2}(y)}{1}$$

Here we have used

$$\frac{d^k L^m_n(x)}{dx^k}=(-1)^k L^{m+k}_{n-k}(x)$$

Besides from the closed form of the polynomials

$$L^m_n(x)=\sum_{k=0}^{n}\frac{(n+m)!}{(n-k)!(m+k)!}\frac{(-x)^k}{k!}$$

We have

$$L^m_n(0)=\frac{(n+m)!}{(n)!(m)!}$$

Therefore

$$L^1_{n-1}(0)=\frac{(n)!}{(n-1)!(1)!}=n, \qquad L^1_{n-2}(0)=\frac{(n-1)!}{(n-2)!(1)!}=n-1$$

The limit becomes

$$\lim_{y\to 0}\frac{L_n(y)-L_{n-1}(y)}{y}=-\lim_{y\to 0}\frac{L^1_{n-1}(y)-L^1_{n-2}(y)}{1}=-(n-(n-1))=-1, \qquad n\geq 1$$

By analogy

$$\lim_{y\to 0}\frac{L_{n-1}(y)-L_{n-2}(y)}{y}=-\lim_{y\to 0}\frac{L^1_{n-2}(y)-L^1_{n-3}(y)}{1}=-\left[n-1-(n-2)\right]=-1, \qquad n\geq 2$$

then

$$\lim_{y\to 0}\frac{\left(S^0_n(y)-S^0_{n-1}(y)\right)}{y}=-1, \qquad n\geq 1 \tag{C.10}$$

and

$$\lim_{y\to 0}\frac{\left(S^0_{n-1}(y)-S^0_{n-2}(y)\right)}{y}=-1, \qquad n\geq 2 \tag{C.11}$$

Now recall

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{2ni}{k\sigma h_\sigma(x-y)^2}\Bigg(2x\left[S^0_{n-1}(y)S^0_n(x)-S^0_{n-1}(x)S^0_n(y)\right]$$
$$-(x-y)\,S^0_{n-1}(x)\left[2n\left(S^0_n(y)-S^0_{n-1}(y)\right)-yS^0_n(y)\right]-(x-y)^2S^0_n(x)\left[\frac{2n}{y}\left(S^0_n(y)-S^0_{n-1}(y)\right)-S^0_n(y)\right]$$
$$+(x-y)\,x\left[\frac{2(n-1)}{y}\left(S^0_{n-1}(y)-S^0_{n-2}(y)\right)-S^0_{n-1}(y)\right]S^0_n(x)\Bigg) \tag{C.12}$$

For $y=0$

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{2ni}{k\sigma^2(x)^2}\Bigg(2xS^0_{n-1}(0)S^0_n(x)-2xS^0_{n-1}(x)S^0_n(0)$$
$$-(x)^2S^0_n(x)\left[-2n-S^0_n(0)\right]+x^2\left[-2(n-1)-S^0_{n-1}(0)\right]S^0_n(x)\Bigg)$$

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{2ni}{k\sigma^2(x)^2}\left(2xS^0_n(x)-2xS^0_{n-1}(x)-(x)^2S^0_n(x)\left[-2n-1\right]+x^2\left[-2(n-1)-1\right]S^0_n(x)\right)$$

Since $S^0_n(0)=S^0_{n-1}(0)=1$ for $n\geq 1$ (The whole component is zero for $n=0$):

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{2ni}{k\sigma^2(x)^2}\left(2xS^0_n(x)-2xS^0_{n-1}(x)+\left(-\left[-2n-1\right]+\left[-2(n-1)-1\right]\right)x^2S^0_n(x)\right)$$

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{2ni}{k\sigma^2(x)^2}\left(2xS^0_n(x)-2xS^0_{n-1}(x)+2x^2S^0_n(x)\right)$$

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{4ni}{k\sigma^2x}\left(S^0_n(x)-S^0_{n-1}(x)+xS^0_n(x)\right)$$

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{-4n}{ik\sigma^2(ik\sigma^2)}\left(S^0_n(ik\sigma^2)-S^0_{n-1}(ik\sigma^2)+(ik\sigma^2)S^0_n(ik\sigma^2)\right)$$

$$(\vec{N}_{n,0}(\vec{r}))_\sigma=\frac{-4n}{x^2}\left(S^0_n(x)-S^0_{n-1}(x)+(x)S^0_n(x)\right) \tag{C.13}$$

C.2 General case

By definition:

$$\frac{-k}{i}\vec{M}_{n,m}(\vec{r}) = \nabla \times S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}\left[\frac{\tau\hat{\mathbf{e}}_\tau - \sigma\hat{\mathbf{e}}_\sigma}{(\tau^2+\sigma^2)^{1/2}}\right]$$

$$\frac{k}{i}\vec{N}_{n,m}(\vec{r}) = \nabla \times \vec{M}_{n,m}(\vec{r})$$

The components of the $\vec{M}_{n,m}(\vec{r})$ vectors are:

$$\begin{aligned}
(\vec{M}_{n,m}(\vec{r}))_\sigma &= \frac{m}{k\sigma h_\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi} \\
(\vec{M}_{n,m}(\vec{r}))_\tau &= \frac{m}{k\tau h_\tau}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi} \\
(\vec{M}_{n,m}(\vec{r}))_\varphi &= \frac{-2\sigma\tau e^{im\varphi}}{\tau^2+\sigma^2}\left[\frac{\partial}{\partial x}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right) - \frac{\partial}{\partial y}\left(S_n^{|m|}(x)S_n^{|m|}(y)\right)\right]
\end{aligned}$$

where $x = ik\sigma^2$ and $y = -ik\tau^2$ in the last term; this notation is going to be used along the whole Appendix to simplify algebra.

Recalling the derivative of the functions:

$$\frac{\mathrm{d}S_n^m(z)}{\mathrm{d}z} = \frac{1}{2}\left[\left(\frac{m+2n}{z} - 1\right)S_n^m(z) - 2\frac{(n+m)}{z}S_{n-1}^m(z)\right]$$

we have for the φ component:

$$\begin{aligned}
(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{\tau^2+\sigma^2}\Bigg(&\left[\left(\frac{|m|+2n}{x} - 1\right)S_n^{|m|}(x) - 2\frac{(n+|m|)}{x}S_{n-1}^{|m|}(x)\right]S_n^{|m|}(y) \\
&- S_n^{|m|}(x)\left[\left(\frac{|m|+2n}{y} - 1\right)S_n^{|m|}(y) - 2\frac{(n+|m|)}{y}S_{n-1}^{|m|}(y)\right]\Bigg)
\end{aligned}$$

which simplifies to:

$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{\tau^2+\sigma^2}\left(\left[\left(\frac{|m|+2n}{x}\right)S_n^{|m|}(x) - 2\frac{(n+|m|)}{x}S_{n-1}^{|m|}(x)\right]S_n^{|m|}(y)\right.$$
$$\left. - S_n^{|m|}(x)\left[\left(\frac{|m|+2n}{y}\right)S_n^{|m|}(y) - 2\frac{(n+|m|)}{y}S_{n-1}^{|m|}(y)\right]\right)$$
$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{xy(\tau^2+\sigma^2)}\left(\left[(|m|+2n)S_n^{|m|}(x) - 2(n+|m|)S_{n-1}^{|m|}(x)\right]yS_n^{|m|}(y)\right.$$
$$\left. - xS_n^{|m|}(x)\left[(|m|+2n)S_n^{|m|}(y) - 2(n+|m|)S_{n-1}^{|m|}(y)\right]\right)$$

$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-e^{im\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)}\left((y-x)(|m|+2n)S_n^{|m|}(x)S_n^{|m|}(y) - 2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right.$$
$$\left. + 2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)\right)$$

Unfortunately there seems to be no way of factoring out $(\tau^2+\sigma^2)$ to cancel with the denominator. This greatly complicates the calculation of the $\vec{N}_{n,m}(\vec{r})$ vector's components and, can make it difficult to apply a orthogonality relation in the boundary conditions if is needed. However, it is possible to express the component in another useful form. Consider:

$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{xy(\tau^2+\sigma^2)}\left(\left[(|m|+2n)S_n^{|m|}(x) - 2(n+|m|)S_{n-1}^{|m|}(x)\right]yS_n^{|m|}(y)\right.$$
$$\left. - xS_n^{|m|}(x)\left[(|m|+2n)S_n^{|m|}(y) - 2(n+|m|)S_{n-1}^{|m|}(y)\right]\right)$$

and

$$(m+n)S_{n-1}^m(z) + (z-m-2n-1)S_n^m(z) + (n+1)S_{n+1}^m(z) = 0$$
$$\Longrightarrow zS_n^m(z) = (m+2n+1)S_n^m(z) - (m+n)S_{n-1}^m(z) - (n+1)S_{n+1}^m(z)$$

Then:

$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{xy(\tau^2+\sigma^2)}\Bigg((|m|+2n)(|m|+2n+1)S_n^{|m|}(x)S_n^{|m|}(y) - (|m|+2n)(|m|+n)S_n^{|m|}(x)S_{n-1}^{|m|}(y)$$
$$-(|m|+2n)(n+1)S_n^{|m|}(x)S_{n+1}^{|m|}(y) - 2(n+|m|)(|m|+2n+1)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+2(n+|m|)(n+|m|)S_{n-1}^{|m|}(x)S_{n-1}^{|m|}(y) + 2(n+|m|)(n+1)S_{n-1}^{|m|}(x)S_{n+1}^{|m|}(y)$$
$$-(|m|+2n)(|m|+2n+1)S_n^{|m|}(x)S_n^{|m|}(y) + (|m|+2n)(|m|+n)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+(|m|+2n)(n+1)S_{n+1}^{|m|}(x)S_n^{|m|}(y) + 2(n+|m|)(|m|+2n+1)S_n^{|m|}(x)S_{n-1}^{|m|}(y)$$
$$-2(n+|m|)(n+|m|)S_{n-1}^{|m|}(x)S_{n-1}^{|m|}(y) - 2(n+|m|)(n+1)S_{n+1}^{|m|}(x)S_{n-1}^{|m|}(y)\Bigg)$$

Canceling terms:

$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{xy(\tau^2+\sigma^2)}\Bigg(-(|m|+2n)(|m|+n)S_n^{|m|}(x)S_{n-1}^{|m|}(y)$$
$$-(|m|+2n)(n+1)S_n^{|m|}(x)S_{n+1}^{|m|}(y) - 2(n+|m|)(|m|+2n+1)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+2(n+|m|)(n+1)S_{n-1}^{|m|}(x)S_{n+1}^{|m|}(y)$$
$$+(|m|+2n)(|m|+n)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+(|m|+2n)(n+1)S_{n+1}^{|m|}(x)S_n^{|m|}(y) + 2(n+|m|)(|m|+2n+1)S_n^{|m|}(x)S_{n-1}^{|m|}(y)$$
$$-2(n+|m|)(n+1)S_{n+1}^{|m|}(x)S_{n-1}^{|m|}(y)\Bigg)$$

Taking common factors:

$$(\vec{M}_{n,m}(\vec{r}))_\varphi = \frac{-\sigma\tau e^{im\varphi}}{xy(\tau^2+\sigma^2)}\Bigg((|m|+2n)(|m|+n)\left[S_{n-1}^{|m|}(x)S_n^{|m|}(y) - S_n^{|m|}(x)S_{n-1}^{|m|}(y)\right]$$
$$+2(n+|m|)(|m|+2n+1)\left[S_n^{|m|}(x)S_{n-1}^{|m|}(y) - S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right]$$
$$+(|m|+2n)(n+1)\left[S_{n+1}^{|m|}(x)S_n^{|m|}(y) - S_n^{|m|}(x)S_{n+1}^{|m|}(y)\right]$$
$$2(n+|m|)(n+1)\left[S_{n-1}^{|m|}(x)S_{n+1}^{|m|}(y) - S_{n+1}^{|m|}(x)S_{n-1}^{|m|}(y)\right]\Bigg)$$

