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CLAUDIA REBOUÇAS LIMA FERNANDES

MEAN CURVATURE FLOW OF GRAPHS WITH ASYMPTOTIC DIRICHLET CONDITIONS IN CARTAN-HADAMARD MANIFOLDS

FORTALEZA

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Orientador: Prof. Dr. Jorge Herbert Soares de Lira.

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I dedicate this work to my father José (*in memoriam*), my mother Raimunda Maria, my husband Alexandre, my son Alexandre and my daugther Marcela.

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"The first gulp from the glass of natural sciences will turn you into an atheist, but at the bottom of the glass **God is waiting for you**."(HEISENBERG, 1974, p. 213)

RESUMO

Este trabalho aborda a evolução pela curvatura média de gráficos de Killing em variedades de Cartan-Hadamard com condições de Dirichlet assintóticas. Para mostrar a existência do fluxo, obtém-se estimativas *a priori*, as quais asseguram o uso da teoria de equações diferenciais parciais parabólicas. Estuda-se a regularidade da solução obtida, construindo-se barreiras nos pontos da fronteira assintótica. Tal construção é possível ao considerar-se um conceito de convexidade no infinito. Esta tese trata ainda, do problema mais geral da evolução de gráficos por uma função de suas curvaturas principais. Neste caso, sob algumas condições, obtém-se uma estimativa *a priori* (interior) de gradiente.

Palavras-chave: variedade de Cartan-Hadamard; fluxo pela curvatura média; fronteira assintótica; gráficos.

ABSTRACT

This work approach the mean curvature evolution of Killing graphs in Cartan-Hadamard manifolds with asymptotic Dirichlet conditions. In order to proof the existence of the flow, *a priori* estimates are obtained, which ensure the use of the theory of parabolic partial differential equations. The regularity of the obtained solution is studied, building barriers at the points of the asymptotic frontier. Such a construction is possible when considering a concept of convexity at infinity. This thesis also deals with the more general problem of the evolution of graphs by a function of their principal curvatures. In this case, under some conditions, an *a priori* (interior) gradient estimate is obtained.

Keywords: Cartan-Hadamard manifold; mean curvature flow; asymptotic boundary; graphs.

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CAPES	Coordenação de Aperfeiçoamento de Pessoal de Nível Superior
f-flow	Flow by the function f
MCF	Mean curvature flow
MMCF	Modified mean curvature flow

CONTENTS

1 INTRODUCTION

Many geometric flows have attracted the attention of mathematicians in recent years. Besides the Ricci flow and the inverse of the mean curvature flow, flows by curvature functions are important examples of the geometric flows.

We say that a positive differentiable concave function f is a curvature function if f is symmetric in λ_i, where $\lambda = (\lambda_1, \cdots, \lambda_n)$ belongs to the domain of the function f. Such a domain has special characteristics, which we present more precisely later. Flow by a curvature function f is the term that we use to describe the evolution of a hypersurface whose normal velocity is given by f.

Given P^n and M^{n+1} Riemannian manifolds and given $\Psi_0 : P \to M$ an immersion and

$$\Psi : P \times [0, T) \to M$$

a one parameter family of immersions, we say that Ψ defines a flow by function f (f-flow for short) of Ψ_0 if it is solution of

$$\begin{cases} \frac{\partial \Psi}{\partial t}(x,t) = f(\kappa(\Psi(x,t)))N \\ \Psi(x,0) = \Psi_0(x), \end{cases} \tag{1}$$

where $N(\cdot, t)$ is the unit normal vector field of the immersion $\Psi_t := \Psi(\cdot, t)$ and $\kappa(\Psi(\cdot, t))$ is the vector which coordinates are the principal curvatures of the Ψ_t.

The main examples of curvature functions are the r-th root of the higher order mean curvature functions

$$S_r(\kappa) = \sum_{i_1 < i_2 < \cdots < i_r} \kappa_{i_1}\kappa_{i_2} \ldots \kappa_{i_r}.$$

Among these examples the one by the mean curvature ($H_1 = S_1(\kappa)$) has stood out for being intensively developed in several directions. For instance, Huisken proved in [17] that every n-dimensional ($n \geq 2$) compact convex hypersurface evolving by mean curvature flow in $\mathbb{R}^{n+1}$ must shrink to a round point in finite time. He also proved in [19] that if N^{n+1} is a Riemannian manifold and Σ is a n-dimensional "convex enough"submanifold then Σ must shrink to a point. Here, the expression "convex enough"is used to indicate that the initial hypersurface must be convex enough to overcome the obstructions imposed by the geometry of N. Other references for the convergence and regularity of the MCF are [12], [15], [29], among others.

The study of the singularities of the flow induces a natural interest for a special type of solutions known as mean curvature flow solitons. This interest is justified by the fact that in the Euclidean space these solutions provided relevant information about the singularities. There is a vast literature about this subject. The reference [2] stands out for the detailed study of solitons in a large class of Riemannian ambient spaces.

Another interesting topic explored has been the evolution of integer graphics. In [13], the authors considered Σ_0 a entire graph above $\mathbb{R}^n$ and they proved that any polynomial growth rate for the height and the gradient of the initial hypersurface Σ_0 is preserved during the evolution by MCF. They also proved that in the case of Lipschitz initial data with linear growth, the problem (1) has a solution for all $t > 0$. Unterberger in [30], considered as initial surfaces Σ_0 which, in the upper half space model of hyperbolic space, $\mathbb{H}^{n+1}$, can be written as entire Euclidean radial graphs above $\mathbb{S}^n_+ = \mathbb{S}^n_+(1)$, the Euclidean upper hemisphere of radius one centered at origin. Then, he proved that

Theorem 1.1 ([30], Theorem 3.2) *If $\Sigma_0 = \Psi_0(B^n)$ is a locally Lipschitz continuous entire radial graph over $\mathbb{S}^n_+ \subset \mathbb{H}^{n+1}$, then the problem* (1) *has a smooth solution $\Sigma_t = \Psi_t(B^n)$ for all $t > 0$. Moreover, each Σ_t is an entire graph over $\mathbb{S}^n_+$.*

Then, assuming a bound for the gradient and the geodesic height of the initial surface, Unterberger used hyperspheres as barriers and he also proved the following convergence result:

Theorem 1.2 ([30],Theorem 3.3) *If Σ_0 has bounded gradient and hyperbolic height over $\mathbb{S}^n_+$, then, under MCF, Σ_t converges in C^∞ to $\mathbb{S}^n_+$.*

We remember that $\mathbb{H}^{n+1}$ is a Cartan-Hadamard manifold, that is, $\mathbb{H}^{n+1}$ is a complete, connected, simply connected Riemannian manifold and its sectional curvature is non-positive. It is well-known that is possible to define a boundary at infinity for a Cartan-Hadamard manifold P by addition of a sphere at infinity, which we denote by $\partial_\infty P$. Then, we define a topology in $\bar{P} = P \cup \partial_\infty P$ such that $\bar{P}$ endowed with this topology is compact. We call $\partial_\infty A$ of asymptotic boundary of A, for every $A \subset P$. In this context, a natural question is what happens to the asymptotic boundary of the initial surface during evolution by mean curvature. Even if we fix the asymptotic boundary during evolution, it is interesting to know what is the regularity in $\bar{P}$ of the solution obtained.

In [23], the authors introduced the modified mean curvature flow (MMCF, for short) in the upper half space model of hyperbolic space, $\mathbb{H}^{n+1}$, and as Unterberger, they considered the entire Euclidean radial graphs above $\mathbb{S}^n_+$ as initial hypersurface Σ_0. In this case, the problem studied is

$$\begin{cases} \frac{\partial \Psi}{\partial t}(x,t) = (H - \sigma) N_{\mathbb{H}}, \quad (x,t) \in \mathbb{S}^n_+ \times [0, +\infty) \\ \Psi(x,0) = \Sigma_0, \quad x \in \mathbb{S}^n_+ \\ \Psi(x,t) = \Omega(x), \quad x \in \partial_\infty \mathbb{S}^n_+,\ t \in [0,\infty) \end{cases} \tag{2}$$

where $\sigma \in (-1,1)$, $N_{\mathbb{H}}$ is the normal of the Ψ_t, H is the scalar mean curvature of the Ψ_t with respect to the hyperbolic metric, and $\Omega = \partial_\infty \Sigma_0$. Under conditions imposed on the initial hypersurfaces Σ_0 and its asymptotic boundary, they showed the existence and unicity of solution for the above problem. Moreover, they proved that the solution is continuous in $\mathbb{S}^n_+ \cup \partial_\infty \mathbb{S}^n_+$. In [24], the authors considered the same problem, but they removed the geometric conditions imposed in the initial surface in [23]. In [24], they proved that the MMCF starting from an entire locally Lipschitz radial graph exists and remains radial graph for every $t > 0$. However, in this case they do not have any information about regularity of the solution in $(x,t) \in \partial_\infty \mathbb{S}^n_+ \times [0, +\infty)$.

In the first part of this thesis, we consider (P^n, g) a Cartan-Hadamard manifold and $M = P \times_\varrho \mathbb{R}$ a Riemannian manifold endowed with the warped metric $\bar{g} = \varrho^2(x)\mathrm{d}s^2 + g$. Given a function $\varphi \in C^\infty(P) \cap C^0(\overline{P})$, under conditions imposed on the geometries of P and M, we study the evolution by mean curvature of the Killing graph of the φ which we denote by Σ_0. In [6], the authors establish some conditions under which the warped product $M = P \times_\varrho \mathbb{R}$ is also a Cartan-Hadamard manifold. In this context, it makes sense to define the Killing graph Γ of φ and, in some cases, it is possible to verify that Γ is the asymptotic boundary $\partial_\infty \Sigma_t$ of each graph Σ_t in the evolving family. Then, our first objective is to solve the following problem

$$\begin{cases} \frac{\partial \Psi}{\partial t}(x,t) = nH(\Psi(x,t)), \quad \text{in} \quad P \times (0,\infty) \\ \Psi(x,0) = \Psi_0(x) = \Phi(x, \varphi(x)), \quad \text{in} \quad P \times \{0\} \\ \Psi(x,t) = \Phi(x, \varphi(x)), \quad \text{on} \quad \partial_\infty P \times [0,\infty), \end{cases} \tag{3}$$

where Φ is the flow map of the Killing vector field $X = \partial_s$. In fact, we will solve the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i, \quad \text{in} \quad P \times [0,\infty) \\ u(x,0) = \varphi(x), \quad \text{in} \quad P \times \{0\} \\ u(x,t) = \varphi(x) \quad \text{if} \quad x \in \partial_\infty P, \quad t \in [0,+\infty) \end{cases} \tag{4}$$

and if u solves (4), then $\Psi(x,t) = \Phi(x, u(x,t))$ solves the problem (3).

In order to investigate the regularity of a solution u of the problem (4) in $\overline{P} \times [0, +\infty)$, we use a concept of convexity at infinity. In [7], the author used the concept of convex neighborhood to study the regularity of solutions of the Laplacian operator at infinity. Unfortunately, the technique used there heavily depends on the linearity of the operator. In view of this, in [28], the authors introduced the notion of strictly convex manifolds. Basically, if P satisfies the strict convexity condition (SC condition), then for any point $x \in \partial_\infty P$ we can extract a neighborhood U of the x in $\overline{P}$ such that $P \setminus U$ is convex. Assuming that P satisfies the SC condition, we use this property for building barriers at infinity and consequently to obtain the regularity of the solution in $\overline{P} \times [0, +\infty)$.

Let us summarize the conditions under which we will prove our main result and its consequences. We consider P^n a Cartan-Hadamard manifold and $M = P \times_\varrho \mathbb{R}$ a warped product, with $\varrho \in C^\infty(P)$ a convex function satisfying (6) and (7). Fixed a point $o \in P$, suppose that the radial sectional curvatures along geodesics rays issuing from o satisfies (11) for $\xi \in C^\infty([0,\infty))$ satisfies (10). In addition, we suppose that there exists positive constants L, L_1 such that

$$\overline{\mathrm{Ric}} \geq -L_1 \bar{g} \qquad \text{and} \qquad \mathrm{Ric}_g + \nabla^2 \log \varrho \geq -Lg.$$

In this context, our main result is the following:

Theorem 1.3 *Let P and M be Riemannian manifolds satisfying the conditions cited in the paragraph above. Suppose that P satisfies the SC condition at infinity and its sectional curvatures satisfies $K_P \leq -\kappa^2 < 0$. If Σ_0 is the Killing graph of $\varphi \in C^\infty(P) \cap C(\bar{P})$, then there exists a unique solution $\Psi \in C^\infty(P \times (0,\infty)) \cap C(\bar{P} \times [0,\infty))$ for the problem*

$$\begin{cases} \frac{\partial \Psi}{\partial t}(x,t) = nH(\Psi(x,t)), & in \quad P \times (0,\infty) \\ \Psi(x,0) = \Psi_0(x) = \Phi(x,\varphi(x)), & in \quad P \times \{0\} \\ \Psi(x,t) = \Phi(x,\varphi(x)), & on \quad \partial_\infty P \times [0,\infty). \end{cases} \tag{5}$$

In [28], Ripoll and Telichevesky showed that if P is rotationally symmetric and satisfieis $K_P \leq -\kappa^2 < 0$, then P satisfies the SC condition. Moreover, they also proved that if there exists positive constants κ and ϵ such that

$$\frac{e^{2\kappa r(x)}}{r(x)^{2+2\epsilon}} \leq K_P(x) \leq -\kappa^2 < 0$$

for every $x \in P$ such that $r(x) = \mathrm{d}(x,o) \geq R^*$, for R^* large enough, P also satisfies the SC condition. This gives us the following consequences of the Theorem 1.3:

Corollary 1.4 *Let us suppose that P is rotationally symmetric and satisfies $K_P \leq -\kappa^2 < 0$. If Σ_0 is the Killing graph of $\varphi \in C^\infty(P) \cap C(\bar{P})$, then there exists a unique solution $\Psi \in C^\infty(P \times (0,\infty)) \cap C(\bar{P} \times [0,\infty))$ for the problem* (5)

Corollary 1.5 *Suppose that*

$$\frac{e^{2\kappa r(x)}}{r(x)^{2+2\epsilon}} \leq K_P(x) \leq -\kappa^2 < 0$$

for every $x \in P$ such that $r(x) = \mathrm{d}(x,o) \geq R^*$, for R^* large enough, where $\kappa, \epsilon > 0$ are constant. If Σ_0 is the Killing graph of $\varphi \in C^\infty(P) \cap C(\bar{P})$, then there exists a unique solution $\Psi \in C^\infty(P \times (0,\infty)) \cap C(\bar{P} \times [0,\infty))$ for the problem (5).

In order to prove the Theorem (1.3), we use a process of exhaustion. To do this, we need to solve the problem in compact parabolic cylinders.

Next, we will describe how the thesis is organized. We divide the text in chapters. The Chapters 2, 3, 4, 5 and 6 are devoted to the prove the Theorem (1.3). In Chapter 2, we present the initial concepts of the problem, the geometric structure and we deduce evolution equation for some important functions. In Chapter 3, we obtain a priori estimates for height, gradient and curvature. For each one, we consider the problem in $B_R(o) \times [0, T)$ and deduce the estimates in the parabolic cylinder $B_{R'}(o) \times [0, T_R)$ with $0 < R' < R$ properly chosen. In Chapter 4, we solve the problem (4) in $B_R(o) \times [0, T)$. In Chapter 5, we show that is possible to build the barriers at infinity. In Chapter 6, we use an exhaustion argument to construct the function whose graph solves the problem (5). Then, we use the barriers for proving the regularity in $\overline{P}$ of the constructed solution. In Chapter 7, we consider $M = P \times \mathbb{R}$ a Riemannian product, with P not necessarily a Cartan-Hadamard manifold, we return to the problem of the flow by general curvature function and we obtain a priori interior gradient estimate using the technique due Korevaar.

2 PRELIMINARIES

In this chapter, we fix notations and concepts used in the whole text. It is also obtained evolution equation for some useful functions.

2.1 The geometric setting

We recall that a Riemannian manifold N is called a Cartan-Hadamard manifold if N is simply conneted, connected, complete and has $K(q,\sigma) \leq 0$ for all $q \in N$ and $\sigma \subset T_qN$. Throughout the text we denote by P a n-dimensional Cartan-Hadamard manifold with sectional curvature $K_P \leq -\kappa^2 < 0$ and Gaussian global coordinates $(r,\vartheta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ defined with respect to a fixed point $o \in P$. The existence of this global coordinates system is ensured by Cartan-Hadamard theorem. We consider a function $\varrho \in C^\infty(P)$ satisfying

$$\varrho(x) = \varrho(r(x)) = \varrho(\operatorname{dist}(o,x)), \text{ for } x \in P. \tag{6}$$

$$\begin{aligned} &\varrho(r) > 0,\ \varrho'(r) > 0 \quad \text{for} \quad r > 0, &(7)\\ &\varrho(0) = 1,\ \varrho^{(2k+1)}(0) = 0, \quad \text{for} \quad k \in \mathbb{N}, &(8)\\ &\liminf_{r\to\infty} \frac{\varrho'(r)}{\varrho(r)} > 0. &(9)\end{aligned}$$

We also consider $\xi \in C^\infty([0,\infty))$ a function satisfying the following conditions

$$\begin{aligned} &\xi(r) > 0, \quad \text{for} \quad r > 0,\\ &\xi'(0) = 1,\\ &\xi^{(2k)}(0) = 0, \quad \text{for} \quad k \in \mathbb{N}\cdot \end{aligned} \tag{10}$$

We suppose that the radial sectional curvatures along geodesics rays issuing from o satisfies

$$K(\partial_r \wedge \mathsf{v}) \geq -\frac{\xi''(r)}{\xi(r)} \tag{11}$$

for all $r > 0$, $\mathsf{v} \in TM, \mathsf{v} \perp \partial_r$. It follows from Hessian comparison theorem [1] that

$$\nabla^P \nabla^P r \leq \frac{\xi'(r)}{\xi(r)} \left(g - \mathrm{d}r \otimes \mathrm{d}r\right). \tag{12}$$

We also suppose that

$$\left|\frac{\partial_r \varrho}{\varrho}\right| \leq \frac{\xi'(r)}{\xi(r)}. \tag{13}$$

Finally, our ambient manifold is the product

$$M = P \times_{\varrho} \mathbb{R}$$

endowed with the warped metric $\bar{g} = \varrho^2(x)\mathrm{d}s^2 + g$ where s is the natural coordinate in $\mathbb{R}$ and g is the induced Riemannian metric in each totally geodesic leaf $P \times \{s\}$, $s \in \mathbb{R}$. The coordinate vector field $X = \partial_s$ is a Killing vector field whose norm $|X| = \varrho$ is preserved along the flow lines. We also assume that there exists constants $L, L_1 > 0$ such that

$$\overline{\mathrm{Ric}} \geq -L_1 \bar{g} \qquad \text{and} \qquad \mathrm{Ric}_g + \nabla^2 \log \varrho \geq -Lg.$$

2.2 The mean curvature flow

In order to define the mean curvature flow of the Killing graphs we recall that the Killing graph of a function $u \in C^2(P)$ is by definition the hypersurface in M given by

$$\Sigma[u] = \{\Phi(x, u(x)) : x \in P\}, \tag{14}$$

where $\Phi : P \times \mathbb{R} \longrightarrow M$ is flow map of vector field X.

As we said before, a one parameter family of functions $u : P \times [0, T) \to \mathbb{R}$, $T > 0$, defines a mean curvature flow of Killing graphs

$$\Psi(x, t) = \Phi(x, u(x, t)) \tag{15}$$

if and only if

$$\partial_t \Psi = n\mathbf{H}, \tag{16}$$

where $\mathbf{H} = HN$ is the mean curvature vector of the Killing graph $\Sigma_t := \Sigma[u(\cdot,\, t)]$. Here, H is the scalar mean curvature of Σ_t calculated with respect to the orientation given by the unit normal vector field

$$N = N|_{\Psi(\cdot,\, t)} = \frac{1}{W}(\varrho^{-2}X - \nabla^P u), \tag{17}$$

where $W = (\varrho^{-2} + |\nabla^P u|^2)^{\frac{1}{2}}$ and ∇^P denotes the Riemannian gradient in (P, g). If (x^i) is a coordinate system in P, then the induced metric in $\Sigma_t = \Sigma[u(\cdot,\, t)]$ and its inverse have components

$$\sigma_{ij} = g_{ij} + \varrho^2(x) u_i u_j \qquad \text{and} \qquad \sigma^{ij} = g^{ij} - \frac{u^i u^j}{W^2},$$

respectively. Moreover, the volume element in $\Sigma_t = \Sigma[u(\cdot,\, t)]$ is given by

$$\mathrm{d}\Sigma_t = \varrho\sqrt{\varrho^{-2} + |\nabla^P u|^2}\, \mathrm{d}P. \tag{18}$$

Given a domain $\Omega \subset P$ we define the constrained area functional

$$\mathcal{A}[u] = \int_\Omega \varrho\sqrt{\varrho^{-2} + |\nabla^P u|^2}\, \mathrm{d}P.$$

For any $v \in C_0^\infty(\Omega)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{A}[u+tv] = -\int_\Omega \Big(\mathrm{div}_P\Big(\frac{\nabla^P u}{W}\Big) + \Big\langle \nabla^P \log\varrho, \frac{\nabla^P u}{W}\Big\rangle\Big) v\varrho\, \mathrm{d}P,$$

where the differential operators ∇^P and div_P are taken with respect to the metric g in P. Then the Euler-Lagrange equation of the functional $\mathcal{A}$ is

$$nH = \mathrm{div}_P\Big(\frac{\nabla^P u}{W}\Big) + \Big\langle \nabla^P \log\varrho, \frac{\nabla^P u}{W}\Big\rangle, \tag{19}$$

where H is the scalar mean curvature of the Killing graph of u. Once differentiating (15) with respect to t we have

$$\partial_t \Psi = \partial_t u\, X,$$

we conclude that (16) is equivalent to

$$\partial_t u\, X = \Big(\mathrm{div}_P\Big(\frac{\nabla^P u}{W}\Big) + \Big\langle \nabla^P \log\varrho, \frac{\nabla^P u}{W}\Big\rangle\Big) N.$$

If we take the normal projection on both sides we get

$$\partial_t u\langle X, N\rangle = \mathrm{div}_P\Big(\frac{\nabla^P u}{W}\Big) + \Big\langle \nabla^P \log\varrho, \frac{\nabla^P u}{W}\Big\rangle.$$

Since $\langle X, N\rangle = 1/W$ we conclude that (15) defines a mean curvature flow if and only if $u(\cdot,\, t)$ satisfies the parabolic equation

$$\partial_t u = \mathcal{Q}[u], \tag{20}$$

where

$$\mathcal{Q}[u] = W\Big(\mathrm{div}_P\Big(\frac{\nabla^P u}{W}\Big) + \Big\langle \nabla^P \log\varrho, \frac{\nabla^P u}{W}\Big\rangle\Big). \tag{21}$$

In general, this non-parametric formulation is equivalent to the mean curvature flow (16) up to tangential diffeomorphisms of the evolving graphs Σ_t. This equivalence follows from the fact that we are assuming a fixed *gauge*, namely the choice of coordinates fixed in (15).

2.3 Some auxiliary facts

In this section we deduce evolution equations for some functions which will be useful in the sequel. In the first result, by abusing of notation, we denote by $\mathrm{s} : \mathrm{P} \times \mathbb{R} \to \mathbb{R}$ the projection on the second factor. That is, s denotes the function $\mathrm{s}(x, s) = s$.

