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EXPONENTIAL-TYPE POTENTIALS DERIVED FROM FIRST PRINCIPLES

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Advisor: Prof. Dr. André Auto Moreira.

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A mi familia...

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RESUMO

Neste trabalho estudamos a mecânica quântica não aditiva induzida por um sistema com massa dependente da posição (PDM) a partir de uma perspectiva geométrica, utilizando o formalismo matemático da geometria diferencial. Estabelecemos o modelo geométrico a partir da dualidade entre uma partícula com PDM que se move no espaço euclidiano e uma partícula com massa constante que se move em um espaço curvo. Nesta abordagem, o momento deformado característico da mecânica quântica não aditiva surge naturalmente quando determinamos as simetrias do tensor métrico que descreve o espaço curvo. Finalmente, como uma aplicação desta abordagem, estudaremos o potencial de Coulomb deformado e sua relação com os potenciais de Hulthen e Manning-Rosen.

Palavras-chave: Mecânica quântica não aditiva. Sistemas PDM. Potencial de Hulthen. Potential de Manning-Rosen.

ABSTRACT

In this work we study the non-additive quantum mechanics induced by a system with position dependent mass (PDM) from a geometric perspective, using the mathematical formalism of differential geometry. We establish the geometric model from the duality between a particle with PDM that moves in Euclidean space and a particle with constant mass that moves in a curved space. In this approach the deformed momentum characteristic of the non-additive quantum mechanics arises naturally when we determine the symmetries of the metric tensor that describes the curved space. Finally, as an application of this approach we shall study the deformed Coulomb potential and its relation with the Hulthen and Manning–Rosen potentials.

Keywords: Non-additive quantum mechanics. PDM systems. Hulthen Potential. Manning-Rosen Potential.

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1 INTRODUCTION

In recent years, there has been a growing interest in the understanding of quantum mechanical systems with position dependent mass (PDM) due to the high applicability in numerous areas of condensed matter physics, for example: electronic properties of semiconductors [2], quantum wells [3], quantum dots [4], quantum liquids [5], polarons [6], etc. In addition to their practical relevance, these systems bring with them interesting conceptual principles, such as the ordering ambiguity of the momentum and mass operators in the kinetic energy term [7], the non-self-adjointness of some potentials [8] and the construction of coherent states [9].

Moreover, previous studies have shown that exist an intimate connection between systems with PDM and other two very different areas in quantum mechanics [10]. The first one, is the area of deformed algebras, wherein modifications of the usual canonical commutation relations are introduced *ad hoc* to describe nonzero minimal uncertainties in position and momentum measurements [11, 12, 13]. This generalizations of the uncertainty principle have been introduced in the framework of string theory [14] and quantum gravity [15].

The second connection is with the problem of quantization in curved spaces, which dates back to Schrödinger himself [16], when he was working on the factorization method to study the hydrogen atom in spherical geometry. The problem of quantization in non-Euclidean spaces is a fundamental topic in gravitation [17], also in other fields, like condensed matter, for example, in the study of the quantum Hall effect in a hyperbolic manifold [18].

In particular, Quesne *et al* have demonstrated that the approaches of PDM, deformed algebras and curved spaces are equivalent under some conditions [10]. As a result, any Schrödinger equation known in one of these three models can be written as an Schrödinger equation in the other two. This brings important consequences, one of which is the possibility of using the mathematical formalism developed to study curved spaces, called differential geometry, to study systems with PDM from a more fundamental perspective based on concepts of symmetry.

Differential geometry is a branch of mathematics that study the intrinsic properties of curves and surfaces using techniques of differential and integral calculus [19]. In physics, it has been widely used in theories such as classical mechanics, electrodynamics, particle physics and general relativity. Moreover, concepts of differential geometry have been used to study the behaviour of systems with PDM [20, 21, 22], leading to the formulation of the method known as Killing vector fields and Noether momenta, in which the dynamics of the particle is determined by the geometric properties of the space, namely the metric tensor and the curvature.

Recently, Costa Filho *et al.* have proposed another approach to describe systems with PDM by introducing a generalized translation operator $\mathscr{T}_\gamma(x)$ which produces infinitesimal non-linear displacement [23]. Through this new operator is possible to obtain a generalized momentum $\hat{p}_\gamma$, that allows the formulation of the Schrödinger equation used to model electrons with effective masses propagating through abrupt interfaces in semiconductor heterostructures [2]. Initially, this approach had some limitations, for example the momentum operator was not Hermitian with regard the usual product in the Hilbert space and therefore the translation operator was not unitary. These problems were corrected by introducing appropriately the metric of the space [24, 25].

In this work we study the PDM problem proposed by Costa Filho *et al.* from a geometric perspective, using the Killing vector fields and Noether momenta approach, with the purpose of obtaining the generalized momentum $\hat{p}_\gamma$ and the non-additive translation operator $\mathscr{T}_\gamma(x)$ from the intrinsic properties of the curved space. This allows to obtain Hermitian and unitarity operators without introducing additional modifications. The geometric concepts behind the PDM systems were analysed using tools of geometric mechanics [26, 27].

Another purpose of this work is to illustrate the advantage of this geometric formalism by solving an specific problem: a particle with radial PDM submitted to the Coulomb potential. Additionally, with the obtained results, we draw a direct connection between the deformed Coulomb potential and other two potentials used to describe short range interactions known as the Hulthen potential and Manning-Rosen potential. This type of connection between two different potentials by means of a PDM system was already realized for the harmonic potential and the Morse potential [25].

The Hulthen potential is one of the most important short range potentials in physics [28], this has been used in different fields such as nuclear physics [29], particle physics [30], atomic physics [31], solid state [32] and chemical physics [33]. This potential behaves like the Coulomb potential for small values of r and is exponentially damped for large distances. The Manning-Rosen potential was proposed by Manning and Rosen in 1933 [34] as an exponential-type potential that is used as an important mathematical model for molecular vibrations and rotations [35, 36]. This potential found considerable applications in several bound states and scattering problems in physics [37].

2 GEOMETRY OF THE PDM SYSTEMS

In this chapter, we discuss the geometric concepts behind the PDM systems, initiating with the simplest problem: a free particle moving in one-dimensional space. We begin with the classical treatment using the Lagrangian formalism to determine the trajectory and the gravitational tidal force. In Section 2.3 we develop the basic geometric formalism needed to describe the PDM system from the properties of the curved space. Finally, in Section 2.4 we quantize the Nother momentum to establish the PDM Schrödinger equation.

2.1 Geometric classical mechanics

In classical mechanics, the generalized coordinates q^i are used to describe the dynamical systems, this correspond to the natural coordinates of the system since they have the information of the constraints and the topology of the region in which the system is free to move. The generalized coordinates lie in the space Q called the configuration manifold and their number is equal to the dimension of Q (number of degrees freedoms of the system) [1].

The Lagrangian $L\left(q^i,\dot{q}^i,t\right)$ is a function of the coordinates q^i and the velocities $\dot{q}^i$ and this determines the equations of motions and the dynamics of the system. The dependence on the velocities makes clear that the Lagrangian is not a function defined over the configuration manifold Q, for this reason is necessary to introduce a higher dimensional space called the tangent manifold TQ, which is composed of all generalized coordinates and its derivatives, then a point in TQ is denoted as $\left(q^i,\dot{q}^i\right)$ [38].

Is possible to show that the equations of motion (the Euler-Lagrange equations) are equivalent to a vector field defined over the tangent manifold TQ which is denoted as Δ_L, solving the equations of motions is equivalent to finding the integral curves of the dynamical vector field [1, 26]. If $F\left(q,\dot{q}\right)$ is a dynamical variable then the time evolution is determined by applying Δ_L:

$$\dot{F}\left(q,\dot{q}\right)=\Delta_L\left(F\left(q,\dot{q}\right)\right), \tag{2.1}$$

where the dynamical vector field is given by

$$\Delta_L=\dot{q}^{\,i}\frac{\partial}{\partial q^i}+\ddot{q}^{\,i}\frac{\partial}{\partial \dot{q}^{\,i}}. \tag{2.2}$$

In the above equation $\dot{q}^{\,i}$ and $\ddot{q}^{\,i}$ are functions on TQ and the basis of the vector field is given by $\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial \dot{q}^{\,i}}\right)$. Then Δ_L can be treated as an operator that transform functions over the tangent manifold. This approach offers an alternative way to obtain the orbits for a given potential, moreover allows us to use elements of vector calculus (divergence and rotational) to characterize them.

Performing a Legendre transformation $L(q,\dot{q}) \longrightarrow H(q,p)$ is possible to construct a dynamical vector field in the Hamiltonian formalism:

$$\Delta_H = \dot{q}^{\,i}\frac{\partial}{\partial q^i} + \dot{p}^{\,i}\frac{\partial}{\partial p^i}, \tag{2.3}$$

this operator lie in the cotangent space of the configuration manifold Q, in classical mechanics is known as the phase space and is denoted as T*Q. Working in the phase space, we can study some important concepts such as the Poisson brackets, canonical transformations, the simplectic form, Liouville and Darboux theorems, etc.

