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OPTIMIZATION OF FLAT ARCHES UNDER UNIFORMLY DISTRIBUTED LOAD CONSIDERING IN-PLANE BUCKLING

Francisco G. Chaves Neto[1], Antônio M. C. de Melo[2]

[1]*Dept. of Structural Engineering and Civil Construction, Federal University of Ceará*
Av. Mister Hull, 60440-900, Ceará, Brazil
francisco_gchavesneto@hotmail.com
[2]*Dept. of Structural Engineering and Civil Construction, Federal University of Ceará*
Av. Mister Hull, 60440-900, Ceará, Brazil
macario@ufc.br

Abstract. The optimization of shallow two hinged arches is studied in this work. Specific problem investigated are sinusoidal or segmental arch, with an arbitrary section, under uniformly distributed load. Arches resist general loading by a combination of axial compression and bending actions and an arch which is braced by lateral restraints that prevents lateral buckling, nevertheless, may still suffer in-plane buckling. Shallow arches stability is studied under an energy method to establish the critical load, considering the shortening of its center line, but simplifying the curvature change. Approximations to the symmetric buckling of shallow arches are demonstrated. Analytical solutions for buckling loads and the compressive stress on the section under an uniaxial stress state are reviewed and used for efficiently evaluation of the constraints in the optimization problem. In a volume minimization model of the arch the design variables are continuous, consisting of the section dimensions and arch rise. Three types of sections are investigated: circular, rectangular and I-shaped. Others shapes can be considered but constraints to prevent local buckling must be applied, as in the case of the I-section. Results for the optimal section dimensions and arch rise are obtained using Microsoft Excel's Solver tool.

Keywords: shallow arch, optimization, in-Plane buckling.

1 Introduction

Arches are planar structures with an initial curvature that support loads providing structures that eliminate tensile stresses in most cases and significant spans can be achieved, but they transfer forces to the base, which need to be restrained. They are widely used in architectures around the world because of its esthetically pleasing shape. Arches can be fixed, hinged or have three hinges and are presented in various shapes [1].

Slender structures, including arches, are subjected to a type of behavior known as buckling, which might have a critical load that when reached, the member suddenly deflects. This bending gives rise to a large deformation, which causes the member to collapse. That phenomenon must be taken in consideration when designing those structures [2].

It is known that arches might buckle in an in-plane symmetric snap-through mode or an in-plane antisymmetric bifurcation mode, as shown on Fig. 1, which must be taken into consideration when studying buckling failure of arches.

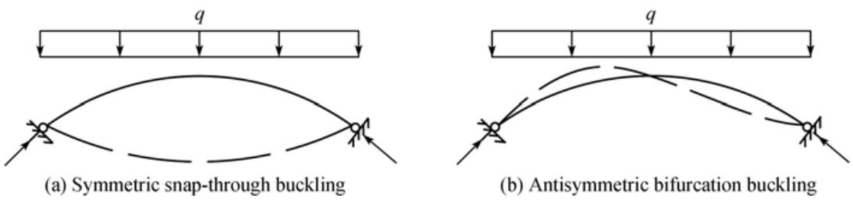

Figure 1. Buckling modes [3]

According to Bazant & Cedolin [4], it is important to distinguish two types of arches: high and flat (or shallow) arches. It is separated in two categories because the first type have a higher height to length ratio, which means the center line of the arch is considered incompressible, but flat arches have a lower ratio and the shortening of its length is important for the analysis.

Arches resist general loading by a combination of axial compression and bending actions and an arch which is braced by lateral restraints that prevents lateral buckling, nevertheless, may still suffer in-plane buckling [5].

This paper is concerned with the optimization of shallow two hinged arches with an arbitrary section under a distributed load, considering the in-plane buckling and the material's yield stress.

2 Analysis of arches

2.1 High arches stability

This topic study a two-hinged perfect high arch, that being a circular arc under uniform radial pressure and the assumption of incompressible center lines, which means the bending moments will be zero. The circular arch was chosen because it is the funicular curve for uniform pressure.

A perfect arch is an arch which the center line before deflection coincides with the compression line, that being the locus of the points of the normal forces resultant within the cross section [4]. Due to the center lines being incompressible, the bending moments provided by boundary conditions due to center line compression are insignificant.