Proposition 2.1 *If (11) holds, then restrictions of the functions r and s to the graphs Σ_t, $t \in [0, T)$, satisfy*

$$(\partial_t - \Delta) r \geq -\frac{\xi'(r)}{\xi(r)} \left(n - |\nabla r|^2\right) - \varrho^2 |\nabla s|^2 \left(\langle \bar{\nabla} \log \varrho, \nabla r \rangle - \frac{\xi'(r)}{\xi(r)} \right) \tag{22}$$

and

$$(\partial_t - \Delta) s = -2 \langle \bar{\nabla} \log \varrho, N \rangle \langle \bar{\nabla} s, N \rangle. \tag{23}$$

In both expressions, ∇ and Δ are the intrinsic Riemannian connection and Laplacian in Σ_t, respectively, whereas $\bar{\nabla}$ denotes the Riemannian connection in M. Besides, given the function

$$\zeta(\Psi(x,t)) = \int_0^{r(\Psi(t,x))} \xi(\varsigma) \, \mathrm{d}\varsigma \tag{24}$$

we get

$$(\partial_t - \Delta) \zeta \geq -n \xi'(r) - \varrho^2 |\nabla s|^2 \xi(r) \left(\langle \bar{\nabla} \log \varrho, \nabla r \rangle - \frac{\xi'(r)}{\xi(r)} \right). \tag{25}$$

Proof. Since $\bar{\nabla} s = \varrho^{-2} X$, we have $\nabla s = \varrho^{-2} X^\top$, where $\top$ denotes the tangential projection onto $T\Sigma_t$. Given a local orthonormal tangent frame $\{\mathsf{e}_i\}_{i=1}^n$ in Σ_t, we get

$$\begin{aligned} \Delta s &= \langle \nabla \varrho^{-2}, X^\top \rangle + \varrho^{-2} \sum_{i=1}^n \langle \bar{\nabla}_{\mathsf{e}_i} X, \mathsf{e}_i \rangle + nH \langle \varrho^{-2} X, N \rangle = \langle \bar{\nabla} \varrho^{-2}, X^\top \rangle + nH \langle \bar{\nabla} s, N \rangle \\ &= -\langle \bar{\nabla} \varrho^{-2}, N \rangle \langle X, N \rangle + nH \langle \bar{\nabla} s, N \rangle = 2 \langle \bar{\nabla} \log \varrho, N \rangle \langle \bar{\nabla} s, N \rangle + nH \langle \bar{\nabla} s, N \rangle. \end{aligned}$$

We also have

$$\partial_t s = \langle \bar{\nabla} s, \partial_t \Psi \rangle = nH \langle \bar{\nabla} s, N \rangle.$$

Thus

$$(\partial_t - \Delta) s = -2 \langle \bar{\nabla} \log \varrho, N \rangle \langle \bar{\nabla} s, N \rangle.$$

Now one has

$$\langle \bar{\nabla}_X \bar{\nabla} r, X \rangle = \langle \bar{\nabla}_{\bar{\nabla} r} X, X \rangle = \frac{1}{2} \partial_r |X|^2 = \frac{1}{2} \partial_r \varrho^2 = \varrho \langle \bar{\nabla} \varrho, \bar{\nabla} r \rangle.$$

Fixed a local orthonormal tangent frame $\{\mathsf{e}_i\}_{i=1}^n$ in Σ_t, we have

$$\begin{aligned}
\Delta r &= \sum_i \langle \nabla_{\mathsf{e}_i} \nabla r, \mathsf{e}_i \rangle = \sum_i \langle \bar\nabla_{\mathsf{e}_i} (\bar\nabla r - \langle \bar\nabla r, N\rangle N), \mathsf{e}_i\rangle \\
&= \sum_i \langle \nabla^P_{\pi_* \mathsf{e}_i} \pi_* \bar\nabla r, \pi_* \mathsf{e}_i \rangle + \frac{1}{\varrho^4} \sum_i \langle \mathsf{e}_i, X\rangle^2 \langle \bar\nabla_X \bar\nabla r, X\rangle - \sum_i \langle \bar\nabla r, N\rangle \langle \bar\nabla_{\mathsf{e}_i} N, \mathsf{e}_i\rangle \\
&= \sum_i \langle \nabla^P_{\pi_* \mathsf{e}_i} \pi_* \bar\nabla r, \pi_* \mathsf{e}_i \rangle + \frac{1}{\varrho^4} |X^\top|^2 \langle \varrho \bar\nabla \varrho, \nabla r\rangle - \sum_i \langle \bar\nabla r, N\rangle \langle \bar\nabla_{\mathsf{e}_i} N, \mathsf{e}_i\rangle \\
&= \sum_i \langle \nabla^P_{\pi_* \mathsf{e}_i} \nabla^P r, \pi_* \mathsf{e}_i \rangle + |\nabla s|^2 \langle \varrho \bar\nabla \varrho, \nabla r\rangle + nH \langle \bar\nabla r, N\rangle
\end{aligned}$$

where $\pi : M = P \times \mathbb{R} \to P$ is the projection on the first factor, that is, $\pi(x, s) = x$ for all $(x, s) \in P \times \mathbb{R}$. It follows from the Hessian comparison theorem (12) that

$$\begin{aligned}
\Delta r &\le \frac{\xi'(r)}{\xi(r)} \sum_i \left(|\pi_* \mathsf{e}_i|^2 - \langle \mathsf{e}_i, \nabla^P r\rangle^2 \right) + |\nabla s|^2 \langle \varrho \bar\nabla \varrho, \nabla r\rangle + nH \langle \bar\nabla r, N\rangle \\
&= \frac{\xi'(r)}{\xi(r)} \left(n - \frac{1}{\varrho^2} |X^\top|^2 - |\nabla r|^2 \right) + \varrho^2 |\nabla s|^2 \langle \bar\nabla \log \varrho, \nabla r\rangle + nH \langle \bar\nabla r, N\rangle \qquad (26) \\
&= \frac{\xi'(r)}{\xi(r)} \left(n - \varrho^2 |\nabla s|^2 - |\nabla r|^2 \right) + \varrho^2 |\nabla s|^2 \langle \bar\nabla \log \varrho, \nabla r\rangle + nH \langle \bar\nabla r, N\rangle.
\end{aligned}$$

Thus,

$$\Delta r \le \frac{\xi'(r)}{\xi(r)} \left(n - |\nabla r|^2 \right) + \varrho^2 |\nabla s|^2 \left(\langle \bar\nabla \log \varrho, \nabla r\rangle - \frac{\xi'(r)}{\xi(r)} \right) + nH \langle \bar\nabla r, N\rangle. \qquad (27)$$

Now, since $\nabla \zeta = \xi(r) \nabla r$ and

$$\Delta \zeta = \xi(r) \Delta r + \xi'(r) |\nabla r|^2. \qquad (28)$$

we have

$$\Delta \zeta \le n\xi'(r) + \varrho^2 |\nabla s|^2 \xi(r) \left(\langle \bar\nabla \log \varrho, \nabla r\rangle - \frac{\xi'(r)}{\xi(r)} \right) + nH \xi(r) \langle \bar\nabla r, N\rangle. \qquad (29)$$

On the other hand

$$\partial_t r = \left\langle \bar\nabla r, \partial_t \Psi \right\rangle = nH \langle \bar\nabla r, N\rangle$$

and

$$\partial_t \zeta = nH \xi(r) \langle \bar\nabla r, N\rangle.$$

Therefore

$$(\partial_t - \Delta) r \ge -\frac{\xi'(r)}{\xi(r)} \left(n - |\nabla r|^2 \right) - \varrho^2 |\nabla s|^2 \left(\langle \bar\nabla \log \varrho, \nabla r\rangle - \frac{\xi'(r)}{\xi(r)} \right) \qquad (30)$$

and

$$(\partial_t - \Delta)\zeta \geq -n\xi'(r) - \varrho^2|\nabla s|^2\xi(r)\left(\langle\bar{\nabla}\log\varrho, \nabla r\rangle - \frac{\xi'(r)}{\xi(r)}\right). \tag{31}$$

□

Proposition 2.2 *If the graphs Σ_t, $t \in [0,T]$, evolve by the mean curvature flow (15)-(16), then*

$$(\partial_t - \Delta)\, W = -W(|A|^2 + \overline{\mathrm{Ric}}(N,N)) - 2W^{-1}|\nabla W|^2, \tag{32}$$

where $W = \langle X, N\rangle^{-1} = (\varrho^{-2} + |\nabla^M u)|^2)^{1/2}$ and A is the Weingarten map of Σ_t.
Proof. We have

$$\nabla\langle X, N\rangle = \langle X, N\rangle(\bar{\nabla}\log\varrho)^\top - \langle\bar{\nabla}\log\varrho, N\rangle X^\top - AX^\top. \tag{33}$$

Note that

$$\begin{aligned}&\langle X, N\rangle(\bar{\nabla}\log\varrho)^\top - \langle\bar{\nabla}\log\varrho, N\rangle X^\top\\ &\quad = \langle X, N\rangle(\bar{\nabla}\log\varrho - \langle\bar{\nabla}\log\varrho, N\rangle N) - \langle\bar{\nabla}\log\varrho, N\rangle(X - \langle X, N\rangle N)\\ &\quad = \langle X, N\rangle\bar{\nabla}\log\varrho - \langle\bar{\nabla}\log\varrho, N\rangle X.\end{aligned}$$

It follows from the second variation formula for the functional $\mathcal{A}$ that

$$\Delta\langle X, N\rangle + |A|^2\langle X, N\rangle + \overline{\mathrm{Ric}}(N,N)\langle X, N\rangle = -n\langle\nabla H, X^\top\rangle,$$

where $|A|$ stands for the norm of the Weingarten map of Σ_t and $\top$ denotes the tangencial projection onto $T\Sigma_t$. On the other hand, since X is a Killing vector field we get

$$\begin{aligned}\partial_t\langle X, N\rangle &= \langle\bar{\nabla}_{\partial_t}X, N\rangle + \langle X, \bar{\nabla}_{\partial_t}N\rangle = nH\langle\bar{\nabla}_N X, N\rangle - n\langle X, \nabla H\rangle\\ &= -n\langle X^\top, \nabla H\rangle,\end{aligned}$$

where $\bar{\nabla}$ denotes the Riemannian connection in $\bar{M}$. Thus

$$(\partial_t - \Delta)\,\langle X, N\rangle = |A|^2\langle X, N\rangle + \overline{\mathrm{Ric}}(N,N)\langle X, N\rangle$$

So, using that $\langle X, N\rangle = 1/W$ we have

$$\partial_t W = -W^2\partial_t W^{-1} = -W^2\partial_t\langle X, N\rangle$$

and

$$\Delta W - \frac{2}{W}|\nabla W|^2 = -W^2\Delta W^{-1} = -W^2\Delta\langle X, N\rangle. \tag{34}$$

Therefore

$$(\partial_t - \Delta)W = -W^2 (\partial_t - \Delta) \langle X, N\rangle - \frac{2}{W}|\nabla W|^2 = -W(|A|^2 + \overline{\mathrm{Ric}}\,(N,N)) - 2W^{-1}|\nabla W|^2.$$

□

2.4 Asymptotic boundary

This section is devoted to define the boundary at infinity of a Cartan-Hadamard manifold P following [10] and we list some important facts and results. For further details, we refer the reader to [10], [11] and also [6], [7] and [28].

Definition 2.1 *Two unit speed geodesic rays $\alpha, \beta : [0, +\infty) \longrightarrow P$ are called* asymptotic *if* $\sup_t \mathrm{dist}(\alpha(t), \beta(t)) < \infty$.

We observe that

(i) If two unit speed asymptotic geodesic rays have a point in common, then they are the same;

(ii) Given a geodesic ray α and a point $p \in P$, there exists a unique geodesic β such that $\beta(0) = p$ and β is asymptotic to α;

(iii) The *asymptotic relation* is an equivalence relation on the set of all unit speed geodesic rays in P. The asymptotic class of α is denoted by $\alpha(\infty)$ and $\alpha(-\infty)$ denotes the asymptotic class of the reverse curve of α.

With this equivalence relation we define the asymptotic boundary $\partial_\infty P$ of P as a smooth manifold given by the set of the asymptotic classes of unit speed geodesic rays in P. From now on we will denote $\bar{P} = P \cup \partial_\infty P$.

We recall that if P is a Cartan-Hadamard manifold, given $x \in P$ and $y \in P \setminus \{x\}$, there exists a unique unit speed geodesic $\gamma^{xy} : \mathbb{R} \to P$ such that $\gamma^{xy}(0) = x$ and $\gamma^{xy}(t) = y$ where $t = \mathrm{dist}(x, y)$. When we have $K_P \leq -\kappa^2 < 0$, [10] proved the following more general result.

Proposition 2.3 *If the Cartan-Hadamard manifold P has the sectional curvature $K_P \leq -\kappa^2 < 0$, then for any $x, y \in \bar{P}$ there exists a unique unit speed geodesic γ^{xy} joining x and y.*

In order to define a convenient topology in $\bar{P}$, we use the following notion of angle between vectors. Given $x \in P$ and $(x, \mathsf{v}), (x, \mathsf{w})$ in the unit tangent bundle of P, we denote the angle between v and w in T_xP as $\angle(\mathsf{v}, \mathsf{w})$. For any $y, z \in \bar{P}$ we define $\angle(y, z) = \angle(\dot{\gamma}^{x,y}(0), \dot{\gamma}^{x,z}(0))$. Then for fixed $\delta > 0$ and $r > 0$, we define the cone of opening angle δ and axis v by

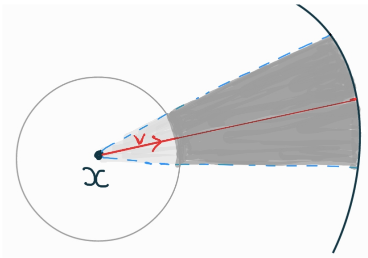
$$C(x, \mathsf{v}, \delta) = \{y \in \bar{P} \setminus \{x\} : \angle(\mathsf{v}, \dot{\gamma}^{x,y}(0)) < \delta\}$$

and the truncated cone of radius r by

$$T(x, \mathsf{v}, \delta, r) = C(x, \mathsf{v}, \delta) \setminus \bar{B}_r(x)$$

where $B_r(x)$ is the geodesic ball of radius r centred at x.

Figura 1: Truncated cone



Source: elaborated by author.

Using these truncated cones in $\bar{P}$ and the geodesic balls in P we define a special topology in $\bar{P}$. More precisely we have the following

Proposition 2.4 *Let $x \in P$ fixed. The set of all truncated cones $T(x, \mathsf{v}, \delta, r)$ with vertice in x and all geodesic balls $B_r(y) = \{z \in P \,|\, \mathrm{dist}(y, z) < r\}$ in P defines a local basis of topology in $\bar{P}$, which is called the cone topology. The cone topology does not depend on the choice of x. With this topology, $\bar{P}$ is a compact manifold. Moreover, under this topology $\bar{P}$ is homeomophic to the closed ball $\bar{B} \subset \mathbb{R}^n$, P to the open ball B and $\partial_\infty P$ to the boundary sphere $\mathbb{S}^{n-1} = \partial \bar{B}$.*

We remember that our ambient manifold is $M = P \times_\varrho \mathbb{R}$. In [4], the authors proved that the warped function ϱ is convex if and only if M Cartan-Hadamard manifold as well. In this case, we can associate $\partial_\infty P$ with a subset of $\partial_\infty M$ in the following way. Given $x \in \partial_\infty P$ and γ be a representative of x, we have γ is also a geodesic in M since P is a totally geodesic submanifold of M. Then there exists $\tilde{x} \in \partial_\infty M$ such that γ is a representative of $\tilde{x}$. In this sense, we can say that $\partial_\infty P$ is a subset of $\partial_\infty M$. Then, following [6], we define the Killing graph of the function $\varphi \in C(\partial_\infty P)$ on $\partial_\infty M$. Given $x \in \partial_\infty P$, we consider the leaf

$$P_{\varphi(x)} := \Phi(P, \varphi(x)) = \{(y, \varphi(x)); y \in P\} \subset P \times \mathbb{R}.$$

If γ^x is a geodesic in P representative of x (that is, $\gamma^x(\infty) = x$), we consider $\tilde{\gamma}^x : \mathbb{R} \to M$ given by

$$\tilde{\gamma}^x(t) = \Phi(\gamma^x(t), \varphi(x)).$$

Since Φ is a isometry, $\tilde{\gamma}^x$ is a geodesic on $P_{\varphi(x)}$ and consequently, on M. Thus, $\tilde{\gamma}^x$ define a point in $\partial_\infty M$ which we will denote by $(x, \varphi(x))$. So, we say that the set

$$\{(x, \varphi(x)); x \in \partial_\infty P\} \subset \partial_\infty M$$

is the Killing graph of φ.

When the warped function ϱ is not convex, we can associate the Killing graph of $\varphi \in C(\partial_\infty P)$ with a subset of $\partial_\infty P \times \mathbb{R}$ as follow.

Given $x \in \partial_\infty P$ and γ^x a geodesic in P representative of x, we consider $\tilde{\gamma}^x : \mathbb{R} \to M$ defined by

$$\tilde{\gamma}^x(t) = \Phi(\gamma^x(t), \varphi(x)).$$

As before, $\tilde{\gamma}^x$ is a geodesic in the leaf $P_{\varphi(x)} := \Phi(P, \varphi(x)) = \{(y, \varphi(x)); y \in P\}$. Hence, $\tilde{\gamma}^x$ define a point in the $\partial_\infty P_{\varphi(x)}$ which we denote by $(x, \varphi(x))$. Thus, we identify the Killing graph of φ with a subset in $\partial_\infty P \times \mathbb{R}$.

3 A PRIORI ESTIMATES

In this chapter, we get the a priori estimates for a solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i, & \text{in} \quad B_R \times [0, T_R) \\ u(x,0) = u_0(x), \quad \text{in} \quad B_R \times \{0\} \\ u(x,t) = \varphi(x) \quad \text{if} \quad x \in \partial B_R,\, t \in [0, T_R), \end{cases} \tag{35}$$

where $R > 0$ is fixed and $B_R(o) \subset P$ is a geodesic ball in P. We want these estimates to be uniform in $B_{R'} \times [0, T_{R'})$ for some appropriate $0 < R' < R$. This fact will be fundamental in the process of exhaustion which we will use in the proof of the Theorem 1.3.

3.1 Height estimate

In order to obtain a height estimate for a solution of the problem (35) we will use graphs as barriers. We can find the construction of these graphs and some facts about them in [26]. However, for a better understanding we chose to repeat it here.

Let us consider P_+ be a complete, non-compact, n-dimensional model manifold with respect to a fixed pole $o_+ \in P_+$ in the sense that the Riemannian metric in P_+ can be expressed in Gaussian coordinates $(r, \vartheta) \in \mathbb{R} \times \mathbb{S}^{n-1}$ centered at o_+ as

$$g_+ = \mathrm{d}r^2 + \xi^2(r)\, \mathrm{d}\vartheta^2 \tag{36}$$

where $\mathrm{d}\vartheta^2$ denotes the round metric in $\mathbb{S}^{n-1}$ and $\xi \in C^\infty([0, \infty))$ is the function mentioned in (10).

We define the warped metric in $P_+ \times \mathbb{R}$ as

$$\varrho^2(r)\mathrm{d}s^2 + \mathrm{d}r^2 + \xi^2(r)\, \mathrm{d}\vartheta^2. \tag{37}$$

We denote

$$A(r) = \varrho(r)\xi^{n-1}(r), \qquad V(r) = \int_0^r \varrho(\varsigma)\xi^{n-1}(\varsigma)\, \mathrm{d}\varsigma. \tag{38}$$

and we also define

$$H(r) = -\frac{1}{n}\frac{A(r)}{V(r)}. \tag{39}$$

Given $x \in P_+$ we denote the geodesic distance between o_+ and x by $r(x) = \mathrm{dist}(o_+, x)$. For $R > 0$, we consider $B_R(o_+)$ be the closed geodesic ball centered at o_+ with radius R. Then $x \in B_R(o_+)$ if and only if $r(x) \le R$. We remember that the mean curvature of the Killing cylinder over the geodesic sphere $\partial B_r(o_+)$ is given by

$$H_{\rm cyl}(r) = \frac{1}{n}\left((n-1)\frac{\xi'(r)}{\xi(r)} + \frac{\varrho'(r)}{\varrho(r)}\right). \tag{40}$$

In this context, we have the following result.

Proposition 3.1 *For each $R>0$, the graph of the function*

$$v_R(x) = \int_R^{r(x)} \frac{nH(R)V(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma)-n^2H^2(R)V^2(\varsigma))^{\frac{1}{2}}}\,{\rm d}\varsigma \tag{41}$$

defined in $B_R(o_+)$ has constant mean curvature $H(R)$ and its boundary is the geodesic sphere $\partial B_R(o_+)$.

Proof. For $R>0$ fixed, we consider v_R the radial solution of the Dirichlet problem for the constant mean curvature equation

$$\begin{cases} {\rm div}_+\left(\frac{\nabla^+ v_R}{W_+}\right) + g_+\left(\nabla^+\log\varrho, \frac{\nabla^+ v_R}{W_+}\right) = nH(R) \ \text{ in } \ B_R(o_+), \\ v_R|_{\partial B_R(o_+)} = 0, \end{cases} \tag{42}$$

where the differential operators ${\rm div}_+$ and ∇^+ are defined with respect to the metric (36) in P_+ and

$$W_+ = (\varrho^{-2}(r) + v_R'^2(r))^{\frac{1}{2}},$$

with $'$ denoting derivatives with respect to r. Then we have

$$\left(\frac{v_R'(r)}{(\varrho^{-2}(r)+v_R'^2(r))^{1/2}}\right)' + \frac{v_R'(r)}{(\varrho^{-2}(r)+v_R'^2(r))^{1/2}}\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right) = nH(R). \tag{43}$$

We can also rewrite (42) in terms of a weighted divergence as

$${\rm div}_{-\log\varrho}\left(\frac{\nabla^+ v_R}{W_+}\right) \doteq \frac{1}{\varrho}{\rm div}_+\left(\varrho\frac{\nabla^+ v_R}{W_+}\right) = nH(R). \tag{44}$$

Integrating with respect to the density $\varrho\,{\rm d}P_+$ yields

$$\begin{aligned} \int_{B_r(o_+)} nH(R)\,\varrho\,{\rm d}P_+ &= \int_{B_r(o_+)} {\rm div}_+\left(\varrho\frac{\nabla^+ v_R}{W_+}\right){\rm d}P_+ \\ &= \int_{\partial B_r(o_+)} g_+\left(\frac{\nabla^+ v_R}{W_+}, \partial_r\right)\varrho\,{\rm d}\partial B(r), \end{aligned} \tag{45}$$

for $r\le R$. Thus v_R is the solution of the first order equation

$$\frac{v_R'(r)}{(\varrho^{-2}(r)+v_R'^2(r))^{1/2}}\,\varrho(r)\xi^{n-1}(r) = \int_0^r nH(R)\varrho(\varsigma)\xi^{n-1}(\varsigma)\,{\rm d}\varsigma, \tag{46}$$

with initial condition $v_R|_{r=R} = 0$. Solving this expression for v_R', we obtain

$$v_R'(r) = \frac{nH(R)V(r)}{\varrho(r)(A^2(r) - n^2H^2(R)V^2(r))^{1/2}}. \tag{47}$$

The graph Σ_R of v_R is a rotationally invariant hypersurface which can be parametrized in terms of coordinates (s, r, ϑ) as $\varsigma \mapsto (s(\varsigma), \vartheta, r(\varsigma))$, where ς can be taken as the arc lenght parameter. If ϕ denotes the angle between the coordinate vector field ∂_r and a given profile curve $\vartheta =$ constant in Σ_R, we have

$$\dot{r} = \cos\phi, \quad \varrho\dot{s} = \sin\phi.$$

It follows from (43) that

$$-\frac{\mathrm{d}}{\mathrm{d}\varsigma}(\varrho\dot{s})\frac{\mathrm{d}\varsigma}{\mathrm{d}r} - \varrho\dot{s}\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right) = nH(R),$$

or yet,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\varsigma} + \sin\phi\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right) = -nH(R).$$

Therefore, a profile curve of Σ_R is given by the solution of the first order system

$$\begin{cases} \dot{r} = \cos\phi, \\ \varrho\dot{s} = \sin\phi, \\ \dot{\phi} = -nH(R) - nH_{\mathrm{cyl}}(r)\sin\phi, \end{cases}$$

with initial conditions $r(0) = R, s(0) = 0, \phi(0) = \frac{\pi}{2}$. Then we can rewrite (46) as

$$\varrho(r)A(r)\dot{s} = -nH(R)V(r)$$

where $\cdot$ indicates derivatives with respect to the parameter ς. We note that when the coordinate r attains its maximum value, that is, when $r = R$, we have $\dot{r} = 0$ and $\varrho\dot{s} = 1$. This is consistent with the choice of $H(R)$ in (39). Moreover, when $r \to 0^+$ we have $\dot{s} \to 0$ and $\dot{r} \to 1$. □

In order to get a suitable one-parameter family of graphs which we will use as barriers, we fix $r_0 > 0$ and we consider for $R \geq r_0$ the variable $\mu = R - r_0$. We note that μ can be considered as the geodesic distance between the geodesic spheres $\partial B_{r_0}(o) = \partial\Sigma_{r_0}$ and $\partial B_R(o) = \partial\Sigma_R$. Thus $\nabla\mu_{|\partial B_R(o)} = \partial_{r|_{r=R}}$. So, we set a time parameter $t \in [0, \infty)$ given by

$$\begin{cases} \frac{\mathrm{d}\mu}{\mathrm{d}t} = -nH(R) = -nH(\mu + r_0), \\ \mu(0) = 0. \end{cases} \tag{48}$$

This means that $\mu = \mu(t)$ is implicitly defined by

$$\int_{r_0}^{\mu(t)+r_0} \frac{V(\varsigma)}{A(\varsigma)}\,\mathrm{d}\varsigma = t. \tag{49}$$

We denote $R(t) = \mu(t) + r_0$. We want to use the one-parameter family of constant mean curvature graphs $\{\Sigma_{R(t)}\}_{t\geq 0}$ as barrier. For this we claim that $\{\Sigma_{R(t)}\}_{t\geq 0}$ evolves by the (negative) mean curvature flow

$$\partial_t \Psi^+ = -nH(R(t))N_t, \tag{50}$$

where

$$N(t) = \frac{1}{W}(\varrho^{-2}(r)X - v_R'(r)\partial_r) = -\frac{\dot{r}}{\varrho}X + \varrho\dot{s}\,\partial_r.$$

In fact, this means that $\Sigma_{R(t)} = \Psi_t^+(\Sigma_{r_0})$. In particular, we must have

$$\partial B_{R(t)} = \partial\Sigma_{R(t)} = \Psi_t^+(\partial\Sigma_{r_0}) = \Psi_t^+(\partial B_{r_0}).$$

In other words, we must choose the time parameter t in a way that the geodesic spheres evolve as $\partial B_{R(t)} = \Psi_t^+(\partial B_{r_0})$. Since $\dot{r} = 0$ and $\varrho\dot{s} = 1$ at $r = R(t)$ it follows from (50) that

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}t} &= \langle \partial_t\Psi^+, \nabla^+\mu\rangle = \langle \partial_t\Psi^+, \partial_r|_{r=R(t)}\rangle = -nH(R(t))\langle N_t, \partial_r\rangle|_{r=R(t)}\\ &= -nH(R(t)) = -nH(r_0+\mu(t))\end{aligned}$$

what means that t coincides with the parameter defined in (48) and then satisfying the condition that $\partial B_{R(t)} = \Psi_t^+(\partial B_{r_0})$. Note that $R(t) \geq r_0$ for $t \geq 0$. So the one-parameter family of functions $u_+(x,t) = v_{R(t)}(r(x))$ defined on the common domain $B_{r_0}(o)$ defines a solution of the geometric flow (50). Thus, we set

$$\Psi^+(x,t) = (x, u_+(x,t)), \quad x \in B_{r_0}(o). \tag{51}$$

We conclude that u_+ satisfies the parabolic equation

$$\begin{aligned}\partial_t u_+ = -(\varrho^{-2}(r) + |\partial_r u_+|^2)^{1/2}\Bigg(\partial_r\left(\frac{\partial_r u_+}{(\varrho^{-2}(r)+|\partial_r u_+|^2)^{1/2}}\right)\\ + \frac{\partial_r u_+}{(\varrho^{-2}(r)+|\partial_r u_+|^2)^{1/2}}\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right)\Bigg).\end{aligned} \tag{52}$$

Now, we use this information for to prove that u_+ is a supersolution to the mean curvature flow in M.