2.2 Position dependent mass problem

2.2.1 Lagrangian Approach

Our starting point is the classical PDM free particle moving in one dimension. The configuration space Q and the tangent space TQ are $q^i = x$ and $(q^i, \dot{q}^i) = (x, \dot{x})$, respectively. In the general case the mass function $m(x)$ is a well behaved and positive function of the position, thus the Lagrangian for this problem can be written as

$$L = \frac{1}{2}m(x)\dot{x}^2, \quad x \in \mathbb{R}, \quad m(x) > 0. \tag{2.4}$$

Applying the Euler-Lagrange equations we obtain the equation of motion for this general problem

$$m(x)\ddot{x} + \frac{1}{2}m'(x)\dot{x}^2 = 0, \tag{2.5}$$

in this work we shall focus only on the particular case when the mass is of the form

$$m(x) = \frac{m_0}{f^2(x)} = \frac{m_0}{(1+\gamma x)^2}, \tag{2.6}$$

where $f(x)$ is the deforming function. This PDM is closely related to the problem of non-additive quantum mechanics proposed by Costa Filho *et al* in [23]. An important property of this deforming function is the possibility of returning to the usual free particle case when $\gamma \to 0$. From (2.5) is easy to see that the motion for a free particle with PDM (2.6) is governed by the following equation of motion

$$\ddot{x}(1+\gamma x) - \gamma \dot{x}^2 = 0, \tag{2.7}$$

then the acceleration of the system can be written as a function on TQ:

$$\ddot{x} = \frac{\gamma \dot{x}^2}{(1+\gamma x)}. \tag{2.8}$$

Now, with the acceleration of the system we are able to define the dynamical vector field Δ_L over the tangent space described by the equation (2.2), which determines the form of the orbits. The vector field for a particle with PDM is:

$$\Delta_L = \dot{x}\frac{\partial}{\partial x} + \frac{\gamma \dot{x}^2}{(1+\gamma x)}\frac{\partial}{\partial \dot{x}}, \tag{2.9}$$

where the basis vectors are $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \dot{x}^i}\right)$. We can write Δ_L in the Cartesian basis of unitary vectors $(\hat{\imath}, \hat{\jmath})$ with the substitution $y = \dot{x}$ as

$$\Delta_L = y\hat{\imath} + \frac{\gamma y^2}{(1+\gamma x)}\hat{\jmath}. \tag{2.10}$$

Representing the integral curves associated to the above dynamical vector field we can see in the Figure 1 that the velocity phase diagram for a particle with PDM has a unstable equilibrium point, namely, if the mass is located near, the force is directed away from the equilibrium point.

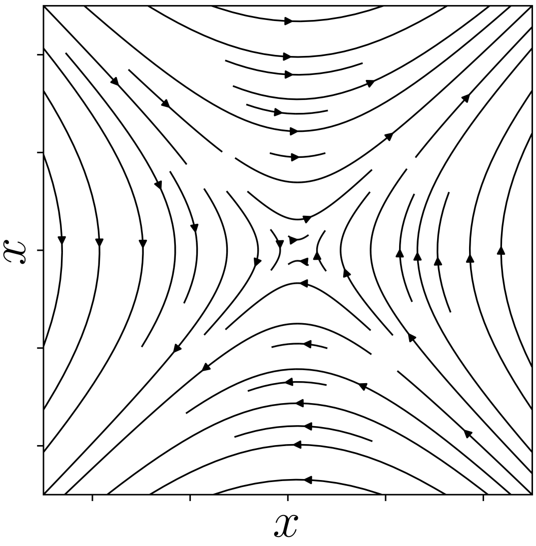

Figure 1 – Integral curves or flow lines of the dynamical vector field in the velocity phase space for a free particle with PDM. This velocity phase diagram is equal to the problem of the inverted harmonic oscillator [1].

Using the equation (2.1), we can determine the evolution of the dynamical variables such as the energy and momentum, along the integral curves. For instance, the conservation of energy can be expressed as:

$$\begin{aligned}\Delta_L(E) &= \dot{x}\frac{\partial E}{\partial x} + \frac{\gamma \dot{x}^2}{(1+\gamma x)}\frac{\partial E}{\partial \dot{x}} \\ &= \dot{x}\frac{\partial}{\partial x}\left(\frac{m_0\dot{x}^2}{(1+\gamma x)^2}\right) + \frac{\gamma \dot{x}^2}{(1+\gamma x)}\frac{\partial}{\partial \dot{x}}\left(\frac{m_0\dot{x}^2}{(1+\gamma x)^2}\right) \\ &= 0.\end{aligned}$$

Similarly, the time evolution of the momentum gives the force that the particle feel when it moves over the integral curves, this geometric force is determined as follows

$$\begin{aligned}\Delta_L(m_0\dot{x}) &= \dot{x}\frac{\partial(m_0\dot{x})}{\partial x} + \frac{\gamma \dot{x}^2}{(1+\gamma x)}\frac{\partial(m_0\dot{x})}{\partial \dot{x}} \\ &= \frac{m_0\gamma\dot{x}^2}{1+\gamma x} = F. \end{aligned} \tag{2.11}$$

This is called gravitational tidal force and arise naturally in spaces with curvature. The equation for the force (2.11) is the same that the expression given in [13]. From the energy of the particle the velocity can be written as

$$\dot{x} = \pm\sqrt{\frac{2E}{m_0}}(1+\gamma x). \tag{2.12}$$

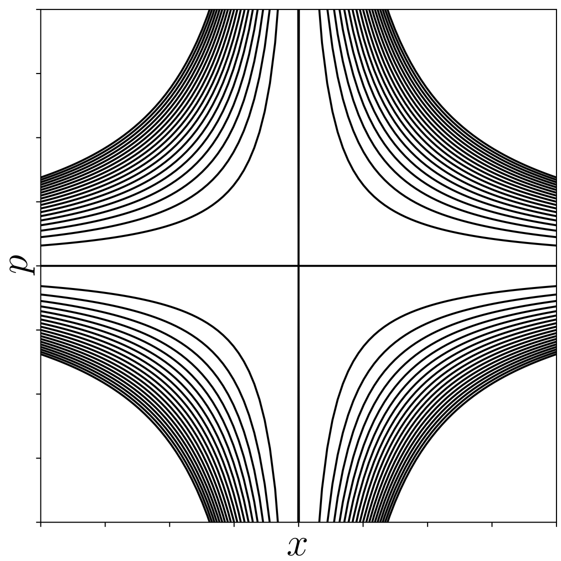

Figure 2 – Phase diagram for 20 different configurations of the energy. Quadrants I and III (II and IV) correspond to the plus (minus) sing of the momentum.

The solution of the above equation is not the usual trajectory followed by the free particle, instead, the solution has a exponential behaviour

$$x(t) = \frac{(1+\gamma x_0)\left(e^{\sqrt{\frac{2E}{m_0}}\gamma t} - 1\right)}{\gamma}. \tag{2.13}$$

This result is somewhat counter-intuitive, because even though the Lagrangian (2.4) describes a free particle, the solution (2.13) is not the common straight-line trajectory. In order to obtain an expression for the trajectories in the phase space we determine the canonical momentum of the system

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m_0 \dot{x}}{(1+\gamma x)}, \tag{2.14}$$

and replacing the velocity from equation (2.12)

$$p = \pm \frac{\sqrt{2m_0 E}}{(1+\gamma x)} \tag{2.15}$$

we achieve an equation for the canonical momentum as a function of the energy. This result allow us to visualise the orbits in the phase space for different values of the energy.

In the Figure 2, is evidenced that the case when $E = 0$ correspond to and unstable equilibrium point represented by two straight lines, which are also the asymptotes of the other hyperbolic trajectories.

On the other hand, as an illustration of the equation (2.13) the Figure 3 reveals the behaviour of the different free particle trajectories in this curved space, this graphic is equal to the Fig. 1 in [13].

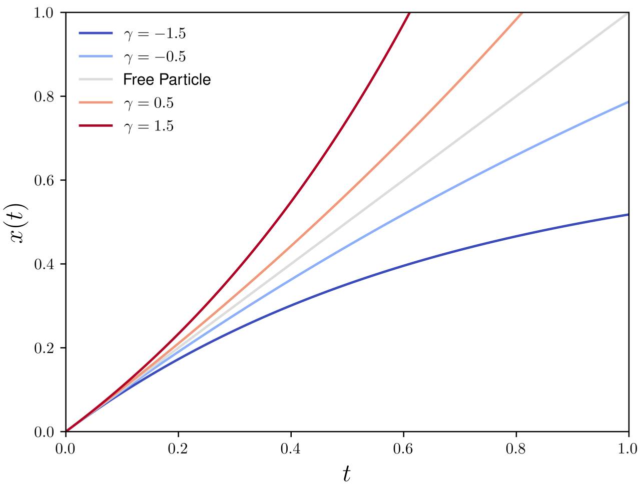

Figure 3 – Geodesic motion for different values of curvature parameter γ. The red line corresponds to the case of zero curvature or flat space, the lines that are above and below corresponds to the cases of positive and negative curvature respectively.

This trajectories are called geodesics in the language of differential geometry and are defined as the shortest path between two given points in a curved space [27]. For example, the red line ($\gamma = 0$) corresponds to the case of Euclidean space, where the shortest path between two points is a straight line. Additionally, for the other trajectories we can see that the sign of γ determines if the particle accelerates $\gamma > 0$ or decelerate $\gamma < 0$.