Now recall:

$$(\vec{N}_{n,m}(\vec{r}))_\tau = \frac{1}{k^2 h_\sigma h_\varphi}\left[\frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\varphi M_\varphi\right) - \frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\sigma M_\sigma\right)\right] \tag{C.14}$$

$$(\vec{N}_{n,m}(\vec{r}))_\sigma = \frac{-1}{k^2 h_\tau h_\varphi}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\varphi M_\varphi\right) - \frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\tau M_\tau\right)\right] \tag{C.15}$$

$$(\vec{N}_{n,m}(\vec{r}))_\varphi = \frac{1}{k^2 h_\tau h_\sigma}\left[\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\sigma M_\sigma\right) - \frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\tau M_\tau\right)\right] \tag{C.16}$$

with

$$\frac{-k}{i}h_\sigma M_\sigma = \frac{im}{\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$
$$\frac{-k}{i}h_\tau M_\tau = \frac{im}{\tau}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$
$$\frac{-k}{i}h_\varphi M_\varphi = \frac{e^{im\varphi}}{x-y}\Bigg((y-x)(|m|+2n)S_n^{|m|}(x)S_n^{|m|}(y) - 2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y) + 2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)\Bigg)$$

The corresponding derivatives are:

$$\frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\sigma M_\sigma\right) = \frac{-m^2}{\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$

$$\frac{\partial}{\partial\varphi}\left(\frac{-k}{i}h_\tau M_\tau\right) = \frac{-m^2}{\tau}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\sigma M_\sigma\right) &= \frac{mk\tau}{\sigma}S_n^{|m|}(ik\sigma^2)e^{im\varphi}\left[\left(\frac{|m|+2n}{y}-1\right)S_n^{|m|}(y) - 2\frac{(n+|m|)}{y}S_{n-1}^{|m|}(y)\right]\\ &= \frac{imy}{\sigma\tau}S_n^{|m|}(ik\sigma^2)e^{im\varphi}\left[\left(\frac{|m|+2n}{y}-1\right)S_n^{|m|}(y) - 2\frac{(n+|m|)}{y}S_{n-1}^{|m|}(y)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\tau M_\tau\right) &= \frac{-mk\sigma}{\tau}S_n^{|m|}(-ik\tau^2)e^{im\varphi}\left[\left(\frac{|m|+2n}{x}-1\right)S_n^{|m|}(x) - 2\frac{(n+|m|)}{x}S_{n-1}^{|m|}(x)\right]\\ &= \frac{imx}{\sigma\tau}S_n^{|m|}(-ik\tau^2)e^{im\varphi}\left[\left(\frac{|m|+2n}{x}-1\right)S_n^{|m|}(x) - 2\frac{(n+|m|)}{x}S_{n-1}^{|m|}(x)\right]\end{aligned}$$

$$\frac{\partial}{\partial\tau}\left(\frac{-k}{i}h_\varphi M_\varphi\right)=\frac{\partial}{\partial\tau}\frac{e^{im\varphi}}{x-y}\Bigg((y-x)(|m|+2n)S_n^{|m|}(x)S_n^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)\Bigg)$$

$$\frac{\partial}{\partial\sigma}\left(\frac{-k}{i}h_\varphi M_\varphi\right)=\frac{\partial}{\partial\sigma}\frac{e^{im\varphi}}{x-y}\Bigg((y-x)(|m|+2n)S_n^{|m|}(x)S_n^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)\Bigg)$$

The φ component reads:

$$(\vec{N}_{n,m}(\vec{r}))_\varphi=\frac{1}{k^2(\sigma^2+\tau^2)}\Bigg(\frac{imy}{\sigma\tau}S_n^{|m|}(ik\sigma^2)e^{im\varphi}\left[\left(\frac{|m|+2n}{y}-1\right)S_n^{|m|}(y)-2\frac{(n+|m|)}{y}S_{n-1}^{|m|}(y)\right]$$
$$-\frac{imx}{\sigma\tau}S_n^{|m|}(-ik\tau^2)e^{im\varphi}\left[\left(\frac{|m|+2n}{x}-1\right)S_n^{|m|}(x)-2\frac{(n+|m|)}{x}S_{n-1}^{|m|}(x)\right]\Bigg)$$

$$(\vec{N}_{n,m}(\vec{r}))_\varphi=\frac{ime^{im\varphi}}{k^2\sigma\tau(\sigma^2+\tau^2)}\Bigg(yS_n^{|m|}(ik\sigma^2)\left[\left(\frac{|m|+2n}{y}-1\right)S_n^{|m|}(y)-2\frac{(n+|m|)}{y}S_{n-1}^{|m|}(y)\right]$$
$$-xS_n^{|m|}(-ik\tau^2)\left[\left(\frac{|m|+2n}{x}-1\right)S_n^{|m|}(x)-2\frac{(n+|m|)}{x}S_{n-1}^{|m|}(x)\right]\Bigg)$$

$$(\vec{N}_{n,m}(\vec{r}))_\varphi=\frac{ime^{im\varphi}}{k^2\sigma\tau(\sigma^2+\tau^2)}\Bigg(S_n^{|m|}(x)\left[(|m|+2n-y)S_n^{|m|}(y)-2(n+|m|)S_{n-1}^{|m|}(y)\right]$$
$$-S_n^{|m|}(y)\left[(|m|+2n-x)S_n^{|m|}(x)-2(n+|m|)S_{n-1}^{|m|}(x)\right]\Bigg)$$

It may be worth to consider another form of the previous component using the identity:

$$(\nu+\alpha)S_{\nu-1}^{\alpha}(z)+(z-\alpha-2\nu-1)S_\nu^\alpha(z)+(\nu+1)S_{\nu+1}^\alpha(z)=0$$

which implies:

$$(m+2n-z)S_n^m(z)+S_n^m(z)-(m+n)S_{n-1}^m(z)-(n+1)S_{n+1}^m(z)=0$$

and therefore:

$$(m+2n-z)S_n^m(z)-(m+n)S_{n-1}^m(z)=(n+1)S_{n+1}^m(z)-S_n^m(z)$$

Then:

$$(\vec{N}_{n,m}(\vec{r}))_\varphi = \frac{ime^{im\varphi}}{k^2\sigma\tau(\sigma^2+\tau^2)}\Bigg(S_n^{|m|}(x)\Big[(n+1)S_{n+1}^{|m|}(y)-S_n^{|m|}(y)-(n+|m|)S_{n-1}^{|m|}(y)\Big]$$
$$-S_n^{|m|}(y)\Big[(n+1)S_{n+1}^{|m|}(x)-S_n^{|m|}(x)-(n+|m|)S_{n-1}^{|m|}(x)\Big]\Bigg)$$

which can be simplified further:

$$(\vec{N}_{n,m}(\vec{r}))_\varphi = \frac{ime^{im\varphi}}{k^2\sigma\tau(\sigma^2+\tau^2)}\Bigg(S_n^{|m|}(x)\Big[(n+1)S_{n+1}^{|m|}(y)-(n+|m|)S_{n-1}^{|m|}(y)\Big]$$
$$-S_n^{|m|}(y)\Big[(n+1)S_{n+1}^{|m|}(x)-(n+|m|)S_{n-1}^{|m|}(x)\Big]\Bigg)$$

$$(\vec{N}_{n,m}(\vec{r}))_\varphi = \frac{ime^{im\varphi}}{k^2\sigma\tau(\sigma^2+\tau^2)}\Bigg((n+1)\Big[(S_n^{|m|}(x)S_{n+1}^{|m|}(y)-S_{n+1}^{|m|}(x)S_n^{|m|}(y)\Big]$$
$$+(n+|m|)\Big[S_{n-1}^{|m|}(x)S_n^{|m|}(y)-S_n^{|m|}(x)S_{n-1}^{|m|}(y)\Big]\Bigg)$$

This form has the advantage that all dependence on the coordinate variables are inside the functions $S_n^{|m|}(z)$ inside the parenthesis.

The τ and σ components are more complex. An interesting result can be obtained if we let a portion of the τ component in terms of a derivative of σ after applying boundary conditions.

$$(\vec{N}_{n,m}(\vec{r}))_\tau = \frac{e^{im\varphi}}{k^2h_\sigma h_\varphi}$$
$$\times\Bigg(\frac{\partial}{\partial\sigma}\frac{1}{x-y}\bigg((y-x)(|m|+2n)S_n^{|m|}(x)S_n^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)$$
$$+2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)\bigg)+\frac{m^2}{\sigma}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)\Bigg)$$

By analogy:

$$(\vec{N}_{n,m}(\vec{r}))_\sigma = \frac{-e^{im\varphi}}{k^2 h_\tau h_\varphi}$$
$$\times \left(\frac{\partial}{\partial \tau} \frac{1}{x-y} \left((y-x)(|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \right.\right.$$
$$\left.\left. + 2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) \right) + \frac{m^2}{\tau} S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) \right)$$

C.3 N_σ component calculation

The derivative:

$$\frac{\partial}{\partial \tau} \frac{1}{x-y} \left((y-x)(|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \right.$$
$$\left. + 2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) \right)$$

can be expressed as:

$$\frac{\partial}{\partial \tau} \frac{1}{x-y} \left(2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \right)$$
$$- \frac{\partial}{\partial \tau} (|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y)$$

Since

$$\frac{\partial}{\partial \tau} = -2ik\tau \frac{\partial}{\partial y}$$

we can take $-2ik\tau$ out as a factor and write:

can be expressed as:

$$\frac{\partial}{\partial y} \frac{1}{x-y} \left(2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \right)$$
$$- \frac{\partial}{\partial y} (|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y)$$

or

$$\frac{1}{(x-y)^2}\left(2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right)$$
$$+\frac{1}{x-y}\left(2x(n+|m|)S_n^{|m|}(x)\frac{\partial S_{n-1}^{|m|}(y)}{\partial y}-2y(n+|m|)S_{n-1}^{|m|}(x)\frac{\partial S_n^{|m|}(y)}{\partial y}-2(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right)$$
$$-(|m|+2n)S_n^{|m|}(x)\frac{\partial S_n^{|m|}(y)}{\partial y}$$

Therefore:

$$\begin{aligned}(\vec{N}_{n,m}(\vec{r}))_\sigma = &\frac{-e^{im\varphi}}{k^2 h_\tau h_\varphi}\\ &\times\Bigg(-2ik\tau\Bigg(\frac{1}{(x-y)^2}\left(2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right)\\ &+\frac{1}{x-y}\left(2x(n+|m|)S_n^{|m|}(x)\frac{\partial S_{n-1}^{|m|}(y)}{\partial y}-2y(n+|m|)S_{n-1}^{|m|}(x)\frac{\partial S_n^{|m|}(y)}{\partial y}-2(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right)\\ &-(|m|+2n)S_n^{|m|}(x)\frac{\partial S_n^{|m|}(y)}{\partial y}\Bigg)\\ &+\frac{m^2}{\tau}S_n^{|m|}(x)S_n^{|m|}(y)\Bigg)\end{aligned}$$

With:

$$\frac{\mathrm{d}S_n^m(y)}{\mathrm{d}y}=\frac{1}{2}\left[\left(\frac{m+2n}{y}-1\right)S_n^m(y)-2\frac{(n+m)}{y}S_{n-1}^m(y)\right]$$

Note that this relation seems to imply that the derivative diverges when $z\to 0$. To avoid this problem the derivative in terms of the Laguerre polynomials is more useful:

$$\frac{\mathrm{d}S_n^m(y)}{\mathrm{d}y}=\frac{y^{\frac{m}{2}}}{2}e^{-\frac{y}{2}}\left[\left(\frac{m}{y}-1\right)L_n^m(y)-2L_{n-1}^{m+1}(y)\right]$$