Proposition 3.2 *The one-parameter family of functions*

$$u_+(x,t) = v_{R(t)}(x) = v_{R(t)}(r(x)), \quad x \in B_{r_0}(o), \quad t \in [0,\infty) \tag{53}$$

is a supersolution of the mean curvature flow in $M = P \times_\varrho \mathbb{R}$.
Proof. Denoting $W = (\varrho^{-2} + |\nabla^P u_+|^2)^{1/2}$ we have

$$\begin{aligned}
\mathcal{Q}[u_+] + \partial_t u_+ &= W\left(\mathrm{div}_P\left(\frac{\nabla^P u_+}{W}\right) + \left\langle \nabla^P \log\varrho, \frac{\nabla^P u_+}{W}\right\rangle\right) + \partial_t u_+ \\
&= (\varrho^{-2} + u_+'^2(r))^{1/2}\Bigg(\partial_r\left(\frac{u_+'(r)}{(\varrho^{-2}+u_+'^2(r))^{1/2}}\right) \\
&+ \frac{u_+'(r)}{(\varrho^{-2}+u_+'^2(r))^{1/2}}\big(\Delta_P r + \langle \nabla^P \log\varrho, \nabla^P r\rangle\big)\Bigg) + \partial_t u_+,
\end{aligned}$$

where Δ_P is the Laplace-Beltrami operator in (P,g). However

$$\langle \nabla^P \log\varrho, \nabla^P r\rangle = \frac{\partial_r \varrho}{\varrho} = \frac{\varrho'(r)}{\varrho(r)}.$$

Furthermore (12) implies that

$$\Delta_P r \le (n-1)\frac{\xi'(r)}{\xi(r)}.$$

Since $u_+' = v_R' \le 0$ we have

$$\begin{aligned}
\mathcal{Q}[u_+] + \partial_t u_+ &\ge (\varrho^{-2} + u_+'^2(r))^{1/2}\Bigg(\partial_r\left(\frac{u_+'(r)}{(\varrho^{-2}+u_+'^2(r))^{1/2}}\right) \\
&+ \frac{u_+'(r)}{(\varrho^{-2}+u_+'^2(r))^{1/2}}\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right)\Bigg) + \partial_t u_+ = 0.
\end{aligned}$$

Thus u_+ is a supersolution of the mean curvature flow in M. □

Proposition 3.3 *If* u *is a solution of* (35), *then we have the following height estimate*

$$|u(x,t)| \le \sup_{B_{r_0}(o)}|u| + v_{R(T)}(o) - v_{r_0}(r(x)). \tag{54}$$

More precisely,

$$\begin{aligned}
|u(x,t)| \le \sup_{B_{r_0}(o)}|u(\cdot,0)| &+ \int_{R(T)}^{0} \frac{nH(R(T))V(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma) - n^2H^2(R(T))V^2(\varsigma))^{\frac{1}{2}}}\,\mathrm{d}\varsigma \\
&- \int_{r_0}^{r(x)} \frac{nH(r_0)V(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma) - n^2H^2(r_0)V^2(\varsigma))^{\frac{1}{2}}}\,\mathrm{d}\varsigma.
\end{aligned} \tag{55}$$

for $(x,t) \in B_{r_0}(o) \times [0,T]$.
Proof. By construction, the graph Σ_{r_0} of $u_+(\cdot,0) = v_{r_0}$ is defined in the geodesic ball

$B_{r_0}(o)$. For $T>0$ we have that $\Psi_T^+(\Sigma_{r_0})$ is the graph $\Sigma_{R(T)}$ of $u_+(\cdot,T) = v_{R(T)}\big|_{B_{r_0}(o)}$ with

$$\int_{r_0}^{R(T)} \frac{V(\varsigma)}{A(\varsigma)}\,\mathrm{d}\varsigma = T.$$

For $\varepsilon>0$ we have

$$-u_+(x,T) + u_+(o,T) + \sup_{B_{r_0}(o)} u + \varepsilon > u(x,0)$$

for all $x \in B_{r_0}(o)$. We also have

$$v_\varepsilon(x,t) := -u_+(x,T-t) + u_+(o,T) + \sup_{B_{r_0}(o)} u + \varepsilon > u(x,t)$$

for all $(x,t) \in \partial B_{r_0}(o) \times [0,T]$. It follows from the Proposition 3.2 that

$$\partial_t v_\varepsilon - \mathcal{Q}[v_\varepsilon] = \partial_t u_+ + \mathcal{Q}[u_+] \geq 0 \tag{56}$$

in the parabolic cylinder $B_{r_0}(o) \times (0,T)$. Then the parabolic maximum principle implies that

$$u(x,t) \leq v(x,t) \leq v(x,T)$$

in $B_{r_0}(o) \times [0,T]$ where

$$v(x,t) = -u_+(x,T-t) + u_+(o,T) + \sup_{B_{r_0}(o)} u. \tag{57}$$

So

$$u(x,t) \leq v(x,T) = u_+(o,T) - u_+(x,0) + \sup_{B_{r_0}(o)} u.$$

Thus

$$u(x,t) \leq \sup_{B_{r_0}(o)} u + v_{R(T)}(o) - v_{r_0}(r(x)) \tag{58}$$

for $(x,t) \in B_{r_0}(o) \times [0,T]$. In a similar way we can prove that

$$u(x,t) \geq w(x,t) \geq w(x,T)$$

in $B_{r_0}(o) \times [0,T]$ where

$$w(x,t) = u_+(x,T-t) - u_+(o,T) + \inf_{B_{r_0}(o)} u. \tag{59}$$

Thus

$$u(x,t) \geq \inf_{B_{r_0}(o)} u - v_{R(T)}(o) + v_{r_0}(r(x)) \tag{60}$$

in $B_{r_0}(o)\times[0,T]$ and we have

$$|u(x,t)|\leq \sup_{B_{r_0}(o)}|u|+v_{R(T)}(o)-v_{r_0}(r(x)).$$

□

Now we obtain an uniform estimate for C^0 bounds of the functions $u_+(\cdot,t)$.

Proposition 3.4 *Let $r_0>0$ be a fixed constant and $R_0:[0,\infty)\to[r_0,\infty)$, be the function implicitly defined in (49). If $\ell_0>0$ satisfies*

$$R_0(t)\leq \ell_0 r_0 \quad \textit{for all} \quad t\in[0,T]$$

then

$$\sup_{B_{r_0}(o)\times[0,T]}u_+(x,t)=\sup_{[0,T]}u_+(o,t)\leq c(r_0,\,\ell_0,\varrho,\xi).$$

Proof. It follows directly from (41) and (53) that

$$u_+(x,t)=v_{R_0(t)}(r(x))=\int_{r(x)}^{R_0(t)}\frac{-nH(R_0(t))V(\varsigma)}{\varrho(\varsigma)\big(A^2(\varsigma)-n^2H^2(R_0(t))V^2(\varsigma)\big)^{\frac{1}{2}}}\mathrm{d}\varsigma$$
$$\leq\int_0^{R_0(t)}\frac{-nH(R_0(t))V(\varsigma)}{\varrho(\varsigma)\big(A^2(\varsigma)-n^2H^2(R_0(t))V^2(\varsigma)\big)^{\frac{1}{2}}}\mathrm{d}\varsigma=v_{R_0(t)}(o).$$

Moreover

$$\begin{aligned}\frac{-nH(R_0(t))V(\varsigma)}{\varrho(\varsigma)\big(A^2(\varsigma)-n^2H^2(R_0(t))V^2(\varsigma)\big)^{\frac{1}{2}}}&=\frac{-nH(R_0(t))V(\varsigma)}{-H(R_0(t))A(\varsigma)\varrho(\varsigma)\Big(\frac{1}{H^2(R_0(t))}-\frac{1}{H^2(\varsigma)}\Big)^{\frac{1}{2}}}\\&=-\frac{1}{H(\varsigma)\varrho(\varsigma)}\bigg(\frac{1}{H^2(R_0(t))}-\frac{1}{H^2(\varsigma)}\bigg)^{-\frac{1}{2}}\\&=-\frac{H^2}{\varrho H'}\frac{H'}{H^3}\bigg(\frac{1}{H^2(R_0(t))}-\frac{1}{H^2}(\varsigma)\bigg)^{-\frac{1}{2}}\\&\leq-\sup_{[0,\ell_0r_0]}\bigg(\frac{H^2}{\varrho H'}\bigg)\frac{H'}{H^3}\bigg(\frac{1}{H^2(R_0(t))}-\frac{1}{H^2(\varsigma)}\bigg)^{-\frac{1}{2}},\end{aligned}$$

where in the right hand side of the inequality above we use that $nH(r)=-\frac{A(r)}{V(r)}$ is an increasing function. Note that

$$nH'=-\frac{A'}{A}\frac{A}{V}+\frac{A}{V}\frac{V'}{V}=-nH\bigg(\frac{V'}{V}-\frac{A'}{A}\bigg).$$

Therefore

$$\frac{\varrho H'}{H^2}=-\frac{\varrho}{H}\bigg(\frac{V'}{V}-\frac{A'}{A}\bigg)$$

Using the change of variables

$$\eta = \left(\frac{1}{H^2(R_0(t))} - \frac{1}{H^2(\varsigma)}\right)^{\frac{1}{2}}$$

one gets

$$\begin{aligned} v_{R_0(t)}(o) &= \int_0^{R_0(t)} \frac{-nH(R_0(t))V(\varsigma)}{\varrho(\varsigma)\big(A^2(\varsigma) - n^2H^2(R_0(t))V^2(\varsigma)\big)^{\frac{1}{2}}}\mathrm{d}\varsigma \\ &\leq -\left(\sup_{[0,\ell_0 r_0]} \frac{H^2}{\varrho H'}\right)\int_0^{R_0(t)} \frac{H'}{H^3}\left(\frac{1}{H^2(R_0(t))} - \frac{1}{H^2(\varsigma)}\right)^{-\frac{1}{2}}\mathrm{d}\varsigma \\ &= -\left(\sup_{[0,\ell_0 r_0]} \frac{H^2}{\varrho H'}\right)\int_{-\frac{1}{H(R_0(t))}}^{0} \mathrm{d}\eta \\ &= -\sup_{[0,\ell_0 r_0]}\left(\frac{H^2}{\varrho H'}\right)\frac{1}{H(R_0(t))} \leq -\left(\sup_{[0,\ell_0 r_0]} \frac{H^2}{\varrho H'}\right)\frac{1}{H(\ell_0 r_0)}. \end{aligned}$$

Thus we have

$$\sup_{B_{r_0}(o)\times[0,T]} |u_+(x,t)| = \sup_{[0,T]} u_+(o,t) \leq c(r_0,\ \ell_0,\ \varrho,\ \xi).$$

□

A consequence of this proposition is an a priori height estimate that does not depend on the maximum time of the solution.

Corollary 3.5 *Let u be a solution of* (35) *in $B_{r_0} \times [0,\epsilon]$ and $R_0 : [0,\infty) \to [r_0,\infty)$ be the function implicitly defined in* (49). *For $\tau > \epsilon$, if $\ell_0 > 0$ satisfies*

$$R_0(t) \leq \ell_0 r_0 \qquad \forall \quad t \in [0,\tau]$$

then

$$|u(x,t)| \leq \sup_{B_{r_0}(o)}|u| + c(r_0,\tau,\ell_0,\varrho,\xi) - v_{r_0}(r(x)).$$

Proof. In fact, for $(x,t) \in B_{r_0} \times [0,\epsilon]$, we have

$$\begin{aligned} |u(x,t)| &\leq \sup_{B_{r_0}(o)}|u| + v_{R(\epsilon)}(o) - v_{r_0}(r(x)) \\ &\leq \sup_{B_{r_0}(o)}|u| + |u_+(o,\epsilon)| - v_{r_0}(r(x)) \\ &\leq \sup_{B_{r_0}(o)}|u| + c(r_0,\tau,\ell_0,\varrho,\xi) - v_{r_0}(r(x)). \end{aligned}$$

□

Now, let's see how the height estimate we obtained in some cases as follows.

Example 3.1 In the case where $M = \mathbb{R}^{n+1}$ with $P = \mathbb{R}^n$ and X is a parallel vector field with $\varrho = 1$ we have

$$H(r) = -\frac{1}{r}$$

and

$$v_R(x) = -\int_R^{r(x)} \frac{R^{-1}\varsigma^n}{(\varsigma^{2(n-1)} - R^{-2}\varsigma^{2n})^{\frac{1}{2}}}\, \mathrm{d}\varsigma = -\int_R^{r(x)} \frac{\varsigma}{(R^2-\varsigma^2)^{\frac{1}{2}}}\, \mathrm{d}\varsigma = (R^2 - r^2(x))^{1/2}.$$

Therefore the suitable time parameter defined by

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{n}{R(t)} = \frac{n}{\mu(t)+r_0}$$

is given explicitly by

$$R(t) = (r_0^2 + 2nt)^{1/2}, \quad t \in [0,\infty).$$

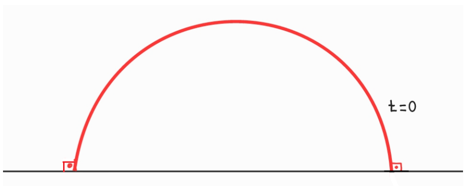
For $T > 0$ fixed and $\ell_0 := \sqrt{1 + \frac{2n}{r_0^2}T}$ we have

$$R(t) \le \ell_0 r_0 \quad \forall \quad t \in [0,T].$$

Hence

$$\sup_{B_{r_0}(o)\times[0,T]} u_+(x,t) = \sup_{[0,T]} u_+(o,t) = v_{R(T)}(o) = R(T) = \ell_0 r_0 := c(\ell_0, r_0). \qquad (61)$$

Figura 2: Graph of $v_{R(0)}$



Source: elaborated by author.

Example 3.2 Now we consider the case where M is the hyperbolic space $\mathbb{H}^{n+1}$ which has been already considered in the references [30] and [24]. We can define a mean curvature flow of geodesic spheres in $\mathbb{H}^{n+1}$ defining a time parameter by the ODE

$$\frac{\mathrm{d}R}{\mathrm{d}t} = n\frac{\cosh(R(t))}{\sinh(R(t))}$$

whose general solution has the form

$$\cosh R(t) = e^{nt}\cosh r_0,$$

where $r_0 > 0$ is the radius of the geodesic sphere at time $t = 0$. For $T > 0$ fixed, if we take

$$\ell_0 := \frac{\operatorname{arccosh}(e^{nT})\cosh(r_0)}{r_0}$$

we have

$$R(t) \le \ell_0 r_0 \quad \forall \quad t \in [0, T].$$

Then

$$\begin{aligned}
u_+(x,t) = v_{R(t)}(r(x)) &= \int_{r(x)}^{R(t)} \frac{-H^2(\zeta)}{\varrho(\zeta)H'(\zeta)} \frac{H'(\zeta)}{H^3(\zeta)} \left(\frac{1}{H^2(R(t))} - \frac{1}{H^2(\zeta)} \right)^{\frac{1}{2}} d\zeta \\
&= \int_{r(x)}^{R(t)} -\cosh(\zeta) \frac{H'(\zeta)}{H^3(\zeta)} \left(\frac{1}{H^2(R(t))} - \frac{1}{H^2(\zeta)} \right)^{\frac{1}{2}} d\zeta \\
&\le \left(\sup_{[0,\ell_0 r_0]} \cosh(t) \right) \int_{r(x)}^{R(t)} -\frac{H'(\zeta)}{H^3(\zeta)} \left(\frac{1}{H^2(R(t))} - \frac{1}{H^2(\zeta)} \right)^{\frac{1}{2}} d\zeta \\
&= \cosh(\ell_0 r_0) \left(\frac{1}{H^2(R(t))} - \frac{1}{H^2(r(x))} \right)^{\frac{1}{2}} \\
&= \cosh(\ell_0 r_0) \Big(\tanh^2(R(t)) - \tanh^2(r(x)) \Big)^{\frac{1}{2}} \\
&\le \cosh(\ell_0 r_0) \tanh(R(t)) \le \sinh(\ell_0 r_0) := c_0(\ell_0, r_0).
\end{aligned}$$

3.2 Gradient estimates

Our task now is to produce a priori gradient estimates for the problem (35). First we will to do that on the boundary $\partial B_R(o) \times [0,T]$. For this we will use barriers of the form $v = \tilde{u_0} + h(d)$ where $\tilde{u_0}$ is a extension of u_0 in a neighborhood of $\partial B_R(o) \times [0,T]$ in which the function $d(x) = R - r(x)$ is a smooth distance function.

3.2.1 *Boundary gradient estimate*

In order to obtain a gradient estimate in $\partial B_R(o) \times [0,T]$, we consider

$$K_R = \{\Psi(x,t); x \in \partial B_R(o), t \in [0, +\infty)\}$$

be the Killing cylinder over $\partial B_R(o)$ and we consider the function $\mathrm{d}(x) = \operatorname{dist}(x, \partial B_R(o)) = R - r(x)$ for $x \in B_R(o)$. Then we define the function d in $B_R(o) \times [0, +\infty)$ as

$$\mathrm{d}(x,t) = \operatorname{dist}(\Psi(x,t), K_R) = \mathrm{d}(x)$$

and $\Omega_\alpha = \{x \in B_R(o); \mathrm{d}(x) < \alpha\}$ for $\alpha > 0$. So we take the neighborhood Ω_ϵ of $\partial B_R(o)$ where $\epsilon > 0$ is such that d is a smooth function in Ω_ϵ.

Proposition 3.6 *Let u be a solution of (35) defined in $B_R(o) \times [0,T]$ for $R > 0$ and $T > 0$. Then there exists a constant $C > 0$ such that*

$$\sup_{\partial B_R(o)\times[0,T]} |\nabla u| \leq C.$$

Proof.

We consider

$$v(x,t) = \widetilde{u}_0(x) + h(d(x))$$

where $d(x) = R - r(x)$ for $x \in B_R(o)$, $\widetilde{u}_0$ is a local extension of u_0 defined for $d < \varepsilon$ and h is a function to be choose latter. Denoting

$$W = \sqrt{\varrho^{-2} + |\nabla^P v|^2} = \sqrt{\varrho^{-2} + h'^2(d) + 2h'(d)\langle \nabla^P d, \nabla^P \widetilde{u}_0\rangle + |\nabla^P \widetilde{u}_0|^2}$$

we have

$$\partial_t v - \mathcal{Q}[v] = \partial_t v - \Delta_P v + \frac{1}{W^2}\langle \nabla^P_{\nabla^P v}\nabla^P v, \nabla^P v\rangle - \left(1 + \frac{1}{\varrho^2 W^2}\right)\langle \nabla^P \log\varrho, \nabla^P v\rangle.$$

Then

$$\begin{aligned}
&W^2(\partial_t v - \mathcal{Q}[v]) = -W^2(\Delta_P \widetilde{u}_0 + h''(d) + h'(d)\Delta_P d)\\
&\quad + (\widetilde{u}_0^i + h'(d)d^i)(\widetilde{u}_0^j + h'(d)d^j)(\langle \nabla^P_{\partial_i}\nabla^P\widetilde{u}_0, \partial_j\rangle + h'(d)\langle \nabla^P_{\partial_i}\nabla^P d, \partial_j\rangle + h''(d)d_i d_j)\\
&\quad - \langle \nabla^P \log\varrho, \nabla^P\widetilde{u}_0 + h'(d)\nabla^P d\rangle(\varrho^{-2} + W^2).
\end{aligned}$$

Rearranging some terms one gets

$$\begin{aligned}
&W^2(\partial_t v - \mathcal{Q}[v]) = -h''(d)W^2 + h''(d)(\widetilde{u}_0^i + h'(d)d^i)(\widetilde{u}_0^j + h'(d)d^j)d_i d_j\\
&\quad - W^2(\Delta_P\widetilde{u}_0 + h'(d)\Delta_P d) + (\widetilde{u}_0^i + h'(d)d^i)(\widetilde{u}_0^j + h'(d)d^j)(\langle \nabla^P_{\partial_i}\nabla^P\widetilde{u}_0, \partial_j\rangle\\
&\quad + h'(d)\langle \nabla^P_{\partial_i}\nabla^P d, \partial_j\rangle) - \langle \nabla^P \log\varrho, \nabla^P\widetilde{u}_0 + h'(d)\nabla^P d\rangle(\varrho^{-2} + W^2).
\end{aligned}$$

We have

$$\begin{aligned}
&W^2 - (\widetilde{u}_0^i + h'(d)d^i)(\widetilde{u}_0^j + h'(d)d^j)d_i d_j = \varrho^{-2} + |\nabla^P\widetilde{u}_0 + h'(d)\nabla^P d|^2\\
&\quad - \langle \nabla^P\widetilde{u}_0 + h'(d)\nabla^P d, \nabla^P\widetilde{u}_0 + h'(d)\nabla^P d\rangle = \varrho^{-2}.
\end{aligned}$$

Since

$$W^2 = \varrho^{-2} + |\nabla^P\widetilde{u}_0|^2 + 2h'(d)\langle\nabla^P d, \nabla^P\widetilde{u}_0\rangle + h'^2(d)$$

and $d^i d^j \langle \nabla^P_{\partial_i} \nabla^P d, \partial_j\rangle = 0$ we also have

$$
\begin{aligned}
&- W^2(\Delta_P \widetilde{u}_0 + h'(d)\Delta_P d) + (\widetilde{u}_0^i + h'(d)d^i)(\widetilde{u}_0^j + h'(d)d^j)(\langle \nabla^P_{\partial_i}\nabla^P \widetilde{u}_0, \partial_j\rangle + h'(d)\langle \nabla^P_{\partial_i}\nabla^P d, \partial_j\rangle)\\
&= -(\varrho^{-2} + h'^2(d))h'(d)\Delta_P d - h'^2(d)(\Delta_P \widetilde{u}_0 + 2\langle \nabla^P d, \nabla^P \widetilde{u}_0\rangle \Delta_P d - d^i d^j \langle \nabla^P_{\partial_i}\nabla^P \widetilde{u}_0, \partial_j\rangle)\\
&- h'(d)(|\nabla^P \widetilde{u}_0|^2 \Delta_P d - \widetilde{u}_0^i \widetilde{u}_0^j \langle \nabla^P_{\partial_i}\nabla^P d, \partial_j\rangle + 2\langle \nabla^P d, \nabla^P \widetilde{u}_0\rangle \Delta_P \widetilde{u}_0 - 2d^i \widetilde{u}_0^j \langle \nabla^P_{\partial_i}\nabla^P \widetilde{u}_0, \partial_j\rangle)\\
&- \varrho^{-2}\Delta_P \widetilde{u}_0 - |\nabla^P \widetilde{u}_0^2|\Delta_P \widetilde{u}_0 + \widetilde{u}_0^i \widetilde{u}_0^j \langle \nabla^P_{\partial_i}\nabla^P \widetilde{u}_0, \partial_j\rangle.
\end{aligned}
$$

Gathering these expressions and requiring that $h' > 0$ and $h'' < 0$, one gets

$$
\begin{aligned}
W^2(\partial_t v - \mathcal{Q}[v]) \geq{}& -h''(d)\varrho^{-2} - (\varrho^{-2} + h'^2(d))h'(d)\Delta_P d\\
&- \langle \nabla^P \log \varrho, \nabla^P \widetilde{u}_0 + h'(d)\nabla^P d\rangle(\varrho^{-2} + W^2)\\
&- h'^2(d)(\Delta_P \widetilde{u}_0 + 2|\nabla^P \widetilde{u}_0||\Delta_P d| + |\nabla^P\nabla^P \widetilde{u}_0|)\\
&- h'(d)(|\nabla^P \widetilde{u}_0|^2|\Delta_P d| + |\nabla^P \widetilde{u}_0|^2|\nabla^P\nabla^P d| + 2|\nabla^P \widetilde{u}_0||\Delta_P \widetilde{u}_0|\\
&+ 2|\nabla^P \widetilde{u}_0||\nabla^P\nabla^P \widetilde{u}_0|) - C(\varrho^{-2} + |\nabla^P \widetilde{u}_0|^2)|\nabla^P\nabla^P \widetilde{u}_0|,
\end{aligned}
$$

where C here and in what follows stands for a positive constant that depends on n and on the first and second derivatives of d. Hence,

$$
\begin{aligned}
&W^2(\partial_t v - \mathcal{Q}[v]) \geq -h''(d)\varrho^{-2} - (\varrho^{-2} + h'^2(d))h'(d)\Delta_P d\\
&- \langle \nabla^P \log \varrho, \nabla^P \widetilde{u}_0 + h'(d)\nabla^P d\rangle(\varrho^{-2} + W^2) - Ch'^2(d)(|\nabla^P \widetilde{u}_0| + |\nabla^P\nabla^P \widetilde{u}_0|)\\
&- Ch'(d)\big(|\nabla^P \widetilde{u}_0|^2 + |\nabla^P \widetilde{u}_0||\nabla^P\nabla^P \widetilde{u}_0|\big) - C(\varrho^{-2} + |\nabla^P \widetilde{u}_0|^2)|\nabla^P\nabla^P \widetilde{u}_0|\\
&= -h''(d)\varrho^{-2} - h'(d)\langle \nabla^P \log \varrho, \nabla^P d\rangle \varrho^{-2} - (\varrho^{-2} + h'^2(d))h'(d)\big(\Delta_P d + \langle \nabla^P \log \varrho, \nabla^P d\rangle\big)\\
&- h'(d)\langle \nabla^P \log \varrho, \nabla^P d\rangle\big(|\nabla^P \widetilde{u}_0|^2 + 2h'(d)\langle \nabla^P d, \nabla^P \widetilde{u}_0\rangle\big) - (\varrho^{-2} + W^2)\langle \nabla^P \log \varrho, \nabla^P \widetilde{u}_0\rangle\\
&- Ch'^2(d)(|\nabla^P \widetilde{u}_0| + |\nabla^P\nabla^P \widetilde{u}_0|) - Ch'(d)\big(|\nabla^P \widetilde{u}_0|^2 + |\nabla^P \widetilde{u}_0||\nabla^P\nabla^P \widetilde{u}_0|\big)\\
&- C(\varrho^{-2} + |\nabla^P \widetilde{u}_0|^2)|\nabla^P\nabla^P \widetilde{u}_0|.
\end{aligned}
$$

It follows from (12) and (13) that

$$
-\big(\Delta_P d + \langle \nabla^P \log \varrho, \nabla^P d\rangle\big) = \Delta_P r + \langle \nabla^P \log \varrho, \nabla^P r\rangle \geq -n\frac{\xi'(r)}{\xi(r)} \geq -nB,
$$

where $B := \sup_{B_R(o)} \frac{\xi'(r(x))}{\xi(r(x))}$.