As noted above the curvature of the non-Euclidean space induced by the position dependent mass is defined by the parameter γ. Then, if we analyze the case when the space is nearly flat, $\gamma \approx 0$, the Taylor expansion of (2.13) gives

$$x(t) \approx C_2 + C_1 t + \frac{1}{2}\gamma (C_2 + C_1 t)^2 + O(\gamma^2). \tag{2.16}$$

From which one can recognize the close resemblance with the equation for a particle moving with constant acceleration.

Having studied the geodesic motion for a particle with position dependent mass, now we shall analyse the symmetries of the Lagrangian

$$L_\gamma = \frac{1}{2}\frac{m_0}{(1+\gamma x)^2}\dot{x}^2. \tag{2.17}$$

The first thing we can point out, is the non-invariance of this free particle Lagrangian under an infinitesimal translation $x' = x + \varepsilon$, consequently the x coordinate is not longer a cyclic variable and the linear momentum is not a constant of motion $\dot{p} \neq 0$. Although, this system has an unexpected symmetry, associated with the following infinitesimal deformed translation

$$\begin{aligned} x' &= x + \varepsilon(1+\gamma x), \\ \dot{x}' &= \dot{x}(1+\varepsilon\gamma). \end{aligned}$$

The demonstration of this invariance is fairly straightforward, replacing the above transformation in (2.17) we obtain:

$$\begin{aligned} L'_\gamma(x', \dot{x}') &= \frac{1}{2}\frac{m_0\dot{x}^2\,(1+\varepsilon\gamma)^2}{(1+\gamma(x+\varepsilon(1+\gamma x)))^2} \\ &= \frac{1}{2}\frac{m_0\dot{x}^2\,\cancel{(1+\varepsilon\gamma)^2}}{(1+\gamma x)^2\cancel{(1+\varepsilon\gamma)^2}} \\ &= L_\gamma(x, \dot{x}). \end{aligned}$$

From the Noether's Theorem we know that if the Lagrangian is invariant under an infinitesimal transformation $q \to q + \varepsilon$, then, exist a law of conservation and a conserved current associated to this invariance [26]. For this PDM problem is not possible determine the conserved quantity using conventional methods of classical mechanics, since as was explained previously this is a curved space. For this reason, in the next section, we shall formulate the problem in a different way, based on the assumption that the space is not Euclidean.

2.2.2 Hamiltonian Approach

Moving to the phase space T*Q has some mathematical advantages, since the generalized coordinates and the canonical momentums are treated as coordinates of T*Q, which allows us to introduce the symplectic structure and define concepts such as the Poisson brakets, canonical transformations, generating functions, etc. In this work we shall study this concepts for the phase space of the position dependent mass problem.

The tangent space TQ and the phase space T*Q are connected through the Legendre transformation. Applying this transformation to the Lagrangian function (2.17) we get the Hamiltonian function for the PDM problem:

$$H_\gamma = \frac{(1+\gamma x)^2 p^2}{2m_0}. \tag{2.18}$$

As a didactic example, we shall determine the dynamical vector field in the phase space Δ_H, defined in the equation (2.3), for this we need to calculated the Hamilton equations:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} & &\longrightarrow & \dot{x} &= \frac{(1+\gamma x)^2 p}{m_0} \\ \dot{p} &= -\frac{\partial H}{\partial x} & &\longrightarrow & \dot{p} &= -\frac{(1+\gamma x)\gamma p^2}{m_0} \end{aligned}$$

then the dynamical vector field is

$$\Delta_H = \frac{(1+\gamma x)^2 p}{m_0}\frac{\partial}{\partial x} - \frac{(1+\gamma x)\gamma p^2}{m_0}\frac{\partial}{\partial p}, \tag{2.19}$$

the dynamical vector field also give us the conserved quantities along the integral curves.

Through a transformation $(q,p) \to (Q(q,p,t),P(q,p,t))$ we can transform the Hamiltonian (2.18) into the usual Hamiltonian for a free particle, but in order to perform a canonical transformation, i.e, a transformation that leaves the volume of the phase space invariant, this has to be of the form:

$$Q = \frac{\ln(1+\gamma x)}{\gamma}, \tag{2.20}$$

$$P = (1+\gamma x)p. \tag{2.21}$$

This is a canonical transformation because leaves the volume ω of the phase space invariant:

$$\omega = dQ \wedge dP = dq \wedge dp \tag{2.22}$$

or we can also say that satisfy the canonical property of the Poisson brackets

$$\{Q,P\} = \{q,p\} = 1. \tag{2.23}$$

Assuming that the generating function of the canonical transformation is of type 3, then $F_3(p,Q)$ satisfy

$$p_i\dot{q}_i - H(q,p) = P_i\dot{Q}_i - K(Q,P) + \frac{dF_3(p,Q)}{dt},$$
$$p\dot{x} - H = P\dot{Q} - K + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial Q}\dot{Q} + \frac{\partial F_3}{\partial t}$$

where $K(Q,P) = H(q(Q,P), p(Q,P))$ is the Kamiltonian. With the canonical relations we get a system of partial differential equation:

$$P = -\frac{\partial F_3}{\partial Q},$$
$$x = \frac{\partial F_3}{\partial p}.$$

Solving the system we obtain the generating function of the canonical transformation:

$$F_3(Q,p) = -\frac{e^{\gamma Q}}{\gamma} + \frac{p}{\gamma} + constant. \tag{2.24}$$

2.2.3 PDM Kepler problem

In this section we shall introduce the problem of a particle with radial PDM subjected to a central potential. We will show the principal elements of the problem, beginning with the reduced Hamiltonian (only the radial part, since the angular momentum is conserved) of a particle under a central potential:

$$H_\gamma = \frac{(1+\gamma r)^2 p_r^2}{2\mu} + V_{eff}(r) \tag{2.25}$$

where μ is the reduced mass. Then with the effective potential the total Hamiltonian is

$$H_\gamma = \frac{(1+\gamma r)^2 p_r^2}{2\mu} + \frac{\ell^2}{2\mu r^2} - \frac{1}{r}. \tag{2.26}$$

If we consider the case of zero angular momentum $\ell = 0$ and the canonical transformation (2.20) we obtain the following Kamiltonian:

$$K_\gamma = \frac{P^2}{2\mu} - \frac{\gamma}{(e^{\gamma Q} - 1)}, \tag{2.27}$$

where (P, Q) are the new variables in the phase space. This is the Hamiltonian for the Hulthen potential [28, 29, 30]

$$K_\gamma = \frac{P^2}{2\mu} - \frac{\gamma e^{-\gamma Q}}{(1 - e^{-\gamma Q})}. \tag{2.28}$$

The dynamical vector field for this space is:

$$\Delta_K = \frac{P}{\mu}\frac{\partial}{\partial Q} - \frac{\gamma^2 e^{\gamma Q}}{(e^{\gamma Q} - 1)^2}\frac{\partial}{\partial P}, \tag{2.29}$$

and defines the integral curves in the phase space, as we can see in Figure 4 there is not closed orbits for the Hulthen potential.

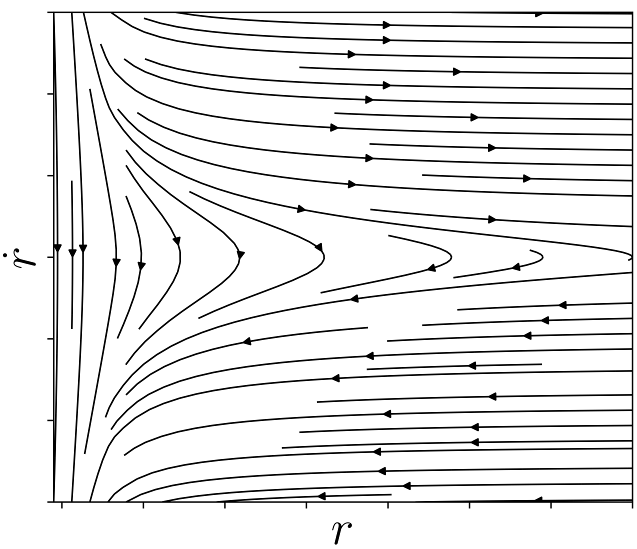

Figure 4 – Integral curves or flow lines of the dynamical vector field in the velocity phase space for the Hulthen potential.

Finally, for the case with angular momentum, applying the canonical transformation to the Hamiltonian (2.26) we obtain the Hamiltonian for the Manning-Rosen potential [34, 37]:

$$K_\gamma = \frac{P^2}{2\mu} + \frac{\ell^2\gamma^2 e^{-2\gamma Q}}{2\mu(1-e^{-\gamma Q})^2} - \frac{\gamma e^{-\gamma Q}}{(1-e^{-\gamma Q})}, \tag{2.30}$$

the dynamical vector field for this system is

$$\Delta_K = \frac{P}{\mu}\frac{\partial}{\partial Q} + \left(\frac{\ell^2\gamma^3 e^{\gamma Q}}{\mu(e^{\gamma Q}-1)^3} - \frac{\gamma^2 e^{\gamma Q}}{(e^{\gamma Q}-1)^2}\right)\frac{\partial}{\partial P} \tag{2.31}$$

and defines the integral curves in the phase space, as we can see in Figure 5 there is closed and parabolic orbits.