C.3.1 Special case $\tau=0$ and $m=0$

Under such conditions:

$$
\begin{aligned}
L_n^m(0) &= \frac{(n+m)!}{n!m!} \\
L_n^0(0) &= 1 \\
L_{n-1}^1(0) &= \frac{n!}{(n-1)!}
\end{aligned}
$$

which implies

$$
\left.\frac{\mathrm{d}S_n^0(y)}{\mathrm{d}y}\right|_{y=0} = -\left[\frac{1}{2} + \frac{n!}{(n-1)!}\right]
$$

and if $n > 0$:

$$
\left.\frac{\mathrm{d}S_{n-1}^0(y)}{\mathrm{d}y}\right|_{y=0} = -\left[\frac{1}{2} + \frac{(n-1)!}{(n-2)!}\right]
$$

otherwise

$$
\left.\frac{\mathrm{d}S_{n-1}^0(y)}{\mathrm{d}y}\right|_{y=0} = 0
$$

On the other hand for $m = 0$:

$$
\begin{aligned}
(\vec{N}_{n,0}(\vec{r}))_\sigma = {} & \frac{2i}{k(\sigma^2+\tau^2)^{1/2}\sigma} \\
& \times \left(\frac{1}{(x-y)^2} \left(2xnS_n^0(x)S_{n-1}^0(y) - 2ynS_{n-1}^0(x)S_n^0(y) \right) \right. \\
& + \frac{1}{x-y} \left(2xnS_n^0(x)\frac{\partial S_{n-1}^0(y)}{\partial y} - 2ynS_{n-1}^0(x)\frac{\partial S_n^0(y)}{\partial y} - 2nS_{n-1}^0(x)S_n^0(y) \right) \\
& \left. - 2nS_n^0(x)\frac{\partial S_n^0(y)}{\partial y} \right)
\end{aligned}
$$

If we set $\tau = 0 (y = 0)$:

$$
\begin{aligned}
(\vec{N}_{n,0}(\vec{r}))_\sigma = &\frac{2i}{k\sigma^2} \\
&\times \left(\frac{1}{x^2} \left(2xnS_n^0(x)H(n-1/2) \right) \right. \\
&+ \frac{1}{x} \left(-2xnS_n^0(x)H(n-1/2) \left[\frac{1}{2} + \frac{(n-1)!}{(n-2)!} \right] - 2nS_{n-1}^0(x) \right) \\
&\left. + 2nS_n^0(x) \left[\frac{1}{2} + \frac{n!}{(n-1)!} \right] \right)
\end{aligned}
$$

The Heaviside function $H(n-1/2)$ was introduced to take into account the fact that $S_{-1}^0(y) = 0$. Nevertheless the whole component is zero for $n = 0$ so we can start evaluating for $n = 1$. Simplifying:

$$
\begin{aligned}
(\vec{N}_{n,0}(\vec{r}))_\sigma = &\frac{4i}{k\sigma^2} \\
&\times \left(\frac{1}{x} \left(nS_n^0(x)H(n-1/2) \right) \right. \\
&- nS_n^0(x)H(n-1/2) \left[\frac{1}{2} + \frac{(n-1)!}{(n-2)!} \right] - \frac{1}{x} nS_{n-1}^0(x) \\
&\left. + nS_n^0(x) \left[\frac{1}{2} + \frac{n!}{(n-1)!} \right] \right)
\end{aligned}
$$

$$
\begin{aligned}
(\vec{N}_{n,0}(\vec{r}))_\sigma = &\frac{4i}{k\sigma^2} \\
&\times \left(\frac{1}{ik\sigma^2} nS_n^0(ik\sigma^2)H(n-1/2) - \frac{1}{ik\sigma^2} nS_{n-1}^0(ik\sigma^2) \right. \\
&\left. + nS_n^0(ik\sigma^2) \left[\frac{n!}{(n-1)!} - H(n-1/2)\frac{(n-1)!}{(n-2)!} \right] \right)
\end{aligned}
$$

C.4 N_τ component calculation

$$(\vec{N}_{n,m}(\vec{r}))_\tau = \frac{e^{im\varphi}}{k^2 h_\sigma h_\varphi} \times \left(\frac{\partial}{\partial\sigma} \frac{1}{x-y} \left((y-x)(|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) + 2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) \right) + \frac{m^2}{\sigma} S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) \right)$$

The derivative:

$$\frac{\partial}{\partial\sigma} \frac{1}{x-y} \left((y-x)(|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) + 2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) \right)$$

can be expressed as:

$$\frac{\partial}{\partial\sigma} \frac{1}{x-y} \left(2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \right) - \frac{\partial}{\partial\sigma} (|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y)$$

Since

$$\frac{\partial}{\partial\sigma} = 2ik\sigma \frac{\partial}{\partial x}$$

we can take $2ik\sigma$ out as a factor and write:

$$\frac{\partial}{\partial x} \frac{1}{x-y} \left(2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \right) - \frac{\partial}{\partial x} (|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y)$$

or

$$\frac{-1}{(x-y)^2}\left(2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right)$$
$$+\frac{1}{x-y}\left(2(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)+2x(n+|m|)\frac{\mathrm{d}S_n^{|m|}(x)}{\mathrm{d}x}S_{n-1}^{|m|}(y)-2y(n+|m|)\frac{\mathrm{d}S_{n-1}^{|m|}(x)}{\mathrm{d}x}S_n^{|m|}(y)\right)$$
$$-(|m|+2n)\frac{\mathrm{d}S_n^{|m|}(x)}{\mathrm{d}x}S_n^{|m|}(y)$$

therefore

$$\begin{aligned}(\vec{N}_{n,m}(\vec{r}))_\tau = &\frac{e^{im\varphi}}{k^2h_\sigma h_\varphi}\\ &\times\Bigg(\frac{-2ik\sigma}{(x-y)^2}\left(2x(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)-2y(n+|m|)S_{n-1}^{|m|}(x)S_n^{|m|}(y)\right)\\ &+\frac{2ik\sigma}{x-y}\Bigg(2(n+|m|)S_n^{|m|}(x)S_{n-1}^{|m|}(y)+2x(n+|m|)\frac{\mathrm{d}S_n^{|m|}(x)}{\mathrm{d}x}S_{n-1}^{|m|}(y)\\ &-2y(n+|m|)\frac{\mathrm{d}S_{n-1}^{|m|}(x)}{\mathrm{d}x}S_n^{|m|}(y)\Bigg)-2ik\sigma(|m|+2n)\frac{\mathrm{d}S_n^{|m|}(x)}{\mathrm{d}x}S_n^{|m|}(y)\\ &+\frac{m^2}{\sigma}S_n^{|m|}(x)S_n^{|m|}(y)\Bigg)\end{aligned}$$

For $m=n=0$ The whole component is zero and for $m=0$ and $n\geq 1$ it can be shown that:

$$Lim_{\tau\to 0}(\vec{N}_{n,0}(\vec{r}))_\tau = 0$$

using the fact that the lowest polynomial term in $L_{n-1}^0(y)-L_n^0(y)$ is y^1. The independent term of the Laguerre polynomials is always 1 which cancels out after taking the difference.

APPENDIX D – EXPANSION OF A PLANE WAVE TRAVELING ALONG THE Z AXIS

In some cases it may be of interest the interaction between a wave traveling along the axis of the paraboloid. For this case the electric field takes the form:

$$\vec{E}(\vec{r}) = E_0 e^{ikz}\hat{\mathbf{e}}_x \tag{D.1}$$

There are two possible approaches, the obvious one is making use of the scalar plane wave expansion and transforming the unitary vector in paraboloid coordinates:

$$e^{i\vec{k}.\vec{r}} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} \frac{i^m \varepsilon_m \Gamma(n+1)}{\Gamma(n+|m|+1)} \frac{tan^{2n+|m|}\left(\frac{\theta_k}{2}\right)}{cos^2\left(\frac{\theta_k}{2}\right)} (-1)^n S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im(\varphi-\varphi_k)}$$

$$\hat{\mathbf{e}}_x = \frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi$$

we see that:

$$\vec{E}(\vec{r}) = E_0 S_0^0(ik\sigma^2) S_0^0(-ik\tau^2)\left(\frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi\right) \tag{D.2}$$

With the use of Maxwell's equations:

$$\vec{E}(\vec{r}) = \frac{i}{k} Z\nabla \times H \tag{D.3}$$

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}\nabla \times E \tag{D.4}$$

where Z is the vacuum impedance. We can obtain the expansion the same way as it was obtained in the previous section. Note however that the expansion only contain one term in this case. In order to obtain the expansion vectors $\vec{M}(\vec{r})$ and $\vec{N}(\vec{r})$ it is necessary to consider a field that can be expressed as an expansion with non-zero coefficients:

$$\begin{aligned}\vec{E}(\vec{r}) = E_0 &\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)} S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im(\varphi)} \\ &\times \left(\frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi\right)\end{aligned} \tag{D.5}$$

We can easily recover the plane wave by defining $G_{n,m}^{(TM)} = \delta_{n,0}\delta_{m,0}$.
Therefore the magnetic field is:

$$Z\vec{H}(\vec{r}) = -\frac{i}{k}E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)} \nabla \times S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im(\varphi)} \times \left(\frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi\right) \tag{D.6}$$

Again we define:

$$\frac{-k}{i}\vec{M}_{n,m}(\vec{r}) = \nabla \times S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im(\varphi)} \times \left(\frac{cos\varphi}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) - sin\varphi\hat{\mathbf{e}}_\varphi\right) \tag{D.7}$$

then:

$$Z\vec{H}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)}\vec{M}_{n,m}(\vec{r}) \tag{D.8}$$

W can use the Maxwell's equations again to obtain $\vec{N}(\vec{r})$:

$$\vec{E}(\vec{r}) = \frac{i}{k}E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)}\nabla \times \vec{M}_{n,m}(\vec{r}) \tag{D.9}$$

or

$$\vec{E}(\vec{r}) = E_0 \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} G_{n,m}^{(TM)}\vec{N}_{n,m}(\vec{r}) \tag{D.10}$$

where:

$$\frac{k}{i}\vec{N}_{n,m}(\vec{r}) = \nabla \times \vec{M}_{n,m}(\vec{r}) \tag{D.11}$$

Even thought each term of the expansion in eq D.5 satisfies the vector Helmholtz equation, they do not satisfies the Maxwell's equations since they are not divergenless. We can make each term divergenless by taking the curl and making use of the fact that the curl commute with the laplacian and as a consequence the Helmholtz operator $\nabla^2 + k^2$. Therefore the rotational of each term in the expansion is both solution of the helmholtz equation and divergenless so each term ($\vec{M}_{n,m}(\vec{r})$) of the expansion in eq D.8 can succesively represent a real field. This allows us to find the expansion of the electric field by taking two times the curl.