Therefore,

$$\begin{aligned} W^2(\partial_t v - \mathcal{Q}[v]) \geq & -h''(d)\varrho^{-2} - h'(d)|\nabla^P \log\varrho|\varrho^{-2} - nBh'^3(d) \\ & - Ch'^2(d)\Big(|\nabla^P \log\varrho||\nabla^P \widetilde{u}_0| + |\nabla^P \widetilde{u}_0| + |\nabla^P\nabla^P \widetilde{u}_0|\Big) \\ & - Ch'(d)\Big(nB\varrho^{-2} + |\nabla^P \log\varrho||\nabla^P \widetilde{u}_0|^2 + |\nabla^P \widetilde{u}_0|^2 + |\nabla^P \widetilde{u}_0||\nabla^P\nabla^P \widetilde{u}_0|\Big) \\ & - C\big(\varrho^{-2} + |\nabla^M \widetilde{u}_0|^2\big)\Big(|\nabla^P \log\varrho||\nabla^P \widetilde{u}_0| + |\nabla^M\nabla^M \widetilde{u}_0|\Big). \end{aligned}$$

For $L > 0$, we fix $d_0 < \frac{1}{L}$ and $d_0 < d^*$ and take

$$A = \frac{L}{1 - Ld_0}\cdot$$

Then

$$L = \frac{A}{1 + Ad_0}.$$

We consider

$$h(d) = \frac{1}{L}\log(1 + Ad)$$

for $d \in [0, d_0]$ and we note that

$$h'(d) = \frac{1}{L}\left(\frac{A}{1 + Ad}\right), \quad \text{and} \quad h''(d) = -Lh'^2(d).$$

Thus

$$\begin{aligned} W^2(\partial_t v - \mathcal{Q}[v]) \geq & \left(\frac{1}{L}\frac{A^2}{(1+Ad)^2} - \frac{1}{L}\frac{A}{1+Ad}|\nabla^M \log\varrho|\right)\varrho^{-2} - nB\frac{1}{L^3}\frac{A^3}{(1+Ad)^3} \\ & - \widetilde{L}\left(\frac{1}{L^2}\frac{A^2}{(1+Ad)^2} + \frac{1}{L}\frac{A}{1+Ad} + 1\right), \end{aligned}$$

where

$$\widetilde{L} = C\Big(n, B, \varrho, |\nabla^P \log\varrho|, |\nabla^P \widetilde{u}_0|, |\nabla^P\nabla^P \widetilde{u}_0|\Big).$$

More precisely,

$$\begin{aligned} W^2(\partial_t v - \mathcal{Q}[v]) \geq \frac{1}{L}\bigg\{ & - nB\frac{1}{L^2}\frac{A^3}{(1+Ad)^3} + \left(\varrho^{-2} - \frac{\widetilde{L}}{L}\right)\frac{A^2}{(1+Ad)^2} \\ & - \left(|\nabla^P \log\varrho|\varrho^{-2} + \widetilde{L}\right)\frac{A}{1+Ad} - L\widetilde{L}\bigg\}, \end{aligned}$$

For $d \in [0, d_0]$ we have

$$L=\frac{A}{1+Ad_0}\leq\frac{A}{1+Ad}\quad\text{and}\quad -L\widetilde{L}\geq-\widetilde{L}\frac{A}{1+Ad}.$$

So

$$\begin{aligned}W^2(\partial_t v-\mathcal{Q}[v])&\geq\frac{1}{L}\frac{A}{1+Ad}\bigg\{-nB\frac{1}{L^2}\frac{A^2}{(1+Ad)^2}+\bigg(\varrho^{-2}-\frac{\widetilde{L}}{L}\bigg)\frac{A}{1+Ad}\\&\qquad-\Big(|\nabla^P\log\varrho|\varrho^{-2}+2\widetilde{L}\Big)\bigg\}\\&\geq\frac{\widetilde{L}}{L}\frac{A}{1+Ad}\bigg\{-\frac{1}{L^2}\bigg(\frac{A}{1+Ad}\bigg)^2+\bigg(\frac{L}{\varrho^2\widetilde{L}}-1\bigg)\frac{1}{L}\bigg(\frac{A}{1+Ad}\bigg)-3\bigg\}\end{aligned}$$

If we choose $L>5\widetilde{L}\sup_{B_R(o)}\varrho^2$ we have

$$D=L^2\bigg(\frac{L}{\varrho^2\widetilde{L}}-1\bigg)^2-12L^2>0.$$

As D is the discriminant of the inequality

$$-z^2+\bigg(\frac{L}{\varrho^2\widetilde{L}}-1\bigg)Lz-3L^2\geq0$$

we can choose $d_0<\frac{1}{L}$ such that

$$-\bigg(\frac{A}{1+Ad}\bigg)^2+\bigg(\frac{L}{\varrho^2\widetilde{L}}-1\bigg)L\bigg(\frac{A}{1+Ad}\bigg)-3L^2\geq0$$

for $d\in[0,d_0]$. Then we get

$$W^2(\partial_t v-\mathcal{Q}[v])\geq\frac{\widetilde{L}}{L}\bigg(\frac{A}{1+Ad}\bigg)\bigg\{-\frac{1}{L^2}\bigg(\frac{A}{1+Ad}\bigg)^2+\bigg(\frac{L}{\varrho^2\widetilde{L}}-1\bigg)\frac{1}{L}\bigg(\frac{A}{1+Ad}\bigg)-3\bigg\}\geq0$$

for all (x,t) with $d(x)\in[0,d_0]$.

Hence $v=\widetilde{u_0}+h(d)$ is an upper barrier. If we take $\omega=-\widetilde{u_0}-h(d)$ we get a lower barrier. Therefore there exists a constant $C>0$ such that

$$\sup_{\partial B_R(o)\times[0,T]}|\nabla u|\leq C.$$

□

3.2.2 *Interior gradient estimate*

In this subsection we will use a technique due to Korevaar and Simon [21], and further developed by Wang [31] to prove an interior gradient estimates.

Given $R>0$, and $T>0$, let

$$\mathcal{C}_{R,T}=\big\{\Psi(x,t);\ \zeta(r(\Psi(x,t)))+t\le\zeta(R),\ x\in B_R(o),\ t\in[0,T]\big\}.$$

If $R'\in(0,R)$ is such that $\zeta(r)<\frac{1}{4}\zeta(R)$ for all $r\le R'$ we get $B_{R'}(o)\times[0,T_{R'}]\subset\mathcal{C}_{R,T}$ with $T_{R'}=\min\left\{\frac{1}{2}\zeta(R),T\right\}$.

Proposition 3.7 *Let u be a positive solution of* (35) *defined in $B_R(o)\times[0,T]$ for $R>0$ and $T>0$. Let $L\ge0$ be a constant such that $\mathrm{Ric}_g-\nabla^2\log\varrho\ge-Lg$ in $B_R(o)$ and suppose that* (11) *holds. Then for (x,t) in $B_{R'}(o)\times[0,T_{R'}]$ either*

$$|\nabla^P u(x,t)|\le\exp\left(128\frac{\left(1+\min_{\overline{B}_R(o)}\varrho\right)^2}{\min_{\overline{B}_R(o)}\varrho}M\sup_{B_R(o)}\frac{\xi(r)}{\zeta(R)}\right)\tag{62}$$

or

$$|\nabla^P u(x,t)|\le\exp\left(64\frac{\left(1+\min_{\overline{B}_R(o)}\varrho\right)^2}{\min_{\overline{B}_R(o)}\varrho}MC_0\right),$$

where $M=\sup_{B_{R'}(o)\times[0,T_{R'}]}u$, and

$$C_0=\sup_{B_R(o)}\frac{\varrho^2}{\mu}\left\{\frac{5}{4}+nM\frac{\xi'(r)}{\zeta(R)}+2\sqrt{1-\beta}\frac{\xi(r)}{\zeta(R)}+\left(M(6-5\beta)\frac{\xi(r)}{\zeta(R)}+2\sqrt{1-\beta}\right)\frac{\varrho'}{\varrho}\right\}.$$

Proof. Suppose initially that u is C^3 positive solution of (35) in $B_R(o)\times(0,T)\subset P\times\mathbb{R}$. Let η be a nonnegative and smooth function with $\eta=0$ in $P\times\mathbb{R}\setminus B_R(o)\times\mathbb{R}$. We consider a function

$$\chi=\eta\gamma(u)\psi(|\nabla u|^2),\tag{63}$$

defined in $\overline{B_R(o)}\times[0,T]$ where η, γ and ψ are functions to be specified later.

If χ attains its maximum value in $B_{R'}(o)\times[0,T_{R'}]$ at point (x_0,t_0), and $\eta(x_0,t_0)\ne0$, we have

$$(\log\chi)_j=\frac{\eta_j}{\eta}+\frac{\gamma'}{\gamma}u_j+2\frac{\psi'}{\psi}u^ku_{k;j}=0\tag{64}$$

at (x_0,t_0). Then

$$2\frac{\psi'}{\psi}u^ku_{k;j}=-\left(\frac{\eta_j}{\eta}+\frac{\gamma'}{\gamma}u_j\right).\tag{65}$$

Moreover, the matrix

$$(\log\chi)_{i;j} = (\log\eta)_{i;j} + \left(\frac{\gamma'}{\gamma}\right)' u_i u_j + \frac{\gamma'}{\gamma} u_{i;j} + 2\frac{\psi'}{\psi}(u^k u_{k;ij} + u^k_{;i} u_{k;j}) + 4\left(\frac{\psi'}{\psi}\right)' u^k u_{k;i} u^\ell u_{\ell;j}$$

is non-positive at (x_0, t_0). Applying the Ricci identities for the Hessian of u we have

$$u_{k;ij} = u_{i;kj} = u_{i;jk} + R^\ell_{kji} u_\ell,$$

and this yields

$$\begin{aligned}(\log\chi)_{i;j} &= \frac{\eta_{i;j}}{\eta} + \frac{\gamma''}{\gamma} u_i u_j + \frac{\gamma'}{\gamma} u_{i;j} + \frac{\gamma'}{\gamma}\left(\frac{\eta_i}{\eta} u_j + \frac{\eta_j}{\eta} u_i\right) \\ &\quad + 2\frac{\psi'}{\psi}(u^k u_{i;jk} + u^k_{;i} u_{k;j}) - 2\frac{\psi'}{\psi} R^\ell_{jki} u^k u_\ell + 4\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2}\right) u^k u_{k;i} u^\ell u_{\ell;j}.\end{aligned}$$

On the other hand, denoting

$$f(x) = \partial_t u - \left\langle \bar\nabla \log\varrho, \bar\nabla u\right\rangle \left(1 + \frac{1}{\varrho^2 W^2}\right) \tag{66}$$

and differentiating both sides in (20) we get

$$\sigma^{ij} u_{i;jk} = f_k - \sigma^{ij}_{;k} u_{i;j}. \tag{67}$$

After contraction of (67) with u^k, we have

$$\begin{aligned}\sigma^{ij} u^k u_{i;jk} &= f_k u^k + \frac{1}{W^2} u^k (u^i_{;k} u^j + u^i u^j_{;k}) u_{i;j} \\ &\quad - \frac{2}{W^4} u^i u^j u_{i;j}\big(- \varrho^{-2} (\log\varrho)_k u^k + u^k u^\ell u_{\ell;k}\big).\end{aligned}$$

Using the previous identity, (106) and noticing that

$$\sigma^{ij} R^\ell_{jki} u^k u_\ell = -\mathrm{Ric}_g(\nabla^P u, \nabla^P u),$$

after some computations we obtain

$$
\begin{aligned}
0 \geq \sigma^{ij}(\log\chi)_{i;j} &= \sigma^{ij}\frac{\eta_{i;j}}{\eta} + \frac{\gamma''}{\gamma}\sigma^{ij}u_iu_j + \frac{\gamma'}{\gamma}\partial_t u - \frac{\gamma'}{\gamma}\left(1+\frac{1}{\varrho^2W^2}\right)\langle\nabla^P\log\varrho, \nabla^P u\rangle \\
&+ 2\frac{\gamma'}{\gamma}\frac{1}{\varrho^2W^2}\left\langle\frac{\nabla^P\eta}{\eta}, \nabla^P u\right\rangle + \quad 2\frac{\psi'}{\psi}u^k\partial_t u_k + \frac{4|\nabla^P u|^2}{\varrho^2W^2}\frac{\psi'}{\psi}\langle\nabla^P\log\varrho, \nabla^P u\rangle^2 \\
&- \frac{4}{\varrho^2W^4}\left\langle\frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u, \nabla^P u\right\rangle\langle\nabla^P\log\varrho, \nabla^P u\rangle - 2\frac{\psi'}{\psi}\left(1+\frac{1}{\varrho^2W^2}\right)\nabla^2\log\varrho(\nabla^P u, \nabla^P u) \\
&\left(1+\frac{1}{\varrho^2W^2}\right)\left\langle\nabla^P\log\varrho, \frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u\right\rangle + 4\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2}\frac{1}{W^2}\frac{\psi'}{\psi}\right)\sigma^{i\ell}u^ju^ku_{k;i}u_{j;\ell} \\
&+ 2\frac{\psi'}{\psi}\mathrm{Ric}_g(\nabla^P u, \nabla^P u) + 2\frac{\psi'}{\psi}\sigma^{i\ell}\sigma^{jk}u_{k;i}u_{j;\ell}.
\end{aligned}
$$

We also have

$$
\partial_t\log\chi = \frac{\partial_t\eta}{\eta} + \frac{\gamma'}{\gamma}\partial_t u + 2\frac{\psi'}{\psi}u^k\partial_t u_k.
$$

Thus

$$
\begin{aligned}
0 &\leq \partial_t\log\chi - \sigma^{ij}(\log\chi)_{i;j} \\
&= \frac{\partial_t\eta}{\eta} - \sigma^{ij}\frac{\eta_{i;j}}{\eta} - \frac{\gamma''}{\gamma}\sigma^{ij}u_iu_j + \frac{\gamma'}{\gamma}\left(1+\frac{1}{\varrho^2W^2}\right)\langle\nabla^P\log\varrho, \nabla^P u\rangle - 2\frac{\gamma'}{\gamma}\frac{1}{\varrho^2W^2}\left\langle\frac{\nabla^P\eta}{\eta}, \nabla^P u\right\rangle \\
&- \frac{4|\nabla^P u|^2}{\varrho^2W^4}\frac{\psi'}{\psi}\langle\nabla^P\log\varrho, \nabla^P u\rangle^2 + \frac{4}{\varrho^2W^4}\left\langle\frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u, \nabla^P u\right\rangle\langle\nabla^P\log\varrho, \nabla^P u\rangle \\
&+ 2\frac{\psi'}{\psi}\left(1+\frac{1}{\varrho^2W^2}\right)\nabla^2\log\varrho(\nabla^P u, \nabla^P u) - \left(1+\frac{1}{\varrho^2W^2}\right)\left\langle\nabla^P\log\varrho, \frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u\right\rangle \\
&- 4\left[\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2W^2}\frac{\psi'}{\psi}\right]\sigma^{i\ell}u^ju^ku_{k;i}u_{j;\ell} - 2\frac{\psi'}{\psi}\sigma^{i\ell}\sigma^{jk}u_{k;i}u_{j;l} - 2\frac{\psi'}{\psi}\mathrm{Ric}_g(\nabla^P u, \nabla^P u)
\end{aligned}
$$

We note that

$$
\begin{aligned}
4\frac{\psi'^2}{\psi^2}\sigma^{i\ell}u^ju^ku_{k;i}u_{j;\ell} &= \left|\frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u\right|^2 \geq \frac{1}{\varrho^2W^2}\left|\frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u\right|_g^2 \\
&= \frac{|\nabla^P u|^2}{\varrho^2W^2}\left|\frac{\nabla^P\eta}{|\nabla^P u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla^P u}{|\nabla^P u|}\right|_g^2.
\end{aligned}
$$

So, the previous estimate yields

$$\frac{\psi^2}{\psi'^2}\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2W^2}\frac{\psi'}{\psi}\right)\frac{|\nabla^P u|^2}{\varrho^2 W^2}\left|\frac{\nabla^P\eta}{|\nabla^P u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla^P u}{|\nabla^P u|}\right|_g^2 + \frac{\gamma''}{\gamma}\sigma^{ij}u_iu_j + 2\frac{\psi'}{\psi}\sigma^{i\ell}\sigma^{jk}u_{k;i}u_{j;\ell}$$
$$\leq \frac{\partial_t\eta}{\eta} - \sigma^{ij}\frac{\eta_{i;j}}{\eta} + \frac{\gamma'}{\gamma}\left(1+\frac{1}{\varrho^2W^2}\right)\langle\nabla^P\log\varrho, \nabla^P u\rangle + 2\frac{\gamma'}{\gamma}\frac{1}{\varrho^2 W}\frac{|\nabla^P u|}{W}\left|\frac{\nabla^P\eta}{\eta}\right|$$
$$-\frac{4|\nabla^P u|^2}{\varrho^2W^4}\frac{\psi'}{\psi}\langle\nabla^P\log\varrho, \nabla^P u\rangle^2 + \frac{4}{\varrho^2 W^4}\left\langle\frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u, \nabla^P u\right\rangle\langle\nabla^P\log\varrho, \nabla^P u\rangle$$
$$-2\frac{\psi'}{\psi}\left[\mathrm{Ric}_g(\nabla^P u, \nabla^P u) - \nabla^2\log\varrho(\nabla^P u, \nabla^P u)\right] + \frac{2}{\varrho^2W^2}\frac{\psi'}{\psi}\nabla^2\log\varrho(\nabla^P u, \nabla^P u)$$
$$-\left(1+\frac{1}{\varrho^2W^2}\right)\left\langle\nabla^P\log\varrho, \frac{\nabla^P\eta}{\eta} + \frac{\gamma'}{\gamma}\nabla^P u\right\rangle$$

Discarding a non-positive term in the right hand side, we get

$$\frac{\psi^2}{\psi'^2}\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2W^2}\frac{\psi'}{\psi}\right)\frac{|\nabla^P u|^2}{\varrho^2 W^2}\left|\frac{\nabla^P\eta}{|\nabla^P u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla^P u}{|\nabla^P u|}\right|_g^2 + \frac{\gamma''}{\gamma}\frac{|\nabla^P u|^2}{\varrho^2W^2}$$
$$\leq \frac{\partial_t\eta}{\eta} - \sigma^{ij}\frac{\eta_{i;j}}{\eta} + 2\frac{\gamma'}{\gamma}\frac{1}{\varrho^2 W}\frac{|\nabla^P u|}{W}\left|\frac{\nabla^P\eta}{\eta}\right| + 4\frac{\psi'}{\psi}\frac{|\nabla^P\log\varrho|^2}{\varrho^2}\frac{|\nabla^P u|^4}{W^4}$$
$$+4\frac{|\nabla^P\log\varrho|}{\varrho^2}\left(\left|\frac{\nabla^P\eta}{\eta}\right|\frac{|\nabla^P u|^2}{W^4} + \frac{\gamma'}{\gamma}\frac{|\nabla^P u|^3}{W^4}\right) - 2\frac{\psi'}{\psi}\left[\mathrm{Ric}_g(\nabla^P u, \nabla^P u) - \nabla^2\log\varrho(\nabla^P u, \nabla^P u)\right]$$
$$+2\frac{\psi'}{\psi}\frac{|\nabla^2\log\varrho|}{\varrho^2}\frac{|\nabla^P u|^2}{W^2} + \left|\nabla^P\log\varrho\right|\left|\frac{\nabla^P\eta}{\eta}\right|\left(1+\frac{1}{\varrho^2W^2}\right).$$

If $|\nabla^P u(x_0,t_0)| \leq \alpha$ for some $\alpha > 0$, we take $\psi(s) = s$ and η, γ such that

$$\eta(x,t) \leq \beta < \infty \quad \text{and} \quad 1 + \min_{\overline{B_R(o)}}\varrho \geq \gamma(x,t) \geq 1 \qquad \text{in} \quad B_R(o)\times[0,T]$$

and we obtain

$$|\nabla u(x,t)|\beta \;\leq\; \chi(x,t) \;\leq\; \chi(x_0,t_0) \;\leq\; \beta(1+\inf_{\overline{B_R(o)}}\varrho)\alpha.$$

Thus,

$$|\nabla u(x,t)| \leq (1+\inf_{\overline{B_R(o)}}\varrho)\alpha \qquad \text{in} \qquad B_R(o)\times(0,T).$$

Then, we supose that $|\nabla^P u(x_0,t_0)|^2 > 1$ and following [31], we set

$$\psi(\tau) = \log\tau, \tag{68}$$

where $\tau = |\nabla^P u|^2$. We have

$$\frac{|\nabla^P u|^2}{W^2}\frac{\psi^2}{\psi'^2}\left(\left(\frac{\psi'}{\psi}\right)' - \frac{\psi'^2}{\psi^2} + \frac{3}{2}\frac{1}{W^2}\frac{\psi'}{\psi}\right) = \frac{\tau}{W^2}\left(\log\tau\frac{\frac{1}{2}\tau - \varrho^{-2}}{\tau+\varrho^{-2}} - 2\right).$$

Now we consider $k>0$ be a constant such that $|\nabla^P u(x_0,t_0)|^2 > e^k$ and we fix a constant

$$\max\left\{\frac{3}{4},\ \frac{\varrho^2 e^k}{1+\varrho^2 e^k}\right\} < \beta < 1.$$

We can suppose that

$$\frac{\tau}{W^2} = \frac{|\nabla^P u|^2}{W^2} \geq \beta \tag{69}$$

at (x_0,t_0). We also consider $\frac{1}{\varrho^2}\frac{\beta}{1-\beta} =: e^{\delta'}$, $\delta = \frac{3}{2}\beta - 1$ and $\mu := 2\beta\frac{\delta\delta'-2}{\delta'}$, and we note that

$$\frac{1}{8} < \delta < \frac{1}{2}, \qquad k < \delta' \leq \log\tau \qquad \mu > \frac{3}{16}\left(\frac{k-16}{k}\right) > 0,$$

if $k > 16$. We get

$$\begin{aligned}
&\mu \log|\nabla^P u| \frac{1}{\varrho^2}\left|\frac{\nabla^P\eta}{|\nabla^P u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla^P u}{|\nabla^P u|}\right|_g^2 + \frac{\gamma''}{\gamma}\frac{|\nabla^P u|^2}{\varrho^2 W^2}\\
&+ 2\frac{\psi'}{\psi}|\nabla^P u|^2\left(\mathrm{Ric}_g\left(\frac{\nabla^P u}{|\nabla^P u|}, \frac{\nabla^P u}{|\nabla^P u|}\right) - \nabla^2\log\varrho\left(\frac{\nabla^P u}{|\nabla^P u|}, \frac{\nabla^P u}{|\nabla^P u|}\right)\right)\\
&\leq \frac{1}{\eta}(\partial_t - \Delta)\eta + 2\sqrt{1-\beta}\frac{1}{\varrho}\frac{\gamma'}{\gamma}\left|\frac{\nabla^P\eta}{\eta}\right| + \frac{4}{\delta'}(1-\beta)|\nabla^P\log\varrho|^2\\
&+ \left|\frac{\nabla^P\eta}{\eta}\right|(6-5\beta)|\nabla^P\log\varrho| + 4\frac{\gamma'}{\gamma}\frac{1}{\varrho}\sqrt{1-\beta}|\nabla^P\log\varrho| + \frac{2}{\delta'}(1-\beta)|\nabla^2\log\varrho|.
\end{aligned}$$

Now, we choose η in $\mathcal{C}_{R,T}$ as

$$\eta = (\zeta(R) - \zeta(r) - t)^2 \tag{70}$$

where $r = d(o,\cdot)$.