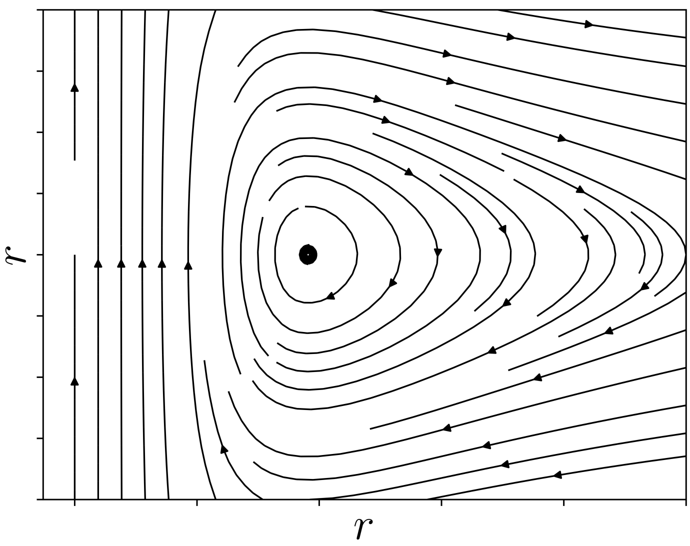

Figure 5 – Integral curves or flow lines of the dynamical vector field in the velocity phase space for PDM radial potential in the curved space.

2.3 Curved space approach

The aim of this section is to describe the PDM problem in a different way, by introducing a more suitable mathematical formalism based on the connection with the curved space. In this approach, the mass of the particle is no longer dependent on the position, instead we have a particle with constant mass m_0 moving in a curved space [10]. We shall describe the dynamics of this particle by the following one-dimensional free particle Lagrangian

$$L = \frac{1}{2} m_0 \, g_{ij}(x) \, \dot{x}_i \dot{x}_j, \tag{2.32}$$

where $g_{ij}(x)$ is the metric tensor that in general is a function of the coordinates and encodes the information of how to measure distances, areas and volumes in the curved space through the associated line element

$$ds^2 = g_{ij}(x)\,dx^i dx^j = g(x)\,dx \otimes dx. \tag{2.33}$$

As we can see in (2.32) the metric tensor allows the definition of dot product between vectors belonging to the tangent space TQ, for this reason this tensor must be symmetric $g_{ij}(x) = g_{ji}(x)$ and non-degenerate $\det(g_{ij}) \neq 0$ [27]. The metric determines the geometry of the space, but different metrics may well determine the same geometry, namely just metrics that are related by a change of variables, for example for three-dimensional Euclidean space we have that the line element can be written in Cartesian and spherical coordinates as

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2, \\ ds^2 &= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \end{aligned}$$

where we can identify the components of the metric tensor for the Euclidean space in both representations coordinates:

$$\begin{aligned} g_{xx} &= 1, \qquad & g_{yy} &= 1, \qquad & g_{zz} &= 1, \\ g_{rr} &= 1, \qquad & g_{\theta\theta} &= r^2, \qquad & g_{\phi\phi} &= r^2\sin^2\theta. \end{aligned}$$

This practical example gives an insight of how the geometry of the space is determined by the metric tensor, nevertheless it does not reveal any information about the curvature of the space.

In order to obtain the PDM Lagrangian (2.17) through the geometric approach, the metric tensor in (2.32) must be of the form:

$$g_{ij}(x) = \frac{\delta_{ij}}{h^2(x)} = \frac{\delta_{ij}}{(1+\gamma x)^2}, \tag{2.34}$$

where $h(x)$ is the deforming function of the metric, at this point is evident the relation between this function and the deforming function of the position dependent mass (2.6). With this metric, we are able to write the equation (2.17) as the Lagrangian for a free particle with constant mass m_0 moving in a curved space:

$$L_\gamma = \frac{1}{2}\frac{m_0\delta_{ij}}{(1+\gamma x)^2}\,\dot{x}_i\dot{x}_j. \tag{2.35}$$

As we know from the previous section the Lagrangian L_γ has a hidden symmetry related to the deformed translation $x \to \varepsilon(1+\gamma x)$, however, this invariance was intuitively determined. In order to determine the conserved quantity analytically, we are going to introduce the concepts of Lie derivative and the Killing vector field [27], which are used in general relativity to obtain symmetries from the metric tensor.

The symmetries of the metric tensor $g(x)$, or so-called isometries, are determined by identifying all possible vector fields along which the metric tensor remains invariant, namely, the metric tensor is invariant under the action of a vector field X if the Lie derivative $\mathscr{L}$ with respect this field vanish

$$\mathscr{L}_X\, g(x) = 0. \tag{2.36}$$

All vector fields that satisfy this property are called Killing vectors or infinitesimal generators of isometries. This vector fields plays a crucial role in our formalism because they are the conserved quantities associated to the infinitesimal deformed translation symmetry and provide the Noether momenta for the system [20, 18].

In order to determine the symmetry of the metric (2.34) we suppose that the most general Killing vector field is of the form

$$X = k(x)\,\frac{d}{dx}, \tag{2.37}$$

where $k(x)$ is a coefficient that depends of the coordinates. Determining the Lie derivative of the metric tensor $g(x)$ along the vector field X we obtain [22]

$$\begin{aligned}\mathscr{L}_X\, g(x) &= k(x)\,\frac{d}{dx}\left(\frac{1}{(1+\gamma x)^2}\,dx\otimes dx\right)\\ &= k(x)\,\frac{d}{dx}\left(\frac{1}{(1+\gamma x)^2}\right)dx\otimes dx + \left(\frac{2}{(1+\gamma x)^2}\right)\frac{dk(x)}{dx}\,dx\otimes dx.\end{aligned}$$

We can see that the unique vector field X that leaves invariant the metric, namely, that satisfy the condition (2.36) is

$$X = (1+\gamma x)\,\frac{d}{dx}, \tag{2.38}$$

this quantity is the conserved current associated to the translational invariance of the Lagrangian (2.17) that we weren't able to determine in the position dependent mass approach. For this reason, is naturally associated with the momentum of the particle in the curved space

$$P_\gamma = (1+\gamma x)\frac{d}{dx}, \tag{2.39}$$

this vector field is called Noether momenta and plays the central role in the transition to the quantum formalism, since the usual quantization prescription can be applied to its components. Additionally, for the quantization process, we need to define an appropriate Hilbert space endowed with an invariant (under the Killing vector action) measure dx_γ [20], this is determined by the metric and given by

$$dx_\gamma = \frac{dx}{1+\gamma x}. \tag{2.40}$$

This measure also provides a natural way to connect the coordinates of the Euclidean space, in which is defined the PDM problem, with the coordinates of the curved space. More precisely, we can solve the differential equation (2.40) with boundary condition $x_\gamma(0) = 0$, and find the relation [25]

$$x_\gamma = \frac{\ln(1+\gamma x)}{\gamma}, \tag{2.41}$$

where the coordinates of the curved space x_γ are written as a function of the Euclidean space x. In the same manner, the inverse transformation is

$$x = \frac{e^{\gamma x_\gamma} - 1}{\gamma}, \tag{2.42}$$

this equation has the same form of the solution for the free particle (2.13), corresponding to the case when $t = 0$. It is not a coincidence, in fact, this relation allows to determine the constants C_1 and C_2 through initial conditions, since as we can see these constants are related to the curved space. Setting the initial position in the curved space as $x_\gamma(x_0) = C_2$ and the initial speed as $V_0 = \sqrt{2E/m_0}$ the solution for the free particle in the Euclidean space is:

$$x(t) = \frac{(1+\gamma x_0)}{\gamma}\left(e^{\sqrt{\frac{2E}{m_0}}\gamma t} - 1\right). \tag{2.43}$$

One last aspect to mention is that the equation of motion (2.7) and the tidal force (2.11) can be obtained from the one-dimensional geodesic equation, in which, the trajectory of a free particle in a curved space is determined using only geometric properties:

$$\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l = 0, \tag{2.44}$$

where Γ^i_{kl} is the Christoffel symbol given by $\Gamma^i_{kl} = g^{kl}\partial_i g_{kl}/2$, and g^{kl} are the components of the inverse metric tensor. This procedure seems more difficult, nevertheless in higher dimensions is the only way to proceed.

In this section we have shown a mathematical formalism based on geometrical concepts of the space, in which symmetries, conserved quantities, equations of motion and forces can be derived from the metric tensor, but until now, this has been a classical treatment. Therefore, in the section that follows, we consider the implications of this formalism in the quantum regime, beginning with the quantization of the Noether momenta (2.39).

2.4 Quantization

The aim of this section is to define the quantum formulation of the PDM using the results of the previous section. First of all, we must define the Hilbert space of square integrable functions $L^2(M)$ with respect to an appropriate measure dx, where M is the curved space in which the functions are defined. This part of the problem is essential because allows to determine the Hermitian and unitary operators that can be defined in this space.

Since the usual Lebesgue measure dx is not invariant under the action of the Killing vector field (2.38), we shall use the invariant measure dx_γ defined in the previous section and described by the equation (2.40). This means that the Hilbert space of the quantum system is the linear space of square integrable functions on M with respect to the appropriate measure $L^2(M, dx_\gamma)$. This automatically implies that the first-order linear operator (2.39) is skewsymmetric and that the operator $\hat{P}_\gamma$ representing the quantum version of the Noether momentum is selfadjoint [21].