The calculation of the components of the new vector fields $\vec{M}_{n,m}(\vec{r})$ and $\vec{N}_{n,m}(\vec{r})$ is tedious. Fortunately there is another way. Recall the components of the $\vec{M}_{n,m}(\vec{r})$ vector based on the Hertz vector along the z axis instead:

$$
\begin{aligned}
(\vec{M}_{n,m}(\vec{r}))_\sigma &= \frac{m}{k\sigma h_\sigma} S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \\
(\vec{M}_{n,m}(\vec{r}))_\tau &= \frac{m}{k\tau h_\tau} S_n^{|m|}(ik\sigma^2) S_n^{|m|}(-ik\tau^2) e^{im\varphi} \\
(\vec{M}_{n,m}(\vec{r}))_\varphi &= \frac{-e^{im\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)} \Bigg((y-x)(|m|+2n) S_n^{|m|}(x) S_n^{|m|}(y) - 2y(n+|m|) S_{n-1}^{|m|}(x) S_n^{|m|}(y) \\
&\quad + 2x(n+|m|) S_n^{|m|}(x) S_{n-1}^{|m|}(y) \Bigg)
\end{aligned}
$$

with $h_\sigma = h_\tau = (\tau^2+\sigma^2)^{1/2}$, $x = ik\sigma^2$ and $y = -ik\tau^2$. Now consider the Vectors $\vec{M}_{0,1}(\vec{r})$ and $\vec{M}_{0,-1}(\vec{r})$:

$$
\begin{aligned}
(\vec{M}_{0,1}(\vec{r}))_\sigma &= \frac{1}{k\sigma(\tau^2+\sigma^2)^{1/2}} S_0^1(ik\sigma^2) S_0^1(-ik\tau^2) e^{i\varphi} \\
(\vec{M}_{0,1}(\vec{r}))_\tau &= \frac{1}{k\tau(\tau^2+\sigma^2)^{1/2}} S_0^1(ik\sigma^2) S_0^1(-ik\tau^2) e^{i\varphi} \\
(\vec{M}_{0,1}(\vec{r}))_\varphi &= \frac{-e^{i\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)} (y-x) S_0^1(ik\sigma^2) S_0^1(-ik\tau^2)
\end{aligned}
$$

$$(\vec{M}_{0,-1}(\vec{r}))_\sigma = \frac{-1}{k\sigma(\tau^2+\sigma^2)^{1/2}} S_0^1(ik\sigma^2) S_0^1(-ik\tau^2) e^{-i\varphi}$$
$$(\vec{M}_{0,-1}(\vec{r}))_\tau = \frac{-1}{k\tau(\tau^2+\sigma^2)^{1/2}} S_0^1(ik\sigma^2) S_0^1(-ik\tau^2) e^{-i\varphi}$$
$$(\vec{M}_{0,-1}(\vec{r}))_\varphi = \frac{-e^{-i\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)} (y-x) S_0^1(ik\sigma^2) S_0^1(-ik\tau^2)$$

We have used the fact that $S^m_{-1}(x) = 0$. Besides $L_0^1(x) = 1$ so $S_0^1(x) = x^{1/2}e^{-x/2}$. Therefore $S_0^1(ik\sigma^2)S_0^1(-ik\tau^2) = \sqrt{k^2\sigma^2\tau^2}e^{\frac{ik(\tau^2-\sigma^2)}{2}}$ and since $z = \frac{(\tau^2-\sigma^2)}{2}$ the vectors take the form:

$$(\vec{M}_{0,1}(\vec{r}))_\sigma = \frac{1}{k\sigma(\tau^2+\sigma^2)^{1/2}} k\sigma\tau e^{ikz} e^{i\varphi}$$
$$(\vec{M}_{0,1}(\vec{r}))_\tau = \frac{1}{k\tau(\tau^2+\sigma^2)^{1/2}} k\sigma\tau e^{ikz} e^{i\varphi}$$
$$(\vec{M}_{0,1}(\vec{r}))_\varphi = \frac{e^{i\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)} ik(\tau^2+\sigma^2) k\sigma\tau e^{ikz}$$

$$(\vec{M}_{0,-1}(\vec{r}))_\sigma = \frac{-1}{k\sigma(\tau^2+\sigma^2)^{1/2}} k\sigma\tau e^{ikz} e^{-i\varphi}$$
$$(\vec{M}_{0,-1}(\vec{r}))_\tau = \frac{-1}{k\tau(\tau^2+\sigma^2)^{1/2}} k\sigma\tau e^{ikz} e^{-i\varphi}$$
$$(\vec{M}_{0,-1}(\vec{r}))_\varphi = \frac{e^{-i\varphi}}{k^2\sigma\tau(\tau^2+\sigma^2)} ik(\tau^2+\sigma^2) k\sigma\tau e^{ikz}$$

Simplifying we have:

$$\vec{M}_{0,1}(\vec{r}) = \left(\frac{1}{(\tau^2+\sigma^2)^{1/2}} (\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + i\hat{\mathbf{e}}_\varphi \right) e^{ikz} e^{i\varphi}$$
$$\vec{M}_{0,-1}(\vec{r}) = \left(\frac{-1}{(\tau^2+\sigma^2)^{1/2}} (\tau\hat{\mathbf{e}}_\sigma + \sigma\hat{\mathbf{e}}_\tau) + i\hat{\mathbf{e}}_\varphi \right) e^{ikz} e^{-i\varphi}$$

Now consider the linear combination:

$$\frac{1}{2}\left(\vec{M}_{0,1}(\vec{r}) - \vec{M}_{0,-1}(\vec{r}) \right)$$

Which is equal to:

$$\frac{1}{2}e^{ikz}\left(\frac{1}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma+\sigma\hat{\mathbf{e}}_\tau)(e^{i\varphi}+e^{-i\varphi})\right.$$
$$\left.+i\hat{\mathbf{e}}_\varphi(e^{i\varphi}-e^{-i\varphi})\right)$$

And finally:

$$\frac{1}{2}\left(\vec{M}_{0,1}(\vec{r})-\vec{M}_{0,-1}(\vec{r})\right)=e^{ikz}\left(\frac{\cos(\varphi)}{(\tau^2+\sigma^2)^{1/2}}(\tau\hat{\mathbf{e}}_\sigma+\sigma\hat{\mathbf{e}}_\tau)-\sin(\varphi)\hat{\mathbf{e}}_\varphi\right)$$

The term inside the parenthesis on the right side is just the $\hat{\mathbf{e}}_x$ unitary vector Therefore:

$$e^{ikz}\hat{\mathbf{e}}_x=\frac{1}{2}\left(\vec{M}_{0,1}(\vec{r})-\vec{M}_{0,-1}(\vec{r})\right)$$

This means that a plane wave traveling along the z axis polarized along the x axis can be expressed as:

$$\vec{E}(\vec{r})=E_0e^{ikz}\hat{\mathbf{e}}_x=E_0\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}G_{n,m}\vec{M}_{n,m}(\vec{r}) \tag{D.12}$$

With the expansion coefficients:

$$G_{n,m}=\frac{1}{2}(\delta_{n,0}\delta_{m,1}-\delta_{n,0}\delta_{m,-1}) \tag{D.13}$$

APPENDIX E – NOTE ON THE ORTHOGONAL RELATION

Recall the orthogonal relation of the Pinney $S_n^{|m|}(x)$ functions:

$$\int_0^\infty S_n^\alpha(x)S_m^\alpha(x)dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}$$

Here x take real positive values. To be useful in practice we would use:

$$\int_0^\infty S_n^\alpha(ik\sigma^2)S_m^\alpha(ik\sigma^2)d(ik\sigma^2) = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}$$

However in practice k and σ are assumed to take only real values which apparently limit the usefulness of the orthogonal relation. Nevertheless if we assume it is valid in practical applications we may actually get good results. For instance we can recover the result of the expansion of scalar plane waves by using this relation.

Before continuing we need the result of another integral:

$$\int_0^\infty e^{-t}t^{\mu/2}L_\nu^\mu(zt)J_\mu(2\sqrt{z't})dt = (z')^{\mu/2}(1-z)^\nu e^{-z'}L_\nu^\mu\left(\frac{-zz'}{1-z}\right)$$

Which according to Pinney is valid for $Re(\mu) > -1$ and $\nu = n \geq 0$ with z unrestricted or other combination of indices values of which we are not interested (PINNEY, 1946). It is important to note that he use analytical continuation and the convergence of a limit to arrive at this result; this will become clearer later.

Next we make the following substitutions $z = 2$, $t = x/2$ and $z' = y/2$:

$$\frac{1}{2}\int_0^\infty e^{-x/2}x^{\mu/2}L_\nu^\mu(x)J_\mu(\sqrt{xy})dx = (y)^{\mu/2}(-1)^\nu e^{-y/2}L_\nu^\mu(y)$$

which can be expressed in terms of Pinney functions as:

$$\int_0^\infty S_\nu^\mu(x)J_\mu(\sqrt{xy})dx = 2(-1)^\nu S_\nu^\mu(y)$$

Now assume we want to find the coefficients of the expansion:

$$e^{ik\sigma\tau\cos(\varphi)} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}A_{n,m}S_n^{|m|}(ik\sigma^2)S_n^{|m|}(-ik\tau^2)e^{im\varphi}$$

by multiplying by $e^{-il\varphi}/2\pi$ with $l \geq 0$, integrating φ form 0 to 2π and using the orthogonal relation of the exponential:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\sigma\tau\cos(\varphi)-il\varphi}d\varphi = \sum_{n=0}^{\infty} A_{n,l}S_n^l(ik\sigma^2)S_n^l(-ik\tau^2)$$

The integral on the left side is just $i^l J_l(k\sigma\tau)$:

$$i^l J_l(k\sigma\tau) = \sum_{n=0}^{\infty} A_{n,l}S_n^l(ik\sigma^2)S_n^l(-ik\tau^2)$$

Now consider $x = ik\sigma^2$ and $y = -ik\tau^2$ then $xy = (k\sigma\tau)^2$ and:

$$i^l J_l(\sqrt{xy}) = \sum_{n=0}^{\infty} A_{n,l}S_n^l(x)S_n^l(y)$$

Multiplying by $S_p^l(x)$ and integrating from 0 to ∞:

$$i^l \int_0^{\infty} S_p^l(x)J_l(\sqrt{xy})dx = \sum_{n=0}^{\infty} A_{n,l}\int_0^{\infty} S_p^l(x)S_n^l(x)dxS_n^l(y)$$

By using the new integral relation and the orthogonality of the Pinney functions it is obtained:

$$i^l 2(-1)^p S_p^l(y) = A_{p,l}\frac{(p+l)!}{p!}S_p^l(y)$$

Therefore:

$$A_{p,l} = \frac{i^l 2(-1)^p p!}{(p+l)!}$$

Which is the same coefficient obtained by the Hardy-Hille formula. For $l < 0$ a similar procedure can be done or just use the relation $J_{-m}(x) = (-1)^m J_m(x)$. The only problem is that the coefficient must be defined as:

$$A_{n,m} = Lim_{\delta^- \to 0}\left(i^m \varepsilon_m \frac{(2-\delta)(-1+\delta)^n n!}{(n+|m|)!}\right)$$

$$\varepsilon_m = \begin{cases} 1 & \text{if} m \geq 0 \\ (-1)^m & \text{if} m < 0 \end{cases}$$

To make sure the expansion converges, which is evident from the Hardy-Hille formula.

APPENDIX F – MULTIPLICATION THEOREM FOR PINNEY FUNCTIONS

A future attempt to solve the boundary conditions equations is going to require a multiplication theorem to express the functions $S_n^m(-ik_2\tau^2)$ in terms of $S_n^m(-ik_1\tau^2)$. As far as the authors known, the only way to obtain one is by using two multiplication formulas known for Laguerre polynomials. These are:

$$t^{n+\alpha+1}e^{(1-t)z}L_n^\alpha(zt) = \sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}\left(1-\frac{1}{t}\right)^{k-n}L_k^\alpha(z), \qquad \left|\frac{t}{1+t}\right| < 1 \tag{F.1}$$

$$\frac{L_n^\alpha(x\lambda)}{L_n^\alpha(0)} = \sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\lambda^l(1-\lambda)^{n-l}\frac{L_l^\alpha(x)}{L_l^\alpha(0)}, \qquad any \quad \lambda \tag{F.2}$$

we start by substituting $z = \frac{x}{2}$ on the first multiplication theorem

$$t^{n+\alpha+1}e^{(1-t)\frac{x}{2}}L_n^\alpha\left(\frac{x}{2}t\right) = \sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}\left(1-\frac{1}{t}\right)^{k-n}L_k^\alpha\left(\frac{x}{2}\right)$$

$$t^{n+\alpha+1}e^{-t\frac{x}{2}}L_n^\alpha\left(\frac{x}{2}t\right) = \sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}\left(1-\frac{1}{t}\right)^{k-n}e^{-\frac{x}{2}}L_k^\alpha\left(\frac{x}{2}\right)$$

$$t^{n+\alpha/2+1}\left[(xt)^{\frac{\alpha}{2}}e^{-t\frac{x}{2}}L_n^\alpha\left(\frac{x}{2}t\right)\right] = \sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}\left(1-\frac{1}{t}\right)^{k-n}\left[x^{\frac{\alpha}{2}}e^{-\frac{x}{2}}L_k^\alpha\left(\frac{x}{2}\right)\right]$$

From the second multiplication theorem we have for $\lambda = \frac{1}{2}$

$$\frac{L_n^\alpha\left(\frac{x}{2}\right)}{L_n^\alpha(0)} = \sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\left(\frac{1}{2}\right)^l\left(1-\frac{1}{2}\right)^{n-l}\frac{L_l^\alpha(x)}{L_l^\alpha(0)}$$