It follows from Proposition 2.1 that

$$\begin{aligned}
\frac{1}{\eta}(\partial_t - \Delta)\eta &= -2\frac{\sqrt{\eta}}{\eta}(\partial_t-\Delta)\zeta - 2\frac{1}{\eta}|\nabla\zeta|^2 - 2\frac{\sqrt{\eta}}{\eta}\\
&\leq \frac{2}{\sqrt{\eta}}\left(n\xi'(r) + \varrho^2|\nabla s|^2\xi(r)\left(\left\langle\bar\nabla\log\varrho, \nabla r\right\rangle - \frac{\xi'(r)}{\xi(r)}\right)\right)\\
&- \frac{2\xi(r)^2}{\eta}|\nabla r|^2 - 2\frac{\sqrt{\eta}}{\eta}.
\end{aligned}$$

Since $\left\langle\bar\nabla\log\varrho,\nabla r\right\rangle \leq \left|\frac{\partial_r\varrho}{\varrho}\right| \leq \frac{\xi'(r)}{\xi(r)}$ implies in

$$\varrho^2|\nabla s|^2\xi(r)\left(\left\langle\bar\nabla\log\varrho,\nabla r\right\rangle - \frac{\xi'(r)}{\xi(r)}\right) \leq 0,$$

we obtain

$$\frac{1}{\eta}\big(\partial_t - \Delta\big)\eta \leq 2n\frac{\xi'(r)}{\sqrt{\eta}}.$$

Furthermore, $\left|\frac{\nabla\eta}{\eta}\right| \leq 2\frac{\xi(r)}{\sqrt{\eta}}$.
Finally, we set as in [31]

$$\gamma(u) = 1 + \frac{1}{S}(\min_{\overline{B}_R(o)} \varrho)u$$

where $S = \sup_{B_R(o)\times[0,T]} u > 0$. Then $\gamma'' = 0$ and thus

$$\begin{aligned}\mu \log|\nabla u|\frac{1}{\varrho^2}\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2 - 2L\frac{\psi'}{\psi}|\nabla u|^2 \leq \frac{1}{S\sqrt{\eta}}\Big[&2nS\xi'(r) + 4\sqrt{1-\beta}\xi(r)\\ &+ \frac{2}{\delta'}(1-\beta)\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big)S\sqrt{\eta} + 2S\xi(r)(6-5\beta)|\nabla\log\varrho|\\ &+ 4\sqrt{\eta}\sqrt{1-\beta}|\nabla\log\varrho|\Big]\end{aligned}$$

where $L \geq 0$ is a constant such that

$$\mathrm{Ric}_g - \nabla^2\log\varrho \geq -Lg \tag{71}$$

in $B_R(o)$. Using that $\frac{1}{\log t} \leq \frac{1}{\delta'}$ and $\eta \leq \zeta(R)^2$ we obtain

$$\begin{aligned}\mu \log|\nabla u|\frac{1}{\varrho^2}\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2 \leq \frac{\zeta(R)}{S\eta}\Big[&\frac{2\zeta(R)SL}{\delta'} + 2nS\xi'(r) + 4\sqrt{1-\beta}\xi(r)\\ &+ \frac{2}{\delta'}S\zeta(R)\big(1-\beta\big)\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big) + 2S\xi(r)(6-5\beta)|\nabla\log\varrho|\\ &+ 4\zeta(R)\sqrt{1-\beta}|\nabla\log\varrho|\Big].\end{aligned}$$

We can to rewrite the inequality above as

$$\begin{aligned}\eta\log|\nabla u|\left|\frac{\nabla\eta}{|\nabla u|\eta} + \frac{\gamma'}{\gamma}\frac{\nabla u}{|\nabla u|}\right|_g^2 \leq \frac{\zeta(R)\varrho^2}{\mu S}\Big\{&\frac{2}{\delta'}LS\zeta(R) + 2nS\xi'(r) + 4\sqrt{1-\beta}\xi(r)\\ &+ \frac{2}{\delta'}(1-\beta)S\zeta(R)\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big) + 2S(6-5\beta)\xi(r)|\nabla\log\varrho|\\ &+ 4\zeta(R)\sqrt{1-\beta}|\nabla\log\varrho|\Big\}.\end{aligned}$$

We consider first the case

$$\left|\frac{\nabla\eta}{|\nabla u|\eta}\right| \leq \frac{\gamma'}{4\gamma}.$$

Then we have

$$\begin{aligned}\eta\log|\nabla u| \le{}& \frac{2S^2\gamma^2}{\min\varrho^2}\frac{\varrho^2\zeta(R)}{\mu S}\bigg\{\frac{2}{\delta'}\zeta(R)LS + 2nS\xi'(r) + 4\sqrt{1-\beta}\xi(r) + 4\zeta(R)\sqrt{1-\beta}|\nabla\log\varrho| \\ &+ \frac{2}{\delta'}(1-\beta)S\zeta(R)\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big) + 2S(6-5\beta)\xi(r)|\nabla\log\varrho|\bigg\} \\ \le{}& 2\left(\frac{1+\min\varrho}{\min\varrho}\right)^2 S\zeta^2(R)\frac{\varrho^2}{\mu}\bigg\{\frac{2}{\delta'}LS + 2nS\frac{\xi'(r)}{\zeta(R)} + 4\sqrt{1-\beta}\frac{\xi(r)}{\zeta(R)} \\ &+ \frac{2}{\delta'}(1-\beta)S\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big) + 2\Big((6-5\beta)S\frac{\xi(r)}{\zeta(R)} + 2\sqrt{1-\beta}\Big)|\nabla\log\varrho|\bigg\}\end{aligned}$$

If $\tau > e^k$ with $k = \max\Big\{SL,\ \sup_{B_R(o)}\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big)\Big\}$, then $\delta' > k$ implies

$$\frac{2}{\delta'}(1-\beta)S\Big(2|\nabla\log\varrho|^2 + |\nabla^2\log\varrho|\Big) \le 2(1-\beta) < \frac{1}{2} \qquad \text{and} \qquad \frac{2}{\delta'}SL \le 2.$$

Thus

$$\eta\log|\nabla u| \le 4\left(\frac{1+\min\varrho}{\min\varrho}\right)^2 S\zeta^2(R)C_0,$$

where

$$C_0 = \sup_{B_R(o)}\frac{\varrho^2}{\mu}\left\{\frac{5}{4} + nS\frac{\xi'(r)}{\zeta(R)} + 2\sqrt{1-\beta}\frac{\xi(r)}{\zeta(R)} + \left(S(6-5\beta)\frac{\xi(r)}{\zeta(R)} + 2\sqrt{1-\beta}\right)\frac{\varrho'}{\varrho}\right\}$$

On the other hand, if

$$\frac{\gamma'}{4\gamma} \le \left|\frac{\nabla\eta}{|\nabla u|\eta}\right|$$

we have

$$\eta|\nabla u| \le \frac{8\gamma}{\gamma'}\sqrt{\eta}|\nabla\zeta| = \frac{8\gamma S\sqrt{\eta}\xi(r)}{\min_{\overline{B_R(o)}}\varrho}$$

and consequently

$$\eta\log|\nabla u| \le \frac{8\gamma S\sqrt{\eta}\xi(r)}{\min_{\overline{B_R(o)}}\varrho} \le 8\left(\frac{1+\min_{\bar{B_R}(o)}\varrho}{\min_{\bar{B}(o,R)}\varrho}\right)S\zeta^2(R)\sup_{B_R(o)}\frac{\xi(r)}{\zeta(R)}.$$

Hence at (x_0, t_0)

$$\eta\log|\nabla u| \le 4\left(\frac{1+\min_{\bar{B_R}(o)}\varrho}{\min_{\bar{B_R}(o)}\varrho}\right)^2 S\zeta^2(R)\max\left\{2\sup_{B_R(o)}\frac{\xi(r)}{\zeta(R)}, C_0\right\}.$$

Since $\eta(x,t) > \frac{1}{16}\zeta^2(R)$ and $\gamma(x,t) \geq 1$ for $(x,t) \in B_{R'}(o) \times [0, T_{R'}]$ we conclude that

$$\begin{aligned}
\log|\nabla^P u(x,t)| &\leq \frac{16}{\zeta^2(R)}\eta(x,t)\gamma(x,t)\log|\nabla^P u(x,t)| \leq \frac{16}{\zeta^2(R)}\gamma(x_0,t_0)\eta(x_0,t_0)\log|\nabla^P u(x_0,t_0)| \\
&\leq \frac{16}{\zeta^2(R)}\left(1+\min_{\overline{B_R(o)}}\varrho\right)4\left(\frac{1+\min_{\bar{B}_R(o)}\varrho}{\min_{\bar{B}(o,R)}\varrho}\right)^2 S\zeta^2(R)\max\left\{2\sup_{B_R(o)}\frac{\xi(r)}{\zeta(R)}, C_0\right\} \\
&= 64\frac{\left(1+\min_{\overline{B_R(o)}}\varrho\right)^2}{\min_{\overline{B_R(o)}}\varrho} S\max\{2\sup_{B_R(o)}\frac{\xi(r)}{\zeta(R)}, C_0\}
\end{aligned}$$

unless $|\nabla u(x_0,t_0)| \leq 1$.
We can deal with $H_{2+\alpha}$ functions using a standard approximation argument. Moreover, we can remove the assumption that $u > 0$ translating u upwards by M.

□

Corollary 3.8 *If* $0 < R' < R_1 < R_2$ *such that* $\zeta(r) < \frac{1}{4}\zeta(R_1) < \frac{1}{4}\zeta(R_2)$ *for all* $r \leq R'$ *and* u *be a solution of* (35) *defined in* $B_{R_2}(o)\times[0,T]$ *for* $T > 0$, *then for* (x,t) *in* $B_{R'}(o)\times[0,T_{R'}]$ *either*

$$|\nabla^P u(x,t)| \leq \exp\left(128\frac{\left(1+\min_{\overline{B}_{R_1}(o)}\varrho\right)^2}{\min_{\overline{B}_{R_1}(o)}\varrho} S\sup_{B_{R_1}(o)}\frac{\xi(r)}{\zeta(R_1)}\right) \tag{72}$$

or

$$|\nabla^P u(x,t)| \leq \exp\left(64\frac{\left(1+\min_{\overline{B}_{R_1}(o)}\varrho\right)^2}{\min_{\overline{B}_{R_1}(o)}\varrho} SC_0\right),$$

where $S = \sup_{B_{R_1}(p)\times[0,T]} u$,

$$C_0 = \sup_{B_{R_1}(o)}\frac{\varrho^2}{\mu}\left\{\frac{5}{4} + nS\frac{\xi'(r)}{\zeta(R_1)} + 2\sqrt{1-\beta}\frac{\xi(r)}{\zeta(R_1)} + \left(S(6-5\beta)\frac{\xi(r)}{\zeta(R_1)} + 2\sqrt{1-\beta}\right)\frac{\varrho'}{\varrho}\right\}$$

and $T_{R'} = \frac{1}{2}\zeta(R_1)$.
Proof. In fact, if we choose

$$\psi(|\nabla^P u|^2) = \log(|\nabla^P u|^2), \quad \gamma(u) = 1 + \frac{1}{S}(\min_{\overline{B}_{R_1}(o)}\varrho)u,$$

with $S = \sup_{B_{R_1}(o)\times[0,T]} u$ and we define $\eta = (\zeta(R_1) - \zeta(r) - t)^2$ in

$$\mathcal{C}_{R_1,T} = \left\{\Psi(x,t);\ \zeta(r(\Psi(x,t))) + t \leq \zeta(R_1),\ x \in B_{R_1}(o),\ t \in [0,T]\right\}$$

we have the estimates announced.

□

3.3 Curvature estimate

Given $R>0$ and $T>0$ we want to estimate $|\nabla^m|A||$ for $m\geq 0$ in the parabolic cylinder $B_{R'}(o)\times[0,T_{R'}]$, where $R'\in(0,R)$ is such that $\zeta(r)<\frac{1}{4}\zeta(R)$ for all $r\leq R'$ and $T_{R'}=\frac{1}{2}\zeta(R)$. For this, we will proceed as Ecker-Huisken in [14] studying the evolution of the function

$$f=\psi(W^2)|A|^2, \tag{73}$$

where

$$\psi(W^2)=\frac{W^2}{\gamma-\delta W^2} \tag{74}$$

with

$$\gamma=\inf_{B_R(o)}\frac{1}{\varrho^2}\qquad\text{and}\qquad\delta=\frac{1}{2}\frac{\gamma}{\sup_{B_R(o)\times[0,T]}W^2}.$$

Initially, we need to deduce evolution equations for the second fundamental form and its squared norm, a variant of the classical Simons' formula.

Lemma 3.9 *The squared norm $|A|^2$ of the second fundamental form of the graphs Σ_t, $t\in[0,T]$, evolve as*

$$\begin{aligned}\frac{1}{2}(\partial_t-\Delta)|A|^2+|\nabla A|^2&=|A|^4+|A|^2\overline{\mathrm{Ric}}(N,N)\\&+g^{k\ell}(\bar\nabla_i\bar R_{kj0\ell}+\bar\nabla_k\bar R_{\ell i0j})a^{ij}+2g^{k\ell}(a_{is}\bar R^s_{kj\ell}+a_{sk}\bar R^s_{\ell ij})a^{ij}.\end{aligned} \tag{75}$$

Proof. We have

$$\partial_t a_{ij}=n\nabla_i\nabla_j H-nHa_{is}a^s_j+nH\bar R_{i00j}.$$

Since

$$\partial_t g^{ij}=2nHa^{ij}$$

we get

$$\begin{aligned}\tfrac{1}{2}\partial_t|A|^2=g^{j\ell}a_{ij}a_{k\ell}\partial_t g^{ik}+g^{ik}g^{j\ell}a_{k\ell}\partial_t a_{ij}&=2nHa^{ik}a^\ell_i a_{k\ell}\\+a^{ij}(n\nabla_i\nabla_j H-nHa_{i\ell}a^\ell_j+nH\bar R_{i00j}).\end{aligned} \tag{76}$$

Then

$$\frac{1}{2}\partial_t|A|^2=nHa^{ik}a^\ell_i a_{k\ell}+na^{ij}\nabla_i\nabla_j H+nHa^{ij}\bar R_{i00j}. \tag{77}$$

We also have

$$\Delta a_{ij}=n\nabla_i\nabla_j H+nHa^s_i a_{sj}-a_{ij}|A|^2-g^{k\ell}\left(\nabla_i L_{kj\ell}+\nabla_k L_{\ell ij}\right)+g^{k\ell}(\bar R^s_{ik\ell}a_{sj}+\bar R^s_{ikj}a_{\ell s})$$

and

$$\begin{aligned}\frac{1}{2}\Delta|A|^2-|\nabla A|^2=a^{ij}\Delta a_{ij}=na^{ij}\nabla_i\nabla_j H+nHa_i^s a_{sj}a^{ij}-|A|^4\\ -g^{k\ell}\left(\nabla_i L_{kj\ell}+\nabla_k L_{\ell ij}\right)a^{ij}+g^{k\ell}(\bar{R}^s_{ik\ell}a_{sj}+\bar{R}^s_{ikj}a_{\ell s})a^{ij}\end{aligned}\tag{78}$$

where L is the $(0,3)$-tensor in Σ_t defined by $L_{ijk}=\langle\bar{R}(\partial_i,\partial_j)N,\partial_k\rangle$. Thus

$$\begin{aligned}\frac{1}{2}(\partial_t-\Delta)|A|^2+|\nabla A|^2=|A|^4+nHa^{ij}\bar{R}_{i00j}\\ +g^{k\ell}\left(\nabla_i L_{kj\ell}+\nabla_k L_{\ell ij}\right)a^{ij}-g^{k\ell}(\bar{R}^s_{ik\ell}a_{sj}+\bar{R}^s_{ikj}a_{\ell s})a^{ij}.\end{aligned}\tag{79}$$

Since

$$\begin{aligned}\nabla_i L_{kj\ell}+\nabla_k L_{\ell ij}=\bar{\nabla}_i\bar{R}_{kj0\ell}+\bar{\nabla}_k\bar{R}_{\ell i0j}+a_{ik}\bar{R}_{0j0\ell}+a_{ij}\bar{R}_{k00\ell}+a_{is}\bar{R}^s_{kj\ell}\\ +a_{k\ell}\bar{R}_{0i0j}+a_{ki}\bar{R}_{\ell 00j}+a_{ks}\bar{R}^s_{\ell ij}\end{aligned}$$

we have

$$\begin{aligned}g^{k\ell}(\nabla_i L_{kj\ell}+\nabla_k L_{\ell ij})a^{ij}=g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell}+\bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij}-a_i^\ell a^{ij}\bar{R}_{j00\ell}+|A|^2\overline{\mathrm{Ric}}(N,N)\\ +a_{is}a^{ij}g^{k\ell}\bar{R}^s_{kj\ell}-nHa^{ij}\bar{R}_{i00j}+a_i^\ell a^{ij}\bar{R}_{\ell 00j}+g^{k\ell}a^{ij}a_{sk}\bar{R}^s_{\ell ij}.\end{aligned}$$

Cancelling and grouping some terms one has

$$\begin{aligned}g^{k\ell}(\nabla_i L_{kj\ell}+\nabla_k L_{\ell ij})a^{ij}=g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell}+\bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij}+|A|^2\overline{\mathrm{Ric}}(N,N)\\ +a_{is}a^{ij}g^{k\ell}\bar{R}^s_{kj\ell}-nHa^{ij}\bar{R}_{i00j}+g^{k\ell}a^{ij}a_{sk}\bar{R}^s_{\ell ij}.\end{aligned}$$

Since

$$-g^{k\ell}(\bar{R}^s_{ik\ell}a_{sj}+\bar{R}^s_{ikj}a_{\ell s})a^{ij}=a^{ij}g^{k\ell}(a_{is}\bar{R}^s_{kj\ell}+a_{sk}\bar{R}^s_{\ell ij})$$

we conclude that

$$\begin{aligned}\frac{1}{2}(\partial_t-\Delta)|A|^2+|\nabla A|^2=|A|^4+|A|^2\overline{\mathrm{Ric}}(N,N)\\ +g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell}+\bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij}+2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell}+a_{sk}\bar{R}^s_{\ell ij})a^{ij}.\end{aligned}\tag{80}$$

□

Now, we consider

$$\mathcal{C}_{R,T}=\{\Psi(x,t);\ \zeta(r(\Psi(x,t)))+t\le\zeta(R),\ x\in B_R(p),\ t\in[0,T]\}$$

and we observe that $B_{R'}(o)\times[0,T_{R'}]\subset\mathcal{C}_{R,T}$ with $T_{R'}=\min\left\{\frac{1}{2}\zeta(R),T\right\}$. We want to prove the following estimate.

Proposition 3.10 *Let u be a solution of* (35) *defined in* $B_R(o)\times[0,T]$. *If there exists* $L_1>0$ *such that* $\overline{\mathrm{Ric}}\geq -L_1\overline{g}$, *then*

$$\sup_{B_{R'}(o)\times[0,T_{R'}]}|A|\leq\frac{4}{\sqrt{\delta}}\left[1+L_1+\widetilde{C}+C+\frac{E_{R'}}{\zeta^2(R)}+\frac{1}{2T}\right]^{\frac{1}{2}}\tag{81}$$

where E_R, C *and* $\widetilde{C}$ *are non-negative constants depending on* ξ,ϱ *and its derivatives. Moreover, for* $m\geq 1$

$$\sup_{B_{R'}(o)\times[0,T_{R'}]}|\nabla^m A|\leq C_m\Big(\sup_{B_R(o)\times[0,T]}W^2,\xi(R),\zeta(R),L_1,C,\widetilde{C},E_{R'}\Big).\tag{82}$$

In order to prove this proposition we will study the evolution of the function $f=\psi(W^2)|A|^2$.

Lemma 3.11 *If there exists constant* $L_1>0$ *such that* $\overline{\mathrm{Ric}}\geq -L_1\overline{g}$, *then*

$$\left(\partial_t-\Delta\right)f\leq -2\delta f^2+2(L_1+\tilde{C})f+2C\sqrt{\psi}\sqrt{f}-\frac{2\gamma}{W^3}\psi\langle\nabla W,\nabla f\rangle-2\delta\psi'|\nabla W|^2 f\tag{83}$$

where C *and* $\widetilde{C}$ *are non-negative constants depending on* ϱ *and its derivatives.*

Proof. We have

$$\begin{aligned}
(\partial_t-\Delta)f &= 2|A|^2\psi' W(\partial_t-\Delta)W+\psi(\partial_t-\Delta)|A|^2-2|A|^2(2\psi''W^2+\psi')|\nabla W|^2\\
&\quad -2\langle\nabla\psi,\nabla|A|^2\rangle\\
&= -2|A|^2\psi' W\left(W(|A|^2+\overline{\mathrm{Ric}}(N,N))+2W^{-1}|\nabla W|^2\right)\\
&\quad +2\psi\left(|A|^2(|A|^2+\overline{\mathrm{Ric}}(N,N))-|\nabla A|^2+\mathcal{R}\right)-2|A|^2(2\psi''W^2+\psi')|\nabla W|^2\\
&\quad -2\langle\nabla\psi,\nabla|A|^2\rangle\\
&= 2\left(\psi|A|^2-|A|^2\psi' W^2\right)\left(|A|^2+\overline{\mathrm{Ric}}(N,N)\right)-2\psi|\nabla A|^2+2\psi\mathcal{R}\\
&\quad -2|A|^2\left(3\psi'+2\psi''W^2\right)|\nabla W|^2-2\langle\nabla\psi,\nabla|A|^2\rangle
\end{aligned}$$

where in the second equality we used (32) and (75) and we denote

$$\mathcal{R}:=g^{k\ell}(\bar{\nabla}_i\bar{R}_{kj0\ell}+\bar{\nabla}_k\bar{R}_{\ell i0j})a^{ij}+2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell}+a_{sk}\bar{R}^s_{\ell ij})a^{ij}.$$

In an appendix of [26] was showed that there exists non-negative constants C and $\widetilde{C}$ depending on ϱ and its derivatives such that $\mathcal{R}\leq C|A|+\widetilde{C}|A|^2$. Then,

$$2\psi\mathcal{R}\leq 2C\sqrt{\psi}\sqrt{f}+2\widetilde{C}f.$$

The Kato's inequality implies to

$$-2\psi|\nabla A|^2 \le -2\psi|\nabla|A||^2.$$

Since

$$\psi|A|^2 - |A|^2\psi' W^2 = \big(1 - \frac{\gamma}{\gamma - \delta W^2}\big) f = -\delta\psi f,$$

we have

$$2\left(\psi|A|^2 - |A|^2\psi' W^2\right)\left(|A|^2 + \overline{\mathrm{Ric}}(N,N)\right) = -2\delta f^2 - 2\delta\psi f \overline{\mathrm{Ric}}(N,N) \le -2\delta f^2 + 2\delta\psi L_1 f.$$

Therefore

$$\begin{aligned}(\partial_t - \Delta) f \quad \le \quad & -2\delta f^2 + 2\delta\psi L_1 f - 2\psi|\nabla|A||^2 + 2C\sqrt{\psi}\sqrt{f} + 2\widetilde{C} f \\ & -\big(6\psi' + 4\psi'' W^2\big)|A|^2|\nabla W|^2 - 2\langle\nabla|A|^2, \nabla\psi\rangle.\end{aligned}$$

We note that

$$\begin{aligned}-2\langle\nabla|A|^2, \nabla\psi\rangle \quad &= \quad -\langle\nabla|A|^2, \nabla\psi\rangle - \psi^{-1}\langle\nabla\psi, \psi\nabla|A|^2\rangle \\ &= \quad -4\psi' W|A|\langle\nabla W, \nabla|A|\rangle - \psi^{-1}\langle\nabla\psi, \nabla f\rangle + \psi^{-1}|A|^2|\nabla\psi|^2 \\ &= \quad -\psi^{-1}\langle\nabla\psi, \nabla f\rangle + 4\psi^{-1}|A|^2\psi'^2 W^2|\nabla W|^2 - 4\psi' W|A|\langle\nabla W, \nabla|A|\rangle.\end{aligned}$$

Using Young's inequality, we obtain

$$4\psi' W|A|\langle\nabla W, \nabla|A|\rangle + 2\psi|\nabla|A||^2 + 2\psi^{-1}\psi'^2 W^2|A|^2|\nabla W|^2 \ge 0.$$

Therefore

$$-2\langle\nabla|A|^2, \nabla\psi\rangle \le -\psi^{-1}\langle\nabla\psi, \nabla f\rangle + 6\psi^{-1}|A|^2\psi'^2 W^2|\nabla W|^2 + 2\psi|\nabla|A||^2$$

hence

$$\begin{aligned}(\partial_t - \Delta) f \quad \le \quad & -2\delta f^2 + 2(\delta\psi L_1 + \widetilde{C}) f + 2C\sqrt{\psi}\sqrt{f} - \psi^{-1}\langle\nabla\psi, \nabla f\rangle \\ & -\left(6\psi'\left(1 - \frac{\psi'}{\psi}W^2\right) + 4\psi'' W^2\right)|A|^2|\nabla W|^2.\end{aligned} \tag{84}$$

Since

$$\psi - \psi' W^2 = \frac{W^2}{\gamma - \delta W^2}\left(1 - \frac{\gamma}{\gamma - \delta W^2}\right) = -\frac{\delta W^2}{\gamma - \delta W^2}\psi = -\delta\psi^2,$$

we have

$$-\left(6\psi'\left(1 - \frac{\psi'}{\psi}W^2\right) + 4\psi'' W^2\right) = -\left(6\frac{\psi'}{\psi}(-\delta\psi^2) + \frac{8\gamma\delta}{(\gamma - \delta W^2)^3}W^2\right) = -2\delta\psi'\psi.$$

Hence

$$-\left(6\psi'\left(1-\frac{\psi'}{\psi}W^2\right)+4\psi''W^2\right)|A|^2|\nabla W|^2=-2\delta\psi' f|\nabla W|^2. \tag{85}$$

Moreover

$$\psi^{-1}\nabla\psi=2\psi'\psi^{-1}W\nabla W=2\frac{\gamma}{(\gamma-\delta W^2)^2}\frac{\gamma-\delta W^2}{W^2}W\nabla W=\frac{2\gamma}{W^3}\psi\nabla W \tag{86}$$

and

$$0\leq\delta\psi=\frac{\delta W^2}{\gamma-\delta W^2}\leq\frac{\frac{\gamma}{2}}{\gamma-\frac{\gamma}{2}}=1. \tag{87}$$

Therefore, using (85), (86) and (87) we can rewrite (84) as

$$(\partial_t-\Delta)f\leq-2\delta f^2+2(L_1+\widetilde{C})f+2C\sqrt{\psi}\sqrt{f}-2\delta\psi'|\nabla W|^2f-\frac{2\gamma}{W^3}\psi\langle\nabla W,\nabla f\rangle.$$

□

Now, we can prove the Proposition 3.10.