To obtain the quantum formulation of the problem, it is important to define the Hamiltonian. This is determined from the Lagrangian (2.17) by first obtaining the canonical momentum

$$\frac{\partial L_\gamma}{\partial \dot{x}} = \frac{m_0 \dot{x}}{(1+\gamma x)^2}, \tag{2.45}$$

and using the Legendre transformation

$$H_\gamma = \dot{x}\frac{\partial L_\gamma}{\partial \dot{x}} - L_\gamma. \tag{2.46}$$

Then is obtained the classical Hamiltonian for the one-dimensional free particle with PDM in Euclidean coordinates

$$H_\gamma = \frac{(1+\gamma x)^2}{2m_0} p^2. \tag{2.47}$$

Defining the momentum $P_\gamma = (1+\gamma x)p$, the kinetic term in the above Hamiltonian take the usual form

$$H_\gamma = \frac{P_\gamma^2}{2m_0}. \tag{2.48}$$

This is an expected result since the Noether momenta (2.39) is the natural generator of translational symmetry (invariance of the metric) in the curved space. Therefore, the quantization of the above Hamiltonian is reduced to finding a method to quantize the Noether momenta. Following the process described in [20] the transition from the classical Noether momenta to the quantum operator is given by

$$P_\gamma \mapsto \hat{P}_\gamma = (1+\gamma x)\left(-i\hbar\frac{d}{dx}\right), \tag{2.49}$$

as we pointed out, this quantum operator is self-adjoin in the Hilbert space $L^2(M, dx_\gamma)$, in other words, the momentum operator $\hat{P}_\gamma$ is Hermitian with regard to the following scalar product:

$$(\phi, \psi) = \int dx_\gamma \phi^*(x_\gamma)\psi(x_\gamma) = \int \frac{dx}{(1+\gamma x)}\phi^*(x)\psi(x). \tag{2.50}$$

Consequently, the Hamiltonian (2.48) is represented by the following hermitian operator

$$\begin{aligned}\hat{H}_\gamma &= -\frac{\hbar^2}{2m_0}\left((1+\gamma x)\frac{d}{dx}\right)\left((1+\gamma x)\frac{d}{dx}\right)\\ &= -\frac{\hbar^2}{2m_0}\left[(1+\gamma x)\left(\gamma\frac{d}{dx}+(1+\gamma x)\frac{d^2}{dx^2}\right)\right]\\ &= -(1+\gamma x)^2\frac{\hbar^2}{2m_0}\frac{d^2}{dx^2} - \gamma(1+\gamma x)\frac{\hbar^2}{2m_0}\frac{d}{dx},\end{aligned}$$

and the Schrödinger equation $\hat{H}_\gamma\psi(x) = E\psi(x)$ becomes

$$-(1+\gamma x)^2\frac{\hbar^2}{2m_0}\frac{d^2}{dx^2}\psi(x) - \gamma(1+\gamma x)\frac{\hbar^2}{2m_0}\frac{d}{dx}\psi(x) + V(x)\psi(x,t) = E\psi(x). \tag{2.51}$$

As we had expected this equation of motion is consistent with the results obtained using the non-additive translation operator [25]. Other important consequence that comes out after the quantization process is the connection with the formalism of deformed algebras [12, 10]. Interestingly, in the geometric approach the deformation of the commutation relations arise naturally when the commutator between the position $\hat{x}$ and the Noether momenta is calculated:

$$[\hat{x}, \hat{P}_\gamma] = i\hbar(1+\gamma x), \tag{2.52}$$

this result makes it possible to address the problem of deformed algebras in quantum mechanics from a geometric perspective, depending only on the properties of the space.

Returning to the Schrödinger equation, it is more convenient to express (2.51) in the canonical coordinates of the curved space x_γ (2.41). Using the measure (2.40) as the connection between the two spaces to find the following relations

$$\frac{d}{dx} = \frac{dx_\gamma}{dx}\frac{d}{dx_\gamma} = \frac{1}{(1+\gamma x)}\frac{d}{dx_\gamma},$$
$$\frac{d^2}{dx^2} = \left(\frac{dx_\gamma}{dx}\right)^2 \frac{d^2}{dx_\gamma^2} + \left(\frac{d^2 x_\gamma}{dx^2}\right)\frac{d}{dx_\gamma} = \frac{1}{(1+\gamma x)^2}\frac{d^2}{dx_\gamma^2} - \frac{\gamma}{(1+\gamma x)^2}\frac{d}{dx_\gamma},$$

then the Schrödinger equation rewritten in terms of the coordinates of the curved space recovers the usual form

$$-\frac{\hbar^2}{2m_0}\frac{d^2}{dx_\gamma^2}\phi(x_\gamma) + V_{eff}(x_\gamma)\phi(x_\gamma) = E\phi(x_\gamma) \tag{2.53}$$

where $\phi(x_\gamma, t) = \psi(x(x_\gamma), t)$ and $V_{eff}(x_\gamma) = V(x(x_\gamma))$ is called the effective potential. Strikingly, we have arrived at the same results reported in [23, 25] using a geometric approach based on symmetry properties. In the next section we will solve two archetypal potentials in quantum mechanics using the Schrödinger equation (2.53).

3 APPLICATIONS

In this chapter we apply the results obtained for the geometric formalism in the quantum regime. We analyse the PDM problem subjected to certain potentials using the Schrödinger equation (2.53) to find the corresponding deformed eigenvectors and eigenfunctions in each case.

3.1 Case $f(x) = (1+\gamma x)$

3.1.1 Free particle

The Hamiltonian for the free particle in the deformed space is

$$-(1+\gamma x)^2 \frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} \psi(x) - \gamma(1+\gamma x) \frac{\hbar^2}{2m_0} \frac{d}{dx} \psi(x) = E\psi(x), \tag{3.1}$$

with the change of variables (2.41) we can write the Schrödinger equation in the familiar form

$$-\frac{\hbar^2}{2m_0} \frac{d^2}{dx_\gamma^2} \phi(x_\gamma) = E\phi(x_\gamma). \tag{3.2}$$

The solution of this differential equation is already known and correspond to

$$\phi(x_\gamma) = A\, e^{\pm ikx_\gamma}, \tag{3.3}$$

where $k = \sqrt{2mE/\hbar^2}$ is a continuous variable regarding the particle's wave vector. In the coordinates of the curved space the solution corresponds to a plane wave moving with energy $E = \frac{\hbar^2 k^2}{2m_0}$ independent of the curvature parameter γ. From the transformation (2.41) the solution in terms of the Euclidean coordinates is given by

$$\phi(x_\gamma) = \psi(x) = A\, e^{\pm \frac{ik}{\gamma} Ln(1+\gamma x)}, \tag{3.4}$$

corresponding to a deformed plane wave. The usual solution for the case of zero curvature, is recovered in the limit when $\gamma \to 0$. Although the wave function is not normalizable, always is possible to construct a localized wave packet by superposition of waves with slightly different wavelengths

$$\psi_{wp}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(k) e^{\frac{ik}{\gamma} Ln(1+\gamma x)} dk, \tag{3.5}$$

where $\varphi(k)$ is the Fourier transform of $\psi_{wp}(x)$ and can be written as

$$\varphi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_{wp}(x) e^{-\frac{ik}{\gamma} Ln(1+\gamma x)} \frac{dx}{(1+\gamma x)}, \tag{3.6}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_{wp}(x_\gamma) e^{-ikx_\gamma} dx_\gamma. \tag{3.7}$$

3.1.2 Infinite Well

The problem of the infinite well potential can be considered in two analogous ways. First, as a particle with mass $m = m_0/(1+\gamma x)^2$ confined to move inside an infinitely deep asymmetric potential $V(x)$, this approach was discussed by Costa *et al* using the non-additive translation operator. Second, as a particle of constant mass m_0, moving in a curved space described by a metric tensor $g(x) = (1+\gamma x)^{-2} dx \otimes dx$ and confined to move inside the potential

$$V(x_\gamma) = \begin{cases} +\infty, & x_\gamma < 0, \\ 0, & 0 \leq x_\gamma \leq L_\gamma, \\ +\infty, & x_\gamma > L_\gamma, \end{cases} \tag{3.8}$$

where $L_\gamma = \ln(1+\gamma L)/\gamma$ is the deformed wide, which can be dilated/contracted depending of the curvature γ of the space. The solution for the Schrödinger equation with potential (3.8) is

$$\phi(x_\gamma) = A\sin(kx_\gamma) + B\cos(kx_\gamma), \tag{3.9}$$

where $k = \sqrt{2mE/\hbar^2}$. Applying the boundary conditions: $\phi(0) = 0$ gives $B = 0$, while the condition $\phi(L_\gamma) = 0$ gives the wave vector

$$k_n = \frac{n\pi}{L_\gamma} = \frac{n\pi\gamma}{\ln(1+\gamma L)} \qquad (n = 1, 2, 3, ...), \tag{3.10}$$

this condition determines the quantization of the energy

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2m_0 L_\gamma^2} = \frac{\hbar^2 n^2 \pi^2 \gamma^2}{2m_0 \ln^2(1+\gamma L)}. \quad (3.11)$$