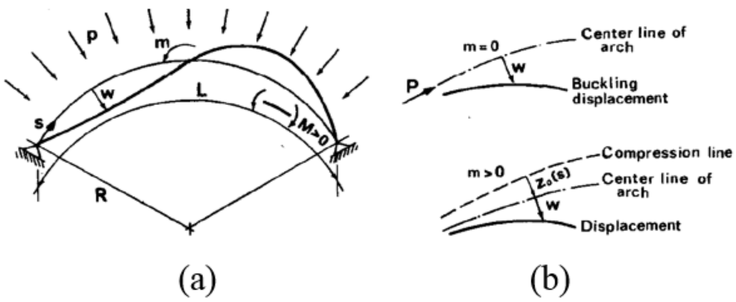

Figure 2. Circular high arch subjected to uniform radial pressure [4]

Another simplification for hinged high arches is that the compression line remains fixed, because the reaction resultants do not move significantly during buckling (Fig. 2-b) and the initial bending moment is zero, otherwise there will be an initial deflection from the compression line to the center line of the arch. A circular high arch under uniform radial pressure is equivalent to a bar under uniform compression.

That being said, the value for the critical buckling load for high arches can be obtained from the differential equation for the deflection curve and it is shown in eq. 1 [4]:

$$P_{cr} = Rp_{cr} = \frac{EI\pi^2}{(\beta L/2)^2}, \tag{1}$$

where is p_{cr} the critical distributed load, EI is the proportionality constant, R is the arch initial curvature radius, L is the arch length and βL is the effective length, where β is given by:

$$\beta = \frac{2}{\sqrt{4 - (L/R\pi)^2}}. \tag{2}$$

According to Pi, Bradford & Tin-Loi [5], when one evaluates an arch subjected to a vertical load uniformly distributed over its entire span, the axial compressive action is relatively high and the bending action is relatively low. That effect it greater when the arch have a lower rise. Therefore, to simplify the problem, an uniform radial pressure is used for high arches methods to make sure the critical load obtained from flat arches method does not surpass the one from high arches.

2.2 Flat arches stability

The analysis of shallow arches is different from high arches, since the simplifications adopted on the previous method may not be valid for shallow arches. The behavior of shallow arches may be non-linear and deformation is significant prior to buckling. These effects may reduce the in-plane buckling resistance of shallow arches, therefore the previous theory might overestimate the in-plane resistance [6]. The theory used here is based on the potential energy of the system and is applied to arches whose rise is small compared to the span.

Experimental and theoretical studies confirm that flat arches may be assumed to fail in a symmetric mode and its center line is shortened during the failure, which is different from high arches that have a negligible center line shortening.

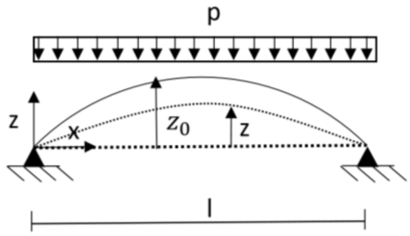

Figure 3. Circular flat arch subjected to uniformly distributed load

Consider the arch in Fig. 3, the initial shape before loading is:

$$z_0 = a\,sin\left(\frac{\pi x}{l}\right), \tag{3}$$

where *a* is the rise of the arch and *l* is its span length. The vertical uniformly distributed load p can be approximated by a Fourier sine expansion. More sinusoidal Fourier components may be considered to obtain a more accurate solution, but the first term has dominant influence [4]:

$$p = \frac{4P}{\pi}\,sin\left(\frac{\pi x}{l}\right). \tag{4}$$

The deflection ordinate $z(x)$ is sinusoidal and can be given by:

$$z = qa\,sin\left(\frac{\pi x}{l}\right), \tag{5}$$

where q is a nondimensional parameter that modifies the initial shape of the arch.

By analyzing the potential energy of the system, one obtains the equilibrium path for the system:

$$P(q) = \frac{\pi^5}{4l^4} EIa(1 + (n-1)q - nq^3), \tag{6}$$

where $n = a^2/4r^2$, which is a nondimensional parameter of arch slenderness and $r^2 = I/A$ is the section radius of gyration.

The equilibrium path may or may not have limit points (*Pcr*), it depends on *dP/dq=0*, then:

$$q_0 = \pm\sqrt{\frac{n-1}{3n}}, \tag{7}$$

where q_0 is the maximum and minimum points of equilibrium of the *P(q)* diagram.