Proof. Let η be a smooth function defined in $\mathcal{C}_{R,T}$ by

$$\eta(\Psi(x,t))=\big(\zeta(R)-\zeta(r(\Psi(x,t)))-t\big)^2.$$

We have by Proposition 2.1 and (13) that

$$\begin{aligned}(\partial_t-\Delta)\eta &= -2\sqrt{\eta}\big(\partial_t-\Delta\big)\zeta-2\sqrt{\eta}-2|\nabla\zeta|^2\\ &\leq 2\sqrt{\eta}\left(n\xi'(r)+\varrho^2|\nabla s|^2\xi(r)\left(\langle\overline{\nabla}\log\varrho,\nabla r\rangle-\frac{\xi'(r)}{\xi(r)}\right)\right)-2\sqrt{\eta}-2|\nabla\zeta|^2\\ &\leq 2n\xi'(r)\sqrt{\eta}.\end{aligned}$$

Then

$$\begin{aligned}(\partial_t-\Delta)(\eta f) &= \eta(\partial_t-\Delta)f+f(\partial_t-\Delta)\eta-2\langle\nabla f,\nabla\eta\rangle\\ &\leq -2\delta\eta f^2+2(L_1+\widetilde{C})\eta f+2C\sqrt{\psi}\sqrt{f}\eta-\frac{2\gamma}{W^3}\psi\eta\langle\nabla W,\nabla f\rangle\\ &\quad -2\delta\psi'|\nabla W|^2\eta f+2n\xi'(r)\sqrt{\eta}f-2\langle\nabla\eta,\nabla f\rangle.\end{aligned}$$

We observe that

$$\begin{aligned}-2\langle\nabla\eta,\nabla f\rangle&=-2\eta^{-1}\langle\nabla\eta,\eta\nabla f\rangle=-2\eta^{-1}\langle\nabla\eta,\nabla(\eta f)\rangle+2\eta^{-1}|\nabla\eta|^2f\\ &=-2\eta^{-1}\langle\nabla\eta,\nabla(\eta f)\rangle+8\xi^2(r)f\end{aligned}$$

and

$$\begin{aligned}-2\gamma W^{-3}\psi\eta\langle\nabla W,\nabla f\rangle &= -2\gamma W^{-3}\psi\langle\nabla W,\eta\nabla f\rangle\\ &= -2\gamma W^{-3}\psi\langle\nabla W,\nabla(\eta f)\rangle+2\gamma W^{-3}\psi f\langle\nabla W,\nabla\eta\rangle\\ &\le -2\gamma W^{-3}\psi\langle\nabla W,\nabla(\eta f)\rangle+2\delta\psi'\eta f|\nabla W|^2+2\frac{\gamma}{\delta W^2}\xi^2(r)f,\end{aligned}$$

in which we used Young's inequality. Therefore

$$\begin{aligned}(\partial_t-\Delta)(\eta f) &\le -2\delta\eta f^2+2(L_1+\widetilde{C})\eta f+2C\sqrt{\psi}\eta\sqrt{f}-2\gamma W^{-3}\psi\langle\nabla W,\nabla(\eta f)\rangle\\ &\quad +2\delta\psi'\eta f|\nabla W|^2+\frac{2\gamma}{\delta W^2}\xi^2(r)f-2\delta\psi'|\nabla W|^2\eta f\\ &\quad 2n\xi'(r)\sqrt{\eta}f-2\eta^{-1}\langle\nabla\eta,\nabla(\eta f)\rangle+8\xi^2(r)f\\ &= -2\delta\eta f^2+2(L_1+\widetilde{C})\eta f+2C\sqrt{\psi}\eta\sqrt{f}-2\Big\langle\frac{\gamma\psi}{W^3}\nabla W+\frac{\nabla\eta}{\eta},\nabla(\eta f)\Big\rangle\\ &\quad +2\Big(\frac{\gamma}{\delta W^2}\xi^2(r)+n\xi'(r)\sqrt{\eta}+4\xi^2(r)\Big)f.\end{aligned}$$

It follows from $\frac{\gamma}{W^2}\le 1$ and $\sqrt{\eta}\le\zeta(R)$ in $\mathcal{C}_{R,T}$ that

$$\Big(\frac{\gamma}{\delta W^2}+4\Big)\xi^2(r)+n\sqrt{\eta}\xi'(r)\le\Big(\frac{1}{\delta}+4\Big)\sup_{B_{R'}(o)\times[0,T_{R'}]}\xi^2(r)+n\zeta(R)\sup_{B_{R'}(o)\times[0,T_{R'}]}|\xi'(r)|:=E_{R'}$$

in $B_{R'}(o)\times[0,T_{R'}]$. Thus

$$(\partial_t-\Delta)(\eta f)\le -2\delta\eta f^2+2(L_1+\widetilde{C})\eta f+\frac{2C}{\sqrt{\delta}}\eta\sqrt{f}+2E_R f-2\Big\langle\frac{\gamma\psi}{W^3}\nabla W+\frac{\nabla\eta}{\eta},\nabla(\eta f)\Big\rangle.$$

So

$$\begin{aligned}(\partial_t-\Delta)(\eta ft) &= t(\partial_t-\Delta)(\eta f)+\eta f\\ &\le -2\delta\eta f^2t+2(L_1+\widetilde{C})\eta ft+\frac{2C}{\sqrt{\delta}}\eta\sqrt{f}t+2E_{R'}ft\\ &\quad -2\Big\langle\frac{\gamma\psi}{W^3}\nabla W+\frac{\nabla\eta}{\eta},\nabla(\eta ft)\Big\rangle+\eta f.\end{aligned}$$

Let (x_0,t_0) be the point where the function ηft attains a maximum value M_R in $B_{R'}(o)\times[0,T_{R'}]$. We can suppose that $t_0\neq 0$ and we note that

$$2\delta\eta f^2t_0\le 2(L_1+\widetilde{C})\eta ft_0+\frac{2C}{\sqrt{\delta}}\eta\sqrt{f}t_0+2E_{R'}ft_0+\eta f.$$

Thus multiplying by $\eta t_0/2\delta$ and grouping the terms we have

$$M_R^2 \leq \frac{L_1+\widetilde{C}}{\delta}\eta t_0 M_R + \frac{C}{\delta^{\frac{3}{2}}}\eta^{\frac{3}{2}} t_0^{\frac{3}{2}}\sqrt{M_R} + \frac{E_R}{\delta} t_0 M_R + \frac{\eta}{2\delta} M_R$$
$$\leq \left[\left(\frac{L_1+\widetilde{C}}{\delta}\zeta^2(R) + \frac{E_R}{\delta}\right)T + \frac{\zeta^2(R)}{2\delta}\right] M_R + \frac{C}{\delta^{\frac{3}{2}}}\zeta^3(R) T^{\frac{3}{2}}\sqrt{M_R}$$

where in the last inequality we used that $t_0 \leq T$ and $\eta \leq \zeta^2(R)$ in $\mathcal{C}_{R,T}$.
Therefore

$$\left(\frac{\sqrt{M_R}}{\zeta(R)}\right)^3 - \frac{1}{\delta}\left[\left(L_1 + \widetilde{C} + \frac{E_R}{\zeta^2(R)}\right)T + \frac{1}{2}\right]\frac{\sqrt{M_R}}{\zeta(R)} - \frac{C}{\delta^{\frac{3}{2}}}T^{\frac{3}{2}} \leq 0$$

or yet

$$\left[\left(\frac{\sqrt{M_R}}{\zeta(R)}\right)^2 - \frac{1}{\delta}\left[\left(L_1 + \widetilde{C} + \frac{E_R}{\zeta^2(R)}\right)T + \frac{1}{2}\right]\right]\frac{\sqrt{M_R}}{\zeta(R)} - \frac{C}{\delta^{\frac{3}{2}}}T_R^{\frac{3}{2}} \leq 0$$

In this case, either

$$\left(\frac{\sqrt{M_R}}{\zeta(R)}\right)^2 - \frac{1}{\delta}\left[\left(L_1 + \widetilde{C} + \frac{E_R}{\zeta^2(R)}\right)T + \frac{1}{2}\right] \leq \frac{CT}{\delta}$$

which leads us to $\frac{\sqrt{M_R}}{\zeta(R)} \leq \frac{1}{\sqrt{\delta}}\left[\left(L_1 + \widetilde{C} + C + \frac{E_R}{\zeta^2(R)}\right)T + \frac{1}{2}\right]^{\frac{1}{2}}$,
or

$$\frac{CT}{\delta}\frac{\sqrt{M_R}}{\zeta(R)} \leq \frac{C}{\delta^{\frac{3}{2}}}T^{\frac{3}{2}}.$$

Thus

$$\frac{\sqrt{M_R}}{\zeta(R)} \leq \max\left\{\frac{1}{\sqrt{\delta}}\left[\left(L_1 + \widetilde{C} + C + \frac{E_R}{\zeta(R)^2}\right)T + \frac{1}{2}\right]^{\frac{1}{2}}, \frac{T^{\frac{1}{2}}}{\sqrt{\delta}}\right\} := C_R.$$

Hence,

$$\frac{\sqrt{\eta f t}}{\zeta(R)}(x,t) \leq \frac{\sqrt{\eta f t}}{\zeta(R)}(x_0,t_0) \leq C_R \qquad \forall \quad (x,t) \in B_{R'}(o) \times [0, T_{R'}].$$

That is

$$\left(1 - \frac{\zeta(r)+t}{\zeta(R)}\right)\sqrt{\psi}|A|t^{\frac{1}{2}} \leq C_R \quad \text{in} \quad B_{R'}(o) \times [0, T_{R'}].$$

Since $\psi = \frac{W^2}{\gamma - \delta W^2} \geq \frac{\varrho^{-2}}{\gamma - \delta\varrho^{-2}} \geq \frac{1}{1-\delta} \geq 1$ and $\zeta(r) + t < \frac{3}{4}\zeta(R)$ in $B_{R'}(o) \times [0, T_{R'}]$ we have

$$\sqrt{T_{R'}}|A|(x,t) \leq |A|\sqrt{t} \leq 4C_R \quad \text{in} \quad B_{R'}(o) \times [0, T'_R],$$

hence

$$\sup_{B_{R'}(o)\times[0,T_{R'}]}|A| \leq \frac{4}{\sqrt{T_{R'}}}C_R.$$

Therefore

$$\begin{aligned}\sup_{B_{R'}(o)\times[0,T_{R'}]}|A| &\leq \frac{4}{\sqrt{\delta}}\max\left\{\left(L+\widetilde{C}+C+\frac{E_R}{\zeta(R)^2}+\frac{1}{2T_{R'}}\right)^{\frac{1}{2}},1\right\}\\ &\leq \frac{4}{\sqrt{\delta}}\left(1+L+\widetilde{C}+C+\frac{E_R}{\zeta(R)^2}+\frac{1}{2T_{R'}}\right)^{\frac{1}{2}}.\end{aligned}$$

For the estimate in (82), we proceed inductively as Ecker-Huisken in [14] and Borisenko-Miquel in [5]. We suppose that for each $k = 0, 1, \ldots, \ell-1$ there exists a constant C_k such that

$$|\nabla^k A| \leq C_k$$

where C_k depends on the bounds of $|\nabla^m A|$, on the tensors $\bar{\nabla}^m \bar{R}$ for $0 \leq m \leq k-1$ and on the geometric data in $B_R(o)\times[0,T]$.
As in [14] and [5] we will use variants of the Simons' inequality for higher order covariant derivatives of A which have the form

$$\frac{1}{2}(\partial_t-\Delta)|\nabla^\ell A|^2 + |\nabla^{\ell+1}A|^2 \leq D_\ell(|\nabla^\ell A|^2+1) \tag{88}$$

where the constant D_ℓ depends on the bounds of $|\nabla^k A|$ and on the tensors $\bar{\nabla}^k\bar{R}$ for $0 \leq k \leq \ell-1$ in $B_R(o)\times[0,T]$. We consider the function

$$h = |\nabla^\ell A|^2 + \beta|\nabla^{\ell-1}A|^2$$

where β is a positive constant to be chosen later. Setting $\beta \geq 2D_\ell$ one has

$$\begin{aligned}\frac{1}{2}\partial_t h &\leq \frac{1}{2}\Delta|\nabla^\ell A|^2 - |\nabla^{\ell+1}A|^2 + D_\ell(|\nabla^\ell A|^2+1)\\ &\quad+\frac{1}{2}\beta\Delta|\nabla^{\ell-1}A|^2 - \beta|\nabla^\ell A|^2 + \beta D_{\ell-1}(|\nabla^{\ell-1}A|^2+1)\\ &\leq \frac{1}{2}\Delta h + (D_\ell-\beta)|\nabla^\ell A|^2 + \beta D_{\ell-1}|\nabla^{\ell-1}A|^2 + D_\ell + \beta D_{\ell-1}\\ &\leq \frac{1}{2}\Delta h - \frac{\beta}{2}|\nabla^\ell A|^2 + \beta D_{\ell-1}|\nabla^{\ell-1}A|^2 + D_\ell + \beta D_{\ell-1}\\ &\leq \frac{1}{2}\Delta h - \frac{\beta}{2}h + \frac{\beta^2}{2}|\nabla^{\ell-1}A|^2 + \beta D_{\ell-1}|\nabla^{\ell-1}A|^2 + D_\ell + \beta D_{\ell-1}.\end{aligned}$$

Choosing $\beta \geq 2D_{\ell-1}$ we obtain

$$(\partial_t-\Delta)h \leq -\beta h + \beta^2\widetilde{C}_\ell + \widetilde{D}_\ell, \tag{89}$$

where $\widetilde{C}_\ell = 2|\nabla^{\ell-1}A|^2$ and $\widetilde{D}_\ell = 2D_\ell + 2\beta D_{\ell-1}$. Again, we consider η defined in $\mathcal{C}_{R,T}$ as $\eta(\psi(x,t)) = (\zeta(R) - \zeta(r(\psi(x,t))) - t)^2$. Then, we have

$$\begin{aligned}(\partial_t - \Delta)\,\eta &= -2\sqrt{\eta}\,(\partial_t - \Delta)\,\zeta - 2|\nabla\zeta|^2 \\ &\le 2\sup_{B_R(o)\times[0,T]} n\xi'(r)\sqrt{\eta} - 2|\nabla\zeta|^2 := 2C_R\sqrt{\eta} - 2|\nabla\zeta|^2\end{aligned}$$

and

$$\begin{aligned}&(\partial_t - \Delta)\,(\eta h) = h\,(\partial_t - \Delta)\,\eta + \eta\,(\partial_t - \Delta)\,h - 2\langle\nabla\eta, \nabla h\rangle \\ &\le 2C_R\sqrt{\eta}h - 2h|\nabla\zeta|^2 + (-\beta h + \beta^2\widetilde{C}_\ell + \widetilde{D}_\ell)\eta - 2\left\langle \eta^{-1}\nabla\eta, \nabla(\eta h) - h\nabla\eta\right\rangle.\end{aligned}$$

Therefore

$$(\partial_t - \Delta)\,(\eta h) + 2\left\langle \eta^{-1}\nabla\eta, \nabla(\eta h)\right\rangle \le (2C_R\sqrt{\eta} - 2|\nabla\zeta|^2 + 2\eta^{-1}|\nabla\eta|^2)h + (-\beta h + \beta^2\widetilde{C}_\ell + \widetilde{D}_\ell)\eta.$$

It follows from $-2|\nabla\zeta|^2 + 2\eta^{-1}|\nabla\eta|^2 = 6|\nabla\zeta|^2$ that

$$(\partial_t - \Delta)\,(\eta h) + 2\left\langle \eta^{-1}\nabla\eta, \nabla(\eta h)\right\rangle \le (2C_R\zeta(R) + 6|\nabla\zeta|^2 - \beta\eta)h + (\beta^2\widetilde{C}_\ell + \widetilde{D}_\ell)\eta.$$

We have at a maximum point of ηh that

$$(\beta\eta - 2C_R\zeta(R) - 6\xi^2(r))h \le (\beta^2\widetilde{C}_\ell + \widetilde{D}_\ell)\zeta^2(R)$$

Since that $\eta \ge \zeta^2(R)/16$ in $B_{R'}(p)\times[0,T_R]$ we have

$$\left(\frac{\beta}{16} - \frac{2C_R}{\zeta(R)} - 6\frac{\xi^2(R)}{\zeta^2(R)}\right)h \le \beta^2\widetilde{C}_\ell + \widetilde{D}_\ell.$$

Choosing

$$\beta \ge \max\left\{2D_\ell, 2D_{\ell-1}, \frac{32}{\zeta^2(R)}\left(C_R\zeta(R) + 6\xi^2(R)\right)\right\}$$

we get

$$h \le \frac{1}{6}\frac{\zeta^2(R)}{\xi^2(R)}\big(\beta^2\widetilde{C}_\ell + \widetilde{D}_\ell\big).$$

Thus

$$|\nabla^\ell A|^2 \le \frac{1}{6}\frac{\zeta^2(R)}{\xi^2(R)}\big(2\beta^2|\nabla^{\ell-1}A|^2 + \widetilde{D}_\ell\big) - \beta|\nabla^{\ell-1}A|^2. \tag{90}$$

A suitable choice of a large enough β yields the desired estimate in (82). □

Corollary 3.12 *Let $L_1 > 0$ be a constant such that $\overline{\mathrm{Ric}} \ge -L_1\bar{g}$. If $0 < R' < R_1 < R_2$ are such that $\zeta(r) < \frac{1}{4}\zeta(R_1) < \frac{1}{4}\zeta(R_2)$ for all $r \le R'$ and u be a solution of* (35) *defined* *in* $B_{R_2}(o) \times [0,T]$ *for* $T > 0$, *then*

$$\sup_{B_{R'}(o)\times[0,T_{R'}]} |A| \leq \frac{4}{\sqrt{\delta}} \left[1 + L_1 + \widetilde{C} + C + \frac{E_{R_1}}{\zeta^2(R_1)} + \frac{1}{2T} \right]^{\frac{1}{2}} \tag{91}$$

where E_{R_1}, C *and* $\widetilde{C}$ *are non-negative constants depending on* ξ, ϱ *and its derivatives. Moreover, for* $m \geq 1$

$$\sup_{B_{R'}(o)\times[0,T_{R'}]} |\nabla^m A| \leq C_m \left(\sup_{B_{R_1}(o)\times[0,T]} W^2, \xi(R_1), \zeta(R_1), L_1, C, \widetilde{C}, E_{R_1} \right). \tag{92}$$

Proof. In fact, if we define

$$h = |\nabla^\ell A|^2 + \beta |\nabla^{\ell-1} A|^2 \qquad \text{in} \qquad B_{R_1}(o) \times [0,T]$$

and

$$\eta = (\zeta(R_1) - \zeta(r) - t)^2 \qquad \text{in} \qquad \mathcal{C}_{R_1,T} = \left\{ \Psi(x,t);\ \zeta(r(\Psi(x,t))) + t < \zeta(R_1),\ x \in B_{R_1}(o),\ t \in [0,T] \right\}$$

we have the estimates announced.

□

4 EXISTENCE OF THE FLOW IN COMPACT CASE

Now, we are ready to solve the problem (35) which we call *R-approximate problem.*

Theorem 4.1 *For $R>0$, let $B_R = B(o,R) \subset P$ be a geodesic ball and $\Psi_0 : B_R \to M$ a smooth immersion. Suppose that $\Psi_0(B_R) = \Sigma_0$ is the graph of $\varphi \in C^\infty(\overline{B_R})$. Then the initial value problem*

$$\begin{cases} \frac{\partial \Psi}{\partial t}(x,t) = H(\Psi(x,t)), \quad \textit{in} \quad B_R \times (0,T_R) \\ \Psi(x,0) = \Psi_0(x) = \Phi(x,\varphi(x)), \quad \textit{in} \quad B_R \times \{0\} \\ \Psi(x,t) = \Phi(x,\varphi(x)), \quad \textit{on} \quad \partial B_R \times [0,T_R] \end{cases} \tag{93}$$

has a unique smooth graph solution in $B_R \times [0,T_R]$ with $T_R = \frac{1}{2}\zeta(R)$.

It's enough to prove that there exists $u \in C^\infty(B_R \times (0,T_R)) \cap C(\overline{B_R} \times [0,T_R])$ such that u solves the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i, \quad \text{in} \quad B_R \times (0,T_R) \\ u(x,0) = \varphi(x), \quad \text{in} \quad B_R \times \{0\} \\ u(x,t) = \varphi(x) \quad \text{if} \quad \text{on} \quad \partial B_R \times [0,T_R]. \end{cases} \tag{94}$$

Then we have that $\Psi(x,t) = \Phi(x,u(x,t))$ solves (93). The uniqueness of Ψ follows from the uniqueness of u.

Proof. We have that the problem (94) with $T_R = \frac{1}{2}\zeta(R)$ is uniformly parabolic, by our a priori gradient estimates. Then there exists $\epsilon > 0$ such that the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i, \quad \text{in} \quad \Omega_\epsilon := B_R \times (0,\epsilon) \\ u(x,0) = u_0(x), \quad \text{in} \quad B_R \times \{0\} \\ u(x,t) = \varphi(x) \quad \text{on} \quad \partial B_R \times [0,\epsilon] \end{cases} \tag{95}$$

has a solution u^ϵ (see Theorem 8.2 in [22]). Moreover $u^\epsilon \in C^\infty(\Omega_\epsilon) \cap C(\overline{\Omega_\epsilon})$ (by Theorem 8.2, Theorem 5.14 in [22] and linear theory). We note that for $\epsilon > 0$ such that the problem 95 has a solution u^ϵ, our a priori gradient estimate gives us a Holder estimate (by [22], Theorem 12.10) for u^ϵ which is independent of ϵ, by Corollary 3.5. Thus, there exists a solution u for the problem (94) (see Theorem 8.3 in [22]). Moreover, this solution u is unique by the parabolic comparison principle and $u \in C^\infty(B_R \times (0,T_R)) \cap C(\overline{B_R} \times [0,T_R])$ by Schauder estimates.

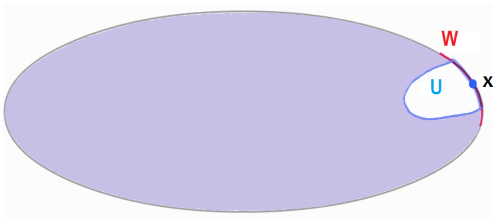
□

5 BARRIER AT INFINITY

In order to study the behavior of the solutions of (4) at infinity we use a notion of regularity at infinity with respect to operator $\partial_t - Q$ for the set $P \times [0,\infty)$. Proceeding as Ripoll-Telichevesky in [28], we prove that when P satisfies the strict convexity condition, $P \times [0,\infty)$ is regular at infinity with respect to $\partial_t - Q$.

We say that P satisfies the strict convexity condition (SC condition for short) if for any $x \in \partial_\infty P$ and a relatively open subset $W \subset \partial_\infty P$ containing x, there exists a C^2 open subset $U \subset \bar{P}$ such that $x \in \operatorname{int}(\partial_\infty U) \subset W$ and $P \setminus U$ is convex.

Figura 3: SC condition



Source: elaborated by author.