On the other hand, the constant A is determined by normalizing the wave function $\phi_n(x_\gamma) = A\sin(n\pi x_\gamma/L_\gamma)$:

$$1 = \int_0^{L_\gamma} |\phi_n(x_\gamma)|^2 dx_\gamma \qquad \Rightarrow \qquad A = \sqrt{\frac{2}{L_\gamma}}, \quad (3.12)$$

hence, the wave function written in both coordinate systems is

$$\begin{aligned} \phi_n(x_\gamma) &= \sqrt{\frac{2}{L_\gamma}} \sin\left(\frac{n\pi x_\gamma}{L_\gamma}\right), \\ \psi_n(x) &= \sqrt{\frac{2\gamma}{\ln(1+\gamma L)}} \sin\left(\frac{n\pi \ln(1+\gamma x)}{\ln(1+\gamma L)}\right). \end{aligned}$$

In the usual infinite square well potential the wave functions are symmetric or antisymmetric about the midpoint $x = L/2$, however, for the potential $V(x_\gamma)$ this kind of symmetry is broken due the curvature of the space. To visualize this asymmetric behaviour in the wave function we first determine the probability density $\rho(x)$ from the normalization condition (3.12)

$$1 = \int_0^{L_\gamma} |\phi_n(x_\gamma)|^2 dx_\gamma = \int_0^{L} \frac{|\psi_n(x)|^2}{(1+\gamma x)} dx \qquad \Rightarrow \qquad \rho(x) = \frac{|\psi_n(x)|^2}{(1+\gamma x)}. \quad (3.13)$$

As a result, in Figure 6 we can see the probability densities of the four states of lowest energy for different values of the curvature parameter γ. The case of zero curvature corresponds to the Euclidean space where the solutions are symmetric and antisymmetric about the midpoint. As the curvature increase, it is more probable to find the particles around the origin. These results seem to be consistent with other research which found the same behaviour through the modification of the non-additive translation operator [39].

The expectation values for a given operator $\hat{O}$ are determined considering that the following correspondence between both spaces holds [25]

$$\langle \hat{O}_x \rangle_x = \langle \hat{O}_{x_\gamma} \rangle_{x_\gamma} = \int \phi_n^*(x_\gamma) \hat{O}_{x_\gamma} \phi_n(x_\gamma) dx_\gamma. \quad (3.14)$$

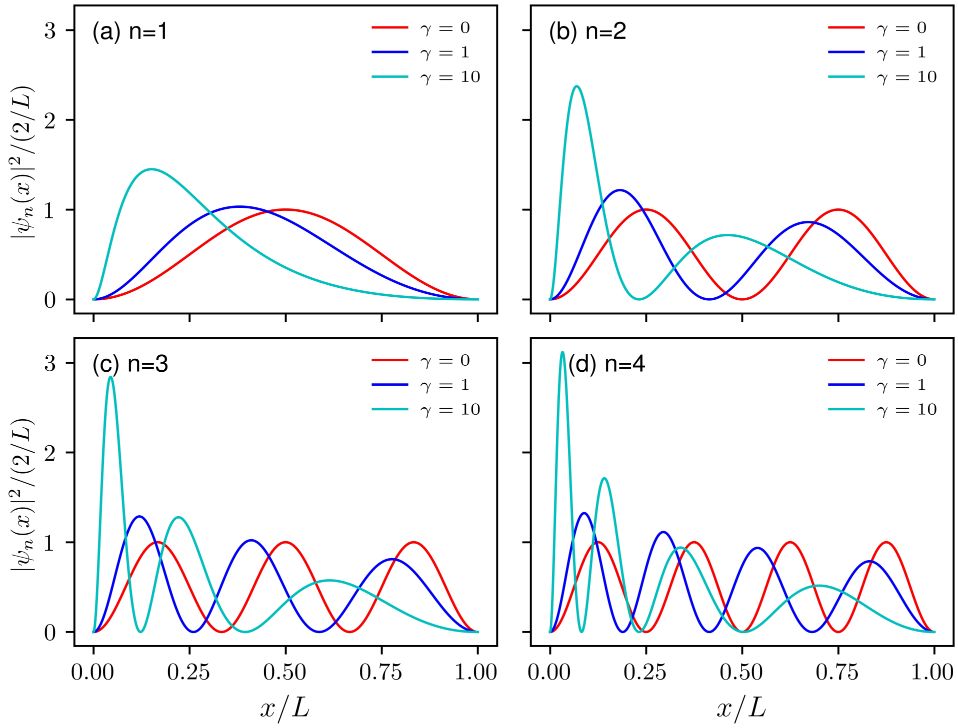

Figure 6 – Probability densities $|\psi_n(x)|^2/(1+\gamma x)$ for a particle confined in an infinite quantum well in curved space for different values of the curvature γ. (a): n=1 (ground state), (b): n=2, (c): n=3, (d): n=4.

For instance, the expectation value of the position can be straightforwardly determined in the curved space as

$$\langle \hat{x}_\gamma \rangle = \int \phi_n^*(x_\gamma)\hat{x}_\gamma \phi_n(x_\gamma) dx_\gamma = \frac{L_\gamma}{2} = \frac{\ln(1+\gamma L)}{2\gamma}. \tag{3.15}$$

However, this result is not in accordance with the reported in previous studies [23, 39, 24] which have suggested that the expectation value of the position depends on the quantum number n. Figure 7 shows the behaviour of the average position $\langle x \rangle$ as a function of the curvature γ, as expected, when the curvature is zero the average position is $L/2$. Also is possible to show that average of the momentum is zero $\langle \hat{p} \rangle = 0$.

As an example of the deformed algebra induced by the PDM we can determine the uncertainty relation between the position and momentum operators. Using the general expression for the uncertainty between two observables:

$$\Delta \hat{x} \Delta \hat{P}_\gamma \geq \frac{1}{2} |\langle [\hat{x}, \hat{P}_\gamma] \rangle|, \tag{3.16}$$

with the value of the commutator (2.52) previously determined the uncertainty for the infinite well potential is

$$\Delta\hat{x}\Delta\hat{P}_{\gamma} \geq \frac{\hbar}{2}\left(1+\frac{\ln(1+\gamma L)}{2}\right), \tag{3.17}$$

the uncertainty depends only of the curvature parameter γ and the usual result $\Delta\hat{x}\Delta\hat{P}_{\gamma} \geq \frac{\hbar}{2}$ is recovered in the limit when $\gamma \to 0$.

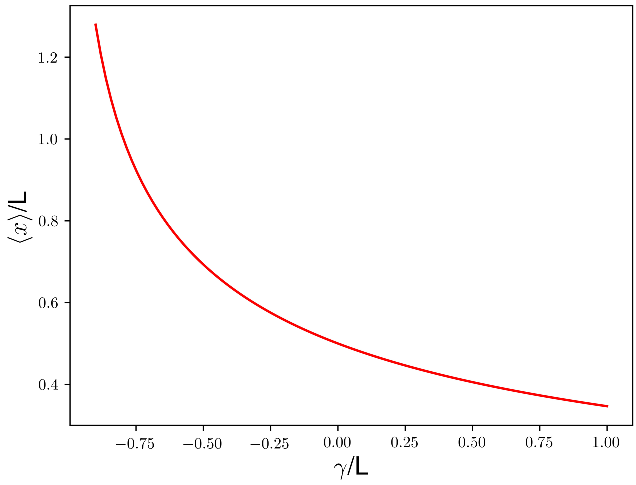

Figure 7 – Average position $\langle x \rangle$ in function of the curvature parameter γ for a particle confined in an infinite well. For the flat space $\gamma = 0$ the usual result $\langle x \rangle = L/2$ is obtained.

The above results can be generalized to two and three dimensions since the position operator still commute in the other directions.

4 DEFORMED SPHERICAL SPACE

In this chapter we are going to generalize the geometric formalism to three dimensions, for the case of spherical coordinates, in order to construct a formalism to study central-type potentials in the context of deformed spaces and PDM systems. We begin applying a radial deformation function to the momentum operator for later define the kinetic operator of the deformed Schrödinger equation.