Which in turn gives:

$$L_k^\alpha\left(\frac{x}{2}\right) = L_k^\alpha(0)\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\left(\frac{1}{2}\right)^l\left(1-\frac{1}{2}\right)^{k-l}\frac{L_l^\alpha(x)}{L_l^\alpha(0)}$$

$$L_n^\alpha\left(\frac{xt}{2}\right) = L_n^\alpha(0)\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\left(\frac{1}{2}\right)^l\left(1-\frac{1}{2}\right)^{n-l}\frac{L_l^\alpha(xt)}{L_l^\alpha(0)}$$

Multiplying by $x^{\frac{\alpha}{2}}e^{-\frac{x}{2}}$ and $(xt)^{\frac{\alpha}{2}}e^{-t\frac{x}{2}}$ respectively

$$\left[x^{\frac{\alpha}{2}}e^{-\frac{x}{2}}L_k^\alpha\left(\frac{x}{2}\right)\right] = L_k^\alpha(0)\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\left(\frac{1}{2}\right)^l\left(1-\frac{1}{2}\right)^{k-l}\frac{\left[x^{\frac{\alpha}{2}}e^{-\frac{x}{2}}L_l^\alpha(x)\right]}{L_l^\alpha(0)}$$

$$\left[(xt)^{\frac{\alpha}{2}}e^{-t\frac{x}{2}}L_n^\alpha\left(\frac{xt}{2}\right)\right] = L_n^\alpha(0)\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\left(\frac{1}{2}\right)^l\left(1-\frac{1}{2}\right)^{n-l}\frac{\left[(xt)^{\frac{\alpha}{2}}e^{-t\frac{x}{2}}L_l^\alpha(xt)\right]}{L_l^\alpha(0)}$$

Which can be expressed in terms of Pinney functions

$$\left[x^{\frac{\alpha}{2}}e^{-\frac{x}{2}}L_k^{\alpha}\left(\frac{x}{2}\right)\right] = L_k^{\alpha}(0)\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\left(\frac{1}{2}\right)^k\frac{\left[S_l^{\alpha}(x)\right]}{L_l^{\alpha}(0)} \tag{F.3}$$

$$\left[(xt)^{\frac{\alpha}{2}}e^{-t\frac{x}{2}}L_n^{\alpha}\left(\frac{xt}{2}\right)\right] = L_n^{\alpha}(0)\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\left(\frac{1}{2}\right)^n\frac{\left[S_l^{\alpha}(tx)\right]}{L_l^{\alpha}(0)} \tag{F.4}$$

Replacing on the first multiplication theorem gives

$$t^{n+\frac{\alpha}{2}+1}\left[L_n^{\alpha}(0)\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\left(\frac{1}{2}\right)^n\frac{\left[S_l^{\alpha}(tx)\right]}{L_l^{\alpha}(0)}\right] = \sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}\left(1-\frac{1}{t}\right)^{k-n} \times\left[L_k^{\alpha}(0)\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\left(\frac{1}{2}\right)^k\frac{\left[S_l^{\alpha}(x)\right]}{L_l^{\alpha}(0)}\right]$$

Simplifying

$$\sum_{l=0}^{n}\frac{n!}{2^n l!(n-l)!}\frac{1}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right] = \frac{t^{-n-\frac{\alpha}{2}-1}}{L_n^{\alpha}(0)}\sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}\left(1-\frac{1}{t}\right)^{k-n}\left[\sum_{l=0}^{k}\frac{k!}{2^k l!(k-l)!}\left(\frac{L_k^{\alpha}(0)}{L_l^{\alpha}(0)}\right)\left[S_l^{\alpha}(x)\right]\right]$$

Lets define

$$C(k,l) = \frac{k!}{2^k l!(k-l)!} \tag{F.5}$$

Then

$$\sum_{l=0}^{n}\frac{C(n,l)}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right] = \frac{t^{-n-\frac{\alpha}{2}-1}}{L_n^{\alpha}(0)}\sum_{k=n}^{\infty}2^k C(k,n)\left(1-\frac{1}{t}\right)^{k-n}\left[\sum_{l=0}^{k}C(k,l)\left(\frac{L_k^{\alpha}(0)}{L_l^{\alpha}(0)}\right)\left[S_l^{\alpha}(x)\right]\right] \tag{F.6}$$

The right side can be simplified further by defining

$$A^{\alpha}(k,n,l,t) = 2^k C(k,n)\left(1-\frac{1}{t}\right)^{k-n}C(k,l)\left(\frac{L_k^{\alpha}(0)}{L_l^{\alpha}(0)}\right) \tag{F.7}$$

so

$$\sum_{l=0}^{n}\frac{C(n,l)}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right] = \frac{t^{-n-\frac{\alpha}{2}-1}}{L_n^{\alpha}(0)}\sum_{k=n}^{\infty}\sum_{l=0}^{k}A^{\alpha}(k,n,l,t)S_l^{\alpha}(x)$$

or

$$t^{n+\frac{\alpha}{2}+1}L_n^{\alpha}(0)\sum_{l=0}^{n}\frac{C(n,l)}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right]=\sum_{k=n}^{\infty}\sum_{l=0}^{k}A^{\alpha}(k,n,l,t)S_l^{\alpha}(x)$$

The sum on the right hand can be expressed as

$$\begin{gathered}
A^{\alpha}(n,n,0,t)S_0^{\alpha}(x)+A^{\alpha}(n,n,1,t)S_1^{\alpha}(x)+A^{\alpha}(n,n,2,t)S_2^{\alpha}(x)+\ldots+A^{\alpha}(n,n,n,t)S_n^{\alpha}(x)+\\
A^{\alpha}(n+1,n,0,t)S_0^{\alpha}(x)+A^{\alpha}(n+1,n,1,t)S_1^{\alpha}(x)+A^{\alpha}(n+1,n,2,t)S_2^{\alpha}(x)+\ldots+A^{\alpha}(n+1,n,n+1,t)S_{n+1}^{\alpha}(x)\\
A^{\alpha}(n+2,n,0,t)S_0^{\alpha}(x)+A^{\alpha}(n+2,n,1,t)S_1^{\alpha}(x)+A^{\alpha}(n+2,n,2,t)S_2^{\alpha}(x)+\ldots+A^{\alpha}(n+2,n,n+2,t)S_{n+2}^{\alpha}(x)\\
\vdots\\
A^{\alpha}(n+i,n,0,t)S_0^{\alpha}(x)+A^{\alpha}(n+i,n,1,t)S_1^{\alpha}(x)+A^{\alpha}(n+i,n,2,t)S_2^{\alpha}(x)+\ldots+A^{\alpha}(n+i,n,n+i,t)S_{n+i}^{\alpha}(x)+\\
\vdots
\end{gathered}$$

This can be rearranged as

$$\begin{gathered}
(A^{\alpha}(n,n,0,t)+A^{\alpha}(n+1,n,0,t)+A^{\alpha}(n+2,n,0,t)+\ldots+A^{\alpha}(n+i,n,0,t)+\ldots)S_0^{\alpha}(x)+\\
(A^{\alpha}(n,n,1,t)+A^{\alpha}(n+1,n,1,t)+A^{\alpha}(n+2,n,1,t)+\ldots+A^{\alpha}(n+i,n,1,t)+\ldots)S_1^{\alpha}(x)+\\
\vdots\\
(A^{\alpha}(n,n,l,t)+A^{\alpha}(n+1,n,l,t)+A^{\alpha}(n+2,n,l,t)+\ldots+A^{\alpha}(n+i,n,l,t)+\ldots)S_l^{\alpha}(x)+\\
\vdots
\end{gathered}$$

Simplifying

$$\left(\sum_{k=n}^{\infty}A^{\alpha}(k,n,0,t)\right)S_0^{\alpha}(x)+\left(\sum_{k=n}^{\infty}A^{\alpha}(k,n,1,t)\right)S_1^{\alpha}(x)+\ldots+\left(\sum_{k=n}^{\infty}A^{\alpha}(k,n,l,t)\right)S_l^{\alpha}(x)+\ldots$$

And finally

$$\sum_{l=0}^{\infty}\left(\sum_{k=n}^{\infty}A^{\alpha}(k,n,l,t)\right)S_l^{\alpha}(x)$$

Therefore

$$t^{n+\frac{\alpha}{2}+1}L_n^{\alpha}(0)\sum_{l=0}^{n}\frac{C(n,l)}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right]=\sum_{l=0}^{\infty}\left(\sum_{k=n}^{\infty}A^{\alpha}(k,n,l,t)\right)S_l^{\alpha}(x)$$

By another definition

$$\tilde{A}_{nl}^{\alpha}(t)=\sum_{k=n}^{\infty}A^{\alpha}(k,n,l,t)$$

so

$$t^{n+\frac{\alpha}{2}+1}L_n^{\alpha}(0)\sum_{l=0}^{n}\frac{C(n,l)}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right]=\sum_{l=0}^{\infty}\tilde{A}_{nl}^{\alpha}(t)S_l^{\alpha}(x) \tag{F.8}$$

A lengthy but not difficult process can be carried out to find that

$$\tilde{A}_{nl}^{\alpha}(t)=\frac{1}{2^n}\frac{(\alpha+1)_n}{(\alpha+1)_l}\frac{1}{\Gamma(n+1-l)}{}_2F_1\left(n+1,n+\alpha+1;n+1-l,\frac{1}{2}\left(1-\frac{1}{t}\right)\right)$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function. This function has no defined value for negative integer values of c. This is overcome in the last formula by the factor $\frac{1}{\Gamma(n+1-l)}$. To avoid this issue it may be more convenient to evaluate the coefficient as:

$$\tilde{A}_{nl}^{\alpha}(t)=\frac{1}{2^n}\frac{(\alpha+1)_n}{(\alpha+1)_l}\sum_{k=0}^{\infty}\frac{(n+1)_k(n+\alpha+1)_k}{\Gamma(k+n+1-l)}\frac{1}{(k)!}\left(\frac{1}{2}-\frac{1}{2t}\right)^k$$

In summary:

$$t^{n+\frac{\alpha}{2}+1}L_n^{\alpha}(0)\sum_{l=0}^{n}\frac{C(n,l)}{L_l^{\alpha}(0)}\left[S_l^{\alpha}(tx)\right]=\sum_{l=0}^{\infty}\tilde{A}_{nl}^{\alpha}(t)S_l^{\alpha}(x) \tag{F.9}$$

$$\tilde{A}_{nl}^{\alpha}(t)=\frac{1}{2^n}\frac{(\alpha+1)_n}{(\alpha+1)_l}\frac{1}{\Gamma(n+1-l)}{}_2F_1\left(n+1,n+\alpha+1;n+1-l,\frac{1}{2}\left(1-\frac{1}{t}\right)\right) \tag{F.10}$$

$$\tilde{A}_{nl}^{\alpha}(t)=\frac{1}{2^n}\frac{(\alpha+1)_n}{(\alpha+1)_l}\sum_{k=0}^{\infty}\frac{(n+1)_k(n+\alpha+1)_k}{\Gamma(k+n+1-l)}\frac{1}{(k)!}\left(\frac{1}{2}-\frac{1}{2t}\right)^k \tag{F.11}$$

APPENDIX G – PROGRAM IN MATHEMATICA

The following pages are the Mathematica notebook code used to calculate the field enhancement

Light Scattering on a Paraboloid of Revolution

Definition of the Laguerre function of second kind. This function is divided in 6 parts called LaguerreU1, LaguerreU2,...,LaguerreU6. Then the final function is defined in LaguerreU. The GammaFactorU5 function is used to speed up the calculation and evade a ∞/∞ indetermination. The Pochhammer factor also has this kind of problem. A new type of Pochhammer Symbol has to be defined to avoid this problem.

```
In[ ]:= LaguerreU1[n_, m_, z_] := ⅈ π LaguerreL[n, m, z];
LaguerreU2[n_, m_, z_] := PolyGamma[n + m + 1] × LaguerreL[n, m, z];
LaguerreU3[n_, m_, z_] :=
  - Sum[(Gamma[n + m + 1] × PolyGamma[k + m + 1])/(Gamma[n - k + 1] × Gamma[k + m + 1]) (-z)^k/Factorial[k], {k, 0, n}];
LaguerreU4[n_, m_, z_] := Log[z] × LaguerreL[n, m, z];
GammaFactorU5[n_] := If[n > 0, PolyGamma[n]/Gamma[n], - (-1)^-n Factorial[-n]];
LaguerreU5[n_, m_, z_] :=
  - (z)^-m Sum[Pochhammer[-n - m, k] × GammaFactorU5[k - m + 1] z^k/Factorial[k], {k, 0, n + m}];
ReducedPochammer[0, j_, x_] := 1;
ReducedPochammer[k_, j_, x_] :=
  If[x + k - 1 == x + j, 1, x + k - 1] × ReducedPochammer[k - 1, j, x];
PochhammerFactor[n_, m_, k_] := Sum[ReducedPochammer[k, j, -n - m], {j, 0, k - 1}];
LaguerreU6[n_, m_, z_] :=
  - (z)^-m Sum[PochhammerFactor[n, m, k]/Gamma[k - m + 1] z^k/Factorial[k], {k, Max[1, m], 20}];
```

Note that the last summation was truncated at k = 20 instead of ∞.