We recall that a function $\eta \in C^0(P \times [0,\infty))$ is said a supersolution of $\partial_t - \mathcal{Q}$ if given any bounded domain $U \subset P \times [0,\infty)$ and $u \in C^0(\bar{U})$ such that $(\partial_t - \mathcal{Q})(u) = 0$ in U with $u|_{\partial U} \leq \eta|_{\partial U}$ we have $u|_U \leq \eta|_U$. If $U \subset M$ is an open set, $v \in C^2(U)$ and $(\partial_t - \mathcal{Q})(v) \geq 0$ then v is a supersolution of $\partial_t - Q$. Then we define a *upper barrier* as follows

Definition 5.1 *Given $(x_0,t_0) \in \partial_\infty P \times [0,\infty)$, a constant $C > 0$ and open subsets $U \subset P$, $I \subset [0,\infty)$ such that $x_0 \in \partial_\infty U$ and $t_0 \in I$, a function $\eta \in C^0(P \times [0,\infty))$ is an upper barrier for $\partial_t - \mathcal{Q}$ relative to (x_0,t_0) and $U \times I$ with height C if*

i. η is a supersolution for $\partial_t - \mathcal{Q}$;

ii. $\eta \geq 0$ and $\eta(x,t) \to 0$ as $(x,t) \to (x_0,t_0)$;

iii. $\eta|_{P\times[0,\infty)\setminus U \times I} \geq C$.

In a similar way, we define subsolutions and lower barries.

We define the regularity at infinity of the operator $P \times [0,\infty)$ as following.

Definition 5.2 *We say that $P \times [0,\infty)$ is regular at infinity with respect to $\partial_t - \mathcal{Q}$ if given a point $(x_0,t_0) \in \partial_\infty P \times [0,\infty)$, a constant $C > 0$ and open subsets $W \subset \partial_\infty P$ $I \subset [0,\infty)$ with $x_0 \in W$, $t_0 \in I$ there exist open subsets $U \subset P$, $J \subset I$ such that $x_0 \in \operatorname{int}(\partial_\infty U) \subset W$, $t_0 \subset J$ and there exist upper and lower barriers $\bar{\eta}, \eta$ for $\partial_t - \mathcal{Q}$ relatives to (x_0,t_0) and $U \times J$ with height C. Here $\operatorname{int}(\partial_\infty U)$ denotes the interior of $\partial_\infty U$ in $\partial_\infty P$.*

In this context, we have

Proposition 5.1 *Let P be a Hadamard manifold with sectional curvature $K_P \leq -\kappa^2 < 0$ satisfying the SC condition and suppose that ϱ satisfies* (7). *Then $P \times [0,\infty)$ is regular at infinity with respect to $\partial_t - Q$.*

Proof. Given $(x_0, t_0) \in \partial_\infty P \times [0,\infty)$, a constant $C > 0$ and open subsets $W \subset \partial_\infty P$ and $I \subset [0,\infty)$ with $x_0 \in W$, $t_0 \in I$, we consider a C^2 open subset $U \subset P$ such that $x_0 \in \operatorname{int}(\partial_\infty U) \subset W$ and $P \setminus U$ is convex. Then we define $d : U \to \mathbb{R}$ as the distance function to ∂U in U which is a C^2 function by regularity of U. Note that for $y = (x,s) \in K_U = \{(x,s) : x \in U, s \in \mathbb{R}\}$ one has

$$\operatorname{dist}((x,s), K_{\partial U}) = \operatorname{dist}(x, \partial U) \doteq d(x).$$

Hence we also denote by d the distance function to the Killing cylinder over ∂U defined in K_U. Recall that $r(x)$ denotes the radial distance $\operatorname{dist}(x,o)$ in P and in the same way, we extend the function r to M as $r(x,s) = r(x)$.

In order to construct a upper barrier for $\partial_t - \mathcal{Q}$ relative to (x_0,t_0) and $U \times I$ with height C, we consider a function

$$w(x,t) = f(d(\Psi(x,t))) = f(d(x))$$

in $U \times I$ where $f(d) = C_1 \exp(-\alpha d)$ where C_1 and α are positive constants to be fixed later. Indicating derivatives with respect to d by $\cdot$ one has $\partial_t w = \dot{f}(d)\partial_t d = 0$. Hence we have to choose constants C_1 and α such that

$$\mathcal{Q}[w] = \Delta w - \frac{1}{W^2}\big\langle \nabla_{\nabla w}\nabla w, \nabla w\big\rangle + \left(1 + \frac{1}{\varrho^2 W^2}\right)\big\langle \nabla \log \varrho, \nabla w\big\rangle \leq 0.$$

It follows from $\nabla w = \dot{f}(d)\nabla d$, $W^2 = \varrho^{-2} + \dot{f}^2(d)$ and $\Delta w = \dot{f}(d)\Delta d + \ddot{f}(d)$ that

$$\langle \nabla \log \varrho, \nabla w\rangle = \dot{f}(d)\frac{\varrho'(r)}{\varrho(r)}\langle \nabla r, \nabla d\rangle$$

We also have

$$\big\langle \nabla_{\nabla w}\nabla w, \nabla w\big\rangle = \dot{f}^2\langle \nabla_{\nabla d}\dot{f}\nabla d, \nabla d\rangle = \dot{f}^2\ddot{f}$$

where we used that $\nabla_{\nabla d}\nabla d = 0$. Hence

$$\begin{aligned}\mathcal{Q}[w] &= \dot{f}\Delta d + \ddot{f} - \frac{\dot{f}^2}{W^2}\ddot{f} + \left(1 + \frac{1}{\varrho^2 W^2}\right)\frac{\varrho'}{\varrho}\langle \nabla r, \nabla d\rangle \dot{f}\\ &= \dot{f}\Delta d + \frac{1}{\varrho^2 W^2}\ddot{f} + \left(1 + \frac{1}{\varrho^2 W^2}\right)\frac{\varrho'}{\varrho}\langle \nabla r, \nabla d\rangle \dot{f}\end{aligned}$$

Since $P \setminus U$ is convex and $K_P \leq -\kappa^2 < 0$, we have $\langle \nabla r, \nabla d \rangle > 0$ and

$$\Delta d \geq (n-1)\kappa \tanh(\kappa d)$$

in U, where in the last inequality we used comparison theorems (see Theorems 4.2 and 4.3 of [7]). Thus,

$$\begin{aligned}\mathcal{Q}[w] &\leq \frac{1}{\varrho^2 W^2}\ddot{f} + (n-1)\kappa \tanh(\kappa d)\dot{f} + \left(1 + \frac{1}{\varrho^2 W^2}\right)\frac{\varrho'}{\varrho}\langle \nabla r, \nabla d \rangle \dot{f} \\ &\leq \ddot{f} + \left((n-1)\kappa \tanh(\kappa d) + \frac{\varrho'}{\varrho}\langle \nabla r, \nabla d \rangle\right)\dot{f} = \left((n-1)\kappa \tanh(\kappa d) + \frac{\varrho'}{\varrho}\langle \nabla r, \nabla d \rangle - \alpha\right)\dot{f}\end{aligned}$$

in $U \times I$. Using that $\liminf_{r\to\infty} \frac{\varrho'(r)}{\varrho(r)} > 0$, we take $d_0 \geq 2$ and $U_0 = \{x \in U; d(x) \geq d_0\}$ such that $\inf_{U_0} \frac{\varrho'(r)}{\varrho(r)}\langle \nabla r, \nabla d \rangle > 0$. Let also consider

$$U_1 = \{x \in U; d(x) > d_0 - 1\} \qquad \text{and} \qquad U_2 = U_1 \setminus U_0.$$

If we choose

$$0 < \alpha \leq \inf_{U_0} \frac{\varrho'(r)}{\varrho(r)}\langle \nabla r, \nabla d \rangle \quad \text{and} \quad C_1 = Ce^{\alpha d_0}$$

we have $\mathcal{Q}[w] \leq 0$ in U_0 and

$$\inf_{U_2 \times I} w(x,t) = \inf_{U_2} Ce^{\alpha d_0} e^{-\alpha d(x)} = C = f(d_0).$$

Setting

$$\widetilde{w}(x,t) = \begin{cases} w(x,t) & \text{if} \quad x \in U_0,\ t \in I \\ C, & \text{if} \quad x \in U_2,\ t \in I \end{cases}$$

one has a continuous function $\widetilde{w}$ in the open subset $U_1 \times I$ that can be extended to $P \times [0,\infty)$ as

$$\eta(x,t) = \begin{cases} \omega(x,t) & \text{if} \qquad x \in U_0, \quad t \in [0,\infty) \\ C, & \text{if} \qquad x \in P \setminus U_0 \quad t \in [0,\infty). \end{cases}$$

which is an upper barrier for $\partial_t - Q$ relative to (x_0,t_0) and $U \times I$ with height C. In a similar way, we obtain a lower barrier relative to (x_0,t_0) and $U \times I$ with height C. Hence, $P \times [0,\infty)$ is regular at infinity with respect to $\partial_t - Q$.

□

Corollary 5.2 *Let P be a Cartan-Hadamard manifold satisfying $K_P \leq -\kappa^2 < 0$ and suppose that ϱ satifies* (7)*. If P is rotationally symmetric, then $P \times [0,\infty)$ is regular at infinity with respect to $\partial_t - Q$.*

Corollary 5.3 *Suppose that P is a Cartan-Hadamard manifold satisfying*

$$\frac{e^{2\kappa r(x)}}{r(x)^{2+2\epsilon}} \leq K_P(x) \leq -\kappa^2 < 0$$

for every $x \in P$ such that $r(x) = \mathrm{d}(x, o) \geq R^*$, for R^* large enough, where $\kappa, \epsilon > 0$ are constant and suppose that ϱ satifies (7)*. Then $P \times [0, \infty)$ is regular at infinity with respect to $\partial_t - Q$.*

As we said before, in [28], the authors proved that under the conditions of the above corollaries, P satisfies SC condition. So, such results are immediate consequences of the Proposition 5.1.

6 EXISTENCE OF THE FLOW

This section is devoted to prove that the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i, & \text{in} \quad P \times [0, \infty) \\ u(x,0) = \varphi(x), \quad \text{in} \quad P \times \{0\} \\ u(x,t) = \varphi(x) \quad \text{if} \quad x \in \partial_\infty P,\ t \in [0,\infty). \end{cases} \tag{96}$$

has a solution $C^\infty(P \times [0,\infty)) \cap C^0(\bar{P} \times [0,\infty))$, when $\varphi \in C^\infty(P) \cap C(\overline{P})$ is given. Then we take $\Psi(x,t) = \Phi(x, u(x,t))$ in $P \times [0,\infty)$ and obtain a solution for (5).

From now on, if $R > 0$ we denote by u^R the solution of the R-approximate problem that is, the problem (94) in $B_R(o) \times [0, T_R)$, which existence is ensured by Theorem 4.1. We also denote

$$\Psi_t^R(x) = \Phi(x, u^R(x,t)) \quad \text{and} \qquad \Sigma_t^R = \Psi_t^R(B_R(o)).$$

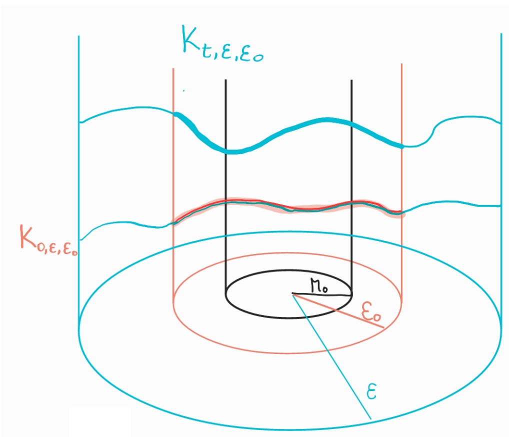
For a fixed $r_0 > 0$, we consider $\epsilon_0 > r_0$ the smallest integer belonging to the set $\{\epsilon; \quad \epsilon > r_0, \quad \zeta(r) < \frac{1}{4}\zeta(\epsilon) \quad \forall \quad r \leq r_0\}$ and we take

$$\mathcal{I}_0 = \left\{\epsilon; \quad \epsilon \geq \epsilon_0, \quad \zeta(r) < \frac{1}{4}\zeta(\epsilon) \quad \forall \quad r \leq r_0\right\}.$$

If $\epsilon \in \mathcal{I}_0$ we denote $u^{\epsilon,0} := u^\epsilon$,

$$K_{t,\epsilon,\epsilon_0} = \Sigma_t^\epsilon \cap B_{\epsilon_0}(o) \times [0,+\infty) \qquad \text{and} \qquad K_{t,\epsilon_0,\epsilon_0} = \Sigma_t^{\epsilon_0} \cap B_{\epsilon_0}(o)) \times [0,+\infty).$$

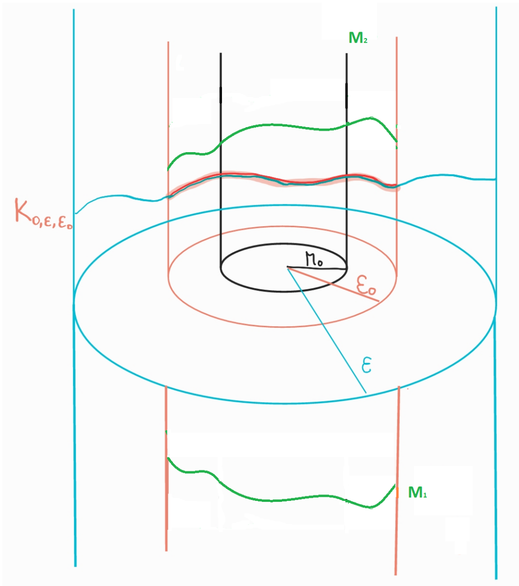
Figura 4: Restriction of the graph to the cylinder $B_{\epsilon_0}(o) \times [0,+\infty]$



Source: elaborated by author.

We note that $K_{0,\epsilon,\epsilon_0} = K_{0,\epsilon_0,\epsilon_0}$ since the initial condition for $r-$approximate problem is $\varphi_{|_{B_r(o)}}$ for all $r > 0$. Since $K_{0,\epsilon,\epsilon_0}$ is compact, there exists hypersurfaces M_1, M_2 such that M_1 is a translation of the Killing graph of the function v_{ϵ_0}, M_2 is a reflection of M_1 with respect to the leaf $P \times \{0\}$ and $K_{0,\epsilon,\epsilon_0}$ is in the strip bounded above and below by M_2 and M_1, respectively.

Figura 5: Initial data in between M_1 and M_2



Source: elaborated by author.

We take $T_0 = \frac{1}{2}\zeta(\epsilon_0)$ and

$$\ell_0 = \min\left\{\ell \in \mathbb{N};\ R_0(t) \leq \ell\epsilon_0 \quad \forall \quad t \in [0, T_0]\right\}$$

where $R_0(t) = \mu(t) + \epsilon_0$ is implicitly defined by (49). By using the comparison principle for the mean curvature flow and the Proposition 3.4 we have

$$\sup_{B_{r_0}(o)\times[0,T_0]} |u^{\epsilon,0}(x,t)| \leq \sup_{B_{\epsilon_0}(o)\times[0,T_0]} |u^{\epsilon,0}(x,t)| \leq c_0,$$

for all $\epsilon \in \mathcal{I}_0$, where the constant c_0 depends of $\epsilon_0, \ell_0, \sup_{B_{\epsilon_0}(o)} |u_0|, \varrho_{|_{B_{\epsilon_0}(o)}}, \xi_{|_{B_{\epsilon_0}(o)}}$ and $\zeta(\epsilon_0)$.

Moreover, it follows from the Corollary 3.8 and Corollary 3.12 that for all $\epsilon \in \mathcal{I}_0$ we get

$$\sup_{B_{r_0}(o)\times[0,T_0]} |\nabla u^{\epsilon,0}(x,t)| \leq c_1,$$

and for $m > 1$

$$\sup_{B_{r_0}(o)\times[0,T_0]} |\nabla^m u^{\epsilon,0}| \leq c_m$$

where c_1 is a constant which depends of c_0 and on the geometric data restrict to $B_{\epsilon_0}(o)$ and c_m is a constant which depends on $c_{m-1}, W^2_{|B_{\epsilon_0}(o)\times[0,T_0]}$ and on the geometric data restricted to $B_{\epsilon_0}(o)$. By using the Arzelà-Ascoli Theorem, we have that there exists a sequence $(\epsilon_\ell)_\ell$ in $\mathcal{I}_0$ with $\epsilon_\ell \to \infty$ as $\ell \to \infty$ and such that $u^{\epsilon_\ell,0}$ converges uniformly in C^∞ to some $v^0 \in C^\infty(B_{r_0} \times [0,T_0])$ which solves (94).

Now, let us consider a sequence $\{r_k\}_{k=0}^\infty$ such that $r_0 < r_1 < \cdots$ and $r_k \to \infty$ as $k \to \infty$. For each $k \geq 1$ we consider ϵ_k the smallest integer belonging to the set $\left\{\epsilon; \quad \epsilon > r_k, \quad \zeta(r) < \frac{1}{4}\zeta(\epsilon) \quad \forall \quad r \leq r_k\right\}$ and we take

$$\mathcal{I}_k = \left\{\epsilon;\, \epsilon \geq \epsilon_k, \quad \zeta(r) < \frac{1}{4}\zeta(\epsilon) \quad \forall \quad r \leq r_k\right\} \qquad \text{and} \qquad T_k = \frac{1}{2}\zeta(\epsilon_k).$$

Note that

$$\mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots \supset \mathcal{I}_k \supset \mathcal{I}_{k+1} \supset \ldots$$

and $T_k \to \infty$ as $k \to \infty$.

We claim that is possible to get functions $v^k \in C^\infty(B_{r_k} \times [0,T_k])$ solving (94) such that v^k is the uniform limit of some sequence $\{u^{\epsilon_j,k}\}_{j=1}^\infty$ and $v^k_{|B_{r_\ell}\times[0,T_l]} = v^\ell$ for all $0 \leq \ell \leq k$.

We will use induction for to prove this claim. For $k = 0$, we was done above. Let us suppose that we have a function $v^k \in C^\infty(B_{r_k} \times [0,T_k])$ solving (94) such that v^k is the uniform limit of some sequence $\{u^{\epsilon_\ell,k}\}_{\ell=1}^\infty$ with $\epsilon_\ell \in \mathcal{I}_k$. Our interior estimates imply that we have uniform bounds of u^ϵ and its derivatives on $B_{r_{k+1}} \times [0,T_{k+1}]$ for all $\epsilon \in \mathcal{I}_{k+1}$. Then we choose a subsequence of $u^{\epsilon_\ell,k}$ (which will also denote by $u^{\epsilon_\ell,k}$) such that $\epsilon_\ell \in \mathcal{I}_{k+1}$. By using the Arzelà-Ascoli Theorem for this subsequence we know that there exist a subsequence $\{u^{\epsilon_\ell,k+1}\}_\ell$ of $\{u^{\epsilon_\ell,k}\}_{\ell=1}^\infty$ such that $u^{\epsilon_\ell,k+1}$ converges uniformly for some $v^{k+1} \in C^\infty(B_{r_{k+1}} \times [0,T_{k+1}])$ as $\ell \to \infty$. Since $B_{r_k} \times [0,T_k] \subset B_{r_{k+1}} \times [0,T_{k+1}]$ and $\{u^{\epsilon_\ell,k+1}\}_\ell$ is a subsequence of $\{u^{\epsilon_\ell,k}\}_\ell$ we must have $v^{k+1}_{|B_{r_k}\times[0,T_k]} = v^k$.

Now, for $(x,t) \in P \times [0,\infty)$, we take $k \geq 0$ such that $(x,t) \in B_{r_k} \times [0,T_k]$ and we define $u(x,t) = v^k(x,t)$. If $(x,t) \in \partial_\infty P \times [0,\infty)$, we define $u(x,t) = \varphi(x)$. It follows from our construction that u is well-defined and $u \in C^\infty(P \times [0,\infty))$. We need to show that u is continuous in (x,t) as $x \in \partial_\infty P$.

Given $(x_0,t_0) \in \partial_\infty P \times [0,\infty)$ and $\epsilon > 0$, there exists an open subset $W \subset \partial_\infty P$ such that $x_0 \in W$ and $\varphi(y) < \varphi(x_0) + \frac{\epsilon}{2}$ for all $y \in W$. Since $P \times [0,\infty)$ is regular at infinity with respect to $\partial_t - Q$ there exists open subsets $U \subset P$ and $J \subset [0,\infty)$ such that $x_0 \in int(\partial_\infty U) \subset W$, $t_0 \in J$ and $\eta : P \times [0,\infty) \longrightarrow \mathbb{R}$ upper barrier with respect to (x_0,t_0) and $U \times J$ with height $C := 2\max_{\overline{P}} |\varphi|$.

Let $\omega(x,t) = \eta(x,t) + \varphi(x_0) + \epsilon$ and $\tilde{\omega}(x,t) = \varphi(x_0) - \eta(x,t) - \epsilon$ be functions defined in $P \times [0,\infty)$. We want to prove that

$$\tilde{\omega}(x,t) \leq u(x,t) \leq \omega(x,t), \quad \text{for} \quad (x,t) \in U \times J.$$

Then we will have

$$|u(x,t) - u(x_0,t_0)| = |u(x,t) - \varphi(x_0)| < \epsilon + \eta(x,t)$$

for all $(x,t) \in U \times J$, which implies that

$$\limsup_{(x,t)\to(x_0,t_0)} |u(x,t) - \varphi(x_0,t_0)| \leq \epsilon.$$

Therefore u is continuous in (x_0,t_0) and consequently, $u \in C^\infty(P\times[0,\infty)) \cap C^0(\bar{P}\times[0,\infty))$.

For to prove that $\tilde{\omega} \leq u \leq \omega$ in $U \times J$ we use the sequence $\{u^{\epsilon_\ell} = u^{\epsilon_\ell,0}\}_\ell$ where each u^{ϵ_ℓ} is solution of

$$\begin{cases} (\partial_t - Q)[u] = 0 \quad \text{in} \quad B_{\epsilon_\ell} \times [0, T_{\epsilon_\ell}] \\ u(x,0) = \varphi(x), \quad x \in B_{\epsilon_\ell} \\ u(x,t) = \varphi(x), \quad x \in \partial B_{\epsilon_\ell} \quad \text{and} \quad t \in [0, T_{\epsilon_\ell}]. \end{cases} \tag{97}$$

Since $\varphi(x)$ is continuous, we can choose $\ell_0 >> 1$ such that $\partial B_{\epsilon_\ell} \cap U \neq \emptyset$ and

$$|\varphi(x) - \varphi(x_0)| < \frac{\epsilon}{2} \qquad \forall \quad x \in \partial B_{\epsilon_\ell} \cap U,$$

when $\ell \geq \ell_0$. We claim that $u^{\epsilon_\ell} \leq \omega \quad \text{in} \quad B_{\epsilon_\ell} \times [0, T_{\epsilon_\ell}] \cap (U \times J)$ for $\epsilon_\ell \geq \ell_0$. In fact, it is enough to prove that $u^{\epsilon_\ell} \leq \omega$ in

$$\partial\big(B_{\epsilon_\ell} \times [0, T_{\epsilon_\ell}] \cap (U \times J)\big) = \left(\partial B_{\epsilon_\ell} \times [0, T_{\epsilon_\ell}] \cap \overline{U \times J}\right) \cup \left(\overline{B_{\epsilon_\ell}} \times [0, T_{\epsilon_\ell}] \cap \partial U \times \overline{J}\right).$$

Since η is a supersolution, we get the inequality in $B_{\epsilon_\ell} \times [0, T_{\epsilon_\ell}] \cap (U \times J)$ for $\epsilon_\ell \geq \ell_0$. For $(x,t) \in \partial B_{\epsilon_\ell} \times [0, T_{\epsilon_\ell}] \cap \overline{U \times J}$ we have

$$u^{\epsilon_\ell}(x,t) = u_0(x) < \varphi(x_0) + \frac{\epsilon}{2} \leq \ \omega(x,t)$$

due to the choise of ℓ_0. If $(x,t) \in \overline{B_{\epsilon_\ell}} \times [0, T_{\epsilon_\ell}] \cap (\partial U \times J)$ we get

$$u^{\epsilon_\ell}(x,t) \leq \max_{\partial B_{\epsilon_\ell}} \widetilde{\varphi} \leq 2 \max_{P \cup \partial_\infty P} |\widetilde{\varphi}| + \varphi(x_0) \leq \varphi(x_0) + \eta(x,t) \leq \omega(x,t).$$

Thus, we conclude that $v^k \leq \omega \quad \text{in} \quad B_{r_k} \times [0, T_{r_k}] \cap U \times J \quad \forall \quad k$ and consequently $u \leq \omega$ in $U \times J$. In a similar way, we prove that $u \geq \tilde{\omega}$ in $U \times J$ and we conclude the proof of the Theorem 1.3.

We note that the Corollary 1.4 is a consequence of the Corollary 5.2 and the Theorem 1.3. The same way, we have that the Corollary 1.5 is a consequence of the Corollary 5.3 and the Theorem 1.3.

7 CURVATURES FUNCTION FLOW

In this chapter we establish a priori interior gradient estimate for the solution of a equation associated to the problem of normal deformation of a hypersurface by a function curvature.