4.1 Deformed Schrödinger equation in spherical coordinates

Generalizing the concept of Noether momenta (2.39) to spherical coordinates with an arbitrary deformation function

$$\hat{P}_\gamma = -i\hbar f(r,\theta,\phi)\nabla, \tag{4.1}$$

where ∇ is the gradient operator in spherical coordinates. Considering the case when the deformation is only in the radial direction we have that the momentum operator is

$$\hat{P}_\gamma = -i\hbar f(r)\left(\vec{r}\frac{\partial}{\partial r} + \vec{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right), \tag{4.2}$$

with $(\vec{r},\vec{\theta},\vec{\phi})$ the unitary vectors. Then the kinetic operator can be written as

$$\frac{\hat{P}_\gamma^2}{2m} = -\hbar^2\left[\vec{r}f(r)\frac{\partial}{\partial r}\cdot\left(\frac{\hat{P}_\gamma}{-i\hbar}\right) + \vec{\theta}\frac{f(r)}{r}\frac{\partial}{\partial\theta}\cdot\left(\frac{\hat{P}_\gamma}{-i\hbar}\right) + \vec{\phi}\frac{f(r)}{r\sin\theta}\frac{\partial}{\partial\phi}\cdot\left(\frac{\hat{P}_\gamma}{-i\hbar}\right)\right]. \tag{4.3}$$

Solving the product in the three terms we obtain:

$$\begin{aligned}\frac{\hat{P}_\gamma^2}{2m} = -\frac{\hbar^2}{2m}\Bigg\{ & f(r)\vec{r}\cdot\left[\vec{r}\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial r} + \vec{r}f(r)\frac{\partial^2}{\partial r^2} + \vec{\theta}\frac{1}{r}\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial\theta} - \vec{\theta}\frac{f(r)}{r^2}\frac{\partial}{\partial\theta} + \vec{\theta}\frac{f(r)}{r}\frac{\partial^2}{\partial r\partial\theta} + \right.\\ & \left.\vec{\phi}\frac{1}{r\sin\theta}\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial\phi} - \vec{\phi}\frac{f(r)}{r^2\sin\theta}\frac{\partial}{\partial\phi} + \vec{\phi}\frac{f(r)}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}\right] + \frac{f(r)}{r}\vec{\theta}\cdot\left[\vec{\theta}f(r)\frac{\partial}{\partial r} + \vec{r}f(r)\frac{\partial^2}{\partial r\partial\theta}\right.\\ & \left.-\vec{r}\frac{f(r)}{r}\frac{\partial}{\partial\theta} + \vec{\theta}\frac{f(r)}{r}\frac{\partial^2}{\partial\theta^2} + \vec{\phi}\frac{f(r)}{r\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\right] + \frac{f(r)}{r\sin\theta}\vec{\phi}\cdot\left[\vec{\phi}f(r)\sin\theta\frac{\partial}{\partial r} + \vec{r}f(r)\frac{\partial^2}{\partial r\partial\phi} + \right.\\ & \left.\vec{\phi}\frac{f(r)\cos\theta}{r}\frac{\partial}{\partial\theta} + \vec{\theta}\frac{f(r)}{r}\frac{\partial^2}{\partial\theta\partial\phi} - \vec{r}\frac{f(r)}{r}\frac{\partial}{\partial\phi} - \vec{\theta}\frac{f(r)\cot\theta}{r}\frac{\partial}{\partial\phi} + \vec{\phi}\frac{f(r)}{r\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]\Bigg\},\end{aligned}$$

using the orthogonality of the unitary vectors only survive the following terms

$$\frac{\hat{P}_{\gamma}^{2}}{2m}=-\frac{\hbar^{2}}{2m}\left[f^{2}(r)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r}\frac{\partial}{\partial r}\right)+f(r)\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial r}+\frac{f^{2}(r)}{r^{2}}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^{2}\theta}\frac{\partial^{2}}{\partial\phi^{2}}\right)\right],$$

knowing the expressions for the radial momentum $\hat{P}_r(r)$ and angular momentum $\hat{L}(\theta,\phi)$ in the normal space, which are given by

$$\begin{aligned}\frac{\hat{P}_r^{2}(r)}{\hbar^{2}}&=-\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r}\frac{\partial}{\partial r}\right),\\ \frac{\hat{L}^{2}(\theta,\phi)}{\hbar^{2}}&=-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^{2}\theta}\frac{\partial^{2}}{\partial\phi^{2}}\right),\end{aligned}$$

we can finally write the deformed kinetic operator as

$$\frac{\hat{P}_{\gamma}^{2}}{2m}=f^{2}(r)\frac{\hat{P}_r^{2}(r)}{2m}-\frac{\hbar^{2}}{2m}f(r)\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial r}+f^{2}(r)\frac{\hat{L}^{2}(\theta,\phi)}{2mr^{2}}. \tag{4.4}$$

Now is possible to define the Schrödinger equation for central potentials in systems with PDM, here the function $f(r)$ define the form of the mass and the curvature of the induced deformed space. The time-independent equation of motion for a particle with radial PDM moving under the action of a central potential is

$$\left(f^{2}(r)\frac{\hat{P}_r^{2}(r)}{2m}-\frac{\hbar^{2}}{2m}f(r)\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial r}+f^{2}(r)\frac{\hat{L}^{2}(\theta,\phi)}{2mr^{2}}\right)\psi(r,\theta,\phi)+V(r)\psi(r,\theta,\phi)=E\psi(r,\theta,\phi), \tag{4.5}$$

writing explicitly the expression for the radial momentum, we obtain the most general Schrödinger equation for a particle under a radial deformation:

$$-\frac{\hbar^{2}}{2m}\left(f^{2}(r)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r}\frac{\partial}{\partial r}\right)+f(r)\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial r}\right)\psi+f^{2}(r)\frac{\hat{L}^{2}(\theta,\phi)}{2mr^{2}}\psi+V(r)\psi=E\psi. \tag{4.6}$$

Using the method of separation of variables in order to solve the equation (4.6), we propose a wave function of the form $\psi(r,\theta,\phi)=R(r)Y_l^m(\theta,\phi)$, where $R(r)$ is a radial wave function and $Y_l^m(\theta,\phi)$ are the spherical harmonics; with the fact that the latter are eigenfunctions of the $\hat{L}^2(\theta,\phi)$ operator we can replace it with its known eigenvalue $\ell(\ell+1)$ and separate into the radial and angular contributions. Since the angular equation is equal that for the non-deformed case, we are going to concentrate only into the radial equation

$$\left[-\frac{\hbar^2}{2m}\left(f^2(r)\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+f(r)f'(r)\frac{d}{dr}\right)+\frac{f^2(r)\ell(\ell+1)\hbar^2}{2mr^2}+V(r)\right]R(r)=ER(r). \tag{4.7}$$

The radial Schrödinger equation simplifies somewhat if we make a change of variables from $R(r)$ to the function

$$R(r)=\frac{U(r)}{r}, \tag{4.8}$$

where $U(r)$ is the known reduced radial wave function [40, 41], this function is used to eliminate the first derivative term in the kinetic operator. Using the chain rule we obtain

$$\begin{aligned}\frac{dR(r)}{dr}&=\frac{1}{r}\left(\frac{dU(r)}{dr}-\frac{U(r)}{r}\right),\\ \frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}&=\frac{1}{r}\frac{d^2U(r)}{dr^2},\end{aligned}$$

replacing in equation (4.7) we have the following radial Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\left(f^2(r)\frac{d^2}{dr^2}+f(r)f'(r)\left(\frac{d}{dr}-\frac{1}{r}\right)\right)+\frac{f^2(r)\ell(\ell+1)\hbar^2}{2mr^2}+V(r)\right]U(r)=EU(r). \tag{4.9}$$

In the next section, we shall explore the instance when the deformation function is $f(r)=(1+\gamma r)$, as we can see in equation (2.41) this particular case is very important since allows us to explore all the central potentials of exponential-type that exist in physics from first principles.

4.2 Case $f(r)=(1+\gamma r)$

We already have studied this deformation for the one-dimensional problem, for this case although the problem is in 3 dimensions, we can reduce the problem to one dimensions using the conservation of angular momentum. For this reason we can replace $f(r) = (1+\gamma r)$ into the equation (4.9) and obtain the deformed radial equation of motion for this particular space

$$\left[-\frac{\hbar^2}{2m}\left((1+\gamma r)^2\frac{d^2}{dr^2}+\gamma(1+\gamma r)\left(\frac{d}{dr}-\frac{1}{r}\right)\right)+\frac{(1+\gamma r)^2\ell(\ell+1)\hbar^2}{2mr^2}+V(r)\right]U(r) = EU(r). \tag{4.10}$$

Following the same procedure for the one-dimensional problem, we can propose a change of variables in the radial variable given by

$$r(\eta) = \frac{e^{\gamma\eta}-1}{\gamma}, \tag{4.11}$$

where η is the canonical radial coordinate in the deformed space. Then, the derivatives are

$$\frac{d}{dr} = \frac{1}{e^{\gamma\eta}}\frac{d}{d\eta},$$
$$\frac{d^2}{dr^2} = \frac{1}{e^{2\gamma\eta}}\frac{d^2}{d\eta^2} - \frac{\gamma}{e^{2\gamma\eta}}\frac{d}{d\eta}.$$

Plugging this change of variables into the Schrödinger equation (4.10) we obtain the following equation of motion in the η variable

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{d\eta^2}+\frac{\hbar^2}{2m}\frac{\gamma^2 e^{\gamma\eta}}{e^{\gamma\eta}-1}+\frac{\hbar^2}{2m}\frac{\ell(\ell+1)\gamma^2 e^{2\gamma\eta}}{(e^{\gamma\eta}-1)^2}+V(\eta)\right)\varphi(\eta) = E\varphi(\eta), \tag{4.12}$$

where $V(\eta) = V(r(\eta))$ and $\varphi(\eta) = U(r(\eta))$ are the external potential and the radial wave function in the coordinates of the deformed space. If we define the effective potential as

$$V_{eff}(\eta) = \frac{\hbar^2}{2m}\frac{\gamma^2 e^{\gamma\eta}}{e^{\gamma\eta}-1}+\frac{\hbar^2}{2m}\frac{\ell(\ell+1)\gamma^2 e^{2\gamma\eta}}{(e^{\gamma\eta}-1)^2}+V(\eta), \tag{4.13}$$

the equation (4.12) reduces to the usual time-independent Schrödinger equation in one dimensions

$$-\frac{\hbar^2}{2m}\frac{d^2}{d\eta^2}\varphi(\eta)+V_{eff}(\eta)\varphi(\eta) = E\varphi(\eta). \tag{4.14}$$

We can identify the second term into the effective potential (4.13) as the related with the centrifugal potential in the non-deformed case. In the next section we are going to study two different potentials used in experimental physics from a first principles approach, drawing a direct connection with the effective potential.