The Second solution of the Laguerre equation is then:

```
In[ ]:= PlusMinusArg[z_] := If[0 < Arg[z] && Arg[z] < π, 1, -1];
```

In[ ]:= LaguerreU[n_, m_, z_] := $\frac{1}{\pi}$ (LaguerreU2[n, m, z] + LaguerreU3[n, m, z] + LaguerreU4[n, m, z] + LaguerreU5[n, m, z] + LaguerreU6[n, m, z]);

The Pinney functions are defined as:

In[ ]:= PinneyS[n_, m_, z_] := $z^{\frac{m}{2}}$ Exp$\left[\frac{-z}{2}\right]$ × LaguerreL[n, m, z];

PinneyV[n_, m_, z_] := $z^{\frac{m}{2}}$ Exp$\left[\frac{-z}{2}\right]$ × LaguerreU[n, m, z];

PinneyV2[n_, m_, z_] := $z^{\frac{m}{2}}$ Exp$\left[\frac{-z}{2}\right]$ × LaguerreU[n, m, z];

With the special case:

In[ ]:= PinneyS[-1, m, z] := 0;

To avoid indeterminations:

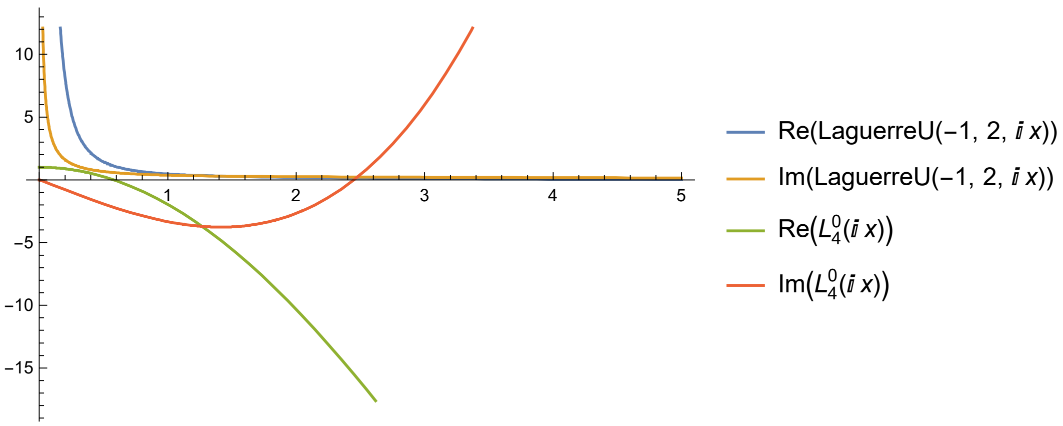
In[ ]:= PinneyV[-1, m, z] := 0;

To simplify the expression of the Nσ component it is convenient to directly define the derivative of the Pinney Functions as anothe function

In[ ]:= DerivPinneyS[n_, m_, z_] :=
$\frac{z^{\frac{m}{2}}}{2}$ Exp$\left[\frac{-z}{2}\right]$ $\left(\left(\frac{m}{z}-1\right)\right.$ LaguerreL[n, m, z] - 2 LaguerreL[n - 1, m + 1, z]$\left.\right)$;

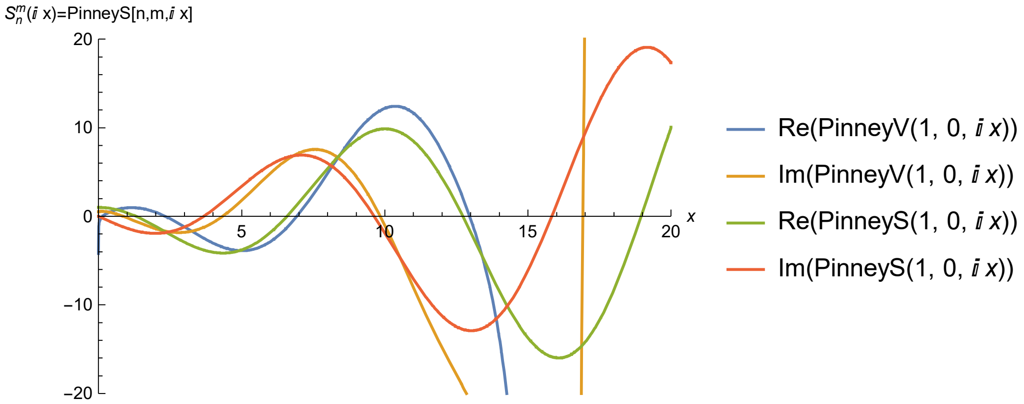
Now we test the functions

In[ ]:= Plot[{Re[LaguerreU[-1, 2, ⅈ x]], Im[LaguerreU[-1, 2, ⅈ x]], Re[LaguerreL[4, 0, ⅈ x]], Im[LaguerreL[4, 0, ⅈ x]]}, {x, 0, 5}, PlotLegends → "Expressions"]

Out[ ]=


In[ ]:=
```
Plot[{Re[PinneyV[1, 0, ⅈ x]], Im[PinneyV[1, 0, ⅈ x]], Re[PinneyS[1, 0, ⅈ x]],
  Im[PinneyS[1, 0, ⅈ x]]}, {x, 0, 20}, PlotRange → {{0, 20}, {-20, 20}},
 PlotLegends → "Expressions", AxesLabel → {x, "S_n^m(ⅈ x)=PinneyS[n,m,ⅈ x]"}]
```

Out[ ]=



The following functions are necessary to simplify the notation in the infinite linear problem. PinneyC is the coefficient Cp(n,m) of the series expansion of PinneyS.

In[ ]:=
```
PinneyC[p_, n_, m_] :=
  Sum[((-1)/2)^l Pochhammer[-n, p - l]/Pochhammer[m + 1, p - l] 1/(Factorial[l] × Factorial[p - l]), {l, 0, p}];
PinneyC[p_, -1, m_] := 0;
PinneyC[p_, -2, m_] := 0;

fs[n_, x_] := (2 n)/x (PinneyS[n, 0, x] - PinneyS[n - 1, 0, x]) - PinneyS[n, 0, x];
hs[n_, x_] := (2 n - x) PinneyS[n, 0, x] - 2 n PinneyS[n - 1, 0, x];
fv[n_, x_] :=
  If[n == 0, 0, (2 n)/x (PinneyV[n, 0, x] - PinneyV[n - 1, 0, x]) - PinneyV[n, 0, x]];
hv[n_, x_] := (2 n - x) PinneyV[n, 0, x] - 2 n PinneyV[n - 1, 0, x];
```

```
In[]:= AuNS[n_, x_, p_] :=
        fs[n, x] (x^2 PinneyC[p, n, 0] - 2 x PinneyC[p - 1, n, 0] + PinneyC[p - 2, n, 0]) +
         fs[n - 1, x] (x PinneyC[p - 1, n, 0] - PinneyC[p - 2, n, 0]) -
         hs[n, x] (x PinneyC[p, n - 1, 0] - PinneyC[p - 1, n - 1, 0]) + 2 PinneyS[n, 0, x] ×
          PinneyC[p - 1, n - 1, 0] - 2 PinneyS[n - 1, 0, x] × PinneyC[p - 1, n, 0];
       AuNV[n_, x_, p_] := fv[n, x] (x^2 PinneyC[p, n, 0] - 2 x PinneyC[p - 1, n, 0] +
            PinneyC[p - 2, n, 0]) + fv[n - 1, x] (x PinneyC[p - 1, n, 0] - PinneyC[p - 2, n, 0]) -
         hv[n, x] (x PinneyC[p, n - 1, 0] - PinneyC[p - 1, n - 1, 0]) +
         2 PinneyV[n, 0, x] × PinneyC[p - 1, n - 1, 0] -
         2 PinneyV[n - 1, 0, x] × PinneyC[p - 1, n, 0];
```

```
In[]:= NS[n_, k_, σ_, p_] := (2 n ⅈ)/k^3 AuNS[n, ⅈ k σ^2, p] k^p;

In[]:= NV[n_, k_, σ_, p_] := (2 n ⅈ)/k^3 AuNV[n, -ⅈ k σ^2, p] (k)^p;
```

```
In[]:= AuMS[n_, x_, p_] :=
        PinneyS[n, 0, x] × PinneyC[p - 1, n, 0] - x PinneyS[n, 0, x] × PinneyC[p, n, 0] -
         PinneyS[n - 1, 0, x] × PinneyC[p - 1, n, 0] + x PinneyS[n, 0, x] × PinneyC[p, n - 1, 0];
       AuMV[n_, x_, p_] := PinneyV[n, 0, x] × PinneyC[p - 1, n, 0] -
         x PinneyV[n, 0, x] × PinneyC[p, n, 0] - PinneyV[n - 1, 0, x] × PinneyC[p - 1, n, 0] +
         x PinneyV[n, 0, x] × PinneyC[p, n - 1, 0];
```

```
In[]:= MS[n_, k_, σ_, p_] := (-2 n)/k^2 AuMS[n, ⅈ k σ^2, p] k^p;
       MV[n_, k_, σ_, p_] := (-2 n)/k^2 AuMV[n, ⅈ k σ^2, p] (k)^p;
```

Here we define the coefficient of the incident field. We take a plane wave traveling along the x axis and polarized along the axis of the paraboloid (z axis). *δ* is defined *δ*>0. This trick is necessary to ensure the expansion of the field converges.

```
In[]:= ϵm[m_] := If[m < 0, (-1)^m, 1];
```

```
In[ ]:= GTM[n_, m_, δ_] := Sin[δ] (Tan[δ/2])^(2 n) / (Cos[δ/2])^2 (-1)^n;
        RealFocusedGTM1[n_, m_, δ_] := (Tan[δ/2])^(2 n) / (Cos[δ/2])^2 (-1)^n;
        RealFocusedGTM2[n_, m_, δ_] := GTM[n, 0, δ];
        RealFocusedGTM[n_, m_, δ_] := NIntegrate[GTM[n, 0, θ], {θ, 0, δ}];
```