7.1 The flow by a curvature function

Now, our ambient manifold is $(M = P \times \mathbb{R}, \bar{g})$ with $\bar{g} = g + \mathrm{d}s^2$ and (P, g) a n-dimensional complete Riemannian manifold. As before, given Ω a bounded domain in P, the Killing graph of a function $u \in C^2(\Omega)$ is the hypersurface in M given by

$$\Sigma[u] = \{\Phi(x, u(x)); x \in \Omega\},$$

where Φ is the flow generated by Killing vector field $X := \partial_s$. Fixed a coordinate system in P the components of the induced metric in $\Sigma[u]$ and of its inverse are given by $\sigma_{ij} = g_{ij} + u_i u_j$ and $\sigma^{ij} = g^{ij} - \frac{1}{W^2} u^i u^j$, respectively. The second fundamental form of $\Sigma[u]$ has components

$$a_{ij} = \langle \bar{\nabla}_{X_j} X_i, N \rangle = \frac{u_{i;j}}{W},$$

where $W = \sqrt{1 + |\nabla^P u|^2}$ and ∇^P denotes the Riemannian connection in P. Moreover, we consider in $\Sigma[u]$ the orientation determined by the unit normal vector field $N = \frac{1}{W}\big(X - \Phi_* \nabla^P u\big)$.

Let us consider Γ an open convex cone with vertex at the origin in $\mathbb{R}^n$, containing the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n; \lambda_i > 0\}$. Suppose that the positive λ_i axes does not belong to $\partial\Gamma$ and

$$\lambda = (\lambda_i) \in \Gamma \Longrightarrow (\lambda_{\pi(i)}) \in \Gamma \quad \forall \quad \pi \in \mathcal{P}_n$$

where $\mathcal{P}_n$ is the set of all permutations of order n. Then we have

$$\Gamma \subset \{\lambda \in \mathbb{R}^n; \sum_{i=1}^{n} \lambda_i > 0\}.$$

We say that a *positive differentiable function* f defined in Γ is a *curvature function* if

$$f(\lambda_i) = f(\lambda_{\pi(i)}) \quad \forall \quad \pi \in \mathcal{P}_n.$$

A one parameter family of functions

$$u : \Omega \times [0, T) \longrightarrow \mathbb{R} \quad T > 0$$

defines a flow by curvature function f,

$$\Psi(x,t) = \Phi(x, u(x,t))$$

if and only if

$$\left(\frac{\partial \Psi}{\partial t}(x,t)\right)^{\perp} = f(k[u])N \tag{98}$$

where $k[u]$ denotes the principal curvatures of $\Sigma_t := \Sigma[u(\cdot,t)]$ calculated with respect to the orientation given by the unit vector field $N = \frac{1}{W}\big(X - \nabla^P u\big)$.
Note that

$$f(k[u])N = \left(\frac{\partial \Psi}{\partial t}(x,t)\right)^{\perp} = \left(\frac{\partial}{\partial t}\Phi(x,u(x,t))\right)^{\perp} = u_t X^{\perp}$$

implies in

$$f(k[u]) = \langle X, N\rangle u_t = \frac{1}{W}u_t.$$

Then (98) defines a flow by f if and only if, u satisfies

$$-u_t + Wf(k[u]) = 0. \tag{99}$$

Following the literature, we say that a function $u \in C^2(\Omega \times [0,T))$ is *admissible* if $k[u] \in \Gamma$ at each point of its graph.

In order to study the equation (99) some conditions must be imposed on f. We suppose that f satisfies the following conditions:

$$\begin{aligned} &f_i = \frac{\partial f}{\partial k_i} > 0 \\ &f \text{ is a concave function} \\ &f_i(k) \geq v_0 \quad \text{for} \quad k \in \Gamma \quad \text{with} \quad k_i < 0 \end{aligned} \tag{100}$$

where v_0 is a positive constant. We note that since f is concave and Γ is convex we have

$$\sum_{i=1}^{n} \frac{\partial f(\lambda)}{\partial \lambda_i}(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda), \quad \forall \quad \lambda, \mu \in \Gamma. \tag{101}$$

7.2 Auxiliary results

In this subsection we list some useful facts about the curvature function f. We can see these results in detail in [16], [8] and [9].

For each t, let $\mathcal{S}_t$ be the space of all symmetric covariant tensors of rank two defined in (Σ_t, σ) and $\mathcal{S}_{\Gamma t}$ be the open subset of those symmetric tensors $a \in \mathcal{S}_t$ for which the eigenvalues with respect σ, are contained in Γ. (For simplicity, we will omit the index t.) We define $F : \mathcal{S}_\Gamma \longrightarrow \mathbb{R}$ by setting

$$F(a) = f(\lambda(a))$$

where $\lambda(a) = (\lambda_1(a), \cdots, \lambda_n(a))$ are the eigenvalues of a. The mapping F is as smooth as f (see [16]) and can be viewed as $F(a^\sharp) = F(a, \sigma)$. In terms of coordinates, we have

$$F(a_i^j) = F(a_{ij}, \sigma_{ij})$$

where $a_i^j = \sigma^{jk} a_{ki}$. We denote the first derivatives of F by

$$F^{ij} = \frac{\partial F}{\partial a_{ij}} \quad \text{and} \quad F_i^j = \frac{\partial F}{\partial a_j^i}$$

and the second derivatives of F are indicated by

$$F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}.$$

Let us to extend the cone Γ to the space of the symmetric matrices of order n which also we denote by $\mathcal{S}$. For $p \in \mathbb{R}^n$ we define

$$\Gamma(p) = \{r \in \mathcal{S} : \lambda(p, r) \in \Gamma\}$$

where $\lambda(p, r)$ denotes the eigenvalues (calculated with respect to the Euclidean inner product) of the matrix

$$A(p, r) = \frac{1}{\sqrt{1 + |p|^2}} \left(I - \frac{p \otimes p}{1 + |p|^2} \right) r. \tag{102}$$

We obtain the matrix $A(p, r)$ from the Weingarten map with (p, r) in place of $(\nabla u, \nabla^2 u)$ and δ^{ij} in place of σ^{ij}. Now, introducing the notation

$$G(p, r) = F(A(p, r)) = f(\lambda(p, r))$$

we can write (99) in the form

$$G(\nabla u, \nabla^2 u) = F[u] = f(k[u]) = \frac{1}{W}\partial_t u. \qquad (103)$$

The next lemma give us informations about the second derivatives of F. We can find its proof in [16] and [9].

Lemma 7.1 *Let $\{e_i\}$ be a local orthonormal basis of eigenvectors for $a \in \mathcal{S}_\Gamma$ with corresponding eigenvalues λ_i. Then, the matrix (F^{ij}) is also diagonal with positive eigenvalues f_i. Moreover, F is concave and its second derivatives are given by*

$$F^{ij,k\ell}\xi_{ij}\xi_{k\ell} = \sum_{k,\ell} f_{k\ell}\xi_{kk}\xi_{\ell\ell} + \sum_{k\neq\ell} \frac{f_k - f_\ell}{\lambda_k - \lambda_\ell}\xi_{k\ell}^2.$$

Note that if we denote $G^i = \frac{\partial G}{\partial p_i}$ and $G^{ij} = \frac{\partial G}{\partial r_{ij}}$, we have

$$G^{ij}(\nabla u, \nabla^2 u) = \frac{\partial G}{\partial u_{ij}} = \frac{\partial F}{\partial a_{\ell k}}\frac{\partial a_{\ell k}}{\partial u_{ij}} = \frac{1}{W}\frac{\partial F}{\partial a_{ij}} = \frac{1}{W}F^{ij}.$$

Hence, $\{G^{ij}(\nabla u, \nabla^2 u)\}$ is a positive-definite matrix.

7.3 Interior gradient estimate

In this subsection we establish a priori interior estimate to the gradient of a solution of (1).

Theorem 7.2 *Let u be a admissible function such that solves the problem*

$$\begin{cases} G(\nabla u, \nabla^2 u) = F[u] = f(k[u]) = \frac{1}{W}\partial_t u \quad in \quad \Omega \times (0,T) \\ u(x,t) = \varphi(x) \quad in \quad \partial\Omega \times [0,T] \end{cases} \tag{104}$$

with Ω a bounded domain of P, $T > 0$ and $\varphi \in C^1(P)$. If u is bounded in $\bar{\Omega} \times [0,T)$ and there exists a constant $c_1 > 0$ such that $\sup_{\partial\Omega\times[0,T]} |\nabla u(x,t)| \leq c_1$, then

$$|\nabla u(x,t)| \leq C$$

for $(x,t) \in \Omega \times (0,T)$, where C is a constant which depends of v_0, c_1 and $c_0 = \sup_{P\times[0,T)} |u|$.

Before to prove this theorem, we need an auxiliar result which gives us a useful formula involving the second and third derivatives of a solution of (99).

Lemma 7.3 *Let u be an admissible solution of (103). Then*

$$\begin{aligned} G^{ij}u_{k;ij} &= WG^{ij}a_j^\ell u_\ell u_{i;k} + WG^{\ell j}a_\ell^i u_j u_{i;k} + \frac{1}{W}G^{\ell j}a_{\ell j}u^i u_{i;k} \\ &- G^{ij}R_{i\ell kj}u^\ell + \frac{1}{W}\partial_t u_k - \frac{1}{W^3}u^\ell u_{\ell;k}\partial_t u. \end{aligned}$$

Proof. Deriving (103) in the $k-th$ direction with respect to the metric σ, we have

$$u_t\partial_k\left(\frac{1}{W}\right) + \frac{1}{W}\partial_k u_t = \frac{\partial G}{\partial u_{i;j}}u_{i;jk} + \frac{\partial G}{\partial u_i}u_{i;k} = G^{ij}u_{i;jk} + G^i u_{i;k}.$$

Since

$$\partial_k\left(\frac{1}{W}\right) = -\frac{\partial_k(W)}{W^2} = -\frac{1}{W^3}u^\ell u_{\ell;k}$$

and the Ricci identity gives us

$$u_{i;jk} = u_{i;kj} + R_{i\ell kj}u^\ell = u_{k;ij} + R_{i\ell kj}u^\ell,$$

we get

$$G^{ij}u_{k;ij} = -\frac{1}{W^3}u^\ell u_{\ell;k}u_t + \frac{1}{W}\partial_k u_t - G^{ij}R_{i\ell kj}u^\ell - G^i u_{i;k}. \tag{105}$$

Using that $F(a_i^j[u]) = G(\nabla u, \nabla^2 u)$ we compute

$$G^i = \frac{\partial G}{\partial u_i} = \frac{\partial F}{\partial a_r^s}\frac{\partial a_r^s}{\partial u_i} = F_s^r \frac{\partial}{\partial u_i}\Big(\sigma^{s\ell} a_{\ell r}\Big) = F_s^r a_{r\ell}\frac{\partial}{\partial u_i}(\sigma^{s\ell}) + F_s^r \sigma^{s\ell}\frac{\partial}{\partial u_i}(a_{r\ell}).$$

Note that

$$F_s^r a_{r\ell}\frac{\partial \sigma^{s\ell}}{\partial u_i} = F^{r\eta}\sigma_{\eta s} a_{r\ell}\frac{\partial \sigma^{s\ell}}{\partial u_i} = WG^{r\eta} a_{r\ell}\sigma_{\eta s}\frac{\partial \sigma^{s\ell}}{\partial u_i}$$

and $\sigma_{\eta s}\sigma^{s\ell} = \delta_\eta^\ell$ implies in

$$\sigma_{\eta s}\frac{\partial \sigma^{s\ell}}{\partial u_i} = -\sigma^{s\ell}\frac{\partial}{\partial u_i}\big(g_{\eta s} + u_\eta u_s\big) = -\sigma^{s\ell}\big(\delta_s^i u_\eta + \delta_\eta^i u_s\big) = -\sigma^{i\ell}u_\eta - \delta_\eta^i \sigma^{s\ell}u_s.$$

Then

$$F_s^r a_{r\ell}\frac{\partial \sigma^{s\ell}}{\partial u_i} = -WG^{r\eta}a_{r\ell}\big(\sigma^{i\ell}u_\eta + \delta_\eta^i\sigma^{s\ell}u_s\big) = -WG^{ij}u_\ell a_j^\ell - WG^{\ell j}u_j a_\ell^i.$$

In addition, it follows from

$$\frac{\partial}{\partial u_i}\Big(\frac{1}{W}\Big) = -\frac{1}{W^2}\frac{\partial}{\partial u_i}\Big(\sqrt{1 + g^{s\ell}u_s u_\ell}\Big) = -\frac{u^i}{W^3},$$

that

$$F_s^r \sigma^{s\ell}\frac{\partial a_{r\ell}}{\partial u_i} = F^{r\ell}u_{r;\ell}\frac{\partial}{\partial u_i}\Big(\frac{1}{W}\Big) = -\frac{1}{W}\frac{F^{r\ell}}{W}\frac{u_{r;\ell}}{W}u^i = -\frac{1}{W}G^{r\ell}a_{r\ell}u^i.$$

Hence,

$$G^i = -WG^{ij}a_j^\ell u_\ell - WG^{\ell j}a_\ell^i u_j - \frac{1}{W}G^{\ell j}a_{\ell j}u^i$$

and consequently,

$$\begin{aligned} G^{ij}u_{k;ij} &= WG^{ij}a_j^\ell u_\ell u_{i;k} + WG^{\ell j}a_\ell^i u_j u_{i;k} + \frac{1}{W}G^{\ell j}a_{\ell j}u^i u_{i;k} \\ &\quad - G^{ij}R_{i\ell kj}u^\ell + \frac{1}{W}\partial_k u_t - \frac{1}{W^3}u^\ell u_{\ell;k}u_t. \end{aligned}$$

□

In order to prove the Theorem 7.2 we will use the technique due to Korevaar for to obtain interior gradient estimate.

Proof. [**of the Theorem 7.2**] We consider $\chi = \gamma(u)\eta(|\nabla u|^2)$ defined in $\overline{\Omega} \times [0,T)$ with $\gamma(s) = \exp(2As)$ and $\eta(s) = s$ where $A > 0$ is a constant to be choose later. Let (x_0, t_0) be a maximum point of χ. If $(x_0,t_0) \in \partial\Omega \times [0,T)$, then we use that $|\nabla u(x_0,t_0)| \leq c_1$ and we obtain a bound for $|\nabla u(x,t)|$ with $(x,t) \in \Omega \times (0,T)$. Hence, we can suppose that $(x_0,t_0) \in \Omega \times (0,T)$. We can also suppose that $|\nabla u(x_0,t_0)| \neq 0$.
Since

$$\chi_i = \gamma'\eta u_i + 2\eta'\gamma u^k u_{k;i} = 2\gamma\big(A\eta u_i + u^k u_{k;i}\big)$$

and

$$\chi_t = \gamma' \eta u_t + 2\eta' \gamma u^k \partial_t u_k = 2\gamma \big(A\eta u_t + u^k \partial_t u_k \big),$$

at (x_0, t_0), we get

$$u^k u_{k;i} = -A\eta u_i \quad \text{and} \quad u^k \partial_t u_k = -A\eta u_t. \tag{106}$$

Computing the second derivatives of χ we have

$$\begin{aligned}
\chi_{i;j} &= \gamma'' \eta u_i u_j + 2\gamma' u_i u^\ell u_{\ell;j} + \gamma' \eta u_{i;j} + 2\gamma' u_j u^k u_{k;i} + 2\gamma u^k_{;j} u_{k;i} + 2\gamma u^k u_{k;ij} \\
&= 2\gamma \Big\{ 2A^2 \eta u_i u_j + 2A u_i u^\ell u_{\ell;j} + A\eta u_{i;j} + 2A u_j u^k u_{k;i} + u^k_j u_{k;i} + u^k u_{k;ij} \Big\}.
\end{aligned}$$

Using (106), we obtain

$$\begin{aligned}
\chi_{i;j}(x_0, t_0) &= 2\gamma \Big\{ 2A^2 \eta u_i u_j - 2A u_i A\eta u_j + A\eta u_{i;j} - 2A u_j A\eta u_i + u^k_j u_{k;i} + u^k u_{k;ij} \Big\} \\
&= 2\gamma \Big\{ A\eta u_{i;j} - 2A^2 \eta u_i u_j + u^k_j u_{k;i} + u^k u_{k;ij} \Big\}.
\end{aligned}$$

Since G^{ij} is definite positive and (x_0, t_0) is a local maximum point of χ, we have

$$0 \geq \frac{1}{2\gamma} G^{ij} \chi_{i;j} = A\eta G^{ij} u_{i;j} - 2A^2 \eta G^{ij} u_i u_j + G^{ij} u^k_j u_{k;i} + G^{ij} u^k u_{k;ij}.$$

It is follows from Lemma 7.3 and the relations in (106) that

$$\begin{aligned}
G^{ij} u^k u_{k;ij} &= W G^{ij} a^\ell_j u_\ell u^k u_{i;k} + W G^{\ell j} a^i_\ell u_j u^k u_{i;k} + \frac{1}{W} G^{\ell j} a_{\ell j} u^i u^k u_{i;k} \\
&\quad - G^{ij} R_{i\ell kj} u^k u^\ell + \frac{1}{W} u^k \partial_t u_k - \frac{1}{W^3} u^\ell u^k u_{\ell;k} \partial_t u \\
&= -2A\eta W G^{ij} a^\ell_j u_\ell u_i - \frac{1}{W} A\eta^2 G^{\ell j} a_{\ell j} - G^{ij} R_{i\ell kj} u^\ell u^k - \frac{1}{W^3} A\eta u_t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &\geq A\eta G^{ij} u_{i;j} - 2A^2 \eta G^{ij} u_i u_j + G^{ij} u^k_j u_{k;i} - 2A\eta W G^{ij} a^\ell_j u_\ell u_i \\
&\quad - \frac{1}{W} A\eta^2 G^{\ell j} a_{\ell j} - G^{ij} R_{i\ell kj} u^\ell u^k - \frac{1}{W^3} A\eta u_t \\
&= \frac{1}{W} A\eta G^{ij} a_{ij} - 2A^2 \eta G^{ij} u_i u_j + G^{ij} u^k_{;j} u_{k;i} - 2A\eta W G^{ij} a^\ell_j u_\ell u_i \\
&\quad - G^{ij} R_{i\ell kj} u^\ell u^k - \frac{1}{W^3} A\eta u_t,
\end{aligned}$$

where in the last step of the inequality above we use that $a_{ij} = \frac{u_{i;j}}{W}$.

Now, we fix a normal coordinate system $\{x^i\}$ in P centered at x_0 such that

$$\frac{\partial}{\partial x^1}\Big|_{(x_0,t_0)} = \frac{\nabla u(x_0,t_0)}{|\nabla u(x_0,t_0)|}.$$

In terms of these coordinates one has

$$u_1(x_0,t_0) = |\nabla u(x_0,t_0)| > 0 \quad \text{and} \quad u_j(x_0,t_0) = 0, \quad j > 1.$$

Since the matrices $\{g_{ij}\}_{|(x_0,t_0)}$ and $\{g^{ij}\}_{|(x_0,t_0)}$ are diagonal in this frame, using (106) one obtains at (x_0,t_0)

$$\begin{aligned} & u^k = u_k, \quad \forall \quad k \\ & u_{1;1} = -A\eta, \qquad u_{1;i} = 0, \quad \text{if} \quad i > 1 \\ & \sigma^{11} = \frac{1}{W^2}, \qquad \sigma^{jk} = \delta^{jk}, \quad \text{if} \quad j > 1. \end{aligned}$$

After a rotation of the coordinates $\{x^2,\ldots,x^n\}$ we may assume that $\nabla^2 u(x_0,t_0) = \{u_{i;j}\}$ is diagonal. Then

$$\begin{aligned} & a_{11} = -\frac{A\eta}{W}, \qquad a_{ij} = a_i^j = 0 \quad \text{if} \quad i \neq j \\ & a_1^1 = -\frac{A\eta}{W^3}, \qquad a_i^i = a_{ii} = \frac{u_{i;i}}{W} \quad \text{if} \quad i > 1. \end{aligned}$$

It is follows from

$$G^{ij} = \frac{\partial G}{\partial u_{i;j}} = \frac{\partial F}{\partial a_{k\ell}}\frac{\partial a_{k\ell}}{\partial u_{i;j}} = \frac{1}{W}\frac{\partial F}{\partial a_{ij}} = \frac{1}{W}\frac{\partial F}{\partial a_\ell^k}\frac{\partial a_\ell^k}{\partial a_{ij}} = \frac{1}{W}\sigma^{ki}F_k^j$$

and the Lemma 7.1 that G^{ij} is also diagonal and we have

$$G^{11} = \frac{1}{W^3}f_1, \quad G^{ii} = \frac{1}{W}f_i \quad \text{if} \quad i \neq 1, \quad G^{ij} = 0 \quad \text{for} \quad i \neq j.$$

Thus,

$$0 \geq \frac{1}{W}A\eta G^{ii}a_{ii} - 2A^2\eta^2 G^{11} + G^{ii}(u_{i;i})^2 - 2A\eta^2 W G^{11} a_1^1 - \frac{1}{W^3}A\eta u_t.$$

As $a_{11} = -\frac{A\eta}{W} < 0$ implies in $G^{11}a_{11} \geq \frac{1}{W}f_1 a_{11}$, we get

$$\frac{1}{W}A\eta G^{ii}a_{ii} - \frac{1}{W^3}A\eta u_t \geq \frac{A\eta}{W^2}\left[\sum_{i=1}^n f_i a_{ii} - \frac{u_t}{W}\right] = \frac{A\eta}{W^2}\left[\sum_{i=1}^n f_i(k)k_i - f(k)\right].$$

Taking $\mu = (\mu_1, 0, \cdots, 0) \in \Gamma$ such that $\mu_1 < 1$ and $f(\mu) \leq v_0$, it follows from (101) that

$$\sum_{i=1}^{n} f_i(k)(k_i - \mu_i) - f(k) \leq -f(\mu) \leq f(\mu).$$

Then we obtain

$$\begin{aligned} \frac{1}{W} A\eta G^{ii} a_{ii} - \frac{1}{W^3} A\eta u_t &\geq \frac{A\eta}{W^2}\left[\sum_{i=1}^{n} f_i(k)(k_i - \mu_i) - f(k) + f(\mu) - f(\mu) + f_1(k)\mu_1\right] \\ &\geq \frac{A\eta}{W^2}\left[f_1(k)\mu_1 - f(\mu)\right] \\ &\geq \frac{A\eta}{W^2}(\mu_1 - 1) f_1(k) \end{aligned}$$

where in the last step of the inequality above we use (100).
So

$$0 \geq -\left(2A^2\eta^2 - (u_{1;1})^2 + 2AW\eta^2 a_1^1\right) G^{11} + \frac{A\eta}{W^2}(\mu_1 - 1) f_1.$$

Now, using that

$$2A^2\eta^2 - (u_{1;1})^2 + 2AW\eta^2 a_1^1 = 2A^2\eta^2 - A^2\eta^2 - 2A\eta^2 \frac{A\eta}{W^2} = \frac{-A^2\eta^3 + A^2\eta^2}{W^2}$$

we have

$$0 \geq \frac{A\left(\eta^2 - \eta\right)}{W^2} A\eta \frac{1}{W^3} f_1 - \frac{A\eta}{W^2} f_1(1 - \mu_1)$$

or equivalent

$$\frac{A(u_1^4 - u_1^2)}{W^3} \leq 1 - \mu_1.$$

If we choose $A \geq 2(1 - \mu_1)$ we obtain

$$u_1^4 - u_1^2 - \frac{1}{2}(1 + u_1^2)^{\frac{3}{2}} \leq 0.$$

Since $u_1 > 0$, we have a bound for u_1 and consequently for $\chi(x_0, t_0)$. Hence, when $(x_0, t_0) \in \Omega \times (0, T)$, there exists a constant $\tilde{C}$ which depends of v_0 and $\sup_{\Omega \times (0,T)} |u|$ such that

$$|\nabla u(x, t)| \leq \tilde{C}$$

for any $(x, t) \in \Omega \times (0, T)$. Consequently, there existe a constant $C > 0$ which depends of v_0, c_0 and c_1 such that

$$|\nabla u(x, t)| \leq C$$

for $(x, t) \in \Omega \times (0, T)$. □

8 CONCLUSION

In this thesis, we considered the problem of the evolution of Killing graphs by a curvature function. In the first part, we restricted to the study of the mean curvature flow in a warped product space $M = P \times_\varrho \mathbb{R}$, where P is a Cartan-Hadamard manifold. More precisely, given a function $\varphi \in C^\infty(P) \cap C(\overline{P})$, we investigated the existence of the mean curvature flow starting from the Killing graph of φ and such that for every time, the solution is also a Killing graph.

Under conditions imposed on the geometry of P and on the geometry of M, we obtained *a priori* estimates for the height of the solution and for its derivatives of all orders in compact parabolic cylinders as well. Such estimates allowed us to use the standart theory of parabolic partial differential equations to solve the problem of mean curvature flow in compact parabolic cylinders with initial data φ. Hence, by using an exhaustion argument, we guaranteed the existence of a solution to the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \left(g^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} + \left(1 + \frac{1}{\varrho^2 W^2}\right) (\log \varrho)^i u_i, & \text{in} \quad P \times [0, \infty) \\ u(x,0) = \varphi(x), \quad \text{in} \quad P \times \{0\} \\ u(x,t) = \varphi(x) \quad \text{if} \quad x \in \partial_\infty P,\ t \in [0,\infty). \end{cases} \tag{107}$$

By using a concept of convexity at infinity introduced in [28], we built barriers that assured us that the solution obtained is continuous on the asymptotic boundary.

In the last part of this thesis, we considered the more general problem of the evolution of Killing graphs by a curvature function. The ambient space considered was a Riemannian product $M = P \times \mathbb{R}$. In this context, we obtained an a priori (interior) gradient estimate.

As next steps, we hope to obtain higher order *a priori* estimates and, under conditions on the geometry of the considered domain, investigate the existence of a solution for the flow by curvature function. We can also ask about the existence of solitons for specific cases of curvature functions.

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