4.2.1 Hulthen Potential

The Hulthén potential [42] is one of the important short-range potentials in physics. The potential has been used in nuclear [43] and particle physics [44], atomic physics [45], solid-state physics [46], and its bound state and scattering properties have been investigated by a variety of techniques [47, 48, 49]. This potential has been used to describe screening problems and is related to the screened Coulomb potential [28]. The Hulthen potential is defined as

$$V_H(r) = -\frac{Zq^2\delta e^{-\delta r}}{1-e^{-\delta r}}, \tag{4.15}$$

where Z is a constant, q is the charge of the electron and δ is the screening parameter. Now, taking the case when the angular momentum vanish $\ell = 0$ and the external potential is zero in the effective potential (4.13), this becomes

$$V_{eff}(\eta) = -\frac{\hbar^2}{2m}\frac{\gamma^2 e^{\gamma\eta}}{1-e^{\gamma\eta}}, \tag{4.16}$$

strikingly, by identifying $Zq^2\delta = \hbar^2\gamma^2/2m$ and $\gamma = -\delta$, we conclude that equation (4.16) corresponds exactly to the expression (4.15) for the Hulthen potential. Since the effective potential (4.16) also can be obtained by writing the Coulomb potential $1/r$ in the η coordinate given by the change of variables (4.11), we can say that the Coulomb potential in deformed space is equivalent to the Hulthen potential in regular space. To the best of our knowledge, this is the first time that a connection based on first principles is provided between this two potentials.

In Figure 8, we can see the behaviour of the Hulthen potential for different values of the screening parameter γ, when this parameter approach to zero we obtain the Coulomb potential. Also we can appreciate that for small values of the radial coordinate r, the Hulthen potential behaves like a Coulombic potential, whereas for large values of r it decreases exponentially, this

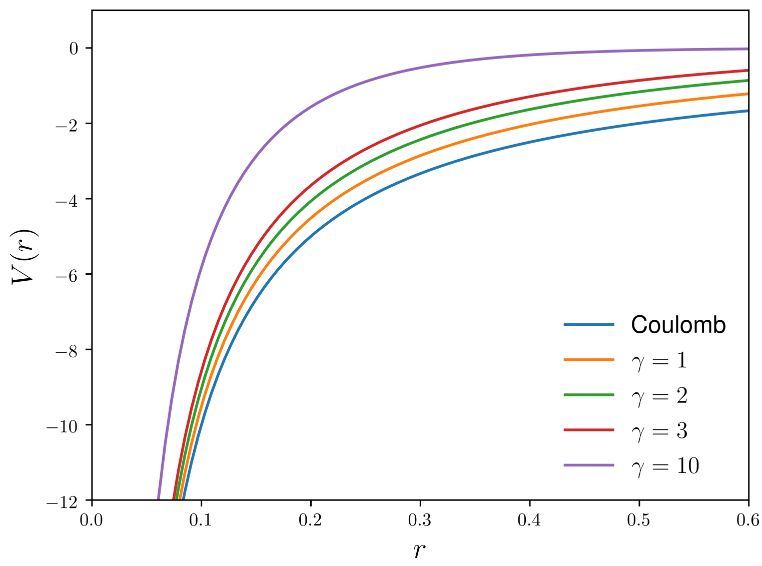

Figure 8 – Hulthen potential as a function of r for different values of the screening parameter γ

can be seen performing a Taylor expansion of the Hulthen potential

$$V_H(r) = -\frac{V_{H0}}{\delta r + \frac{\delta^2 r^2}{2!} + \frac{\delta^3 r^3}{3!} + \ldots}, \tag{4.17}$$

where $V_{H0} = Zq^2\delta$ is a constant.

4.2.2 Manning-Rosen Potential

The Manning-Rosen potential was proposed as an excellent description of the interaction between two atoms in diatomic molecules [34]. The potential has been used to describe systems like optical properties in spherical quantum dots [46] and pseudospin symmetry in nuclear physics [50]. This potential has also motivated numerous studies in the field of approximation techniques (perturbation calculations) of solution to the Schrödinger equation for central potentials with $\ell > 0$, and its bound state and scattering properties have been investigated by a variety of this techniques, as for example the differential equation approach [51], asymptotic iteration method (AIM) [52], Pekeris-type approximation [53], etc.

The Manning-Rosen potential is defined as

$$V_{M-R}(r) = \frac{\beta(\beta-1)\alpha^2 e^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{A\alpha^2 e^{-\alpha r}}{(1-e^{-\alpha r})}, \tag{4.18}$$

where A and β are two parameters, and α is the screening parameter. Now, taking the case when the angular momentum is $\ell > 0$ and the external potential is zero in the effective potential (4.13), this becomes

$$V_{eff}(\eta) = \frac{\hbar^2}{2m}\frac{\gamma^2 e^{\gamma\eta}}{e^{\gamma\eta}-1} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)\gamma^2 e^{2\gamma\eta}}{(e^{\gamma\eta}-1)^2}, \tag{4.19}$$

strikingly, by identifying $\beta(\beta-1) = \hbar^2\ell(\ell+1)/2m$, $A = \hbar^2/2m$ and $\gamma = -\alpha$, we conclude that the equation (4.19) corresponds exactly to the expression (4.18) for the Manning-Rosen potential.

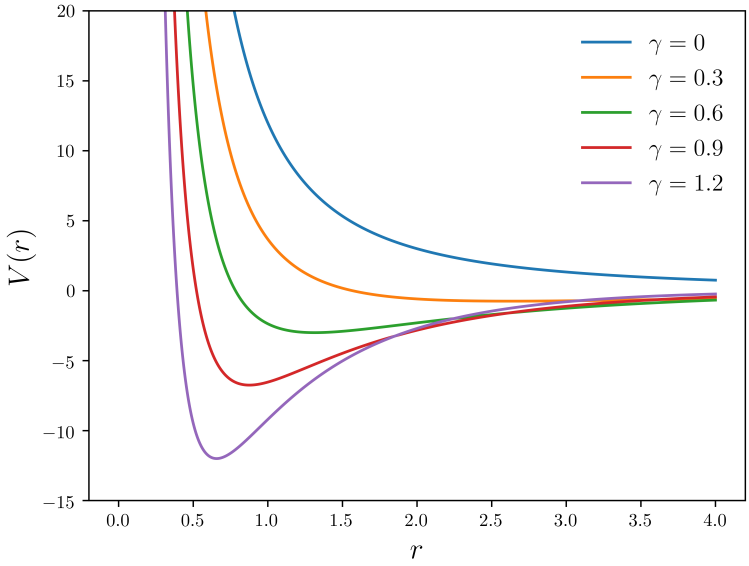

Figure 9 – Manning-Rosen potential as a function of r for different values of the screening parameter α, with $\beta = 4$ and $A = 20$. For this configuration of β and A the Manning-Rosen potential presents bound states.

We can see from Figure 9 that for certain configurations of the parameters β and A the Manning-Rosen potential presents bound states in contrast to Hulthen potential. Since for the case when $\beta = 0$ the Manning-Rosen potential reduces to Hulthen potential, we can conclude that equation (4.19) for the effective potential is equivalent to the Coulomb potential plus the centrifugal term in the deformed space.

As a final comment we see that Figure 5 and Figure 9 represents the same behaviour, since the first describe a classical PDM particle under the influence of the Coulomb potential (with the centrifugal potential) and the second describe an analogous behaviour in the quantum regime.

5 CONCLUSIONS

This work had two main purposes. First: to study the non-additive quantum mechanics from a geometric perspective, using the fact that the deformed translation operator $\hat{T}_\gamma(x)$ induces a non-euclidean space and therefore allows the introduction of concepts of differential geometry and geometric mechanics. Second: to define a spherical deformed space to study central exponential-type potentials from a first principles approach, specifically, the Hulthen and Manning-Rosen potentials.

In Chapter 2, we studied the PDM systems using geometric concepts employed to describe curved spaces, the connection between this two approaches allows us to formulate the problem from the Lagrangian and Hamiltonian perspective. We demonstrate that the kinetic term of the Hamiltonian for a system with PDM can be mapped to the kinetic term of the usual free particle through a canonical transformation. Using the curved space approach we find that the Noether momenta associated to the translational symmetry of the metric tensor (2.34) is the momentum operator $\hat{p}_\gamma$ of the Non-additive quantum mechanics proposed by R. N. Costa *et al* [23].

In Chapter 3, we use the deformed Schrödinger equation to describe two archetypical problems like the free particle and the infinite well potential. We find that the expectation value of the position in the infinite well problem is independent of the quantum number n, in contrast with the results reported in previous studies [23, 39, 24].

Finally, in Chapter 4, we perform a radial deformation in the three dimensional space in spherical coordinates, defining a natural space to study exponential-type potentials. In this space we study the Hulthen and Manning-Rosen potential from a first principles approach, finding a direct connection with the Coulomb problem, additionally, we obtain a natural quantum description of the screening phenomena.

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