Therefore, the equilibrium path has limit points only if > 1 , i.e., if rise is greater than twice the radius of gyration.

2.3 Section stress of shallow arches

By analyzing the problem under an uniaxial stress mechanic, one of the ways arches are able to resist general loading is by bending actions, which must be investigated so its bending stress does not exceed the material's critical load. In this particular problem (shallow arches), the curvature change (Δk) in any section can be approximately given by the second derivative of the displacement function $w(x)$, which is presented in eq. (8) for uniformly distributed load.

$$\Delta k = \frac{d^2w}{dx^2} = \frac{\mathrm{d}^2 z}{dx^2} - \frac{d^2 z_0}{dx^2} = \frac{\pi^2}{l^2} a(1-q)\sin\left(\frac{\pi x}{l}\right). \tag{8}$$

Parameter q can be obtained for any distributed load by isolating from the equilibrium path formulae presented on eq. (6). Therefore, it is possible to obtain the bending moment by replacing the curvature change and evaluating the value of q.

$$M(x) = EI\frac{\pi^2}{l^2} a(1-q)\sin\left(\frac{\pi x}{l}\right). \tag{9}$$

At the top, $x = l/2$, one obtains the value for the highest bending moment of the arch. The extreme values for the moment can be verified by the classical expression:

$$\sigma_b = -\frac{M_B y}{I}, \tag{10}$$

where *y* is the vertical offset from neutral axis.

For the normal stress resultant on the section, it is important to determine the support forces for the supports on the deformed configuration (Fig. 4).

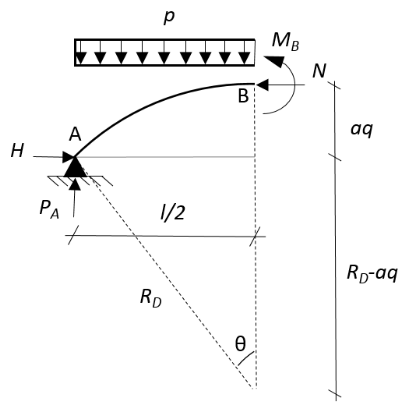

Figure 4. Reaction forces on half of the structure

The deformed configuration radius (R_D), the angle (θ) and total length (L) of the arch can be written as a function of its span (l), rise (a) and the parameter q.

$$R_D = \frac{l^2}{8aq} + \frac{aq}{2} \quad \theta = cos^{-1}\left(\frac{R_D - aq}{R}\right) \quad L = 2\theta R_D. \tag{11}$$

From the equilibrium conditions, one obtains:

$$P_A = \frac{pl}{2} \quad H = \frac{pl^2}{8aq} - \frac{M_B}{aq}, \tag{12}$$

and, at the support A:

$$N_A = -\big(Hcos(\theta) + P_A sen(\theta)\big). \tag{13}$$

3 The optimization procedure

Optimization deals with selecting the best option among a number of possible choices that are feasible or do not violate constraints. Many researches have studied the optimization process of arches under different methods and conditions.

Taysi, Gögüs & Özakça [7] presented an optimization procedure for the minimum weight and strain energy optimization for arch structures using finite element and genetic algorithm, but did not take into consideration the buckling load. Trentadue et al. [8] presented an optimization process for perfectly-constrained plane circular arches under uniformly distributed vertical load and self-weight, but the research did not take into consideration the buckling load either.

Zhang & Wang [9] developed a formulation for shape optimization of pinned-pinned circular arches under uniform radial pressure for maximum buckling capacity using a Hencky bar-chain model. Wang, Pan & Zhang [10] presented a formulation for the optimal design of triangular arches for maximum buckling capacity given a volume of material. The buckling criteria of triangular arches was derived analytically by using stability functions.

No research so far has studied the optimization of sinusoidal or segmental shallow arch, with an arbitrary section, under uniformly distributed load using energy method to establish the critical buckling load analytical function.

Excel's Solver can be used to optimize parameters in a model to best fit data, increase profitability of a potential engineering design, or meet some other type of objective that can be described mathematically with variables and equations.