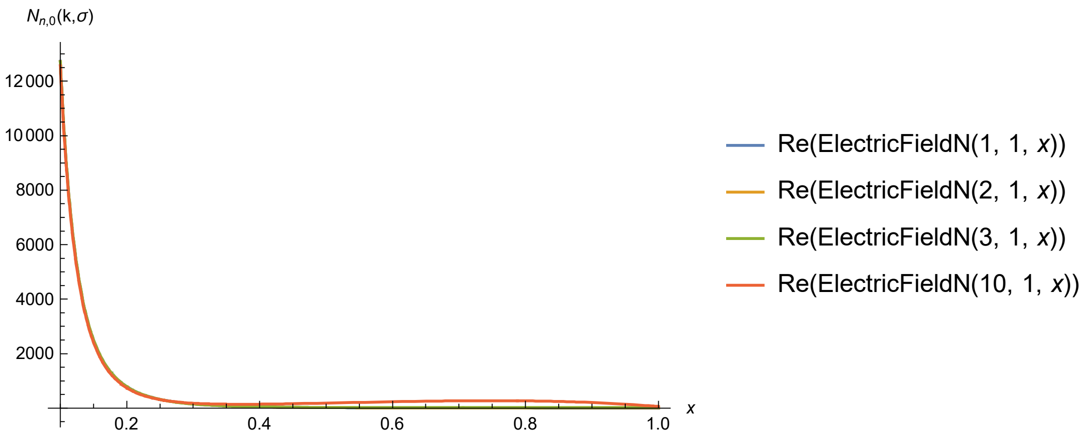
Since our goal is to investigate the dependance of the electric field on the z axis ($\tau = 0$), the only component of the N vector function which contributes to the field is the Nσ component (Both Nϕ and Nτ can be shown to be zero). Thus, it is the only component implemented (along the z axis). Besides only the terms with m = 0 contribute since on the z axis there is no dependance with ϕ component.

```
In[ ]:= ElectricFieldN[n_, k_, σ_] := -4 n / (ⅈ k σ^2)^2 (PinneyV[n, 0, ⅈ k σ^2] - PinneyV[n - 1, 0, ⅈ k σ^2]
            + (ⅈ k σ^2) PinneyV[n, 0, ⅈ k σ^2]);

In[ ]:= ElectricFieldN[0, k_, σ_] := 0;

In[ ]:= IncElectricFieldN[n_, k_, σ_] :=
          -4 n / (ⅈ k σ^2)^2 (PinneyS[n, 0, ⅈ k σ^2] - PinneyS[n - 1, 0, ⅈ k σ^2]
            + (ⅈ k σ^2) PinneyS[n, 0, ⅈ k σ^2]);
        IncElectricFieldN[0, k_, σ_] := 0;
```

Plot of the functions

In[ ]:=
```
Plot[{Re[ElectricFieldN[1, 1, x]], Re[ElectricFieldN[2, 1, x]],
  Re[ElectricFieldN[3, 1, x]], Re[ElectricFieldN[10, 1, x]]}, {x, 0.1, 1},
 PlotRange → Full, PlotLegends → "Expressions", AxesLabel → {x, "Nn,0(k,σ)"}]
```

Out[ ]=



Due to similarity and the amount of information given in the article, the article Theory of Nanometric Optical Tweezer of Novotny and Et al. was chosen as a reference to replicate it's results. We have in this case:
index of refraction of gold: 0.15659+*i*4.8808
Index of refraction of water: 1.3290+*i*1.4780e-7
Wavelenght: 810 nm
Tip radius: 5 nm
Predicted enhancement at the tip: 3000 for plane wave polarized along the paraboloid axis

In[ ]:=
```
σ0 = (10)^(1/2);
n1 = 1.3300 + ⅈ 1.5600 × 10^-7;
n2 = 0.13883 + ⅈ 4.4909;
k1 = ((2 π) / 750) n1;
k2 = ((2 π) / 750) n2;
```

Here starts the method for solving

the boundary conditions

```
In[•]:= FilaN[p_] := {NS[1, k2, σ0, p], -NV[1, k1, σ0, p], NS[2, k2, σ0, p], -NV[2, k1, σ0, p],
          NS[3, k2, σ0, p], -NV[3, k1, σ0, p], NS[4, k2, σ0, p], -NV[4, k1, σ0, p],
          NS[5, k2, σ0, p], -NV[5, k1, σ0, p], NS[6, k2, σ0, p], -NV[6, k1, σ0, p],
          NS[7, k2, σ0, p], -NV[7, k1, σ0, p], NS[8, k2, σ0, p], -NV[8, k1, σ0, p],
          NS[9, k2, σ0, p], -NV[9, k1, σ0, p], NS[10, k2, σ0, p], -NV[10, k1, σ0, p]
          , NS[11, k2, σ0, p], -NV[11, k1, σ0, p], NS[12, k2, σ0, p], -NV[12, k1, σ0, p],
          NS[13, k2, σ0, p], -NV[13, k1, σ0, p], NS[14, k2, σ0, p], -NV[14, k1, σ0, p],
          NS[15, k2, σ0, p], -NV[15, k1, σ0, p], NS[16, k2, σ0, p], -NV[16, k1, σ0, p],
          NS[17, k2, σ0, p], -NV[17, k1, σ0, p], NS[18, k2, σ0, p], -NV[18, k1, σ0, p],
          NS[19, k2, σ0, p], -NV[19, k1, σ0, p], NS[20, k2, σ0, p], -NV[20, k1, σ0, p]};

In[•]:= Length[FilaN[0]]

Out[•]= 40

In[•]:= FilaM[p_] :=
         {k2 MS[1, k2, σ0, p], -k1 MV[1, k1, σ0, p], k2 MS[2, k2, σ0, p], -k1 MV[2, k1, σ0, p],
          k2 MS[3, k2, σ0, p], -k1 MV[3, k1, σ0, p], k2 MS[4, k2, σ0, p], -k1 MV[4, k1, σ0, p],
          k2 MS[5, k2, σ0, p], -k1 MV[5, k1, σ0, p], k2 MS[6, k2, σ0, p], -k1 MV[6, k1, σ0, p],
          k2 MS[7, k2, σ0, p], -k1 MV[7, k1, σ0, p], k2 MS[8, k2, σ0, p], -k1 MV[8, k1, σ0, p],
          k2 MS[9, k2, σ0, p], -k1 MV[9, k1, σ0, p], k2 MS[10, k2, σ0, p], -k1 MV[10, k1, σ0, p]
          , k2 MS[11, k2, σ0, p], -k1 MV[11, k1, σ0, p], k2 MS[12, k2, σ0, p], -k1 MV[12, k1, σ0, p],
          k2 MS[13, k2, σ0, p], -k1 MV[13, k1, σ0, p], k2 MS[14, k2, σ0, p], -k1 MV[14, k1, σ0, p],
          k2 MS[15, k2, σ0, p], -k1 MV[15, k1, σ0, p], k2 MS[16, k2, σ0, p], -k1 MV[16, k1, σ0, p],
          k2 MS[17, k2, σ0, p], -k1 MV[17, k1, σ0, p], k2 MS[18, k2, σ0, p], -k1 MV[18, k1, σ0, p],
          k2 MS[19, k2, σ0, p], -k1 MV[19, k1, σ0, p], k2 MS[20, k2, σ0, p], -k1 MV[20, k1, σ0, p]};

In[•]:= Length[FilaM[0]]

Out[•]= 40

In[•]:= BvectorN[θ_, p_] := Sum[RealFocusedGTM[n, 0, θ] × NS[n, k1, σ0, p], {n, 1, 20}];
        BvectorM[θ_, p_] := Sum[RealFocusedGTM[n, 0, θ] k1 MS[n, k1, σ0, p], {n, 1, 20}];
```

In[ ]:=
```
Matrixm := {FilaN[0], FilaM[1], FilaN[2], FilaM[3], FilaN[4], FilaM[5],
   FilaN[6], FilaM[7], FilaN[8], FilaM[9], FilaN[10], FilaM[11], FilaN[12],
   FilaM[13], FilaN[14], FilaM[15], FilaN[16], FilaM[17], FilaN[18], FilaM[19]
   , FilaN[20], FilaM[21], FilaN[22], FilaM[23], FilaN[24], FilaM[25], FilaN[26],
   FilaM[27], FilaN[28], FilaM[29], FilaN[30], FilaM[31], FilaN[32], FilaM[33],
   FilaN[34], FilaM[35], FilaN[36], FilaM[37], FilaN[38], FilaM[39]};

rtMatrixm = Rationalize[Matrixm] // MatrixForm

Bvec[θ_] := {BvectorN[θ, 0], BvectorM[θ, 1], BvectorN[θ, 2], BvectorM[θ, 3],
   BvectorN[θ, 4], BvectorM[θ, 5], BvectorN[θ, 6], BvectorM[θ, 7],
   BvectorN[θ, 8], BvectorM[θ, 9], BvectorN[θ, 10], BvectorM[θ, 11],
   BvectorN[θ, 12], BvectorM[θ, 13], BvectorN[θ, 14], BvectorM[θ, 15],
   BvectorN[θ, 16], BvectorM[θ, 17], BvectorN[θ, 18], BvectorM[θ, 19]
   , BvectorN[θ, 20], BvectorM[θ, 21], BvectorN[θ, 22], BvectorM[θ, 23],
   BvectorN[θ, 24], BvectorM[θ, 25], BvectorN[θ, 26], BvectorM[θ, 27],
   BvectorN[θ, 28], BvectorM[θ, 29], BvectorN[θ, 30], BvectorM[θ, 31],
   BvectorN[θ, 32], BvectorM[θ, 33], BvectorN[θ, 34], BvectorM[θ, 35],
   BvectorN[θ, 36], BvectorM[θ, 37], BvectorN[θ, 38], BvectorM[θ, 39]};
```