The Solver tool in Excel has an evolutionary algorithm, which attempts to imitate the process of natural selection where the fittest individuals are selected for reproduction in order to produce offspring of the next generation. It performs a multi-directional search, reducing the probability to find a local optimum, maintaining a population of potential solutions and encourages information formation and exchange between these directions [11]. It is widely used around the world and could be applied to find the optimal design for shallow arches. The mathematical optimization problem can be organized as:

$$Minimize\ f(\mathbf{x})\ such\ that \begin{cases} g(\mathbf{x}) \leq 0 \\ g_{eq}(\mathbf{x}) = 0 \\ \mathbf{A\,x} \leq \mathbf{b} \\ \mathbf{A_{eq}\,x} = \mathbf{b_{eq}} \\ \mathbf{x_l} \leq \mathbf{x} \leq \mathbf{x_u} \end{cases}, \tag{14}$$

where $\mathbf{b}$ and $\mathbf{b_{eq}}$ are vectors, $\mathbf{A}$ and $\mathbf{A_{eq}}$ are matrices, $g(\mathbf{x})$ is for inequality and $g_{eq}(\mathbf{x})$ is for equality, those are functions that return vectors, and $f(\mathbf{x})$ is a function that returns a scalar. It's important to mention that $f(\mathbf{x})$, $g(\mathbf{x})$, and $g_{eq}(\mathbf{x})$ can be nonlinear functions.

The vectors $\mathbf{x_l}$ and $\mathbf{x_u}$ are lower and upper boundaries of search for the variables.

One can reorganize the functions for shallow arches to find the minimum volume for an arch under a pre-defined uniformly distributed load.

The project variables are the characteristic section dimensions (depending on the adopted section) and the arch rise (a). The lower and upper boundaries for arch rise were 0.001 and 0.5, for the other variables these values can be observed in Fig. 5.

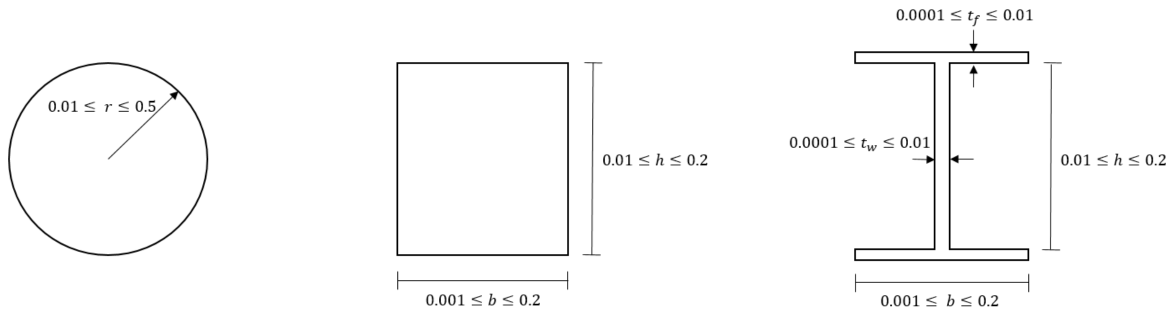

Figure 5. Arches sections

The objective function is the volume of material on the arch, which is given by eq. (15).

$$f(A,a) = L\,A \rightarrow \begin{cases} f(r,a) = L\pi r^2 \rightarrow \text{Circular Section} \\ f(b,h,a) = LbH \rightarrow \text{Rectangular Section} \\ f(b,h,t_f,t_w,a) = L\left(\left(2t_f b\right) + \left(t_w h\right)\right) \rightarrow \text{I Section} \end{cases}. \tag{15}$$

The design variables are subjected to constraints, such as: critical buckling load must be greater than the applied load (eq. 16); normal stress at point A and normal stress with bending stress at point B must be lower than the yield stress of the material (eq. 17 and eq. 18); the parameter n must be greater than one (eq. 19), to make sure the structure have a critical buckling load.

Equation (20) is a constraint to make sure the solution is considered as one of a shallow arch, where P_{SA} and and P_{HA} are the critical buckling loads obtained from shallow arch and high arch methods, respectively.

$$g_1(x) = \frac{p}{Pcr} - 1 \leq 0\,. \tag{16}$$

$$g_2(x) = \frac{\sigma_c^A}{\sigma_y} - 1 \leq 0\,. \tag{17}$$

$$g_3(x) = \frac{|\sigma_c^B + \sigma_b^{B+}|}{\sigma_y} - 1 \leq 0 \qquad g_4(x) = \frac{|\sigma_c^B + \sigma_b^{B-}|}{\sigma_y} - 1 \leq 0. \tag{18}$$

$$g_5(x) = 1 - n \leq 0\,. \tag{19}$$

$$g_6(x) = \frac{P_{SA}}{P_{HA}} - 1 \leq 0\,. \tag{20}$$

Equations (21) and (22) are constraints to prevent local buckling of the flange and web, respectively, for I-sections [12].