Out[ ]//MatrixForm=

$8.33949\times10^{-12}+2.52702\times10^{-13}\,i$	$-2.0065\times10^{-11}-1.50488\times10^{-11}\,i$	$3.33579\times10^{-11}+1.01$
$0.341234-0.0231104\,i$	$0.627503+0.119232\,i$	$1.15181-0.081$
$103.93-0.779522\,i$	$1072.69-373.781\,i$	$354.753-3.52$
$0.0005805-0.0000554998\,i$	$0.0000564965+0.0000107799\,i$	$0.00319537-0.00$
$0.0882327-0.00308486\,i$	$0.0983578-0.0213748\,i$	$0.512189-0.01$
$-3.57572\times10^{-8}+5.72678\times10^{-9}\,i$	$2.67061\times10^{-10}+5.09773\times10^{-11}\,i$	$-3.71193\times10^{-7}+6.0$
$-5.75595\times10^{-6}+5.79489\times10^{-7}\,i$	$4.65536\times10^{-7}-9.56531\times10^{-8}\,i$	$-0.0000661724+7.0$
$6.3602\times10^{-13}-1.42989\times10^{-13}\,i$	$4.01411\times10^{-16}+7.66338\times10^{-17}\,i$	$9.64551\times10^{-12}-2.21$
$1.04482\times10^{-10}-1.72065\times10^{-11}\,i$	$7.0004\times10^{-13}-1.40737\times10^{-13}\,i$	$1.77243\times10^{-9}-3.03$
$-5.32149\times10^{-18}+1.5487\times10^{-18}\,i$	$2.9182\times10^{-22}+5.5716\times10^{-23}\,i$	$-1.06037\times10^{-16}+3.1$
$-8.84624\times10^{-16}+2.02975\times10^{-16}\,i$	$5.0903\times10^{-19}-1.01158\times10^{-19}\,i$	$-1.97962\times10^{-14}+4.6$
$2.57809\times10^{-23}-9.26975\times10^{-24}\,i$	$1.2455\times10^{-28}+2.3781\times10^{-29}\,i$	$6.36158\times10^{-22}-2.32$
$4.32306\times10^{-21}-1.27903\times10^{-21}\,i$	$2.17285\times10^{-25}-4.28709\times10^{-26}\,i$	$1.20044\times10^{-19}-3.64$
$-8.12697\times10^{-29}+3.50398\times10^{-29}\,i$	$3.49081\times10^{-35}+6.66543\times10^{-36}\,i$	$-2.39074\times10^{-27}+1.6$
$-1.37261\times10^{-26}+5.00521\times10^{-27}\,i$	$6.09047\times10^{-32}-1.1958\times10^{-32}\,i$	$-4.54932\times10^{-25}+1.6$
$1.79663\times10^{-34}-9.10257\times10^{-35}\,i$	$6.91188\times10^{-42}+1.3198\times10^{-42}\,i$	$6.13602\times10^{-33}-3.14$
$3.05413\times10^{-32}-1.33329\times10^{-32}\,i$	$1.206\times10^{-38}-2.35947\times10^{-39}\,i$	$1.17607\times10^{-30}-5.23$
$-2.93466\times10^{-40}+1.72267\times10^{-40}\,i$	$1.01786\times10^{-48}+1.9436\times10^{-49}\,i$	$-1.14108\times10^{-38}+6.7$
$-5.01947\times10^{-38}+2.57206\times10^{-38}\,i$	$1.77606\times10^{-45}-3.46542\times10^{-46}\,i$	$-2.2016\times10^{-36}+1.14$
$3.68052\times10^{-46}-2.47852\times10^{-46}\,i$	$1.15817\times10^{-55}+2.21156\times10^{-56}\,i$	$1.60495\times10^{-44}-1.09$
$6.33359\times10^{-44}-3.75723\times10^{-44}\,i$	$2.02096\times10^{-52}-3.93496\times10^{-53}\,i$	$3.11644\times10^{-42}-1.87$
$-3.64982\times10^{-52}+2.80062\times10^{-52}\,i$	$1.04868\times10^{-62}+2.00251\times10^{-63}\,i$	$-1.76375\times10^{-50}+1.3$
$-6.31985\times10^{-50}+4.29858\times10^{-50}\,i$	$1.82996\times10^{-59}-3.55702\times10^{-60}\,i$	$-3.44675\times10^{-48}+2.3$
$2.92878\times10^{-58}-2.54981\times10^{-58}\,i$	$7.73513\times10^{-70}+1.47708\times10^{-70}\,i$	$1.55326\times10^{-56}-1.36$
$5.10436\times10^{-56}-3.95462\times10^{-56}\,i$	$1.34982\times10^{-66}-2.62007\times10^{-67}\,i$	$3.0555\times10^{-54}-2.40$
$-1.93738\times10^{-64}+1.90955\times10^{-64}\,i$	$4.73723\times10^{-77}+9.04618\times10^{-78}\,i$	$-1.11857\times10^{-62}+1.1$
$-3.40007\times10^{-62}+2.98816\times10^{-62}\,i$	$8.2669\times10^{-74}-1.60277\times10^{-74}\,i$	$-2.21583\times10^{-60}+1.9$
$1.07254\times10^{-70}-1.19654\times10^{-70}\,i$	$2.44728\times10^{-84}+4.67334\times10^{-85}\,i$	$6.69564\times10^{-69}-7.5$
$1.89666\times10^{-68}-1.88703\times10^{-68}\,i$	$4.2708\times10^{-81}-8.2719\times10^{-82}\,i$	$1.33647\times10^{-66}-1.34$
$-5.03097\times10^{-77}+6.36352\times10^{-77}\,i$	$1.08078\times10^{-91}+2.06388\times10^{-92}\,i$	$-3.37614\times10^{-75}+4.3$
$-8.9726\times10^{-75}+1.01051\times10^{-74}\,i$	$1.88612\times10^{-88}-3.65001\times10^{-89}\,i$	$-6.7959\times10^{-73}+7.76$
$2.01981\times10^{-83}-2.90747\times10^{-83}\,i$	$4.12703\times10^{-99}+7.88111\times10^{-100}\,i$	$1.44963\times10^{-81}-2.16$
$3.6374\times10^{-81}-4.64567\times10^{-81}\,i$	$7.20236\times10^{-96}-1.39276\times10^{-96}\,i$	$2.94608\times10^{-79}-3.81$
$-6.99678\times10^{-90}+1.15323\times10^{-89}\,i$	$1.37614\times10^{-106}+2.62794\times10^{-107}\,i$	$-5.34643\times10^{-88}+8.9$
$-1.2744\times10^{-87}+1.85308\times10^{-87}\,i$	$2.40162\times10^{-103}-4.64113\times10^{-104}\,i$	$-1.09877\times10^{-85}+1.6$
$2.10417\times10^{-96}-4.00722\times10^{-96}\,i$	$4.04158\times10^{-114}+7.71801\times10^{-115}\,i$	$1.70493\times10^{-94}-3.28$
$3.8851\times10^{-94}-6.47244\times10^{-94}\,i$	$7.05338\times10^{-111}-1.36228\times10^{-111}\,i$	$3.55122\times10^{-92}-6.01$
$-5.51601\times10^{-103}+1.22956\times10^{-102}\,i$	$1.0534\times10^{-121}+2.01163\times10^{-122}\,i$	$-4.72168\times10^{-101}+1.0$
$-1.03581\times10^{-100}+1.99555\times10^{-100}\,i$	$1.83841\times10^{-118}-3.54885\times10^{-119}\,i$	$-9.99984\times10^{-99}+1.9$
$1.26239\times10^{-109}-3.35497\times10^{-109}\,i$	$2.45306\times10^{-129}+4.68453\times10^{-130}\,i$	$1.13756\times10^{-107}-3.06$

In[ ]:=
```
Dimensions[Matrixm]
```

Out[ ]= {40, 40}

In[ ]:=
```
SquareMatrixQ[Matrixm]
```

Out[ ]= True

```
In[•]:= Length[Matrixm]

Out[•]= 40

In[•]:= CoefList[θ_] := LinearSolve[Matrixm, Rationalize[Bvec[θ]]];
        CoefList2[θ_] := Dot[Inverse[rtMatrixm], Rationalize[Bvec[θ]]];
```

In[ ]:= **Coeficientes = CoefList$\left[\frac{\pi}{4}\right]$ // MatrixForm**

LinearSolve: Result for LinearSolve of badly conditioned matrix {{$8.33949\times10^{-12}+2.52702\times10^{-13}\,i$, $-2.0065\times10^{-11}-1.50488\times10^{-11}\,i$, $3.33579\times10^{-11}+1.01081\times10^{-12}\,i$, $7.06047\times10^{-18}+2.0065\times10^{-11}\,i$, $2.5163\times10^{-11}-7.96515\times10^{-13}\,i$, ≪30≫, $2.31149\times10^{-8}-1.44469\times10^{-9}\,i$, $-1.05509\times10^{-8}+4.10369\times10^{-9}\,i$, $-5.36596\times10^{-16}-1.52494\times10^{-9}\,i$, $-2.19658\times10^{-8}+5.98613\times10^{-9}\,i$, $-3.2104\times10^{-9}+3.2104\times10^{-9}\,i$}, ≪38≫, {≪1≫}} may contain significant numerical errors.

Out[ ]//MatrixForm=

$$\begin{pmatrix}
0.0463882+0.0633506\,i \\
-0.00127062-0.00491427\,i \\
-0.0190405-0.0279924\,i \\
0.465151-0.0426629\,i \\
0.00732267+0.0113766\,i \\
-1.58145+0.299323\,i \\
-0.00261656-0.00421591\,i \\
2.49371-0.62011\,i \\
0.000860844+0.00140842\,i \\
-2.63856+0.667052\,i \\
-0.000258046-0.000417233\,i \\
2.18277-0.468672\,i \\
0.0000695581+0.000106664\,i \\
-1.5376+0.288751\,i \\
-0.0000165582-0.00002228\,i \\
0.93756-0.222788\,i \\
3.38296\times10^{-6}+3.25124\times10^{-6}\,i \\
-0.459957+0.177923\,i \\
-5.61589\times10^{-7}-6.4125\times10^{-8}\,i \\
0.146329-0.0965055\,i \\
6.52626\times10^{-8}-1.56512\times10^{-7}\,i \\
-0.00073784+0.0190576\,i \\
-1.51962\times10^{-9}+6.53886\times10^{-8}\,i \\
-0.0309958+0.0134363\,i \\
-1.63011\times10^{-9}-1.73573\times10^{-8}\,i \\
0.0196806-0.0111524\,i \\
5.14445\times10^{-10}+3.49552\times10^{-9}\,i \\
-0.00668929+0.00198373\,i \\
-9.75643\times10^{-11}-5.54262\times10^{-10}\,i \\
0.00123997+0.00168704\,i \\
1.33291\times10^{-11}+6.89959\times10^{-11}\,i \\
-0.0000488066-0.00138101\,i \\
-1.33803\times10^{-12}-6.56006\times10^{-12}\,i \\
-0.0000314069+0.00050622\,i \\
9.49726\times10^{-14}+4.50193\times10^{-13}\,i \\
6.7489\times10^{-6}-0.000107999\,i \\
-4.29512\times10^{-15}-1.99243\times10^{-14}\,i \\
-4.26696\times10^{-7}+0.0000130482\,i \\
9.35764\times10^{-17}+4.28025\times10^{-16}\,i \\
-6.66177\times10^{-9}-6.97651\times10^{-7}\,i
\end{pmatrix}$$

In[ ]:= **Matrixm // Det**

Out[ ]= 0. + 0. i

In[ ]:=

Coefficients test. Boundary conditions

In[ ]:= **Coef = Coeficientes[[1]]**

Out[ ]= {0.0463882 + 0.0633506 i, -0.00127062 - 0.00491427 i, -0.0190405 - 0.0279924 i, 0.465151 - 0.0426629 i, 0.00732267 + 0.0113766 i, -1.58145 + 0.299323 i, -0.00261656 - 0.00421591 i, 2.49371 - 0.62011 i, 0.000860844 + 0.00140842 i, -2.63856 + 0.667052 i, -0.000258046 - 0.000417233 i, 2.18277 - 0.468672 i, 0.0000695581 + 0.000106664 i, -1.5376 + 0.288751 i, -0.0000165582 - 0.00002228 i, 0.93756 - 0.222788 i, $3.38296\times10^{-6} + 3.25124\times10^{-6}$ i, -0.459957 + 0.177923 i, $-5.61589\times10^{-7} - 6.4125\times10^{-8}$ i, 0.146329 - 0.0965055 i, $6.52626\times10^{-8} - 1.56512\times10^{-7}$ i, -0.00073784 + 0.0190576 i, $-1.51962\times10^{-9} + 6.53886\times10^{-8}$ i, -0.0309958 + 0.0134363 i, $-1.63011\times10^{-9} - 1.73573\times10^{-8}$ i, 0.0196806 - 0.0111524 i, $5.14445\times10^{-10} + 3.49552\times10^{-9}$ i, -0.00668929 + 0.00198373 i, $-9.75643\times10^{-11} - 5.54262\times10^{-10}$ i, 0.00123997 + 0.00168704 i, $1.33291\times10^{-11} + 6.89959\times10^{-11}$ i, -0.0000488066 - 0.00138101 i, $-1.33803\times10^{-12} - 6.56006\times10^{-12}$ i, -0.0000314069 + 0.00050622 i, $9.49726\times10^{-14} + 4.50193\times10^{-13}$ i, $6.7489\times10^{-6} - 0.000107999$ i, $-4.29512\times10^{-15} - 1.99243\times10^{-14}$ i, $-4.26696\times10^{-7} + 0.0000130482$ i, $9.35764\times10^{-17} + 4.28025\times10^{-16}$ i, $-6.66177\times10^{-9} - 6.97651\times10^{-7}$ i}

In[ ]:= **Coef[[1]]**

Out[ ]= 0.0463882 + 0.0633506 i

Plot of the scattered field along the z axis (this may take a while)

In[ ]:=
```
ScatteredField[σ_] := Re[Sum[Coef[[2 n]] × ElectricFieldN[n, k1, σ], {n, 1, 20}]];
IncidentField[σ_] :=
  Re[Sum[RealFocusedGTM[n, 0, π/4] × IncElectricFieldN[n, k1, σ], {n, 1, 20}]];
```

In[ ]:= $\left(\frac{\text{ScatteredField}[\sigma_0] + \text{IncidentField}[\sigma_0]}{\text{IncidentField}[\sigma_0]}\right)^2$

Out[ ]= 119.927

In[ ]:=
```
data = Table[{x, ((ScatteredField[x] + IncidentField[x]) / IncidentField[x])^2},
    {x, Range[σ0, 20, 0.1]}];
Export["data.dat", data, "Table"]
```

Out[ ]= data.dat

In[ ]:= Plot[((ScatteredField[x] + IncidentField[x])/IncidentField[x])^2, {x, σ_0, 12}, PlotRange → All]

Out[ ]=