$$g_7(x) = \frac{\left(\frac{b}{2t_f}\right)}{0.56\sqrt{\frac{E}{\sigma_y}}} - 1 \leq 0\,. \tag{21}$$

$$g_8(x) = \frac{\left(\frac{h}{t_w}\right)}{1.49\sqrt{\frac{E}{\sigma_y}}} - 1 \leq 0\,. \tag{22}$$

4 Applications

The formulation is applied to an arch with arbitrary section and a homogeneous material and a pre-defined uniformly distributed load of 20 kN/m. The material chosen is steel and the arch span is 5.00 m. The maximum value for rise must be low, since the method presented on this paper is applied only for shallow arches.

Population size was set as 100, the maximum percentage difference in objective values for the top 99% of the population in order to stop searching was 0.0001% and mutation rate had a 0.075 value.

The optimization problem is solved on Excel using the Solver Evolutionary algorithm add-in. The optimal solutions with minimum volume for the all sections types and the constraints values are presented on Tab. 1.

Table 1. Optimal values for design variables and its constraints

Section	Variables	Value	Volume (m^3)	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
Circular	Rise (m)	0.121	0.02410	-0.105	-0.439	-9.0E-8	-0.881	-8.52	-3.2E-8	-	-
	r (m)	0.039									
Rectangular	Rise (m)	0.249	0.00821	-0.700	-0.367	-8.8E-6	-0.756	-8.30	-5.7E-7	-	-
	b (m)	0.011									
	h (m)	0.141									
I-Shaped	Rise (m)	0.361	0.00497	-0.836	-0.282	-1.1E-6	-0.618	-8.00	-1.8E-6	-0.676	-1.3E-04
	b (m)	0.050									
	h (m)	0.144									
	t_f (m)	0.004									
	t_w (m)	0.003									

Results have been verified using Diana Finite Element Analysis software with a geometrically nonlinear analysis. Figure 6 features the curves for critical buckling loads for the optimal sections and various a/l ratio values using both high and shallow arch methods. It is possible to observe that, for the optimal value of rise, the flat arch method presents a lower critical load than high arches, which demonstrate that the optimal values found correspond to a shallow arch.

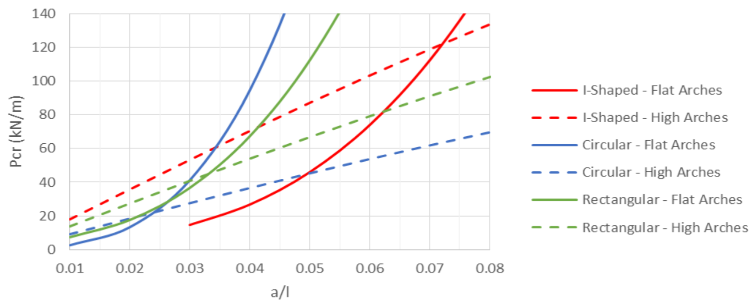

Figure 6. Flat and high arches comparison of critical buckling load for optimal sections

5 Conclusions

This paper has studied the optimization of shallow hinged uniform circular arches with an arbitrary cross-section subjected to a vertical uniformly distributed load considering the in-plane buckling load and stress constraints.

It has been verified that the applied loads are most resisted through axial compression forces, which demands more attention when implementing the constraints, especially at the top of the arch, where the constraint for normal stress in all optimum solution is active. The constraints for shallow arch verification and web local buckling for I-sections are also active, therefore they limited the solution, being important for the optimization process and demands more attention when implemented.

The I-shaped section presented a lower total volume than circular and rectangular sections, requiring less section area than the other two, but a greater length for the arch.

It was possible to observe that high arches theory overestimates the symmetric snap-through buckling load of shallow arches. As well as shallow arch theory overestimates the asymmetric buckling load of higher arches.

Further studies on arches optimization is suggested to focus on issues as the elastoplastic properties of the material, to implement the high arches method on Excel's Solver and other measures to enhance the buckling load of arches in general.

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